Strings of Minimal 6d SCFTs

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Babak Haghighat,[*] Albrecht Klemm,[†] Guglielmo Lockhart,[*] Cumrun Vafa[*]

[*]*Jefferson Physical Laboratory, Harvard University, Cambridge, MA 02138, USA*

[†]*Bethe Center for Theoretical Physics, Universität Bonn, Nussallee 12, D-53115 Bonn*

Abstract: We study strings associated with minimal 6d SCFTs, which by definition have only one string charge and no Higgs branch. These theories are labelled by a number n with $1 \leq n \leq 8$ or $n = 12$. Quiver theories have previously been proposed which describe strings of SCFTs for $n = 1, 2$. For $n > 2$ the strings interact with the bulk gauge symmetry. In this paper we find a quiver description for the $n = 4$ string using Sen's limit of F-theory and calculate its elliptic genus with localization techniques. This result is checked using the duality of F-theory with M-theory and topological string theory whose refined BPS partition function captures the elliptic genus of the SCFT strings. We use the topological string theory to gain insight into the elliptic genus for other values of n.

Contents

1 Introduction

Six-dimensional superconformal theories have light strings as their basic building blocks. One approach to a better understanding of these theories involves unlocking the mysteries associated with these strings. In particular one would like to describe the free propagation of such strings and the degrees of freedom on their worldsheet. Recently, many advances have been made in our understanding of 6d SCFTs [1–3] including in many cases an effective description of their strings' worldsheet QFT [4–6]. The goal of this paper is to study the strings associated with minimal 6d SCFTs. These are (1, 0) SCFTs in six dimensions which have only one string charge (i.e. a one dimensional tensor branch), and are non-Higgsable. They are labelled by an integer $2 \leq n \leq 12$ excluding $n = 9, 10, 11$, and are realized within F-theory as elliptic fibrations over a base $\mathcal{O}(-n) \to \mathbb{P}^1$ [7]. It is also natural to include the $n = 1$ case here although strictly speaking it is not part of the non-Higgsable family. The cases $n = 2, 3, 4, 6, 8, 12$ also arise in the F-theory context as simple orbifolds of $\mathbb{C}^2 \times T^2$ [8] where we rotate each plane of $\mathbb{C}^2$ by an n-th root of unity ω and compensate by rotating the elliptic fiber by ω^{-2}.

For the $n = 1$ and $n = 2$ cases, quivers have been found which describe the worldsheet dynamics of the corresponding strings [4–6]. The $n = 1$ case corresponds to the exceptional CFT with E_8 global symmetry describing an M5 brane near the M9 boundary wall [9–12]; the $n = 2$ case, on the other hand, corresponds to the $(2, 0)$ SCFT of type A_1. In this paper we extend this list by finding the quiver for the $n = 4$ case. This is one of the orbifold cases for which the elliptic fiber can have arbitrary complex modulus τ, as the only symmetry required in the fiber is $\mathbb{Z}_2$, which does not fix the modulus of the torus. To find the quiver describing the strings of this theory, we use Sen's limit of F-theory [13], which corresponds to taking the modulus of the torus $\tau_2 \gg 1$. Following this approach we are able in particular to compute the elliptic genus of these strings, which we do explicitly for the first few string numbers.

If we compactify the theory on a circle, the elliptic genus computes the BPS degeneracies of the wrapped strings. Following the duality between F-theory and M-theory and the relation between M-theory, BPS counting in five dimensions, and topological strings, we find that the elliptic genera are encoded in the topological string partition function defined on the corresponding elliptically-fibered Calabi-Yau, similar to the observation for the $n = 1$ case in [12]. Within topological string theory the genus zero BPS invariants can be easily calculated using mirror symmetry even for high degree k in the base, which corresponds to k times wrapped strings. However, since the boundary conditions are only known to some extent [14], the higher genus theory cannot be completely solved with the generalized holomorphic anomaly equations; on the other hand, the elliptic genus computation provides the all genus answer. In particular we can use this relation to successfully check our answer for the elliptic genus of the $n = 4$ strings.

For the other values of n, no worldsheet description of the associated strings is known. For these cases we employ topological string techniques to obtain BPS invariants of the corresponding geometry, which can be related to an expansion of the elliptic genus of small numbers of strings for specific values of fugacities.

The organization of this paper is as follows: In Section 2 we review the classification of minimal 6d SCFTs and their F-theory realization. We also review the quivers describing the worldsheet dynamics of the strings of the $n = 1$ and $n = 2$ models. In Section 3 we derive the quiver for the strings of the $n = 4$ theory by exploiting its orbifold realization. Furthermore, using the quiver we obtain an integral expression for the elliptic genus of k strings which we evaluate explicitly for the cases $k = 1$

and $k = 2$. We then discuss how one can extract from this the BPS degeneracies associated to the strings. In Section 4 we construct explicitly the elliptic Calabi-Yau manifolds corresponding to $n = 1, \ldots, 12$ as hypersurfaces in toric ambient spaces, solve the topological string theory and calculate the genus zero BPS invariants associated to these Calabi-Yau manifolds, from which one can obtain BPS degeneracies associated to the strings. In Appendix A we give a description of the local mirror geometry for some of these elliptic Calabi-Yau threefolds in terms of non-compact Landau-Ginzburg models.

2 Minimal SCFTs in six-dimensions

Six-dimensional SCFTs can be classified in the context of F-theory by considering compactifications on an elliptically fibered Calabi-Yau threefold X with non-compact base B. In the case where all fiber components are blown down the fibration $\pi : X \to B$ can be described in terms of the Weierstrass form

$$y^2 = x^3 + fx + g, \tag{2.1}$$

where f and g are sections of the line bundles $\mathcal{O}(-4K_B)$ and $\mathcal{O}(-6K_B)$. The discriminant locus, along which the elliptic fibers are singular, is a section of $\mathcal{O}(-12K_B)$ and has the following form:

$$\Delta = 4f^3 + 27g^2. \tag{2.2}$$

The discriminant locus corresponds to the location of seven-branes in the system. More precisely, each component of the discriminant locus is identified with a seven-brane wrapping a divisor $\Sigma \subset B$. Each seven-brane supports a gauge algebra g_Σ which is determined by the singularity type of the elliptic fiber along Σ [16, 17].

In the maximally Higgsed phase (that is, when all hypermultiplet vevs that can be set to non-zero value are turned on) one can classify the resulting models in terms of the base geometry B only [7]. Non-Higgsability requires that the divisor $\Sigma \subset B$ be rigid. This implies that Σ must be a $\mathbb{P}^1$ curve with self-intersection $-n < 0$ for a positive number n (in the following we will refer to this as a $(-n)$ curve), and the local geometry is the bundle $\mathcal{O}(-n) \to \mathbb{P}^1$. Furthermore, it can be shown that n is only allowed to take the values $1 \leq n \leq 8$ or $n = 12$ [7, 16, 17]. In the $n = 1$ case, corresponding to the E-string $(1, 0)$ SCFT [9–11], the discriminant vanishes along the non-compact fiber over isolated points on the $\mathbb{P}^1$. In this case instead of a gauge symmetry one finds an E_8 global symmetry. In the $n = 2$ case the fiber is everywhere non-singular, and one finds the A_1 $(2, 0)$ SCFT which corresponds to the world-volume theory of M5 branes in flat space. For $n > 2$, the seven-branes wrap the compact $\mathbb{P}^1$, and therefore the 6d SCFT has non-trivial gauge symmetry. In the non-Higgsable case this gauge symmetry is completely determined by the integer n. We summarize the list of possibilities in the following table:

7-brane	3	4	5	6	7	8	12
g_Σ	$SU(3)$	$SO(8)$	F_4	E_6	E_7	E_7	E_8
Hyper	–	–	–	–	$\frac{1}{2}\mathbf{56}$	–	–

In the $n = 7$ case, one finds that in addition to E_7 gauge symmetry the 6d theory also contains a half-hypermultiplet. The cases $n = 9$, $n = 10$ and $n = 11$ lead to E_8 gauge symmetry but additionally contain "small instantons"; these cases can be reduced to chains of the more fundamental geometries summarized in the table, as discussed in [1].

These geometries (excluding the cases $n = 1, 5, 7$) can equivalently be realized as orbifolds of the form $(T^2 \times \mathbb{C}^2)/\mathbb{Z}_n$, $n = 2, 3, 4, 6, 8, 12$. Here, $\mathbb{Z}_n$ acts on the z_i, $i = 1, 2, 3$, coordinates of T^2 and $\mathbb{C}^2$ as

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \mapsto \begin{pmatrix} \omega^{-2} & & \\ & \omega & \\ & & \omega \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} , \tag{2.3}$$

with $\omega^n = 1$. This construction will be in particular useful when we study the $n = 4$ SCFT, as it will enable us to find a weak coupling description for the corresponding model.

2.1 Strings of the $\mathcal{O}(-1) \to \mathbb{P}^1$ and $\mathcal{O}(-2) \to \mathbb{P}^1$ models

Let us next discuss the strings that appear on the tensor branch of 6d SCFTs. From the point of view of F-theory these strings arise from D3 branes which wrap the $\mathbb{P}^1$ curve in the base B in the limit of small $\mathbb{P}^1$ size. Let us first review the 'M-strings' that arise in the $n = 2$ case. Since in this case the orbifold acts trivially on the torus, its modulus τ can be taken to be arbitrary, and in particular one can take the weak coupling limit $\tau \to i\infty$ and study this system from the point of view of Type IIB string theory compactified on B. It turns out [4, 5] that the dynamics of M-strings are captured by the two-dimensional quiver gauge theory depicted in Figure 1. For k strings, this quiver describes a two-dimensional $\mathcal{N} = (4, 0)$ theory with gauge group $U(k)$ and the following $(2, 0)$ field content: Q and $\widetilde{Q}$ are chiral multiplets in the fundamental representation of $U(k)$, while Λ^Q and $\Lambda^{\widetilde{Q}}$ are fundamental Fermi multiplets. Furthermore, the Fermi multiplet Λ^ϕ and vector multiplet Υ combine into a $(4, 0)$ vector multiplet, and the adjoint chiral multiplets $B, \widetilde{B}$ combine into a $(4, 0)$ hypermultiplet. One intuitive way to see how this comes about is to look at the configuration of (-2) curves which captures the local geometry $\mathcal{O}(-2) \to \mathbb{P}^1$ [3] and is pictured in Figure 2.

The left and right (-2) curves are non-compact, whereas the curve in the middle is a compact $\mathbb{P}^1$. Choosing the elliptic fiber to be trivial would lead upon circle compactification to $U(2)$ $\mathcal{N} = 4$ gauge theory; it is in fact possible to deform this theory to $\mathcal{N} = 2^*$ by letting the elliptic fiber degenerate over each curve to an I_1

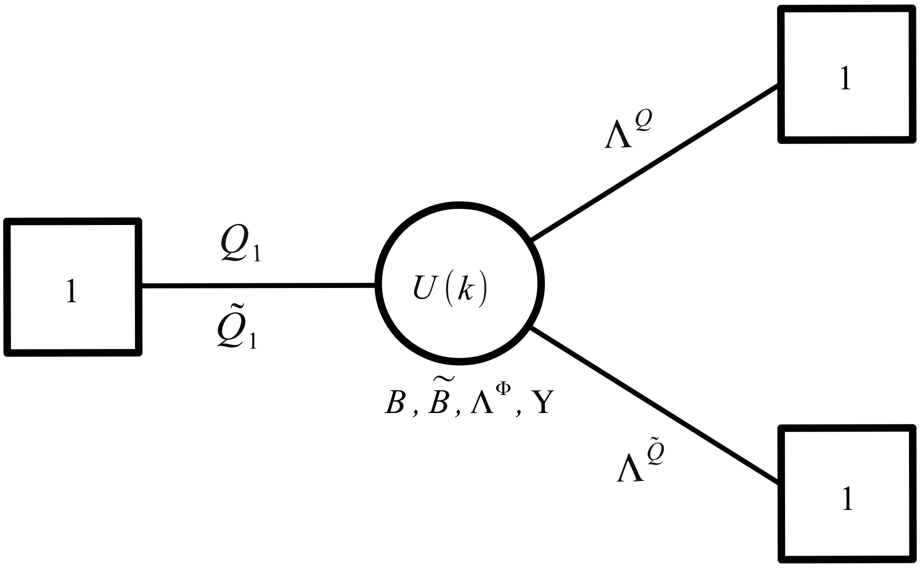


Figure 1: The quiver describing the $\mathcal{O}(-2) \to \mathbb{P}^1$ strings.

