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Strings in AdS_3 and the SL(2,R) WZW model. II: Euclidean black hole

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We consider the one-loop partition function for Euclidean BTZ black hole backgrounds or equivalently thermal AdS_3 backgrounds which are quotients of H_3 (Euclidean AdS_3). The one-loop partition function is modular invariant and we can read off the spectrum which is consistent to that found in hep-th/0001053. We see long strings and discrete states in agreement with the expectations. © *2001 American Institute of Physics.* [DOI: 10.1063/1.1377039]

I. INTRODUCTION

In this paper we continue the investigation started in Ref. 1 of the SL(2,R) WZW model describing string theory on $AdS_3 \times \mathcal{M}$. For other work on this model, see Ref. 2. Our motivation is to understand string theories in curved spacetimes where the metric component g_{00} is nontrivial, of which AdS_3 is the simplest example. Moreover, it is possible to construct black hole solutions as quotients of AdS_3,[3] so understanding string theory on AdS_3 would lead to an understanding of strings moving near black hole horizons.

In Ref. 1 the spectrum of the SL(2,R) WZW model was studied, using spectral flow to generate new representations from the standard ones. These new representations include states corresponding to long strings,[5,6] with a continuous energy spectrum, as well as discrete states. The existence of spectral flow as a symmetry of the theory was argued on the basis of classical and semi-classical analysis. Further support was given by the fact that the seemingly arbitrary upper bound on the mass of string states in AdS_3 was removed, thus recovering the infinite tower of masses one expects from string theory. We would like to verify these results by an explicit calculation of the one-loop partition function. As shown in Ref. 4, the Euclidean black hole background is equivalent to the thermal AdS_3 background. So we will consider string theory on AdS_3 at a finite temperature, which is described by strings moving on a Euclidean AdS_3 background with the Euclidean time identified. The calculation of the partition function for this geometry is a minor variation on the calculation of Gawedzki in Ref. 7. From this we can read off the spectrum of the theory in Lorentzian signature by interpreting the result as the free energy of a gas of strings.

This paper is organized as follows. In Sec. II we review the spectrum found in Ref. 1. In Sec. III we compute the one-loop partition function on thermal AdS_3. In Sec. IV we read off the spectrum from the one-loop calculation. First we present a qualitative analysis, which is then followed by a precise calculation. We explain how the different parts of the spectrum arise from

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this calculation. We further show how the one-loop result contains information about the SL(2,R) and Liouville reflection amplitudes.

II. THE SPECTRUM

We begin by briefly summarizing the results of Ref. 1, where a concrete proposal for the spectrum of AdS_3 string theory was made. We consider a critical bosonic string theory on $AdS_3\times\mathcal{M}$. The Hilbert space of the SL(2,R) WZW model is generated by the action of the left-moving and right-moving current algebra $\widehat{\mathrm{SL}(2,R)}_L\times\widehat{\mathrm{SL}(2,R)}_R$, and all the states form representations of this algebra. The simplest representations are built by first choosing representations for the zero modes, then regarding them as the primary states annihilated by $J^{3,\pm}_{n>0}$. The raising operators $J^{3,\pm}_{n<0}$ are then used to generate the representations of the current algebra. From harmonic analysis, i.e. quantum mechanical limit, it is known that the left–right symmetric combinations $\mathcal{C}^{\alpha}_{j=1/2+is}\times\mathcal{C}^{\alpha}_{j=1/2+is}$ and $\mathcal{D}^{\pm}_{j>1/2}\times\mathcal{D}^{\pm}_{j>1/2}$ form a complete basis in $\mathcal{L}^2(AdS_3)$, where $\mathcal{C}^{\alpha}_{j=1/2+is}$ is the principal continuous representation and $\mathcal{D}^{\pm}_{j>1/2}$ the principal discrete representation of SL(2,R). These representations are unitary, but the resulting current algebra representations $\hat{\mathcal{C}}^{\alpha}_{j=1/2+is}\times\hat{\mathcal{C}}^{\alpha}_{j=1/2+is}$ and $\hat{\mathcal{D}}^{\pm}_{j>1/2}\times\hat{\mathcal{D}}^{\pm}_{j>1/2}$, constructed as explained above, are not. This is not a surprise, for even in flat Minkowski space it is not until one imposes the Virasoro constraints,

$$(L_n+\mathcal{L}_n-\delta_{n,0})|\mathrm{physical}\rangle=0,\quad n\geqslant 0, \tag{1}$$

that a unitary spectrum is obtained. Here $\mathcal{L}_n$ is the Virasoro generator for the internal conformal field theory corresponding to $\mathcal{M}$. The proposal of Ref. 1 is that one should consider not just these representations but also those obtained by the spectral flow

$$\begin{aligned} J^3_n\to\tilde{J}^3_n&=J^3_n-\frac{k}{2}w\,\delta_{n,0},\\ J^+_n\to\tilde{J}^+_n&=J^+_{n+w},\\ J^-_n\to\tilde{J}^-_n&=J^-_{n-w}. \end{aligned} \tag{2}$$

The Virasoro generators, given by the Sugawara form, then become $\tilde{L}_n=L_n+wJ^3_n-k/4w^2\,\delta_{n,0}$. Imposing on $\hat{\mathcal{D}}^{\pm}_{j>1/2}\times\hat{\mathcal{D}}^{\pm}_{j>1/2}$ the condition (1) with $\tilde{L}_n$ one finds that these states have a discrete energy spectrum,

$$\begin{aligned} E&=J^3_0+\overline{J}^3_0=q+\overline{q}+kw+2\tilde{j}\\ &=1+q+\overline{q}+2w+\sqrt{1+4(k-2)(N_w+h-1-\tfrac{1}{2}w(w+1))}; \end{aligned} \tag{3}$$

here N_w is defined to be the level of the current algebra after spectral flow by the amount w, $N_w=\tilde{N}-wq$, and $\tilde{N}$ is the level before spectral flow. The state with energy (3) is obtained from a lowest weight state by acting with the SL(2,R) currents $\Pi\tilde{J}^{\pm}_{n\leqslant 0}|\tilde{j},\tilde{j}\rangle$, with q the net number of $\pm$ signs in this expression. In other words, q is the number of spacetime energy raising operators J^+_a minus the number of spacetime energy lowering operators J^-_a that we have to apply to the lowest weight, lowest energy state $|\tilde{j},m=\tilde{j}\rangle$ to get to the state whose spacetime energy is (3). $\overline{q}$ is the corresponding quantity for the generators $\overline{J}^{\pm}_a$. We also have a level matching condition of the form

$$N_w+h=\overline{N}_w+\overline{h}, \tag{4}$$

which implies that the angular momentum in AdS_3, $l=J_0^3-\bar{J}_0^3=q-\bar{q}$, is an integer. We argued in Ref. 1 that $\tilde{j}$ is further restricted to the range

$$\frac{1}{2}<\tilde{j}<\frac{k-1}{2}, \tag{5}$$

which implies

$$\frac{k}{4}w^2+\frac{1}{2}w<N_w+h-1+\frac{1}{4(k-2)}<\frac{k}{4}(w+1)^2-\frac{1}{2}(w+1). \tag{6}$$