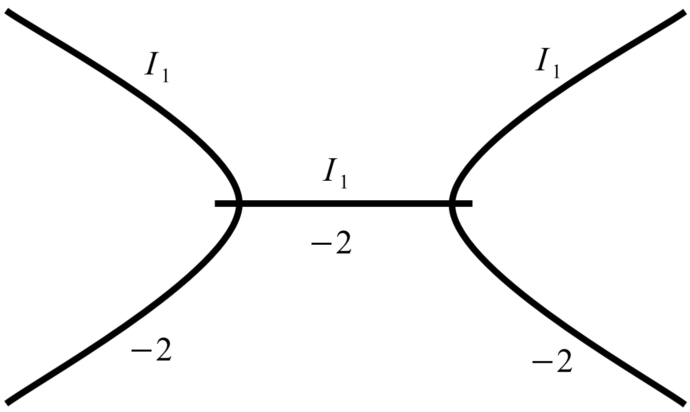


Figure 2: Configuration of (-2) curves that gives rise to the $O(-2) \to \mathbb{P}^1$ local geometry. We have also indicated the degeneration of the elliptic fiber over each curve that gives rise to the M-string geometry.

singularity (that is, by wrapping a D7 brane over each curve). D3 branes wrapping the compact (-2) curve give rise to the strings of the resulting 6d SCFT, and upon circle compactification their BPS degeneracies then capture the BPS particle content of the $U(2)$ $\mathcal{N} = 2^*$ theory. It is easy to understand how the field content of the quiver in Figure 1 arises from strings that end on the D3 branes: D3-D3 strings give rise to a $(4, 4)$ vector multiplet in the adjoint of $U(k)$ consisting of the (2, 0) multiplets $\Upsilon, \Lambda^{\Phi}, B, \tilde{B}$; strings stretching from the D3 branes to the D7 brane wrapping the same compact $\mathbb{P}^1$ give rise to the chiral multiplets Q and $\tilde{Q}$; finally, strings stretching between the D3 branes and the D7 branes that wrap the non-compact (-2) curves give rise to the Fermi multiplets Λ^{Q} and $\Lambda^{\widetilde{Q}}$. Whether D3-D7 strings give rise to chiral or Fermi multiplets is determined by the number of dimensions that are not

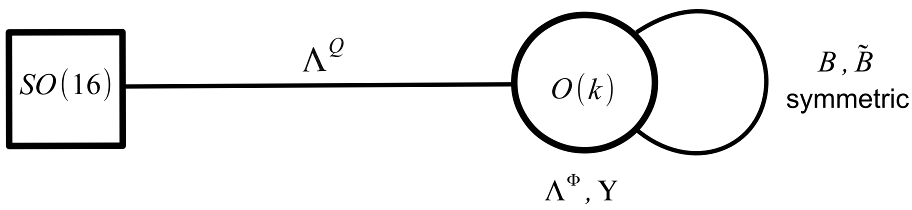


Figure 3: The quiver for $\mathcal{O}(-1) \to \mathbb{P}^1$ strings.

shared by the D3 and D7 branes (four for the D3-D7 strings leading to $Q, \widetilde{Q}$, eight for the ones leading to $\Lambda^Q, \Lambda^{\widetilde{Q}}$).

Recently, a quiver gauge theory was also found that describes the dynamics of E-strings, corresponding to the $\mathcal{O}(-1) \to \mathbb{P}^1$ case [6]. In terms of $(2,0)$ multiplets, the theory of k E-strings was found to have the following field content: a vector multiplet Υ and a Fermi multiplet Λ^ϕ in the adjoint representation of $O(k)$, two chiral multiplets $B, \tilde{B}$ in the symmetric representation of $O(k)$, and a Fermi multiplet Λ^Q in the bifundamental representation of $O(k)$ and of a $SO(16)$ flavor group, which enhances to E_8 at the superconformal point. The relevant quiver is shown in Figure 3.

2.2 From strings of 6d SCFTs to topological strings

In cases where a quiver gauge theory description is available for the strings of minimal six-dimensional SCFTs, one can use the methods of [18–20] to compute the elliptic genus for an arbitrary number of strings. The elliptic genus will depend on the complex structure τ of the torus as well as a number of fugacities corresponding to various $U(1)$ symmetries enjoyed by the two-dimensional quiver theory. In particular, it will always depend on two parameters ϵ_1, ϵ_2 that correspond to rotating the $\mathbb{C}^2$ transverse to the strings' worldsheet in the six-dimensional worldvolume of the SCFT. In addition to this, the elliptic genus will depend on a number of fugacities m_i parametrizing the Cartan of the flavor symmetry group of the worldsheet theory. In the F-theory picture these fugacities correspond to Kähler parameters of the resolved elliptic fiber of the Calabi-Yau.

The elliptic genus encodes detailed information about the spectrum of the strings. Being able to reproduce this information with an alternative method is therefore an important check of the validity of the quiver theory. This can be achieved by exploiting duality between F-theory and M-theory [15, 16], and in particular the relation between D3 branes on one side and M2 branes on the other [12]. This duality relates F-theory on $X \times T^2 \times \mathbb{R}^4$ (where X is an elliptically fibered Calabi-Yau threefold) to M-theory on $X \times S^1 \times \mathbb{R}^4$; under this duality the complex structure τ of the T^2 gets mapped to the Kähler parameter of the elliptic fiber of X on the M-theory side. D3 branes wrapping the base $\mathbb{P}^1$ as well as T^2 correspond to strings wrapped on the torus. It turns out [12] that the BPS states of a configuration of k strings with m units of momentum along a circle get mapped to BPS M2 branes wrapping the base $\mathbb{P}^1$ k times and the elliptic fiber m times; furthermore, if a string BPS state has nonzero flavor symmetry charges, the corresponding BPS M2 brane will also wrap additional curves in X.

The precise relation between the counting of BPS states on the two sides turns out to be [4]:

$$Z^{\text{top}}(X, \epsilon_1, \epsilon_2, \tau, t_b, m_i) = Z_0(\tau, \epsilon_1, \epsilon_2, m_i) \left(1 + \sum_{k=1}^{\infty} Q^k Z_k(\epsilon_1, \epsilon_2, \tau, m_i)\right), \tag{2.4}$$

where $Z^{\text{top}}(\epsilon_1, \epsilon_2, \tau, t_b, m_i)$ is the topological string partition function that counts BPS configurations of M2 branes on the M-theory side (or, equivalently, 5d BPS states of the theory arising from M-theory compactification on X), and Z_k is the elliptic genus of k strings of the six-dimensional SCFT. Furthermore t_b is the Kähler class of the base $\mathbb{P}^1$ and Q is proportional to $e^{-t_b} = Q_b$[1]. In other words, the topological string partition function is given by a sum over elliptic genera of the six-dimensional strings, except for a simple piece Z_0 which captures contributions coming from vector multiplets and can be obtained straightforwardly.

In the next section we will discuss the case of the $\mathcal{O}(-4)$ theory and determine the quiver describing its strings. Furthermore, we will find an integral expression for the elliptic genera of these strings; we will evaluate these integrals explicitly for one and two strings and present an answer in a form from which BPS degeneracies may be readily extracted. In Section 4.2.3 we will compute the topological string partition function of the corresponding Calabi-Yau geometry and extract BPS invariants which can be shown to agree with the elliptic genus computations.

3 Quiver for the $\mathcal{O}(-4) \to \mathbb{P}^1$ model

We now turn to the strings of the $n = 4$ (1,0) SCFT in 6d and construct a quiver theory that describes their dynamics. Recall that the six-dimensional theory is obtained by compactifying F-theory on the following orbifold geometry:

$$CY_3 = \left(T^2 \times \mathbb{C} \times \mathbb{C}\right) / \mathbb{Z}_4, \tag{3.1}$$

where the orbifold action $\mathbb{Z}_4$ on the complex coordinates (z_1, z_2, z_3) of $T^2 \times \mathbb{C} \times \mathbb{C}$ is given by:

$$\begin{pmatrix} z_1 \\ z_1 \\ z_3 \end{pmatrix} \mapsto \begin{pmatrix} \omega^{-2} & & \\ & \omega & \\ & & \omega \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}, \tag{3.2}$$

[1]The proportionality factor will be a combination of Kähler classes in the resolution of the elliptic fiber and its exact form can be determined by requiring Q to be invariant under the monodromy associated to $SL(2, \mathbb{Z})$, as in [21].

and $\omega = i$. To obtain a F-theory construction in terms of a non-compact elliptic Calabi-Yau one has to first blow up the singularity at $\mathbb{C}^2/\mathbb{Z}_4$. The resulting space is described by the bundle

$$\mathcal{O}(-4) \longrightarrow \mathbb{P}^1, \tag{3.3}$$

with the singular elliptic fiber $T^2/\mathbb{Z}_2$ over the $\mathbb{P}^1$ base. The resolution of this fiber leads to the I_0^* fiber in the Kodaira classification of elliptic fibrations. In fact, one can obtain an infinite family of six-dimensional theories by taking the singular fiber to be of type I_p^*, with $p \geq 0$. Lowering p corresponds in physical terms to Higgsing. This geometry can be equivalently viewed in the weak coupling limit as a type IIB orientifold of the $\mathbb{C}^2/\mathbb{Z}_2$ singularity [13]. In this limit the singular elliptic fiber over $\mathbb{P}^1$ can be interpreted as the presence of $4+p$ D7-branes wrapping the $\mathbb{P}^1$ together with an orientifold 7-plane. This gives rise to a $\mathcal{N} = (1,0)$ $SO(8+2p)$ gauge theory in the six non-compact directions parallel to the branes. Furthermore, D3-branes wrapping the $\mathbb{P}^1$ give rise to strings in the six-dimensional theory.

In the following we study the worldsheet theory of these strings and obtain a quiver gauge theory description for it. The particular orientifold we are interested in has been studied in some detail in [22] and we shall describe it here briefly. The theory we want to study is type IIB theory on $\mathbb{C}^2/\mathbb{Z}_2$, modded out by $\Omega\Pi$, where Ω is world-sheet parity and Π acts as

$$\Pi : z_1 \to z_2, \quad z_2 \to -z_1, \tag{3.4}$$

with z_1, z_2 parametrizing the two complex planes in $\mathbb{C}^2/\mathbb{Z}_2$. The D7-branes wrapping the $\mathbb{P}^1$ can, in the singular limit,be thought of as D5-branes probing $\mathbb{C}^2/\mathbb{Z}_2$ together with an orientifold 5-plane at $z_1 = z_2 = 0$. Similarly, D3-branes become D1-branes whose worldvolume theory we wish to determine. We start by describing the brane system probing $\mathbb{C}^2$ and successively add the $\mathbb{Z}_2$ orbifold and $\mathbb{Z}_2$ orientifold actions. Before the orbifolding, the theory living on the D1-branes is a $\mathcal{N} = (4,4)$ $U(k)$ gauge theory with one adjoint and N fundamental hypermultiplets, where N denotes the number of D5-branes [5]. To summarize, we have the following massless field content on the worldvolume:

multiplet	bosons	fermions
vector	$b^{AY}, A_{\pm\pm}$	$\psi_-^{A'Y}, \psi_+^{AA'}$
adjoint hyper	$b^{A'\tilde{A}'}$	$\psi_-^{A\tilde{A}'}, \psi_+^{\tilde{A}'Y}$
fundamental hyper	$H^{A'}$	χ_-^{A}, χ_+^{Y}

(3.5)

where the indices $(A'\tilde{A}')$ represent the fundamental representations of the two $SU(2)$ groups rotating the directions X^2, X^3, X^4, X^5 (the directions orthogonal to D1 but parallel to D5) while (A, Y) are indices for the $SU(2)$'s rotating X^6, X^7, X^8, X^9 (the directions orthogonal both to D1 and D5). Next, we embed the $\mathbb{Z}_2$ orbifold action generated by

$$\begin{pmatrix} \omega & \\ & \omega^{-1} \end{pmatrix}, \quad \omega \in \mathbb{Z}_2 \tag{3.6}$$

into $SU(2)_Y$ and hence obtain the following action on fields with Y-index

$$(b^{AY}, \psi_+^{\tilde{A}'Y}, \psi_+^{\tilde{A}'Y}, \chi_+^Y) \mapsto (\omega b^{AY}, \omega\psi_-^{A'Y}, \omega\psi_+^{\tilde{A}'Y}, \omega\chi_+^Y). \tag{3.7}$$

The resulting theory has $\mathcal{N} = (4,0)$ supersymmetry and its field content can equally well be described in terms of $\mathcal{N} = (2,0)$ chiral superfields Σ, Φ, B, $\tilde{B}$, Q, $\tilde{Q}$, a $(2,0)$ gauge superfield Υ, and $(2,0)$ Fermi superfields $\Lambda^\Phi, \Lambda^B, \Lambda^{\tilde{B}}, \Lambda^Q, \Lambda^{\tilde{Q}}$. The decomposition of $\mathcal{N} =$ (4,0) fields in terms of $(2,0)$ components is as follows:

$$b^{AY} \leftrightarrow (\Sigma, \Phi), \quad b^{A'\tilde{A}'} \leftrightarrow (B, \tilde{B}), \quad H^{A'} \leftrightarrow (Q, \tilde{Q}), \tag{3.8}$$

$$\psi_+^{AA'} \leftrightarrow (\Lambda^\Phi, \Upsilon), \quad \psi_+^{\tilde{A}'Y} \leftrightarrow (\Lambda^B, \Lambda^{\tilde{B}}), \quad \chi_+^Y \leftrightarrow (\Lambda^Q, \Lambda^{\tilde{Q}}). \tag{3.9}$$

Following [23], the theory one obtains after the orbifold (3.7) is a quiver gauge theory whose gauge nodes correspond to the nodes of the affine A_1 quiver (for more general $\mathbb{Z}_N$ orbifolds one would obtain the affine A_{N-1} quiver). The fields that do not carry a Y index are localized at each node, while those with a Y index connect adjacent nodes [5]. In order to turn this D1 quiver into a D3 quiver one needs furthermore to turn off D1 brane charge and instead introduce D3 branes wrapped around blow-up cycles of the resolved A_1 singularity. This transformation corresponds to removing the last node of the inner quiver as well as all links ending on it. Correspondingly, in the case of the A_1 singularity which is of interest here, the single remaining $U(k)$ gauge node contains an $\mathcal{N} = (4,0)$ vector multiplet (Υ, Λ^ϕ) and an adjoint hypermultiplet $(B, \tilde{B})$.

Next, we come to the orientifolding. Orientifolds of orbifolds were discussed in [22]. There it was found that for $\mathbb{C}^2/\mathbb{Z}_n$ orbifolds the gauge group in the ω^l-twisted $(l = 0, \cdots, n-1)$ D-brane sector is of Sp-type if l is even and of SO-type if l is odd. This implies for our case that we have an orthogonal gauge group on the D5-branes in the untwisted sector and a symplectic one on the D5-branes of the ω-twisted sector. Furthermore, anomaly cancellation in six dimensions fixes the ranks of the gauge groups such that the allowed configurations are $SO(8+2p) \times Sp(p)$ [22]. This corresponds to having $4+p$ D5-branes together with an O5-plane at the Orbifold singularity. Uplifting this to F-theory one finds that the $p = 0$ case is obtained from the I_0^* fiber while the $p > 0$ cases come from I_p^* fiber types.

In fact, in the F-theory setup the six-dimensional theory has $SO(8+2p)$ gauge group and *two* $Sp(p)$ *flavor* nodes. The situation here is analogous to the $\mathcal{O}(-2) \to \mathbb{P}^1$ case: the two flavor nodes correspond to non-compact D7 branes intersecting the compact curve as shown in Figure 4 (see for example [3] for more details about the geometry).

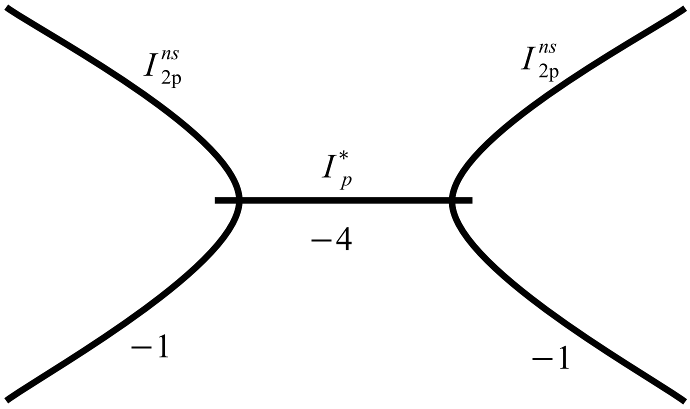


Figure 4: The local geometry that gives rise to the $SO(8+2p)$ 6d SCFT.

From the point of view of the two-dimensional theory living on the strings the $SO(8+2p)$ gauge node and the $Sp(p)$ flavor nodes descend to flavor nodes. Furthermore, orientifolding implies that (Υ, Λ^ϕ) transform in the symmetric (that is, adjoint) representation of $Sp(k)$[2], while $(B, \widetilde{B})$ transform in the antisymmetric representation [24]. It is interesting to note that the introduction of two $Sp(p)$ nodes is also necessary from gauge anomaly cancellation in two dimensions which will be reviewed later. The resulting two-dimensional quiver is the one depicted in Figure 5.

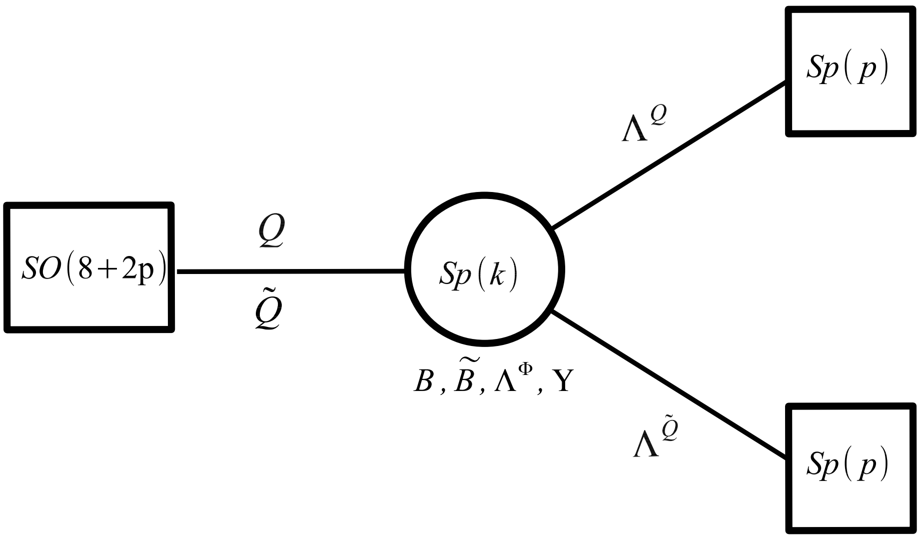


Figure 5: The $\mathbb{C}^2/\mathbb{Z}_4$ quiver.

The various fields in the quiver have different charges with respect to the two $U(1)_{\epsilon_1} \times U(1)_{\epsilon_2} \subset SO(4)$ that rotates the X^2, X^3, X^4, X^5 directions. We denote the fugacities by ϵ_1, ϵ_2, as they are the same parameters that appear in the Nekrasov partition function. For completeness we also present the charges of the fields of

[2]Orientifolding amounts to projecting the gauge group from $U(2k)$ to $Sp(k)$.

the quiver under the different $U(1)$'s and gauge groups; these charges are obtained directly by the orbifolding construction, as in [5].

	Λ^Φ	B	$\tilde{B}$	Q	$\tilde{Q}$	Λ^Q	$\Lambda^{\tilde{Q}}$
$Sp(k)$	symmetric	anti-symmetric	anti-symmetric	$\square$	$\overline{\square}$	$\square$	$\overline{\square}$
$U(1)_{\epsilon_1}$	-1	1	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0
$U(1)_{\epsilon_2}$	-1	0	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0

We have arrived at the conclusion that the theory for k strings is an $Sp(k)$ gauge theory with a (2,0) vector multiplet Υ and a Fermi multiplet Λ^Φ in the adjoint (i.e. symmetric) representation, two chiral multiplets $B, \widetilde{B}$ in the antisymmetric representation, and two chiral multiplets $Q, \widetilde{Q}$, each in the bifundamental representation of $Sp(k) \times SO(8+2p)$. If $p > 0$ one also has Fermi multiplets $\Lambda^Q, \Lambda^{\widetilde{Q}}$ in the bifundamental of $Sp(k) \times Sp(p)$. One can pick a basis $\{e_i\}_{i=1}^k$of the weight lattice of $Sp(k)$ in which the fundamental representation has weights $\pm e_i$ $(i = 1, \dots, k)$. In this basis, the symmetric representation has weights $e_i \pm e_j$ $(\forall i, j)$, while the antisymmetric representation has weights $e_i \pm e_j$ $(i \neq j)$. We also pick $\pm m_i$ to be the Cartan parameters dual to the weights of the fundamental representation of $SO(8+2p)$, and $\pm\mu_i$ and $\pm\tilde{\mu}_i$ to be the Cartan parameters for the two $Sp(p)$ flavor groups.

Let us next comment on gauge anomaly cancellation in two dimensions. The contribution of chiral fermions running in the loop to the anomaly is proportional to the index of their representation $T(R)$ defined as:

$$\mathrm{Tr}(T_R^a T_R^b) = T(R)\delta^{ab}. \tag{3.10}$$

Furthermore, left-moving fermions contribute with a positive sign to the anomaly while right-moving ones contribute with a negative sign. Thus, for our particular quiver we obtain the following result:

$$\begin{aligned} a(L) - a(R) &= T_B(\text{anti-sym}) + T_{\tilde{B}}(\text{anti-sym}) + (n+4)(T_Q(\square) + T_{\tilde{Q}}(\square) \\ &\quad -T_\Upsilon(\text{sym}) - T_{\Lambda^\phi}(\text{sym}) - n(T_{\Lambda^Q}(\square) + T_{\Lambda^{\tilde{Q}}}(\square) \\ &= 2k-2+2k-2+(n+4)(1+1) - (2k+2) - (2k+2) - n(1+1) \\ &= 0, \end{aligned} \tag{3.11}$$

where use has been made of the identities

$$T(\text{anti-sym}) = 2k-2, \quad T(\text{sym}) = 2k+2, \quad T(\square) = 1, \tag{3.12}$$

and the fact that the fundamental fields transform in real representations and therefore only have half the number of degrees of freedom.

3.1 Localization computation

Having written down the field content of the two-dimensional theory of k strings, it is straightforward to compute its elliptic genus, following the localization computation of [19, 20]. The elliptic genus is given by a contour integral of a one-loop determinant:

$$Z_{k\text{ strings}} = \int Z^{1-loop}_{k\text{ strings}}(z_i, m_j, \mu_i, \tilde{\mu}_i, \epsilon_1, \epsilon_2), \tag{3.13}$$

where $Z^{1-loop}_{k\text{ strings}}$ is a k-form on the k complex-dimensional space of flat $Sp(k)$ connections on T^2, which is a complex torus parametrized by variables $\zeta_i = \frac{1}{2\pi i}\log z_i$, and the contour of integration is determined by the Jeffrey-Kirwan prescription [25]. The one-loop determinant is obtained by multiplying together the contributions of all multiplets and takes the following form:

$$Z^{1-loop}_{k\text{ strings}} = Z_\Upsilon\, Z_{\Lambda^\Phi}\, Z_B Z_{\tilde{B}}\, Z_Q Z_{\tilde{Q}}\, Z_{\Lambda^Q}\, Z_{\Lambda^{\tilde{Q}}},$$

where[3]

$$Z_\Upsilon = \left(\prod_{i=1}^{k}\frac{d\zeta_i\,\theta_1'(0)\theta_1(z_i^2)\theta_1(z_i^{-2})}{\eta^3}\right)\left(\prod_{i<j}^{k}\prod_{s_1=\pm1,\,s_2=\pm1}\frac{\theta_1(z_i^{s_1}z_j^{s_2})}{\eta}\right) \tag{3.15}$$

$$Z_{\Lambda^\Phi} = \left(\prod_{i=1}^{k}\frac{\theta_1(dt)\theta_1(dtz_i^2)\theta_1(dtz_i^{-2})}{\eta^3}\right)\left(\prod_{i<j}^{k}\prod_{s_1=\pm1,\,s_2=\pm1}\frac{\theta_1(dtz_i^{s_1}z_j^{s_2})}{\eta}\right) \tag{3.16}$$

$$Z_B Z_{\tilde{B}} = \left(\frac{\eta^2}{\theta_1(d)\theta_1(t)}\right)^k\left(\prod_{i<j}^{k}\prod_{s_1=\pm1,\,s_2=\pm1}\frac{\eta^2}{\theta_1(dz_i^{s_1}z_j^{s_2})\theta_1(tz_i^{s_1}z_j^{s_2})}\right) \tag{3.17}$$

$$Z_Q Z_{\tilde{Q}} = \prod_{i=1}^{k}\prod_{j=1}^{p+4}\frac{\eta^4}{\theta_1(\sqrt{dt}z_iQ_{m_j})\theta_1(\sqrt{dt}z_i^{-1}Q_{m_j})\theta_1(\sqrt{dt}z_iQ_{m_j}^{-1})\theta_1(\sqrt{dt}z_i^{-1}Q_{m_j}^{-1})} \tag{3.18}$$

$$Z_{\Lambda^Q}Z_{\Lambda^{\tilde{Q}}} = \prod_{i=1}^{k}\prod_{j=1}^{p}\frac{\theta_1(z_iQ_{\mu_j})\theta_1(z_i^{-1}Q_{\mu_j})\theta_1(z_iQ_{\tilde{\mu}_j})\theta_1(z_i^{-1}Q_{\tilde{\mu}_j})}{\eta^4}, \tag{3.19}$$

and $Q_{m_i} = e^{2\pi i m_i}$, $Q_{\mu_i} = e^{2\pi i\mu_i}$, $Q_{\tilde{\mu}_i} = e^{2\pi i\tilde{\mu}_i}$, $d = e^{2\pi i\epsilon_1}$, $t = e^{2\pi i\epsilon_2}$. The integral itself is then obtained by computing a sum over Jeffrey-Kirwan residues of the one-loop determinant:

$$Z_{k\text{ strings}} = \frac{1}{|\mathrm{Weyl}[Sp(k)]|}\sum_{\alpha}\text{JK-res}(\alpha,\mathfrak{q})Z^{1-loop}_{k\text{ strings}}, \tag{3.20}$$

---

[3] We use the following definitions for the Dedekind eta function $\eta(\tau)$ and Jacobi theta function $\theta_1(z,\tau)$:

$$\eta(\tau) = q^{1/24}\prod_{j=1}^{\infty}(1-q^j); \qquad \theta_1(z,\tau) = -iq^{1/8}\sqrt{z}\prod_{j=1}^{\infty}(1-q^j)(1-z^{-1}q^j)(1-z^{-1}q^j), \tag{3.14}$$

where $q = e^{2\pi i\tau}$.

where α labels poles of $Z^{1-loop}_{k \text{ strings}}$ and the role of $\mathfrak{q}$ will be clarified shortly. In the following sections we will compute the residue sum for one and two strings, in which case the evaluation of Jeffrey-Kirwan residues turns out to be straightforward and we do not need to resort to the full-fledged formalism.