A similar analysis on $\hat{\mathcal{C}}^{\alpha}_{j=1/2+is}\times\hat{\mathcal{C}}^{\alpha}_{j=1/2+is}$ yields a continuous spectrum,

$$E=\frac{k}{2}w+\frac{1}{w}\left(\frac{2s^2+\frac{1}{2}}{k-2}+\tilde{N}+h+\tilde{\bar{N}}+\bar{h}-2\right), \tag{7}$$

where s takes values over the real numbers and is interpreted as the momentum in the radial direction for the long strings. These states satisfy the level matching condition

$$\tilde{N}+h=\tilde{\bar{N}}+\bar{h}+w\times(\text{integer}). \tag{8}$$

In the rest of the paper we will do an independent calculation which will reproduce this single string spectrum.

III. ONE-LOOP PARTITION FUNCTION

In this section we compute the worldsheet one-loop partition function. First we explain the relation between various useful coordinate systems. Then we consider thermal $AdS_3=H_3/Z$ and show how the identification of Euclidean time in the global coordinates translates into particular boundary conditions for the target space fields. The partition function is then calculated by an explicit evaluation of the functional integral following Ref. 7.

A. Coordinates on H_3 and thermal AdS_3

The natural metric on H_3 is given by

$$ds^2=\frac{k}{y^2}(dy^2+dw\,d\bar{w}), \tag{9}$$

which is the Euclidean continuation of the Poincaré metric on AdS_3. By the coordinate transformation,

$$w=\tanh\rho e^{t+i\theta},\quad \bar{w}=\tanh\rho e^{t-i\theta},\quad y=\frac{e^t}{\cosh\rho}, \tag{10}$$

we obtain the cylindrical coordinates on Euclidean AdS_3,

$$\frac{ds^2}{k}=\cosh^2\rho\,dt^2+d\rho^2+\sinh^2\rho\,d\theta^2. \tag{11}$$

For the purpose of calculating the partition function, however, it is convenient to use coordinates in which the metric reads[7] as

$$\frac{ds^2}{k}=d\phi^2+(dv+v\,d\phi)(d\bar{v}+\bar{v}\,d\phi), \tag{12}$$

which corresponds to the parametrization of H_3 as

$$g=\begin{bmatrix} e^{\phi}(1+|v|^2) & v \\ \overline{v} & e^{-\phi} \end{bmatrix}. \tag{13}$$

The coordinate transformation from (11) to (12) is

$$\begin{aligned} v &= \sinh \rho e^{i\theta} \\ \overline{v} &= \sinh \rho e^{-i\theta} \\ \phi &= t - \log \cosh \rho. \end{aligned} \tag{14}$$

Thermal AdS_3 is given by the identification

$$t+i\theta \sim t+i\theta+\hat{\beta}, \tag{15}$$

where $\hat{\beta}$ represents the temperature T and the imaginary chemical potential $i\mu$ for the angular momentum,

$$\hat{\beta}=\beta+i\mu\beta=\frac{1}{T}+i\frac{\mu}{T}. \tag{16}$$

The corresponding identifications in the coordinates (12) are

$$\begin{aligned} v &\sim v e^{i\mu\beta} \\ \overline{v} &\sim \overline{v} e^{-i\mu\beta} \\ \phi &\sim \phi+\beta, \end{aligned} \tag{17}$$

which is a consistent symmetry of the WZW action,

$$S=\frac{k}{\pi}\int d^2z(\partial\phi\,\overline{\partial}\phi+(\partial\overline{v}+\partial\phi\,\overline{v})(\overline{\partial}v+\overline{\partial}\phi v)). \tag{18}$$

B. Computation of the partition function on thermal AdS_3

In this subsection we compute the partition function for string theory on thermal AdS_3. We consider a conformal field theory on a worldsheet torus with modular parameter τ ($z \sim z+2\pi \sim z+2\pi\tau$). The two-dimensional conformal field theory on the worldsheet is the sum of three parts: the conformal field theory on H_3, the internal conformal field theory on $\mathcal{M}$, and the (b, c) ghosts. First we start with the computation of the partition function for the conformal field theory describing the three dimensions of thermal AdS_3 and then we will multiply the result by the partition function of the ghosts and the internal conformal field theory.

Due to the identification (17), the string coordinates now satisfy the following boundary conditions:

$$\begin{aligned} &\phi(z+2\pi)=\phi(z)+\beta n, \quad \phi(z+2\pi\tau)=\phi(z)+\beta m, \\ &v(z+2\pi)=v(z)e^{in\mu\beta}, \quad v(z+2\pi\tau)=v(z)e^{im\mu\beta}. \end{aligned} \tag{19}$$

The thermal circle is noncontractible and therefore we get two integers (n,m) characterizing topologically nontrivial embeddings of the worldsheet in spacetime. In order to implement these boundary conditions it is convenient to define new fields $\hat{\phi},\hat{v}$ such that they are periodic:

$$\phi = \hat{\phi} + \beta f_{n,m}(z,\overline{z}),$$
$$v = \hat{v}\,\exp(i\mu\beta f_{n,m}(z,\overline{z})), \tag{20}$$

with

$$f_{n,m}(z,\overline{z}) = \frac{i}{4\pi\tau_2}[z(n\overline{\tau}-m) - \overline{z}(n\tau-m)]. \tag{21}$$

When we substitute this into the action (18), we get

$$S = \frac{k\beta^2}{4\pi\tau_2}|n\tau-m|^2 + \frac{k}{\pi}\int d^2z\left(|\partial\hat{\phi}|^2 + \left|\left(\partial + \frac{1}{2\tau_2}U_{n,m} + \partial\hat{\phi}\right)\hat{v}\right|^2\right), \tag{22}$$

where

$$U_{n,m}(\tau) = \frac{i}{2\pi}(\beta - i\mu\beta)(n\overline{\tau}-m). \tag{23}$$

We are interested in the functional integral

$$\mathcal{Z}(\beta,\mu;\tau) = \int \mathcal{D}\phi\,\mathcal{D}\overline{v}\,e^{-S}. \tag{24}$$

This integral is evaluated as explained in Ref. 7. We can first do the integral over $\hat{v},\hat{\overline{v}}$ which is quadratic, giving the determinant

$$\det\left|\partial + \frac{1}{2\tau_2}U_{n,m} + \partial\hat{\phi}\right|^{-2}. \tag{25}$$