One string

For a single string, the one-loop determinant is given by a one-form:

$$
\begin{aligned}
Z^{1-loop}_{1 \text{ string}} = d\zeta & \frac{2\pi i \eta^2 \theta(z^2)\theta(z^{-2})\theta(dt)\theta(dtz^2)\theta(dtz^{-2})}{\theta(d)\theta(t)\eta^3} \\
& \times \prod_{i=1}^{p} \frac{\theta(Q_{\mu_i} z)\theta(Q^{-1}_{\mu_i} z)\theta(Q_{\tilde{\mu}_i} z)\theta(Q^{-1}_{\tilde{\mu}_i} z)}{\eta^4} \\
& \times \prod_{i=1}^{4+p} \frac{\eta^4}{\theta(\sqrt{dt}Q_{m_i} z)\theta(\sqrt{dt}Q^{-1}_{m_i} z)\theta(\sqrt{dt}Q_{m_i} z^{-1})\theta(\sqrt{dt}Q^{-1}_{m_i} z^{-1})}.
\end{aligned}
\tag{3.21}
$$

One first needs to identify the singular loci of the integrand. Each of the theta functions in the second line of (3.21) determines a (0-dimensional) singular hyperplane within the one complex dimensional space $\mathfrak{M}_{1 \text{ string}}$ spanned by $\zeta = \log z$, for a total of $4 \cdot (4 + p)$ distinct singular points at

$$
\pm \zeta + \frac{\epsilon_1 + \epsilon_2}{2} \pm m_i = 0, \;\; i = 1, \dots 4 + p.\text{[4]}
\tag{3.22}
$$

To determine which poles contribute to the residue sum, one needs to consider the normal vectors to the singular hyperplanes. In this case, the normal vector is simply $\pm\partial_\zeta$, where the sign is the one multiplying ζ in (3.22). The data that enters the Jeffrey-Kirwan residue computation corresponds of two quantities: the position of the pole in the ζ plane and a choice of a vector $\mathfrak{q} \in \mathrm{T}\mathfrak{M}_{1 \text{ string}}$. In this case, we can choose either $\mathfrak{q} = \pm\partial\zeta$; let us pick $\mathfrak{q} = -\partial_\zeta$. For two-dimensional theories, it can be argued that once the sum over residues is performed the answer is independent of the choice of $\mathfrak{q}$. Next, one picks the poles satisfying the property that $\mathfrak{q}$ lies within the one-dimensional cone spanned by the vector normal to the corresponding hyperplane. In this trivial example one finds that only the following poles contribute to the integral:

$$
-\zeta + \frac{\epsilon_1 + \epsilon_2}{2} \pm m_i = 0.
\tag{3.23}
$$

Evaluating the Jeffrey-Kirwan residues in this situation corresponds to summing over the ordinary residues at these poles. Summing over the eight residues and dividing

[4] Since $\zeta \sim \zeta + 1 \sim \zeta + \tau$, each theta function leads to a single pole.

by Weyl$[Sp(1)] = \mathbb{Z}_2$ leads to the following answer:

$$Z_{1\text{ string}} = \frac{1}{2}\frac{\eta^2}{\theta(d)\theta(t)} \times \sum_{i=1}^{4+p}\left[\frac{\theta(dtQ_{m_i}^2)\theta(d^2t^2Q_{m_i}^2)}{\eta^2}\prod_{j\neq i}\prod_{s=\pm 1}\frac{\eta^2}{\theta(Q_{m_i}Q_{m_j}^s)\theta(dtQ_{m_i}Q_{m_j}^s)}\right.$$
$$\left.\times\prod_{j=1}^{p}\prod_{s=\pm 1}\frac{\theta(\sqrt{dt}Q_{m_i}Q_{\mu_j}^s)\theta(\sqrt{dt}Q_{m_i}Q_{\tilde{\mu}_j}^s)}{\eta^2} + (Q_{m_i}\to Q_{m_i}^{-1})\right]. \tag{3.24}$$

Note some features of this expression: The existence of theta functions in the denominator which depend on $SO(8+2p)$ fugacities suggests that the $SO(8+2p)$ continues to be carried by some bosonic degrees of freedom in the IR. Also, the fact that the expressions include a mixture of ϵ_i (captured by t, d) and m_i suggests a non-trivial structure for the theory which makes it unlikely to correspond to a free theory in the IR. It would be interesting to identify the non-trivial $(4,0)$ CFT whose elliptic genus is given by the above expression. Perhaps ideas similar to the ones employed in [26] can be used to do this.

In Section 3.3 we will explain how to extract from this expression the BPS degeneracies corresponding to a single string; for the $p = 0$ case, one finds a precise match with the BPS invariants of the geometry that engineers the $\mathcal{O}(-4) \to \mathbb{P}^1$ SCFT, to be discussed in Section 4.

Two strings

The computation for two strings proceeds analogously; first, one should identify the hyperplanes in the two-dimensional space $\mathfrak{M}_{2\text{ strings}}$ along which the denominator of $Z_{2\text{ strings}}^{1-loop}$ vanishes. There are $8(p+5)$ such hyperplanes:

$$\pm\zeta_j + \frac{\epsilon_1+\epsilon_2}{2} \pm m_i = 0,\ \ i = 1,\ldots 4+p,\ \ j = 1,2; \tag{3.25}$$
$$\pm\zeta_1 \pm \zeta_2 + \epsilon_1 = 0; \tag{3.26}$$
$$\pm\zeta_1 \pm \zeta_2 + \epsilon_2 = 0, \tag{3.27}$$

where $\zeta_i = \log(z_i)$. For concreteness, let us focus from now on to the case where $p = 0$, keeping in mind that the computation for arbitrary p proceeds analogously. We display the vectors normal to the hyperplanes, as well as our choice of $\mathfrak{q} \in \mathrm{T}\mathfrak{M}_{2\text{ strings}}$, in Figure 6. The next step is to identify the points at which hyperplanes intersect. The computation of Jeffrey-Kirwan residues is simplified by the fact that for generic values of $m, \epsilon_1, \epsilon_2$ at most two hyperplanes intersect at the same time. The poles whose residues contribute to the elliptic genus are those for which $\mathfrak{q}$ lies within the cone spanned by the vectors normal to the corresponding hyperplanes. For example, since $\mathfrak{q}$ lies in the cone spanned by $-\partial_{\zeta_1}$ and $-\partial_{\zeta_1} - \partial_{\zeta_2}$, but not in the one spanned by $-\partial_{\zeta_1}$ and $-\partial_{\zeta_1} + \partial_{\zeta_2}$, the residue evaluated at

$$-\zeta_1 + \frac{\epsilon_1+\epsilon_2}{2} + m_1 = 0; \qquad -\zeta_1 - \zeta_2 + \epsilon_1 = 0, \tag{3.28}$$

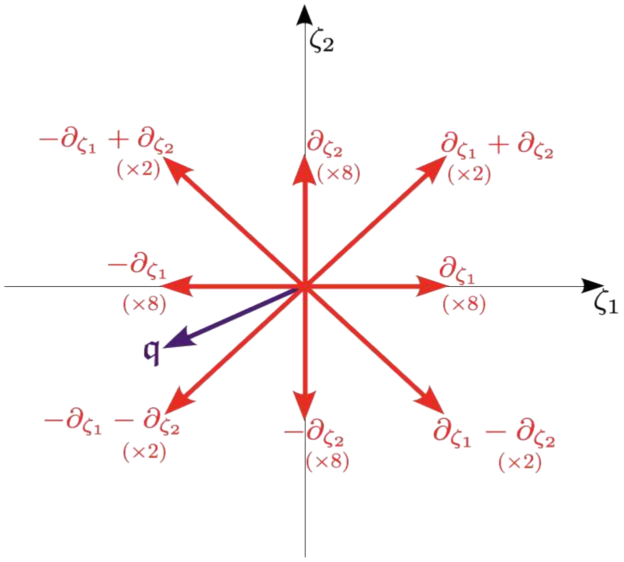


Figure 6: Singular hyperplane configuration for the two-string elliptic genus. The vectors normal to the singular hyperplanes are displayed, along with the multiplicity with which they occur. Our choice of $\mathfrak{q}$ is also displayed here.

will contribute, while the one at

$$-\zeta_1+\frac{\epsilon_1+\epsilon_2}{2}+m_1=0; \qquad -\zeta_1+\zeta_2+\epsilon_1=0 \tag{3.29}$$

will not. Following this prescription, one arrives at the following list of poles whose residues contribute to the computation:

$$\alpha_{i,j,s}: \quad \zeta_1=\frac{\epsilon_1+\epsilon_2}{2}+sm_i, \quad \zeta_2=\zeta_1+\epsilon_j; \qquad (i=1,\ldots,4,\ j=1,2,\ s=\pm1) \tag{3.30}$$

$$\alpha'_{i,j,s}: \quad \zeta_2=\frac{\epsilon_1+\epsilon_2}{2}+sm_i, \quad \zeta_1=\zeta_2+\epsilon_j; \qquad (i=1,\ldots,4,\ j=1,2,\ s=\pm1) \tag{3.31}$$

$$\alpha''_{i,j,s}: \quad -\zeta_2=\frac{\epsilon_1+\epsilon_2}{2}+sm_i, \quad \zeta_1=-\zeta_2+\epsilon_j; \qquad (i=1,\ldots,4,\ j=1,2,\ s=\pm1) \tag{3.32}$$

$$\beta_{i,j,s}: \quad \zeta_1=\frac{\epsilon_1+\epsilon_2}{2}+sm_i, \quad \zeta_2=-\zeta_1+\epsilon_j; \qquad (i=1,\ldots,4,\ j=1,2,\ s=\pm1) \tag{3.33}$$

$$\gamma_{i,j,s_1,s_2}: \quad \zeta_1=\frac{\epsilon_1+\epsilon_2}{2}+s_1m_i, \quad \zeta_2=\frac{\epsilon_1+\epsilon_2}{2}+s_2m_j. \qquad (i=1,\ldots,4,\ j\neq i, s_1=\pm1,\ s_2=\pm1) \tag{3.34}$$

The prescription outlined above also picks up some additional poles, but they do not contribute to the elliptic genus since the numerator of $Z_{2\text{ strings}}^{1-loops}$ turns out to vanish for them. Therefore, the elliptic genus of two strings is obtained by summing over the residues that correspond the 112 poles listed in Equations (3.30)–(3.34). In practice, one can exploit $Sp(2)$ Weyl symmetry to show that the residues of poles (3.31) and (3.32) are identical to the ones of (3.30). For the same reason, one can set $j < i$ in (3.34) and multiply the corresponding 24 residues by a factor of 2.

After these considerations, we are ready to write down the elliptic genus of two strings:

$$Z_{2\text{ strings}} = \frac{1}{8}\left[3\sum_{\alpha_{i,j,s}} \text{Res}_{\alpha_{i,j,s}} Z_{2\text{ str.}}^{1-l.} + \sum_{\beta_{i,j,s}} \text{Res}_{\beta_{i,j,s}} Z_{2\text{ str.}}^{1-l.} + 2\sum_{\substack{\gamma_{i,j,s_1,s_2}\\ j<i}} \text{Res}_{\gamma_{i,j,s_1,s_2}} Z_{2\text{ str.}}^{1-l.}\right], \tag{3.35}$$

where we have divided by an overall factor of $8 = |\text{Weyl}[Sp(2)]|$, and the residues have the following explicit form:

$$\begin{aligned}\text{Res}_{\alpha_{i,1,s}} Z_{2\text{ strings}}^{1-loop} = {} & \frac{\theta_1(d^2t\, Q_{m_i}^{2s})\theta_1(d^3t\, Q_{m_i}^{2s})\theta_1(d^3t^2\, Q_{m_i}^{2s})\theta_1(d^2t^2\, Q_{m_i}^{2s})}{\theta_1(d)\theta_1(t)\theta_1(d^2)\theta_1(t/d)} \\ & \times \prod_{j\neq i}\prod_{r=\pm1} \frac{\eta^4}{\theta_1(Q_{m_i}^s Q_{m_j}^r)\theta_1(d\, Q_{m_i}^s Q_{m_j}^r)\theta_1(dt\, Q_{m_i}^s Q_{m_j}^r)\theta_1(d^2t\, Q_{m_i}^s Q_{m_j}^r)};\end{aligned} \tag{3.36}$$

$$\begin{aligned}\text{Res}_{\alpha_{i,2,s}} Z_{2\text{ strings}}^{1-loop} = {} & \frac{\theta_1(dt^2\, Q_{m_i}^{2s})\theta_1(dt^3\, Q_{m_i}^{2s})\theta_1(d^2t^3\, Q_{m_i}^{2s})\theta_1(d^2t^2\, Q_{m_i}^{2s})}{\theta_1(d)\theta_1(t)\theta_1(t^2)\theta_1(d/t)} \\ & \times \prod_{j\neq i}\prod_{r=\pm1} \frac{\eta^4}{\theta_1(Q_{m_i}^s Q_{m_j}^r)\theta_1(t\, Q_{m_i}^s Q_{m_j}^r)\theta_1(dt\, Q_{m_i}^s Q_{m_j}^r)\theta_1(dt^2\, Q_{m_i}^s Q_{m_j}^r)};\end{aligned} \tag{3.37}$$