We calculate the $\hat{\phi}$ dependence on the determinants by realizing that we can view (25) as an inverse of two fermion determinants. We can then remove $\hat{\phi}$ from the determinants by a chiral gauge transformation and using the formulas for chiral anomalies. The result is

$$\det\left|\partial + \frac{1}{2\tau^2}U_{n,m} + \partial\hat{\phi}\right|^{-2} = e^{2/\pi\int d^2z\,\partial\hat{\phi}\tilde{\partial}\hat{\phi}}\det\left|\partial + \frac{1}{2\tau^2}U_{n,m}\right|^{-2}. \tag{26}$$

The remaining integral over $\hat{\phi}$ gives the usual result for a free boson, except that $k\to k-2$ due to (26). The functional integral for the thermal AdS_3 partition function then gives

$$\mathcal{Z}(\beta,\mu;\tau) = \frac{\beta(k-2)^{1/2}}{8\pi\sqrt{\tau_2}}$$
$$\times\sum_{n,m}\frac{e^{-k\beta^2|m-n\tau|^2/4\pi\tau_2 + 2\pi(\mathrm{Im}\,U_{n,m})^2/\tau_2}}{|\sin(\pi U_{n,m})|^2|\Pi_{r=1}^{\infty}(1-e^{2\pi ir\tau})(1-e^{2\pi ir\tau}+2\pi iU_{n,m})(1-e^{2\pi ir\tau-2\pi iU_{n,m}})|^2}$$
$$= \frac{\beta(k-2)^{1/2}}{2\pi\sqrt{\tau_2}}(q\overline{q})^{-3/24}\sum_{n,m}\frac{e^{-k\beta^2|m-n\tau|^2/4\pi\tau_2 + 2\pi/(\mathrm{Im}\,U_{n,m})^2/r_2}}{|\vartheta_1(\tau,U_{n,m})|^2}, \tag{27}$$

where ϑ_1 is the elliptic theta function and $q=e^{2\pi i\tau}$. The factor $\beta(k-2)^{1/2}$ comes from the length of the circle in the ϕ direction. This partition function is explicitly modular invariant after summing over (n, m). [In our previous paper, there was a puzzle about the apparent lack of modular invariance of the SL(2,R) partition functions with J^3 insertions (see Appendix B of Ref. 1). Here

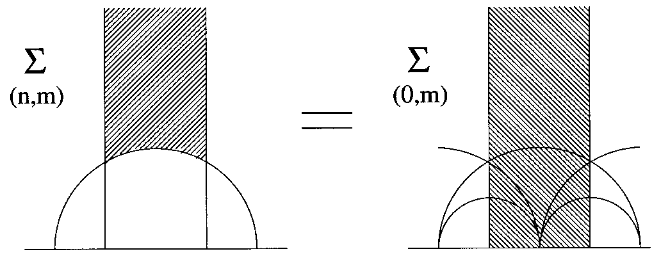

FIG. 1. The sum over n is traded for the sum over copies of the fundamental domain.

we have found that, if we introduce the twist by considering the physical set-up of thermal AdS_3, the result (27) turns out to be manifestly modular invariant. This resolves the puzzle raised in Ref. 1.]

We also need to include the contribution of the (b, c) ghosts and the internal CFT. A partition function of the latter will be of the form

$$\mathcal{Z}_{\mathcal{M}}=(q\bar{q})^{-c_{\mathrm{int}}/24}\sum_{h,\bar{h}} D(h,\bar{h})q^{h}\bar{q}^{\bar{h}}, \tag{28}$$

where $D(h,\bar{h})$ is the degeneracy at left-moving weight h and right-moving weight $\bar{h}$, and c_{int} is the central charge of the internal CFT. Modular invariance requires that $h-\bar{h}\in Z$, a fact which will be useful in the next section. Vanishing of the total conformal anomaly gives

$$c_{\mathrm{SL}(2,R)}+c_{\mathrm{int}}=26. \tag{29}$$

We can calculate now the total contribution to the ground state energy. We found a ground state energy of $-3/24$ in (27), the ghosts contribute with 2/24 and the internal CFT with $-c_{\mathrm{int}}/24 =(c_{\mathrm{SL}(2,R)}-26)/24$. Using $c_{\mathrm{SL}(2,R)}=3+6/(k-2)$, we find the overall factor,

$$(q\bar{q})^{-(1+c_{\mathrm{int}})/24}=e^{4\pi\tau_2(1-1/4(k-2))}. \tag{30}$$

[Note that $c_{\mathrm{int}}\geqslant 0$, $k>2$, and (29) imply that there will always be a tachyon in the theory.]

After multiplying (27) by the (b, c) ghosts and the internal CFT partition functions, we should integrate the resulting expression over the fundamental domain F_0 of the modular parameter τ. The computation is much facilitated by the trick invented in Ref. 8 to trade the sum over n in (27) for the sum over copies of the fundamental domain. See Fig. 1. This is possible since (n, m) transforms as a doublet under the modular group SL(2,Z). If $(n,m)\neq(0,0)$, it can be mapped by an SL(2,Z) transformation to $(0,m), m>0$. The SL(2,Z) transformation also maps the fundamental domain into the strip Im $\tau\geqslant 0$, $|\mathrm{Re}\,\tau|\leqslant 1/2$. On the other hand, $(n,m)=(0,0)$ is invariant under the SL(2,Z) transformation, and the corresponding term still has to be integrated over the fundamental domain F_0. This term represents the zero temperature contribution to the cosmological constant, or the zero temperature vacuum energy. In addition to the usual tachyon divergence of bosonic string theory at large τ_2, it is also divergent due to the $\sin^{-1}$ factor in (27); this divergence can be interpreted as coming from the infinite volume of AdS_3. Since the temperature dependence of this term is trivial we will ignore it from now on. The final result then is that we fix $n=0$ in (27) and we integrate over the entire strip shown in Fig. 1. The one-loop partition function of bosonic string theory on $H_3/Z\times\mathcal{M}$ is then

$$Z(\beta,\mu)=\frac{\beta(k-2)^{1/2}}{8\pi}\int_0^\infty\frac{d\tau_2}{\tau_2^{3/2}}\int_{-1/2}^{1/2}d\tau_1\,e^{4\pi\tau_2(1-1/4(k-2))}\sum_{h,\bar{h}}D(h,\bar{h})q^h\bar{q}^{\bar{h}}$$
$$\times\sum_{m=1}^{\infty}\frac{e^{-(k-2)m^2\beta^2/4\pi\tau_2}}{|\sinh(m\hat{\beta}/2)|^2}\left|\prod_{n=1}^{\infty}\frac{1-e^{2\pi in\tau}}{(1-e^{m\hat{\beta}+2\pi in\tau}(1-e^{-m\hat{\beta}+2\pi in\tau})}\right|^2. \tag{31}$$