$$\begin{aligned}\text{Res}_{\beta_{i,1,s}} Z_{2\text{ strings}}^{1-loop} = {} & \frac{\theta_1(d^2t)\theta_1(t\, Q_{m_i}^{2s})\theta_1(t/d\, Q_{m_i}^{2s})\theta_1(d^2\, Q_{m_i}^{-2s})\theta_1(dt^2\, Q_{m_i}^{2s})\theta_1(d^2t^2\, Q_{m_i}^{2s})}{\theta_1(d)\theta_1(t)^2\theta_1(d^2)\theta_1(t/d)\theta_1(Q_{m_i}^{2s})} \\ & \times \prod_{j\neq i}\prod_{r=\pm1} \frac{\eta^4}{\theta_1(Q_{m_i}^s Q_{m_j}^r)\theta_1(d\, Q_{m_i}^s Q_{m_j}^r)\theta_1(t\, Q_{m_i}^s Q_{m_j}^r)\theta_1(dt\, Q_{m_i}^s Q_{m_j}^r)};\end{aligned} \tag{3.38}$$

$$\begin{aligned}\text{Res}_{\beta_{i,2,s}} Z_{2\text{ strings}}^{1-loop} = {} & \frac{\theta_1(dt^2)\theta_1(d\, Q_{m_i}^{2s})\theta_1(d/t\, Q_{m_i}^{2s})\theta_1(t^2\, Q_{m_i}^{-2s})\theta_1(d^2t\, Q_{m_i}^{2s})\theta_1(d^2t^2\, Q_{m_i}^{2s})}{\theta_1(d)^2\theta_1(t)\theta_1(t^2)\theta_1(d/t)\theta_1(Q_{m_i}^{2s})} \\ & \times \prod_{j\neq i}\prod_{r=\pm1} \frac{\eta^4}{\theta_1(Q_{m_i}^s Q_{m_j}^r)\theta_1(d\, Q_{m_i}^s Q_{m_j}^r)\theta_1(t\, Q_{m_i}^s Q_{m_j}^r)\theta_1(dt\, Q_{m_i}^s Q_{m_j}^r)};\end{aligned} \tag{3.39}$$

$$
\begin{aligned}
\mathrm{Res}_{\gamma_{i,j,s_1,s_2}} Z^{1-loop}_{2\text{ strings}} &= \frac{\theta_1(dt\,Q^{2s_1}_{m_i})\theta_1(dt\,Q^{2s_2}_{m_j})\theta_1(d^2t^2Q^{2s_1}_{m_i})\theta_1(d^2t^2Q^{2s_2}_{m_j})}{\theta_1(d)^2\theta_1(t)^2}\frac{\theta_1(d^2t^2Q^{s_1}_{m_i}Q^{s_2}_{m_j})}{\theta_1(Q^{s_1}_{m_i}Q^{s_2}_{m_j})} \\
&\times \eta^8\Big[\theta_1(d\,Q^{s_1}_{m_i}Q^{s_2}_{m_j})\theta_1(t\,Q^{s_1}_{m_i}Q^{s_2}_{m_j})\theta_1(d\,Q^{s_1}_{m_i}Q^{-s_2}_{m_j})\theta_1(t\,Q^{s_1}_{m_i}Q^{-s_2}_{m_j}) \\
&\times \theta_1(d\,Q^{-s_1}_{m_i}Q^{s_2}_{m_j})\theta_1(t\,Q^{-s_1}_{m_i}Q^{s_2}_{m_j})\theta_1(d^2t\,Q^{s_1}_{m_i}Q^{s_2}_{m_j})\theta_1(dt^2\,Q^{s_1}_{m_i}Q^{s_2}_{m_j})\Big]^{-1} \\
&\times \prod_{k\neq i,j}\prod_{r=\pm1}\frac{\eta^4}{\theta_1(Q^{s_1}_{m_i}Q^{r}_{m_k})\theta_1(Q^{s_2}_{m_j}Q^{r}_{m_k})\theta_1(dt\,Q^{s_1}_{m_i}Q^{r}_{m_j})\theta_1(dt\,Q^{s_2}_{m_j}Q^{r}_{m_j})}.
\end{aligned}
\tag{3.40}
$$

After summing over the 56 residues, one is left with a weight zero meromorphic elliptic function with modular parameter τ and six elliptic parameters (ϵ_1, ϵ_2 and the SO(8) fugacities $(m_1, \ldots, m_4)$). In Section 4.2.3 we will check the validity of our answer in the unrefined limit $\epsilon_2 = -\epsilon_1$ by verifying that it exactly reproduces the genus 0 BPS invariants of the Calabi-Yau threefold that engineers the six-dimensional theory under consideration. If ϵ_1, ϵ_2 are left arbitrary, Equation (3.35) can be used to compute arbitrary genus refined BPS invariants of this geometry.

For higher numbers of strings the computation of the elliptic genus from Equation (3.20) proceeds analogously, but for simplicity and clarity of exposition we limit our discussion to the cases of one and two strings.

3.2 Modular anomaly

In this section we wish to study the behavior of the elliptic genus under $SL(2,\mathbb{Z})$ transformations

$$
\gamma : (t_b, \tau, m_i, \mu_i, \epsilon_i) \to \left(t_b, \frac{a\tau+b}{c\tau+d}, \frac{m_i}{c\tau+d}, \frac{\mu_i}{c\tau+d}, \frac{\epsilon_i}{c\tau+d}\right), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}). \tag{3.41}
$$

The modular properties of the elliptic genus can be best understood starting from the integral expression (3.13), where the integration variables ζ_i also transform as elliptic parameters: $\zeta_i \to \zeta_i/(c\tau+d)$. Each $(2,0)$ multiplet contributes to the integrand a factor of the form $\left(\frac{\theta_1(y,\tau)}{\eta(\tau)}\right)^{\pm1}$. Recall that under S and T transformations

$$
\frac{\theta_1(e^{2\pi i\zeta}, \tau+1)}{\eta(\tau+1)} = \frac{e^{\pi i/4}\theta_1(e^{2\pi i\zeta}, \tau)}{e^{\pi i/12}\eta(\tau)} = e^{\pi i/6}\frac{\theta_1(e^{2\pi i\zeta}, \tau)}{\eta(\tau)}, \tag{3.42}
$$

$$
\frac{\theta(e^{2\pi i\zeta/\tau}, -1/\tau)}{\eta(-1/\tau)} = \frac{e^{-3\pi i/4}e^{\frac{\pi i}{\tau}\zeta^2}\theta_1(e^{2\pi i\zeta}, \tau)}{e^{-\pi i/4}\eta(\tau)} = e^{-\pi i/2}e^{\frac{\pi i}{\tau}\zeta^2}\frac{\theta_1(e^{2\pi i\zeta}, \tau)}{\eta(\tau)}. \tag{3.43}
$$

Using this, one can easily check that the elliptic genus of k strings is invariant under $\tau \to \tau+1$:

$$
Z_{k\text{ strings}}(\tau+1) = Z_{k\text{ strings}}(\tau); \tag{3.44}
$$

on the other hand, under $\tau \to -1/\tau$ one can show that the integrand picks up a z_i-independent phase:

$$\frac{Z_{k\text{ strings}}(-1/\tau)}{Z_{k\text{ strings}}(\tau)} = \exp\Bigg[-\frac{\pi i}{\tau}\Bigg(\epsilon_1\epsilon_2\left(4k^2-2k\right) - \left(\epsilon_1+\epsilon_2\right)^2 k(2+p) \\ - 4k\sum_{j=1}^{4+p} m_j^2 + 2k\sum_{j=1}^{p}(\mu_j^2+\widetilde{\mu}_j^2)\Bigg)\Bigg]. \qquad (3.45)$$

In other words, $Z_{k\text{ strings}}$ transforms as a modular function up to an anomalous phase factor. The origin of this factor can be easily understood by considering the following representation of the theta function:

$$\theta(z,\tau) = \eta(\tau)^3(2\pi\zeta)\exp\left(\sum_{k\geq 1}\frac{B_{2k}}{(2k)(2k)!}E_{2k}(\tau)(2\pi i\zeta)^{2k}\right), \qquad (3.46)$$

where the dependence on the modular parameter τ is expressed in terms of the Eisenstein series

$$E_{2k}(\tau) = \frac{1}{2\zeta(2k)}\sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)}\frac{1}{m\tau+n}, \qquad k\geq 1 \qquad (3.47)$$

and $\zeta(z)$ is the Riemann zeta function. For any $k\geq 2$, $E_{2k}(\tau)$ is a modular form of weight $2k$. On the other hand, $E_2(\tau)$ transforms anomalously:

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2E_2(\tau) - \frac{6ic}{\pi}(c\tau+d). \qquad (3.48)$$

In other words, the phase factors appearing in Equation (3.45) are completely determined by the $E_2(\tau)$-dependence of the integrand, and in lieu of (3.45) we might as well have written:

$$\partial_{E_2}Z_{k\text{ strings}} = -\frac{1}{24}(2\pi)^2\Bigg(\epsilon_1\epsilon_2\left(4k^2-2k\right) - \left(\epsilon_1+\epsilon_2\right)^2 k(2+p) \\ - 4k\sum_{j=1}^{4+p} m_j^2 + 2k\sum_{j=1}^{p}(\mu_j^2+\widetilde{\mu}_j^2)\Bigg)Z_{k\text{ strings}}. \qquad (3.49)$$

This expression is very similar to the E-string and M-string modular anomaly equations found in [4, 14, 27]: in the E-string ($\mathcal{O}(-1)\to\mathbb{P}^1$) case, one has:

$$\frac{1}{Z_E^d}\partial_{E_2}Z_E^d = -\frac{1}{24}(2\pi)^2(\epsilon_1\epsilon_2(k^2+k) - k(\epsilon_1+\epsilon_2)^2 + k(\sum_i m_i^2)),$$

while in the M-string ($\mathcal{O}(-2)\to\mathbb{P}^1$) case, one finds:

$$\frac{1}{Z_M^k}\partial_{E_2}Z_M^d = -\frac{1}{12}(2\pi)^2(\epsilon_1\epsilon_2k^2 - \frac{k}{4}(\epsilon_1+\epsilon_2)^2 + k\,m^2).$$

As discussed in Section 2, in all these cases the elliptic genera of the strings capture part of the topological string partition function of the corresponding Calabi-Yau X:

$$Z^{\text{top}}(X) = Z_0(X) \cdot \left(1 + \sum_{k=1}^{\infty} Q^k Z_{k \text{ strings}}(X)\right). \tag{3.50}$$

In all cases, X is elliptically fibered, and the topological string partition function is expected to be invariant under modular transformations (3.45). However, this is in contradiction with the fact that $Z_{k \text{ strings}}$ is only invariant up to a phase. The resolution to this apparent contradiction is well known: in the topological string expression the second Eisenstein series $E_2(\tau)$ should be replaced by its modular completion

$$\widehat{E}_2(\tau, \overline{\tau}) = E_2(\tau) - \frac{6i}{\pi(\tau - \overline{\tau})}, \tag{3.51}$$

which under the $SL(2,\mathbb{Z})$ action transforms as follows:

$$\widehat{E}_2\left(\frac{a\tau + b}{c\tau + d}, \frac{a\overline{\tau} + b}{c\overline{\tau} + d}\right) = (c\tau + d)^2 \widehat{E}_2(\tau, \overline{\tau}). \tag{3.52}$$

This implies that the topological string partition function is a well defined modular function of τ, but no longer depends holomorphically on it:

$$\partial_{\overline{\tau}} Z_{top}(X) = \frac{6i}{\pi(\tau - \overline{\tau})^2} \partial_{\widehat{E}_2} Z_{top}(X) \neq 0. \tag{3.53}$$

3.3 Refined BPS invariants

Let us now explain how to extract refined BPS invariants from the elliptic genera of k strings Z_k, again specializing to the case $p = 0$ where the global symmetry on the worldsheet is just $SO(8)$. In order to proceed note that the full partition function of the topological string is given by

$$Z^{\text{top}} = e^F = Z_0\left(1 + \sum_{k=1}^{\infty} Q^k Z_k\right), \tag{3.54}$$

where Z_k is the elliptic genus of k strings and Q is a combination of exponentiated Kähler moduli of the elliptic Calabi-Yau geometry to be determined later. Furthermore, we perform the following change of basis which replaces the mass parameters Q_{m_i} by the parameters Q_i corresponding to a choice of simple roots of $SO(8)$:

$$Q_{m_1} = Q_1 Q_c \sqrt{Q_2}\sqrt{Q_3}, \quad Q_{m_2} = Q_c \sqrt{Q_2}\sqrt{Q_3}, \quad Q_{m_3} = \sqrt{Q_2}\sqrt{Q_3}, \quad Q_{m_4} = \frac{\sqrt{Q_3}}{\sqrt{Q_2}}. \tag{3.55}$$

In addition to these parameters, let us also define the parameter Q_4 corresponding to the affine node of the extended Dynkin diagram of type D_4 as shown in Figure

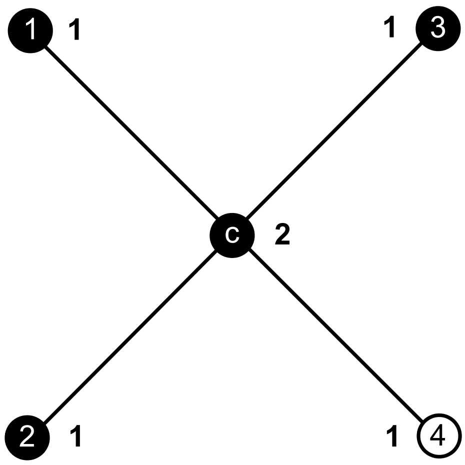


Figure 7: Extended Dynkin diagram for D_4.