IV. READING OFF THE SPECTRUM

We will now extract the spectrum of Lorentzian string theory on AdS_3 by interpreting the one-loop partition function in the spacetime theory. The one-loop partition function is the single particle contribution to the spacetime thermal free energy, $Z(\beta,\mu)=-\beta F$. To this each string state makes a contribution $\beta^{-1}\log(1-e^{-\beta(E+i\mu l)})$, where E and l are the energy and the angular momentum of the state. The total free energy is the sum over all such factors:

$$F(\beta,\mu)=\frac{1}{\beta}\sum_{\text{string}\in\mathcal{H}}\log(1-e^{-\beta(E_{\text{string}}+i\mu l_{\text{string}})})=\sum_{m=1}^{\infty}f(m\beta,m\mu), \tag{32}$$

where

$$f(\beta,\mu)=\frac{1}{\beta}\sum_{\text{string}\in\mathcal{H}}e^{-\beta(E_{\text{string}}+i\mu l_{\text{string}})}. \tag{33}$$

Here $\mathcal{H}$ is the physical Hilbert space of single string states. In both (31) and (32), we have the sums over functions of $(m\beta,m\mu)$. It is therefore sufficient to compare the $m=1$ terms in these expressions. In other words, we want to verify that E_{string} and l_{string} extracted from the identification,

$$f(\beta,\mu)=\sum_{\text{string}\in\mathcal{H}}\frac{1}{\beta}e^{-\beta(E_{\text{string}}+i\mu l_{\text{string}})}$$
$$=\frac{(k-2)^{1/2}}{8\pi}\int_0^\infty\frac{d\rho_2}{\tau_2^{3/2}}\int_{-1/2}^{1/2}d\tau_1\,e^{4\pi\tau_2(1-1/4(k-2))}\sum_{h,\bar{h}}D(h,\bar{h})q^h\bar{q}^{\bar{h}}$$
$$\times\frac{e^{-(k-2)\beta^2/4\pi\tau_2}}{|\sinh(\hat{\beta}/2)|^2}\left|\prod_{n=1}^{\infty}\frac{1-e^{2\pi in\tau}}{(1-e^{\hat{\beta}+2\pi in\tau})(1-e^{-\hat{\beta}+2\pi in\tau})}\right|^2, \tag{34}$$

agree with the string spectrum found in our previous paper.[1] We will see that the sum over the Hilbert space breaks up into a sum over the discrete states and an integral over the continuous states, with the expressions for the energies that were reviewed in Sec. II. Since the one-loop calculation presented here is independent of the assumptions made in Ref. 1 about strings in Lorentzian AdS_3, we can regard this as a derivation of the spectrum starting from the well-defined Euclidean path integral.

A. Qualitative analysis

In this subsection we will analyze (34) in a qualitative way and explain where the different contributions to the spectrum come from. To keep the notation simple, we set $\mu=0$ in this subsection, leaving the exact computation for the next subsection.

As expected, in (34) there is an exponential divergence as $\tau_2\to\infty$, coming from the tachyon. This is just as in the flat space case, where $(\text{mass})^2<0$ of the tachyon causes its contribution to be weighted with a positive exponential. We will disregard this divergence. [A skeptical reader could think that we are really doing the superstring partition function (the fermions included in the internal CFT, etc.). Then the tachyon divergence will disappear but one would still find the

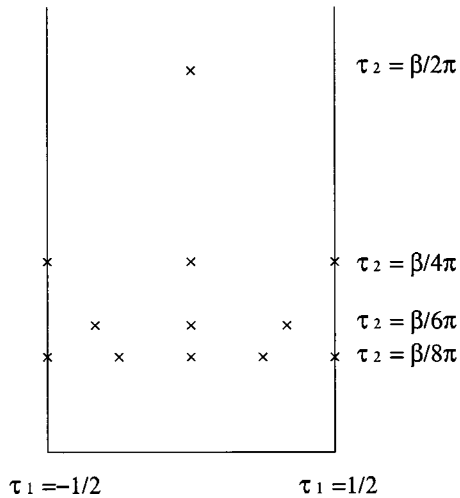

FIG. 2. Poles in the τ-plane, shown for $w=1$ to 4.

divergences that we discuss below. Of course, the one-loop partition function is nonvanishing even in the supersymmetric case since the thermal boundary conditions break supersymmetry.] However, rather unexpectedly, the expression above has additional divergences at finite values of τ. In string theory one might naively expect that divergences come only from the corners of the fundamental domain in the τ-plane, but in this case the divergence is coming from points in the interior of the fundamental domain. Overcoming the initial panic, one realizes that these divergences are related to the presence of long strings. In fact, as with any other string divergence, it can be interpreted as an IR effect. This divergence is due to the fact that long strings feel a flat potential as they go to infinity and therefore we get an infinite volume factor. To see this, note that near the pole (see Fig. 2),

$$\tau=\tau_{\rm pole}+\epsilon, \tag{35}$$

where

$$\tau_{\rm pole}=\frac{r}{w}+i\,\frac{\beta}{2\pi w}, \tag{36}$$

we can expand the partition function and replace τ in all terms by its value at the pole, except in the one term that has the pole. If we integrate (34) near the pole, i.e. in the region $\epsilon<|\tau-\tau_{\rm pole}|\ll 1$, we find that it diverges as $\log\epsilon$ with the coefficient

$$\frac{1}{\sqrt{w\beta^3}}\exp\left[-\beta\left(\frac{k}{2}w+\frac{1}{w}\left(\tilde N+h+\tilde{\bar N}+\bar h-2+\frac{1}{2(k-2)}\right)\right)+\frac{2\pi i r}{w}(\tilde N+h-\tilde{\bar N}-\bar h)\right]. \tag{37}$$

We now sum over r, with $|r/w|\leq 1/2$, since these are the ones corresponding to the poles in the strip. (If some poles are on the boundaries of the strip, $\tau_1=\pm 1/2$, then we only count them once.) This sum constrains $\tilde N+h-\tilde{\bar N}-\tilde h$ to be an integer multiple of w, as in (8), and it introduces an additional factor of w in (37). The log divergence in the τ-integral can therefore be expressed as

$$f(\beta,\mu)\sim\frac{1}{\beta}\log\epsilon\int_0^\infty ds\,e^{-\beta E(s)}+\cdots, \tag{38}$$

where $E(s)$ is the energy spectrum given by (7). Note that the s-integral and the sum over r we mentioned above give the factor $\sqrt{w/\beta}$ needed to match the prefactor in (37) to that in (38). This reproduces the expected contribution from the long strings on the left hand side of (34). The logarithmic divergence should be interpreted as a volume factor due to the fact that the long string can be at any radial position. In the next subsections, we will see more precisely that it is indeed associated to the infinite volume in spacetime by relating ϵ to a long distance cutoff.