7. The elliptic genera Z_k can then be expanded in positive powers of Q_1, Q_2, Q_3, Q_4 and Q_c upon replacing $Q_\tau = e^{2\pi i\tau}$ by the following combination:

$$Q_\tau = Q_1 Q_2 Q_3 Q_4 Q_c^2, \tag{3.56}$$

where the powers are determined by the Coxeter labels of the nodes in Figure 7.

Taking the logarithm of Equation (3.54), the free energy F can be expanded as:

$$F = \log Z = \log(Z_0) + Z_1 Q + \left(-\frac{1}{2}Z_1^2 + Z_2\right) Q^2 + \left(\frac{Z_1^3}{3} - Z_1 Z_2 + Z_3\right) Q^3 + \mathcal{O}(Q^4). \tag{3.57}$$

In order to make contact with the computation of the Calabi-Yau BPS invariants from Section 4 we identify Q with the combination

$$Q = Q_b \frac{Q_4}{Q_1 Q_2 Q_3 Q_c^2}, \tag{3.58}$$

where $Q_b = e^{-t_b}$ and t_b is the Kähler class of the base of the elliptic fibration. The refined BPS invariants are encoded in the free energy F as follows [28, 29]

$$F = \sum_{\substack{j_l, j_R = 0\\ m=1}}^{\infty} \sum_{\beta \in H_2(M,\mathbb{Z})} \frac{n_\beta^{j_L,j_R}}{m} \frac{(-1)^{2(j_L+j_R)}\sqrt{d^m t^m}\left(\sum_{n=-j_L}^{j_L}(d/t)^{mn}\right)\left(\sum_{n=-j_R}^{j_R}(dt)^{mn}\right) e^{m(\beta,\underline{t})}}{(1-d^m)(1-t^m)}, \tag{3.59}$$

where $\underline{t}$ denote the Kähler moduli of the Calabi-Yau. The above free energy encodes BPS degeneracies $n_\beta^{j_L,j_R}$ of short multiplets of the five-dimensional quantum field theory arising from circle compactification of the six-dimensional SCFT. In this context the labels j_L and j_R refer to the spins of the two $SU(2)$ subgroups of the little group $SO(4)$ in the decomposition $SO(4) = SU(2)_L \times SU(2)_R$ and β labels the string charge as well as the various flavor charges. Denoting by $\underline{t_f}$ the collection of the Kähler moduli of the resolved elliptic fiber and making use of the expansion

$$F(\epsilon_1, \epsilon_2, \underline{t}) = F_0(\epsilon_1, \epsilon_2, \underline{t_f}) + \sum_{i=1}^{\infty} F_i(\epsilon_1, \epsilon_2, \underline{t_f}) Q^i, \tag{3.60}$$

we find

$$\begin{aligned} F_1(\epsilon_i, m_i, \tau) &= Z_1(\epsilon_i, m_i, \tau) \\ F_2(\epsilon_i, m_i, \tau) &= Z_2(\epsilon_i, m_i, \tau) - \frac{1}{2} Z_1(\epsilon_i, m_i, \tau)^2 \\ &\vdots \end{aligned} \tag{3.61}$$

Since for F_1 there is no multi-wrapping, we can set $m = 1$ in Equation (3.59) and extract the invariants $n_\beta^{j_L, j_R}$ immediately from the expression (3.24) for the elliptic genus of one string. Let us specify β in terms of the following basis of $H^2(X, \mathbb{Z})$: $J_b, J_1, J_2, J_3, J_4, J_c$; that is, we write:

$$n_\beta^{j_L, j_R} = n_{n_b, n_1, n_2, n_3, n_4, n_c}^{j_L, j_R}. \tag{3.62}$$

In the following tables we present a sample of invariants for some specific choices of low degree curves.

$2j_L \backslash 2j_R$	0	1	2	3	4	5	6	7
0	0	33	0	28	0	9	0	1
1	2	0	3	0	1	0	0	0

$n_{1,2,2,1,1,3}^{j_L, j_R}$

$2j_L \backslash 2j_R$	0	1	2	3	4	5	6	7
0	0	28	0	42	0	29	0	9
1	1	0	3	0	3	0	1	0

$n_{1,2,2,1,1,4}^{j_L, j_R}$

$2j_L \backslash 2j_R$	0	1	2	3	4	5	6	7
0	0	10	0	11	0	6	0	1
1	0	0	0	0	0	0	0	0

$n_{1,3,3,1,1,3}^{j_L, j_R}$

$2j_L \backslash 2j_R$	0	1	2	3	4	5	6	7
0	0	41	0	47	0	28	0	9
1	2	0	4	0	3	0	1	0

$n_{1,3,3,1,1,4}^{j_L, j_R}$

Analogously, we can extract all refined invariants for two strings, that is for base wrapping number $n_b = 2$. For example:

$2j_L \backslash 2j_R$	0	1	2	3	4	5	6	7
0	0	1	0	2	0	1	0	0
1	0	0	0	0	0	0	0	0

$n^{j_L,j_R}_{2,1,1,0,0,1}$

$2j_L \backslash 2j_R$	0	1	2	3	4	5	6	7
0	0	2	0	2	0	1	0	0
1	0	0	0	0	0	0	0	0

$n^{j_L,j_R}_{2,2,2,0,0,1}$

In order to extract unrefined invariants from these one has to sum over the right-moving spin of the multiplets as follows:

$$n^{j_L}_\beta = \sum_{j_R} (-1)^{2j_R}(2j_R+1) n^{j_L,j_R}_\beta. \tag{3.63}$$

Furthermore, in order compare with the genus expansion of the topological string, the $SU(2)_L$ representations have to be organized into

$$I^n_L = \left[(\frac{1}{2}) + 2(0) \right]^{\otimes n}, \tag{3.64}$$

and the BPS invariants n^g_β can be obtained by comparing the two sides of the identity

$$\sum n^{j_L,j_R}_\beta (-1)^{2j_R}(2j_R+1)[j_L] = \sum_g n^g_\beta I^g_L. \tag{3.65}$$

The expansion coefficients in $I^n_L = \sum_j c^{2n}_j [j/2]$ can be found for example in [30]. Using these results we can compute unrefined invariants. For $n_b = 1$, for example, for the curves considered above one has:

$$\begin{aligned}
n^0_{1,2,2,1,1,3} &= -272 \qquad & n^1_{1,2,2,1,1,3} &= 16, \\
n^0_{1,2,2,1,1,4} &= -534 \qquad & n^1_{1,2,2,1,1,4} &= 32, \\
n^0_{1,3,3,1,1,3} &= -108 \qquad & n^1_{1,3,3,1,1,3} &= 0, \\
n^0_{1,3,3,1,1,4} &= -582 \qquad & n^1_{1,3,3,1,1,4} &= 36,
\end{aligned} \tag{3.66}$$

and for the $n_b = 2$ curves considered above we obtain

$$\begin{aligned}
n^0_{2,1,1,0,0,1} &= -16 \qquad & n^1_{2,1,1,0,0,1} &= 0, \\
n^0_{2,2,2,0,0,1} &= -18 \qquad & n^1_{2,2,2,0,0,1} &= 0.
\end{aligned} \tag{3.67}$$

For these classes all invariants with $g \geq 2$ vanish.

Following the procedure outlined above we have extracted an extensive list of BPS invariants corresponding to one and two strings. The genus zero invariants can be computed independently by employing the mirror symmetry and topological string techniques presented in the next section. When comparing the elliptic genus results to the topological string computation presented in Section 4.2.3 we find a perfect agreement.

4 The Calabi-Yau geometries with elliptic singularities

In this section we construct the local elliptic Calabi-Yau geometries corresponding to the different minimal 6d SCFTs. Our strategy will be to first find a minimal compact elliptic Calabi-Yau 3-fold with the right type of elliptic fiber degeneration over the rigid divisor Σ in the base B and subsequently take the local limit by decompactifying the normal direction to Σ in B. The resulting space will be the non-compact Calabi-Yau 3-fold.

Non-compact Calabi-Yau geometries played an important role in the development of topological string theory, which can frequently be completely solved on these geometries, by well understood relations to matrix models, integrable models and gauge theories. One wide class of examples are the non-compact toric Calabi-Yau spaces; another one with some overlap to the first consists of the local (almost) Fano varieties $\mathcal{O}(-K_S) \to S$, where S is an (almost) Fano variety. That is, one considers a rigid divisor S in the Calabi-Yau 3-fold and decompactifies the normal direction to S. In these cases local mirror symmetry leads to a mirror curve. In the present case, instead, we decompactify the normal direction to a rigid divisor Σ in the base B of an elliptic Calabi-Yau 3-fold. Then, the mirror geometry does not reduce to a curve. In cases where there is an orbifold description one can describe the local mirror geometry as a non-compact Landau-Ginzburg model as we exemplify for the $\mathbb{Z}_3$ orbifold in Appendix A.

4.1 The local geometries

The new local geometries we consider arise in Calabi-Yau threefolds, where we zoom close to the elliptic singularity of an elliptic fibration over a divisor Σ in a two-dimensional base B. The divisor Σ corresponds to the 7-brane locus in F-theory with gauge symmetry g_Σ and the exceptional divisors that resolve the elliptic singularity intersect with the negative Cartan matrix $C_{\hat{g}_\Sigma}$ of the affine Lie algebra $\hat{g}_\Sigma$ associated to g_Σ.

We consider non-Higgsable singularities; in other words, the divisor Σ has to be rigid and the Calabi-Yau space has no complex structure deformations which could resolve the singularity. The simplest example for a non compact threefold of this type are elliptic fibrations over $B = (\mathcal{O}(-n) \to \mathbb{P}^1)$. We will start with a compact threefold M_3 constructed as elliptic fibration over a Hirzebruch surface $B = \mathbb{F}_n$ [17]. The $(-n)$ section of the rational fibration of $\mathbb{F}_n$ is then the rigid gauge symmetry divisor Σ, with $\mathcal{O}(-n)$ as its normal bundle.

This setup allows us to solve the topological string using mirror symmetry with normalizable intersections and instanton actions. By decompactifying the normal direction we can easily decouple six-dimensional gravity.

Hirzebruch surfaces as base

Let us recall that the Hirzebruch surfaces $\mathbb{F}_n$ are rational $\mathbb{P}^1$ fibrations over $\mathbb{P}^1$, where n parametrizes the twisting of the fiber. They can be constructed torically or as gauged linear sigma models with four chiral fields Φ_i, $i = 1, \ldots, 4$ and two $U(1)$'s under which the fields have charges $l_i^{(1)}$ and $l_i^{(2)}$. We also use the description in terms of a toric fan in which each field Φ_i corresponds to a primitive vector ν_i spanning the fan in the integer lattice $\mathbb{Z}^2$ and summarize the base data as follows:

$$\begin{array}{c} Div \\ D_0 = K \\ D_1 = S \\ D_2 = F \\ D_3 = S' \\ D_4 = F \end{array} \begin{array}{c} \nu_i^* \\ \left|\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & -n \end{array}\right. \end{array} \begin{array}{cc} l^{(1)} & l^{(2)} \\ \left|\begin{array}{cc} -2 & n-2 \\ 1 & 0 \\ 0 & 1 \\ 1 & -n \\ 0 & 1 \end{array}\right| \end{array} . \tag{4.1}$$

Here we added the inner point $\nu_0 = (0, 0)$ and promoted the points ν_i to $\nu_i^* = (1, \nu_i) \in \mathbb{Z}^3$. This is useful for describing the non-compact Calabi-Yau as the anti-canonical bundle over $\mathbb{F}_n$ (Φ_0 is the noncompact direction), but it could be omitted for the discussion of the compact Hirzebruch surface. Each point ν_i correponds to a toric divisor $D_i = \{\Phi_i = 0\}$ and within the surface the homological relations between these divisors are $S = S' + nF$. The nonvanishing intersections are $S^2 = n$, $FS = 1$ and $(S')^2 = -n$; therefore, S' becomes the gauge theory divisor.

Geometrically the $l^{(k)}$, $k = 1, 2$ represent curve classes $[C_k]$ and the intersection with the toric divisors D_i is given by

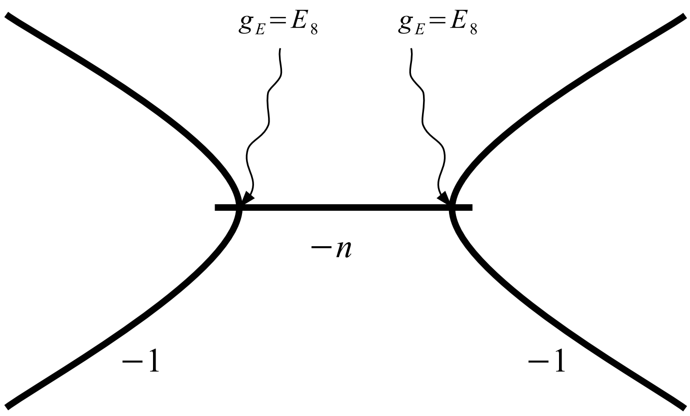
$$[C_k] \cdot [D_i] = l_i^{(k)} \ . \tag{4.2}$$

The $l^{(k)}$ are also called Mori vectors; in the present example, $k = 1$ represents the base $\mathbb{P}^1$ while $k = 2$ represents the fiber $\mathbb{P}^1$ of $\mathbb{F}_n$.

The elliptic fiber types

Next, we want to construct the relevant Calabi-Yau spaces as elliptic fibrations over the Hirzebruch surfaces $\mathbb{F}_n$. From these we will finally obtain the local $\mathcal{O}(-n) \to \mathbb{P}^1$ models by taking the size of the $\mathbb{P}^1$ fiber specified by the class F of the Hirzebruch surface $\mathbb{F}_n$ to infinity. However, it turns out that there are multiple ways to realize the elliptic fiber singularity of the appropriate type leading to different Mordell-Weyl groups. In this paper we will be interested in a rank one Mordell-Weyl group, that is an elliptic fibration with a single section. In the following we describe how to achieve this desired fibration structure.

Generically, for $n > 2$, the situation is such that the discriminant vanishes on the base $\mathbb{P}^1$ called S' (we therefore have $\Sigma = S'$) and on isolated points of the fiber $\mathbb{P}^1$ denoted by F. Let us describe the non-compact geometry which arises when we take the size of F to be infinite. We will denote curves on which the discriminant vanishes point-wise by (-1)-curves borrowing the terminology from E-strings. In fact, the analogy goes even further in that a subset of the E_8 Weyl symmetry of E-strings can act on sections of this elliptic fibration. The exact subset is determined by the elliptic fiber singularity type at the intersection points of the non-compact limit of F with S'. We will denote the corresponding Kodaira group by g_E which should not be confused with g_Σ discussed in Section 2 which labels the fiber degeneration on S'. A consistency condition is that g_Σ should be a subgroup of g_E, that is $g_\Sigma \subset g_E$. In order to restrict to one section for each model we only consider the fiber type $g_E = E_8$ in this paper which leads to the following schematic picture of curve configurations:



For other choices of g_E (simplest cases are $g_E = E_n$, $n = 3, \dots, 8$, where $\{E_n\}_{n=3}^{8} = A_1 \times A_2, A_4, D_4, E_6, E_7, E_8$) the subset of E_8 which becomes the Mordell-Weyl group of the elliptic Calabi-Yau is determined by the commutant of the Weyl group g_E in the Weyl group of E_8. Note that gluing together the two non-compact (-1)-curves gives back the compact Calabi-Yau M_3 with base $B = \mathbb{F}_n$. M_3 can equivalently be viewed as a K3 fibration over the $(-n)$-curve as the elliptic fibration over F has second Chern class 24 due to the two E_8-type degenerations of the elliptic fiber shown in the above figure. For pure 6d gauge theories the Euler number of M_3 is given purely in terms of group theory data as [31]

$$\chi_{pG}(M_3) = -2C(g_E) \int_B c_1^2(B) - \text{rank}(g_\Sigma) C(g_\Sigma) \int_\Sigma c_1(\Sigma) \ . \tag{4.3}$$

Since we are interested in having a single section, we take $g_E = E_8$. In this case the generic elliptic fiber can be given as a degree 6 hypersurface in the weighted projective space $\mathbb{P}^2(1,2,3)$. For the different models labeled by n the corresponding dual Coxeter number $C(g_\Sigma)$ and the Euler numbers of the minimal compact Calabi-Yau manifolds are given in Table 1.

7-brane	n=1,2	3	4	5	6	7	8	9	10	11	12
g_Σ	-	A_2	D_4	F_4	E_6	$E_7^{(\frac{1}{2}HM)}$	E_7	$E_8^{(3)}$	$E_8^{(2)}$	$E_8^{(1)}$	E_8
C_g	-	3	8	12	12	18	18	30	30	30	30
$-\chi(M_3)$	480	492	528	576	624	676	732	780	840	900	960
$h_{11}(M_3) - 1$	2	4	6	6	8	9	9	13	12	11	10

Table 1: Table of Coxeter numbers C_g and Euler numbers $\chi(M_3)$ for the different minimal SCFT Calabi-Yau threefolds.

Note that, compared to the local geometry associated to the 6d SCFT, the compact geometry leads to a larger number of hypermultiplets and one additional vector multiplet.

In the table, we also list the E_8 cases with a non-zero number $n_I = 12-n$ of small instantons as $E_8^{(n_I)}$. Each instanton corresponds to an additional tensor multiplet in the 6d theory. Each of the latter contains one additional modulus, so h_{11} increases by n_I. The 6d anomaly cancellation condition [32, 33] moreover enforces the relation $\#\mathrm{HM} - \#\mathrm{VM} = 273 - 29 n_I$, which implies $\chi(M_3) = \chi_{pG}(M_3) - 2C_{E_{8E}} n_I$. This yields the Hodge numbers $E_8^{(1,2,3)}$ in Table 1. The corresponding toric hypersurfaces are specified in Section 4.2.7.

Tate's algorithm for elliptic fiber singularities and toric constructions

If the generic fiber is the elliptic curve $X_6(3,2,1)$, the elliptically fibered Calabi-Yau threefold over a base B for this fiber type takes the Tate form

$$y^2 + x^3 + a_6(\underline{u})z^6 + a_4(\underline{u})xz^4 + a_3(\underline{u})yz^3 + a_2(\underline{u})z^2x^2 + a_1(\underline{u})zxy = 0 \,, \qquad (4.4)$$

where coordinates on the base B are denoted generically by $\underline{u}$.

The construction of gauge singularities inside an elliptically fibered Calabi-Yau n-fold with the necessary toric data to solve the topological string proceeds as follows [34, 35]. One constructs reflexive polyhedra such that the Calabi-Yau is given by the anti-canonical hypersurface $W_\Delta(Y) = 0$ in the corresponding toric variety, with $W_\Delta(Y)$ being in a generic Tate form (4.4). Then one chooses the divisor Σ in B, restricts the coefficients of $W_\Delta(Y)$ so that at Σ one has the suitable Tate singularity [36], constructs the Newton polytope Δ_r to the restricted polyhedron and its dual

Δ_r^*, and finally resolves all non-toric divisors by modifying Δ_r, without changing the singularity at Σ.

For instance, if B is a $\mathbb{P}^1$ fibration over Σ, one splits the coordinates of the Tate form into $\{Y_k\} = \{z, x, y, u_1, u_2, w, v\}$, where u_i are coordinates of Σ and w, v are homogeneous coordinates of the $\mathbb{P}^1$ fiber. For example, for $w = 0$ the whole Σ becomes a gauge divisor and by setting the coefficients of the monomials in $a_i(\underline{u}, v, w, \underline{z})$ to zero (that is, choosing a specialization of the complex structure moduli), one can put the $a_i(\underline{u}, v, w, \underline{z})$ in the following form:

$$a_1 = \alpha_1 w^{[a_0]}\,, \quad a_2 = \alpha_2 w^{[a_2]}\,, \quad a_3 = \alpha_3 w^{[a_3]}\,, \quad a_4 = \alpha_4 w^{[a_4]}\,, \quad a_6 = \alpha_6 w^{[a_6]}\,, \qquad (4.5)$$

where $\alpha_i(\underline{u}, v, w, \underline{z})$ are of order zero in w. Choosing this leading behavior at $w = 0$ leads by Tate's algorithm to singular fibers and hence results in a gauge group along Σ. The association of the leading powers of $[a_i]$ with the singularity is given by Tate's algorithm [36]. The discussion applies to the Hirzebruch surfaces as bases B which can be viewed as a $\mathbb{P}^1$ fibration over $\Sigma = \mathbb{P}^1$. As already mentioned the divisor Σ becomes S' in this case.

In the following sections we construct the minimal compact Calabi-Yau threefolds with the prescribed local geometries as hypersurfaces in a toric ambient space. Minimal means that they have just one additional modulus, whose decompactification leads to the local geometry. The cases of main interest have the Euler number (4.3) as indicated in Table 1.

4.2 Solution of the topological string on the toric hypersurface Calabi-Yau spaces

Generically the Calabi-Yau geometries under consideration can be described as anti-canonical hypersurfaces H given by

$$W_\Delta(Y) = \sum_{\nu_i \in \Delta} a_i \sum_{\nu_k^* \in \Delta^*} Y_k^{\langle \nu_i, \nu_k^* \rangle + 1} = 0 \qquad (4.6)$$

in $\mathbb{P}_{\Delta^*}$, where (Δ, Δ^*) are reflexive polyhedra.

We denote by $\nu_i^* \in \mathbb{Z}^4$ the relevant points of Δ^* whose complex hull in $\mathbb{R}^4$ is Δ^* and by $l^{(k)}$ the charges or Mori vectors, which fulfill

$$\sum_i l_i^{(k)} \bar{\nu}_i^* = 0\ , \qquad (4.7)$$

where $\bar{\nu}_i^* = (1, \nu_i^*)$. The Mori vectors span the Mori cone, which is dual to the Kähler cone. The possible choices of Mori cones constitute the secondary fan whose data are encoded in the possible star triangulations of Δ^*. Some of them are redundant, because the Calabi-Yau manifold still has the same Mori cones. Others correspond truly to different topological phases of the gauged linear sigma model. According to the theorem of C.T.C. Wall [37] the topological type of the Calabi-Yau threefold M_3 is fixed by the independent Hodge numbers, which for an $SU(3)$ holonomy manifold are h_{11} and h_{21}, the triple intersection numbers $J_i \cdot J_j \cdot J_k$ and evaluation of the second Chern class on the basis J_i of divisors dual to the basis of the Kähler cone. Only the $J_i \cdot J_j \cdot J_k$ change in a non-trivial way in the transitions. Given the $l^{(k)}$ and the C.T.C. Wall topological data one can use toric mirror symmetry [38] to predict the genus zero BPS numbers for all toric hypersurfaces following [39]. We review the formalism that leads to the genus zero BPS numbers in Appendix A, see (A.18)-(A.21).

4.2.1 $\mathcal{O}(-n) \to \mathbb{P}^1$ geometries with $n = 1, 2$

The cases $n = 1, 2$ have only Kodaira fibers of type I_1 over codimension one in the base and hence no gauge theory divisor.

To fix the notation used in the following sections we review the $n = 1$ case, which is of particular interest as the local geometry is the $\frac{1}{2}$K3 on which F-theory compactification yields the E-string theory. The refined BPS spectrum of the E-string has an interpretation as the refined stable pair invariants on the local geometry. The data associated to this geometry is summarized by the following table:

$$\begin{array}{ccccccc|ccc|ccc|} Div. & & & \bar{\nu}_i^* & & & & l_I^{(e)} & l_I^{(f)} & l_I^{(b)} & l_{II}^{(e')} & l_{II}^{(h)} & l_{II}^{(-b)} \\ D_0 & 1 & 0 & 0 & 0 & 0 & & -6 & 0 & 0 & -6 & 0 & 0 \\ D_1 & 1 & -1 & 0 & 0 & 0 & & 2 & 0 & 0 & 2 & 0 & 0 \\ D_2 & 1 & 0 & -1 & 0 & 0 & & 3 & 0 & 0 & 3 & 0 & 0 \\ S' & 1 & 2 & 3 & 0 & -1 & & 0 & 1 & -1 & -1 & 0 & 1 \\ K & 1 & 2 & 3 & 0 & 0 & & 1 & -2 & -1 & 0 & -3 & 1 \\ F & 1 & 2 & 3 & -1 & -1 & & 0 & 0 & 1 & 1 & 1 & -1 \\ S & 1 & 2 & 3 & 0 & 1 & & 0 & 1 & 0 & 0 & 1 & 0 \\ F & 1 & 2 & 3 & 1 & 0 & & 0 & 0 & 1 & 1 & 1 & -1 \end{array} . \tag{4.8}$$

The polyhedron Δ^* has two star triangulations, denoted by subscripts I and II, which lead to different Mori cones in the secondary fan. Such different Mori or Kähler cones can be understood as different geometrical phases of the 2d sigma model, which can have non-geometrical phases as well [40].