Now we would like to calculate the short string spectrum. Since the long string spectrum gives a divergent result, while the short string spectrum gives a finite one, it might appear at first that extracting the contributions due to the short strings from a divergent expression such as (34) will be problematic. Fortunately we can get around this difficulty since the temperature dependence of the long string free energy is different from that of the short string free energy. In the next subsection we will explain how to do this precisely and reproduce the short string spectrum which agrees with Ref. 1. One of the more puzzling aspects of the short string spectrum found there is that there is a cutoff $1/2<\tilde{j}<(k-1)/2$ in the value of the SL$(2,R)$ spin $\tilde{j}$. In the remainder of this section we will explain in a qualitative way how this cutoff arises by doing the calculation for large k.

If we were to evaluate the right hand side of (34) naively (and incorrectly), we would expand the integrand in powers of $q=e^{2\pi i\tau}$ and then perform the τ-integral. If we did this, we would obtain the short string spectrum with $w=0$ and no upper bound on the value of $\tilde{j}$. However this expansion is not correct. How we can expand the integrand in (34) depends on the value of τ_2. When we cross the poles at $\tau_2=\beta/2\pi w$, a different expansion should be used for the denominator:

$$\frac{1}{1-e^{\beta+2\pi i w\tau}}=\sum_{q=0}^{\infty} e^{q(\beta+2\pi i w\tau)}\left(\tau_2>\frac{\beta}{2\pi w}\right),$$
$$=-\sum_{q=0}^{\infty} e^{-(q+1)(\beta+2\pi i w\rho)}\left(\tau_2<\frac{\beta}{2\pi w}\right). \tag{39}$$

When τ_2 is in the range

$$\frac{\beta}{2\pi(w+1)}<\tau_2<\frac{\beta}{2\pi w}, \tag{40}$$

the product over n in the first term in the denominator in (34) is broken up into two factors, a product in $1\leqslant n\leqslant w$ and a product in $w+1\leqslant n$. The first factor is expanded in powers of $e^{(-\beta+2\pi i n\tau)}$ and the second factor is expanded in powers of $e^{\beta+2\pi i n\tau}$. Combining them together with the terms coming from the expansion of the remaining products in (34), we get an exponent of the form[6]

$$-(\tfrac{1}{2}+q+w)\beta+2\pi i\tau(N_w-\tfrac{1}{2}w(w+1)), \tag{41}$$

for some integers q and N_w. [The first term $-\beta/2$ comes from expanding $1/\sinh(\beta/2)$ in (34).] There is a similar term for $\tau\to\bar{\tau}$. We are then to do the τ-integral of the form

$$\int\frac{d^2\tau}{\tau_2^{3/2}}$$
$$\times e^{4\pi\tau_2(1-1/4(k-2))-(k-2)(\beta^2/4\pi\tau_2)-\beta(1+q+\bar{q}+2w)+2\pi i\tau(N_w+h-(1/2)w(w+1))-2\pi i\bar{\tau}(\bar{N}_w+\bar{h}-(1/2)w(w+1))}, \tag{42}$$

over the region (40). The integral over τ_1 produces the level matching condition (4). Now we evaluate the integral over τ_2 using the saddle point approximation. We find that the saddle point is at

$$\tau_{\rm saddle}=\frac{(k-2)\beta}{2\pi\sqrt{1+4(k-2)(N_w+h-1-\frac{1}{2}w(w+1))}}, \tag{43}$$

and the integral gives

$$\frac{1}{\beta}\exp\left[-\beta\left(1+q+\bar{q}+2w+\sqrt{1+4(k-2)\left(N_w+h-1-\frac{1}{2}w(w+1)\right)}\right)\right]. \tag{44}$$

This is the correct form of the contributions due to the short strings on the left hand side of (34). Moreover we obtain the bound on $\tilde{j}$ precisely, because $\tau_{\rm saddle}$ has to be in the range (40) in order for the saddle point approximation to give a nonzero result. By (43), this condition is the same as the bound on the spectrum (6), which is equivalent to $1/2<\tilde{j}<(k-1)/2$. (It is a bit surprising that we get all factors precisely right from the saddle point approximation.) Notice then that the cutoff in $\tilde{j}$ is associated to the fact that we expand the integrand in (34) in different ways depending on the value of τ. The value of τ making the biggest contribution to the integral depends on the values of N and h of the string state.

B. A precise evaluation of the τ-integral

Now let us study the partition function (34) more systematically. In this subsection, we go back to the general case with $\mu\neq 0$. From what we saw in the previous subsection, we expect to find the discrete states from the integral over the range (40), and the continuous states from the poles after a suitable regularization.

In order to evaluate the τ-integral exactly, it is useful to introduce a new variable c by

$$e^{-(k-2)(\beta^2/4\pi\tau_2)}=-\frac{8\pi i}{\beta}\left(\frac{\tau_2}{k-2}\right)^{3/2}\int_{-\infty}^{\infty}dc\ \ c\,e^{-[4\pi\tau_2/(k-2)]c^2+2i\beta c}. \tag{45}$$

Now suppose τ_2 is in the range

$$\frac{\beta}{2\pi(w+1)}<\tau_2<\frac{\beta}{2\pi w}, \tag{46}$$

and expand the integrand in (34) as explained in the previous subsection. The right hand side of (34) becomes a sum of terms like

$$\frac{4}{\beta(k-2)i}\int_{-\infty}^{\infty}dc\ \ c\int_{\beta/2\pi(w+1)}^{\beta/2\pi w}d\tau_2\int_{-1/2}^{1/2}d\tau_1\exp\left[-\hat{\beta}\left(q+w+\frac{1}{2}\right)-\bar{\hat{\beta}}\left(\bar{q}+w+\frac{1}{2}\right)\right.$$
$$\left.+2\pi i\tau_1(N_w+h-\bar{N}_w-\bar{h})+2ic\beta-2\pi\tau_2\left(h+\bar{h}+N_w+\bar{N}_w+\frac{2c^2+\frac{1}{2}}{k-2}-w(w+1)-2\right)\right]. \tag{47}$$

The integral over τ_1 gives a delta function enforcing $N_w+h=\bar{N}_w+\bar{h}$, which is the level-matching condition (4). Integrating over τ_2 in the range (46) gives

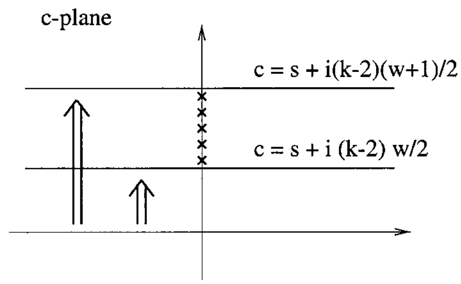

FIG. 3. Shifting the contour of integration picks up the pole residues corresponding to the short string spectrum.

$$\frac{1}{\beta\pi i}\int_{-\infty}^{\infty}dc\quad c\,\frac{\exp[2ic\beta-\hat{\beta}(q+w+\frac{1}{2})-\hat{\bar{\beta}}(\bar{q}+w+\frac{1}{2})]}{c^2+\frac{1}{4}+(k-2)(N_w+h-1-\frac{1}{2}w(w+1))}$$
$$\times\left\{-\exp\left[-\frac{\beta}{w}\left(2N_w+2h-2+\frac{2c^2+\frac{1}{2}}{k-2}-w(w+1)\right)\right]\right.$$
$$\left.+\exp\left[-\frac{\beta}{w+1}\left(2N_w+2h-2+\frac{2c^2+\frac{1}{2}}{k-2}-w(w+1)\right)\right]\right\}, \tag{48}$$

where we used (4).

Let us first look at the first term (the second line) in (48). We note that the exponent can be expressed in the form of a complete square if we set $c=s+(i/2)(k-2)w$. As it will become clear shortly, it is natural to shift the contour of the c-integral from $\mathrm{Im}\,c=0$ to $\mathrm{Im}\,c=1/2(k-2)w$ so that s becomes real. During this process the contour crosses some poles in the integrand, picking up the residues of the poles in the range $0<\mathrm{Im}\,c<1/2(k-2)w$. See Fig. 3. The poles are located at

$$-\frac{c^2}{(k-2)}=N_w+h-\frac{1}{2}w(w+1)-1+\frac{1}{4(k-2)}<\frac{k-2}{4}w^2. \tag{49}$$

Similarly, for the second exponential term (the third line) in (48) we shift the contour to $c=s+(i/2)(k-2)(w+1)$ with s real. This picks up the poles at

$$-\frac{c^2}{(k-2)}=N_w+h-\frac{1}{2}w(w+1)-1+\frac{1}{4(k-2)}<\frac{k-2}{4}(w+1)^2. \tag{50}$$

It is important to note that the residues of these poles have a sign opposite to that of the residues of the poles obeying (49). So the result is that we are left with only those poles in the range

$$\frac{k-2}{2}w<\mathrm{Im}\,c<\frac{k-2}{2}(w+1), \tag{51}$$

with residues

$$\frac{1}{\beta}\exp[-\hat{\beta}q-\hat{\bar{\beta}}\bar{q}-\beta(1+2w+\sqrt{1+4(k-2)(N_w+h-1-\frac{1}{2}w(w+1))})]. \tag{52}$$

This is the expected contribution of the short strings to the right hand side of (34), and we see also that (51) translates into the correct bound on $\tilde{j}$ (5).

It remains to examine the resulting integral over s. Notice that the term coming from just above the pole at $\tau=\hat{\beta}/2\pi w$ has a very similar w dependence in the exponent as that coming from just below the pole. In other words, we combine the first term of (48) with the second term of an expression similar to (48) but with $w \to w-1$ and we find, after shifting the contours as above,

$$\frac{1}{2\pi i\beta}\int_{-\infty}^{\infty} ds\left(\frac{2s}{w(k-2)}+i\right)\left(\frac{\exp\left[-\hat{\beta}q-\bar{\hat{\beta}}\bar{q}-\beta\left(\frac{k}{2}w+\frac{2}{w}\left(\frac{s^2+1/4}{k-2}+N_{w-1}+h-1\right)\right)\right]}{\frac{1}{2}+is-\frac{k}{4}w+\frac{1}{w}\left(N_{w-1}+h-1+\frac{s^2+1/4}{k-2}\right)}\right.$$
$$\left.-\frac{\exp\left[-\hat{\beta}q-\bar{\hat{\beta}}\bar{q}-\beta\left(\frac{k}{2}w+\frac{2}{w}\left(\frac{s^2+1/4}{k-2}+N_w+h-1\right)\right)\right]}{-\frac{1}{2}+is-\frac{k}{4}w+\frac{1}{w}\left(N_w+h-1+\frac{s^2+1/4}{k-2}\right)}\right). \tag{53}$$

Let us concentrate for now on the third line of (53). We first note that the sum of such terms over all states gives rise to the log divergence. To see this, it is useful to notice that the combinations

$$\tilde{N}=qw+N_w, \quad \bar{\tilde{N}}=\bar{q}w+\bar{N}_w, \tag{54}$$

that appear in the exponent of the third line of Eq. (53) are the levels before spectral flow. Thus, for a given state $|\psi\rangle$, states of the form $(\tilde{J}_0^+\bar{\tilde{J}}_0^+)^n|\psi\rangle$ all have the same value of $\tilde{N}$ and $\bar{\tilde{N}}$. Acting with $\tilde{J}_0^+\bar{\tilde{J}}_0^+$ on $|\psi\rangle$ does not change the exponent in (53), but it does change the denominator by one. This implies that when we sum over all the states of this type, we will find a divergent sum of the form

$$\sum_{n=0}^{\infty}\frac{1}{A-n}.$$

This divergence has the same origin as the divergence of the right hand side of (34) at the pole $\tau_{\rm pole}=\hat{\beta}/2\pi w$. In fact, if we regularize the τ-integral by removing a small region near the pole as $|\tau-\tau_{\rm pole}|>\epsilon$, we find an additional factor $e^{-n\epsilon}$ in the sum. In the next subsection, we will give the spacetime interpretation of this procedure. With this regularization, the sum behaves as $\log\epsilon$. More precisely we have

$$-\sum_{n=0}^{\infty}\frac{1}{A-n}e^{-n\epsilon}=\log\epsilon+\frac{d}{dA}\log\Gamma(-A)+\mathcal{O}(\epsilon), \tag{55}$$

where

$$A=-\frac{1}{2}+is-\frac{k}{4}w+\frac{1}{w}\left(\frac{s^2+\frac{1}{4}}{k-2}+\tilde{N}+h-1\right). \tag{56}$$

Here we have assumed that

$$\bar{\tilde{N}}+\bar{h}\leq\tilde{N}+h, \tag{57}$$

but it can be seen that the other case gives the same result.

Now we turn our attention to the second line of (53). In those terms we have one less unit of spectral flow, as compared to the third line in (53) that we analyzed above. In other words, now we will have that $(w-1)q+N_{w-1}=\tilde{N}'$. These states are in the spectral flow image of $\mathcal{D}_j^+$. Since we want to combine these states with the states coming from the third line in (53) it is convenient to do spectral flow one more time and think of these states as in the spectral flow image of $\mathcal{D}_j^-$ under w units of spectral flow. In this case we find that $q+\tilde{N}'=\tilde{N}$ where now $\tilde{N}$ is the level of the $\mathcal{D}_j^-$

representation before spectral flow. From now on the discussion is very similar to what we had above. The states with $(\tilde{J}_0^- \tilde{\tilde{J}}_0^-)^n|\psi\rangle$ all have the same energies but they will contribute to the denominator of the second line in (53) with

$$\sum_{n=0}^{\infty} \frac{1}{B+n} e^{-ne} = \log \epsilon - \frac{d}{dB} \log \Gamma(B) + \mathcal{O}(\epsilon), \tag{58}$$

where

$$B = \frac{1}{2} + is - \frac{k}{4} w + \frac{1}{w}\left(\frac{s^2 + \frac{1}{4}}{k-2} + \tilde{\tilde{N}} + \bar{h} - 1 \right). \tag{59}$$

again assuming (57).

After we perform these two sums, we find that (53) can be written in the form

$$\frac{2}{\beta} \int_0^\infty ds\, \rho(s) \exp\left[-\beta\left(E(s) + i\frac{\mu}{w}(\tilde{N} + h - \tilde{\tilde{N}} - \bar{h}) \right) \right], \tag{60}$$

with $E(s)$ the energy of long strings (7) and $\rho(s)$ the density of states. We will later see that the physical momentum p of a long string in the ρ direction is equal to $p=2s$. The angular momentum $l = (\tilde{N} + h - \tilde{\tilde{N}} - \bar{h})/w$ is an integer since the states in (53) were obeying (4) and the definition (54) ensures that (8) is satisfied. The density of states $\rho(s)$ derived from this analysis is

$$\rho(s) = \frac{1}{2\pi} 2 \log \epsilon + \frac{1}{2\pi i} \frac{d}{2ds} \log\left(\frac{\Gamma(\frac{1}{2} - is + \tilde{\tilde{m}})\Gamma(\frac{1}{2} - is - \tilde{m})}{\Gamma(\frac{1}{2} + is + \tilde{\tilde{m}})\Gamma(\frac{1}{2} + is - \tilde{m})} \right), \tag{61}$$

where

$$\tilde{m} = -\frac{k}{4} w + \frac{1}{w}\left(\frac{s^2 + \frac{1}{4}}{k-2} + \tilde{N} + h - 1 \right), \quad \tilde{\tilde{m}} = -\frac{k}{4} w + \frac{1}{w}\left(\frac{s^2 + \frac{1}{4}}{k-2} + \tilde{\tilde{N}} + \bar{h} - 1 \right). \tag{62}$$

Note that, despite appearances to the contrary, (61) is actually symmetric under $\tilde{m} \leftrightarrow \tilde{\tilde{m}}$ since $\tilde{m} - \tilde{\tilde{m}} = l$ is an integer. In the next subsection we will show that this density of states (61) is what is expected from the spacetime meaning of the cutoff ϵ. In going from (53) to (60) we have states which could be interpreted as coming from the spectral flow of the discrete representations $\mathcal{D}_j^+$ and $\mathcal{D}_j^-$, with the zero modes essentially stripped off since they were explicitly summed over in (55) and (58). This implies that the states we have in the end belong to the continuous representation. Note also that the integral over s in (60) has only half the range in (53). We rewrote it in this way using the fact that the exponent is invariant under $s \to -s$, and that is the reason why we have four gamma functions in (61). In going from (53) to (60) we have also used that $d/dA = (1/d/dA(s)/ds)(d/ds)$ in (56) and similarly in (59).

Combining Eqs. (52) and (60), we have, finally,

$$f(\beta,\mu) = \frac{1}{\beta} \sum D(h, \bar{h}, \tilde{N}, \tilde{\tilde{N}}, w) \left[\sum_{q,\bar{q}} e^{-\beta(E + i\mu l)} + \int_0^\infty ds \rho(s) e^{-\beta(E(s) + i\mu l)} \right] \tag{63}$$

which is the free energy due to the short strings and the long strings, respectively.

C. The density of long string states

What remains to be shown is the interpretation of $\rho(s)$ given by (61) as the density of long string states. Whenever we have a continuous spectrum the density of states may be calculated by first introducing a long distance cutoff which will make the spectrum discrete, and then removing the cutoff. If the cutoff is related to the volume of the system then the density of states will have a divergent part, proportional to the volume and dependent only on the bulk physics, and a finite part which encodes information about the scattering phase shift and also has some dependence on the precise cutoff procedure. To see this, let us consider a one-dimensional quantum mechanical model on the half line, $\rho>0$, with a potential $V(\rho)$. We assume that $V(\rho)$ vanishes sufficiently fast for large ρ, and that there is continuous spectrum above a certain energy level. To define the density of states, it is convenient to introduce a long distance cutoff at large ρ so that the spectrum becomes discrete. Let us first consider a cutoff by an infinite wall at $\rho=L$. If L is sufficiently large, an energy eigenfunction $\psi(\rho)$ near the wall can be approximated by the plane wave,

$$\psi(\rho)\sim e^{-ip\rho}+e^{ip\rho+i\delta(p)}, \tag{64}$$

where $\delta(p)$ is the phase shift due to the original potential $V(\rho)$. Imposing Dirichlet boundary condition $\psi(L)=0$ at the wall, we have

$$2pL+\delta(p)=2\pi(n+\tfrac{1}{2}), \tag{65}$$

for some integer n. If L is sufficiently large, there is a unique solution $p=p(n)$ to this equation for a given n. As we remove the cutoff by sending $L\to\infty$, the spectrum of p becomes continuous. We then define the density of states $\rho(p)$ by

$$dn=\rho(p)dp. \tag{66}$$

From (65), we obtain

$$\rho(p)=\frac{1}{2\pi}\left(2L+\frac{d\delta}{dp}\right). \tag{67}$$

Thus the finite part of the density of states is given by the derivative of the phase shift.

Instead of the infinite wall at $\rho=L$, we may consider a more general potential $V_{\text{wall}}(\rho-L)$ which vanishes for $\rho<L$ but rises steeply for $L<\rho$ to confine the particle. Let us denote by $\delta_{\text{wall}}(p)$ the phase shift due to scattering from $V_{\text{wall}}(\rho)$. We then obtain the condition on the allowed values of momenta by matching these two wavefunctions and their derivatives at $\rho=L$ as

$$\psi(\rho)\sim e^{-ip\rho}+e^{ip\rho+i\delta(p)}\sim A[e^{-ip(\rho-L)}+e^{ip(\rho-L)+i\delta_{\text{wall}}(p)}] \quad (\rho\sim L). \tag{68}$$

It follows that

$$pL+\delta(p)=-pL+\delta_{\text{wall}}(p)+2\pi n. \tag{69}$$

In the limit $L\to\infty$, the density of states given by $dn=\rho(p)dp$ is then

$$\rho(p)=\frac{1}{2\pi}\left(2L+\frac{d\delta}{dp}-\frac{d\delta_{\text{wall}}}{dp}\right). \tag{70}$$

When we have the infinite wall, the phase shift due to the wall is independent of $p(\delta_{\text{wall}}=\pi)$, and (70) reduces to (67).

In order to apply this observation to our problem, it is useful to first identify the origin of the logarithmic divergence in the one-loop amplitude $Z(\beta,\mu)$ by examining the functional integral (24) near the boundary of AdS_3. In the cylindrical coordinates (11), the string worldsheet action (18) for large ρ takes the form

$$S\sim\frac{k}{\pi}\int d^2z\left(\partial\rho\,\bar{\partial}\rho+\frac{1}{4}e^{2\rho}|\bar{\partial}(\theta-it)|^2+\cdots\right). \tag{71}$$

Because of the factor $e^{2\rho}$, the functional integral for large ρ restricts (t,θ) to be a harmonic map from the worldsheet to the target space. Since (t,θ) are coordinates on the torus,

$$\theta - it \sim \theta - it + 2\pi n + i\hat{\beta} m \quad (n,m \quad \text{integers}), \tag{72}$$

the harmonic map from the torus to the torus is

$$\begin{aligned}\theta - it &= (2\pi w + i\hat{\beta} m)\sigma^t + (2\pi r + i\hat{\beta} n)\sigma^2\\ &= [(2\pi w + i\hat{\beta} m)\tau - (2\pi r + i\hat{\beta} n)]\frac{\bar{z}}{2i\tau_2}\\ &\quad - [(2\pi w + i\hat{\beta} m)\bar{\tau} - (2\pi r + i\hat{\beta} n)]\frac{z}{2i\tau_2},\end{aligned} \tag{73}$$

where $z = \sigma^1 + \rho\sigma^2$ is the worldsheet coordinate and (r, w, n, m) are integers. In particular, the map $(\theta - it)$ with $(n,m) = (1,0)$ becomes w–to–1 and *holomorphic* when τ takes the special value

$$\tau_{\text{pole}} = \frac{r}{w} + i\frac{\hat{\beta}}{2\pi w}. \tag{74}$$

On the other hand, if τ is not at one of these points, $\bar{\partial}(\theta - it)$ cannot be set to zero. [For any τ, we also have a trivial holomorphic map $(t,\theta) = \text{const}$. The functional integral around such a map gives a result independent of β and we can neglect it in the following discussion.] This gives rise to an effective potential $e^{2\rho}$ for ρ, which keeps the worldsheet from growing towards the boundary. If τ is near τ_{pole},

$$\tau = \tau_{\text{pole}} + \epsilon, \tag{75}$$

the harmonic map (73) with $(n,m) = (1,0)$ gives

$$|\bar{\partial}(\theta - it)|^2 \sim \left(\frac{2\pi^2 w^2}{\beta}\right)^2 \epsilon^2. \tag{76}$$

Thus the action (71) generates the Liouville potential $\epsilon^2 e^{2\rho}$. When we computed the one-loop amplitude in Secs. IV A and IV B, we regularized the τ-integral by removing a small disk $|\tau - \tau_{\text{pole}}| < \epsilon$ around each of these special points. Near $\tau = \tau_{\text{pole}}$, this is equivalent to adding the infinitesimal Liouville potential $\epsilon^2 e^{2\rho}$ to the worldsheet action. For $|\tau - \tau_{\text{pole}}| \gg \epsilon$, the worldsheet can never grow large enough and the effect of the Liouville term is negligible. To be precise, the Gaussian functional integral of (t,θ) shifts $k \to (k-2)$ as in (26) and the effective action for ρ near $\tau = \tau_{\text{pole}}$ is

$$S_{\text{Liouville}} = \frac{k-2}{\pi}\int d^2 z(\partial\rho\,\tilde{\partial}\rho + \epsilon^2 e^{2\rho}). \tag{77}$$

Therefore, we find that our choice of regularization in (55) and (58) amounts to introducing the Liouville wall which prevents the longs strings from going to very large values of ρ. By looking at the potential in (77), we see that the effective length of the interval is $L \sim \log\epsilon$. The central charge of this Liouville theory is such that the $e^{2\rho}$ term has conformal weight one,

$$c_{\text{Liouville}} = 1 + 6\left(b + \frac{1}{b}\right)^2, \quad b \equiv \frac{1}{\sqrt{k-2}}. \tag{78}$$

The finite part of the density of states will be given through (70) by $\delta(s)$, the phase shift in the SL(2,R) model, and $\delta_{\rm wall}(s)$, the corresponding quantity in Liouville theory. The first one was calculated in Refs. 9 and 10,

$$i\,\delta(s)=\log\left(\frac{\Gamma\left(\frac{1}{2}+is-\tilde{m}\right)\Gamma\left(\frac{1}{2}+is+\tilde{\bar{m}}\right)\Gamma(-2is)\Gamma\left(\frac{2is}{k-2}\right)}{\Gamma\left(\frac{1}{2}-is-\tilde{m}\right)\Gamma\left(\frac{1}{2}-is+\tilde{\bar{m}}\right)(2is)\Gamma\left(\frac{-2is}{k-2}\right)}\right), \tag{79}$$

while the second one was obtained in Refs. 11 and 12,

$$i\,\delta_{\rm wall}(s)=\log\left(\frac{\Gamma(-2is)\Gamma\left(\frac{2is}{k-2}\right)}{\Gamma(2is)\Gamma\left(\frac{-2is}{k-2}\right)}\right). \tag{80}$$

[In order to compare with the expressions in Refs. 11, 12, we use the value of b given in (78) and note that the relevant values of α are $\alpha=Q/2+$isb.] Using these two formulas we can check that indeed the density of states (61) is given by (70). We can view this as an independent calculation of (79) or as an overall consistency check. Notice that the physical momentum p of a long string along the ρ direction is $p=2s$. This can be seen by comparing the energy of a long string (7) with the energy expected from (77) with spacetime momentum p along the radial direction, $p=(k-2)w\dot{\rho}$. We have chosen the variable s since it is conventional to denote by $j=1/2+is$ the SL(2,R) spin of a continuous representation.

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