We give the C.T.C. Wall data for the first phase, namely phase I, which corresponds to the E-string geometry. Both phases have $h_{21} = 243$ and $h_{11} = 3$, and hence Euler number $\chi = -480$. The topological data in the phase marked with I in (4.8) are encoded in

$$\mathcal{R}_I = 8J_e^3 + 3J_e^2 J_f + J_e J_f^2 + 2J_e^2 J_b + J_e J_f J_b \ , \tag{4.9}$$

whose coefficients are the classical triple intersection numbers $c_{ijk} = \int J_i J_j J_k$. The evaluation of the second Chern class is $\int c_2 J_e = 92$, $\int c_2 J_f = 36$, and $\int c_2 J_b = 24$. We can see from (4.2) that $l_I^{(b)}$ corresponds to the section $[C_b] = [S']$ of the base in

$\mathbb{F}_1$, the (-1) curve, while $l_I^{(f)}$ corresponds to the fiber $[C_f] = [F]$ in $\mathbb{F}_1$, a (0) curve. Over the (-1) curve one has a $\frac{1}{2}$-K3, which is the divisor J_f dual to $[C_f]$ in M_3, while over $[C_f]$ one has an elliptically fibered K3, which is the divisor J_b dual to $[C_b]$ in M. According to Oguiso's criterion [41] we see that the latter is a fibration of the geometry M as the K3 does not intersect $J_b^2 = 0$ and $\int c_2 J_b = 24$. The class $[C_e]$ represents the elliptic fiber.

The E-string partition function has the structure $Z = \exp(\lambda^{2g-2} F^{(g)}(Q_\tau, Q_b))$ where the free energies have the form $F^{(g)} = \sum_{n=0}^{\infty} \tilde{F}_n^{(g)}(Q_\tau) Q_b^n$. Here $Q_\tau = \exp(2\pi i \tau)$ with τ the modular parameter and $\tilde{F}_n^{(g)}(Q_\tau) = \frac{Q_\tau^{\frac{n}{2}}}{\eta(Q_\tau)^{12n}} P_n^{(g)}$ with $P_n^{(g)}(\hat{E}_2, E_4, E_6)$ an almost holomorphic modular form of weight $2g - 6n - 2$, e.g. $P_1^{(0)} = E_4$ etc. One has hence to redefine the Kähler parameters so that $Q = Q_b Q_\tau^{\frac{1}{2}}$ and

$$F^{(g)} = \sum_{n=0}^{\infty} F_n^{(g)}(Q_\tau) Q^n. \tag{4.10}$$

In the above formula $F_n^{(g)}(Q_\tau)$ are truly $SL(2,\mathbb{Z})$ invariant coefficients. The analogous redefinition has been made in (3.58) for the D_4 string. The analysis of the monodromies of M_3 that yield an $SL(2,\mathbb{Z})$ action on τ and a non-trivial decoupling limit fix the combination Q. This was discussed in detail in [21] in a similar context and applies to geometries discussed below.

The second phase is obtained by flopping the base $[C_b]$ out of the half K3, which becomes thereby an elliptic pencil, the del Pezzo surface with degree one. The latter can be obtained by eight blow ups of $\mathbb{P}^2$ and is called therefore $d_8\mathbb{P}^2$. Note that the intersections are

$$\mathcal{R}_{II} = 8J_{e'}^3 + 3J_{e'}^2 J_h + J_{e'} J_h^2 + 9J_{e'}^2 J_{-b} + 3J_{e'} J_h J_{-b} + J_h^2 J_{-b} + 9J_{e'} J_{-b}^2 + 3J_h J_{-b}^2 + 9J_{-b}^3,$$

while the evaluation of the second chern class is given by $\int c_2 J_i = \{92, 36, 102\}$. The transformation of the basis $l_{II}^{(e')} = l_I^{(e)} + l_I^{(b)}$, $l_{II}^{(h)} = l_I^{(f)} + l_I^{(b)}$ and $l_I^{(-b)} = -l_I^{(b)}$ gives already almost the intersection ring $\mathcal{R}_{II}$ except that one gets $8J_{-b}^3$ instead of $9J_{-b}^3$, i.e. in a coordinate independent formulation one observes that the triple intersection of the divisors dual to the rational curve that gets flopped increases by $+1$. This can be argued in general in various ways, see e.g. [8].

The case $n = 2$ has only one phase:

$$\begin{array}{c|ccccc|ccc|} Div. & & & \bar{\nu}_i^* & & & l_I^{(e)} & l_I^{(f)} & l_I^{(b)} \\ D_0 & 1 & 0 & 0 & 0 & 0 & -6 & 0 & 0 \\ D_1 & 1 & -1 & 0 & 0 & 0 & 2 & 0 & 0 \\ D_2 & 1 & 0 & -1 & 0 & 0 & 3 & 0 & 0 \\ S' & 1 & 2 & 3 & 0 & -1 & 0 & 1 & -2 \\ K & 1 & 2 & 3 & 0 & 0 & 1 & -2 & 0 \\ F & 1 & 2 & 3 & -1 & -1 & 0 & 0 & 1 \\ S & 1 & 2 & 3 & 0 & 1 & 0 & 1 & 0 \\ F & 1 & 2 & 3 & 1 & 0 & 0 & 0 & 1 \end{array} . \tag{4.11}$$

In this phase one has a $K3$ and an elliptic fibration and the intersection ring is in general

$$\mathcal{R} = 8J_e^3 + 4J_e^2 J_f + 2J_e J_f^2 + 2J_e^2 J_b + J_e J_f J_b, \tag{4.12}$$

with

$$\int c_2 J_e = 92, \qquad \int c_2 J_f = 48, \qquad \int c_2 J_b = 24 . \tag{4.13}$$

The $n = 2$ geometry corresponds to the A_1 $\mathcal{N} = (2, 0)$ SCFT; by making the elliptic fiber singular over the (-2) curve, one obtains the M-string geometry which was studied in detail in [4, 5].

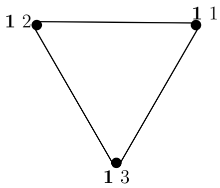
4.2.2 $\mathcal{O}(-3) \to \mathbb{P}^1$ geometry with $\hat{A}_2$ resolution

The easiest example with a non-Higgsable gauge symmetry is the A_2 case, which has the following polyhedron Δ^*:

$$\begin{array}{c|cccc|ccccc|ccccc|} Div. & & \nu_i^* & & & l^{(1)} & l^{(2)} & l^{(3)} & l^{(4)} & l^{(5)} & l_{T^2} & l_{\mathbb{P}^2}^{(1)} & l_{\mathbb{P}^2}^{(2)} & l_{\mathbb{P}^2}^{(3)} & l_{de} \\ D_0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -6 & -3 & 0 & 0 & 0 \\ D_1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 2 & 1 & 1 & 0 & 0 \\ D_2 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 3 & 3 & 0 & 0 & 0 \\ D_3 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 2 & 0 & -3 & 0 & 0 & 0 \\ D_4 & 1 & 2 & 0 & -1 & 1 & -3 & 0 & 3 & 0 & 0 & 0 & -3 & 0 & 0 \\ S' & 2 & 3 & 0 & -1 & 0 & 0 & 1 & -3 & 1 & 0 & 0 & 0 & -3 & 0 \\ K & 2 & 3 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 1 & 0 & 0 & 1 & -\frac{5}{3} \\ F & 2 & 3 & -1 & -3 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 1 & 1 & \frac{1}{3} \\ S & 2 & 3 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ F & 2 & 3 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 1 & 1 & \frac{1}{3} \end{array} . \tag{4.14}$$

We study the basis[5] which is appropriate to exhibit the curve classes that exhibit the affine $\hat{A}_2$ singularity over the divisor S', which are depicted in the figure below:

[5] The intersection ring and an alternate basis appropriate to the Landau-Ginzburg description is given in Appendix A.



This basis corresponds to the following choice of vectors:

$$\begin{aligned}
l^1_{\hat{A}_2} &= 4l^{(1)} + l^{(2)} + l^{(5)} = (-4, 1, 3, -2, \quad 1, \quad 1, 0, 0, 0, 0),\\
l^2_{\hat{A}_2} &= l^{(1)} + l^{(2)} + l^{(5)} = (-1, 1, 0, \quad 1, -2, \quad 1, 0, 0, 0, 0),\\
l^3_{\hat{A}_2} &= l^{(1)} + l^{(2)} + l^{(4)} + l^{(5)} = (-1, 0, 0, \quad 1, \quad 1, -2, 1, 0, 0, 0),\\
l_b &= -(l^{(1)} + l^{(5)}) = (\ \ 1, 0, 0, -1, -1, -1, 0, 1, 0, 1),\\
l_{de} &= 5l^{(1)} + \tfrac{8}{3}l^{(2)} + l^{(3)} + \tfrac{10}{9}l^{(4)} + \tfrac{8}{3}l^{(5)} = (-5, \tfrac{14}{9}, \tfrac{7}{3}, \ \tfrac{1}{3}, \ \tfrac{1}{3}, \ \tfrac{1}{3}, -\tfrac{8}{9}, 0, 1, 0))\ .
\end{aligned} \tag{4.15}$$

Note that we have flopped the $\mathbb{P}^1$ represented by the vector $l^{(1)}+l^{(5)}$ in the Mori cone in order to arrive at the appropriate $\mathbb{P}^1$ base for the affine $\hat{A}_2$ singularity. The $\mathbb{P}^1$ base is represented by the Mori vector l_b, which intersects the three rational components of the degenerate elliptic curve with (-1). The decompactification direction can be specified as a rational element in $H_2(M_3)$ so that the intersection form of the compact two-dimensional part becomes

$$\frac{1}{3^3} J_b \sum_{i,j=1}^{3} C_{ij} J^{(i)}_{\hat{A}_2} J^{(j)}_{\hat{A}_2}\ . \tag{4.16}$$

We note that the Coxeter labels a^i have the property that for C_{ij} the affine Cartan matrix

$$\sum_{j=0}^{r} a^i C_{ij} = 0\ . \tag{4.17}$$

A first check on our identification is therefore that the $\mathbb{P}^1$ curve classes called $l^{(i)}_{\hat{D}_4}$ add up to the class of the elliptic fiber with the Coxeter lables indicated at the affine Dynkin diagram, i.e.

$$l_{T^2} = \mathbf{1} l^{(1)}_{\hat{A}_2} + \mathbf{1} l^{(2)}_{\hat{A}_2} + \mathbf{1} l^{(3)}_{\hat{A}_2}\ .$$

This is geometrically required, because the curve class of the elliptic fiber has self intersection 0. We list in the following some of the BPS invariants $n^{(0)}_{d_b, d^1_{\hat{A}_2}, d^2_{\hat{A}_2}, d^3_{\hat{A}_2}}$, where the degree in the decompactified direction is zero. The number d_b corresponds to the base wrapping number and therefore indicates the string charge in the 6d SCFT whereas the other numbers correspond to the flavor fugacity charges. The numbers $n^{(0)}_{d_b, d^1_{\hat{A}_2}, d^2_{\hat{A}_2}, d^3_{\hat{A}_2}}$ are symmetric in $d^1_{\hat{A}_2}, d^2_{\hat{A}_2}, d^3_{\hat{A}_2}$. Since the emphasis of this paper is on the strings of the 6d SCFTs, we focus on the BPS invariants corresponding to $n_b \geq 1$. For example, the following tables display BPS invariants corresponding to $n_b = 1, 2$ and small values of $d^i_{\hat{A}_2}$.

$d^1_{\hat{A}_2} \backslash d^2_{\hat{A}_2}$	0	1	2	3	4	5
0	1	3	5	7	9	11
1	3	4	8	12	16	20
2	5	8	9	15	21	27
3	7	12	15	16	24	32
4	9	16	21	24	25	35
5	11	20	27	32	35	36

$n^{(0)}_{d_b=1,d^1_{\hat{A}_2},d^2_{\hat{A}_2},0}$

$d^1_{\hat{A}_2} \backslash d^2_{\hat{A}_2}$	1	2	3	4	5
1	16	36	60	84	108
2	36	56	96	144	192
3	60	96	120	180	252
4	84	144	180	208	288
5	108	192	252	288	320

$n^{(0)}_{d_b=1,d^1_{\hat{A}_2},d^2_{\hat{A}_2},1}$

$d^1_{\hat{A}_2} \backslash d^2_{\hat{A}_2}$	2	3	4	5
2	149	288	465	651
3	288	456	735	1080
4	465	735	954	1371
4	651	1080	1371	-

$n^{(0)}_{d_b=1,d^1_{\hat{A}_2},d^2_{\hat{A}_2},2}$

$d^1_{\hat{A}_2} \backslash d^2_{\hat{A}_2}$	3	4
3	1012	1788
4	1788	-

$n^{(0)}_{d_b=1,d^1_{\hat{A}_2},d^2_{\hat{A}_2},3}$

$d^1_{\hat{A}_2} \backslash d^2_{\hat{A}_2}$	0	1	2	3	4	5
0	0	0	-6	-32	-110	-288
1	0	0	-10	-70	-270	-770
2	-6	-10	-32	-126	-456	-1330
3	-32	-70	-126	-300	-784	-2052
4	-110	-270	-456	-784	-1584	-3360
5	-288	-770	-1330	-2052	-3360	-6076

$n^{(0)}_{d_b=2,d^1_{\hat{A}_2},d^2_{\hat{A}_2},0}$

$d^1_{\hat{A}_2} \backslash d^2_{\hat{A}_2}$	1	2	3	4	5
1	-8	-60	-360	-1432	-4280
2	-60	-216	-850	-3164	-9720
3	-360	-850	-2176	-6084	-16960
4	-1432	-3164	-6084	-13000	-29526
5	-4280	-9720	-16960	-29526	-

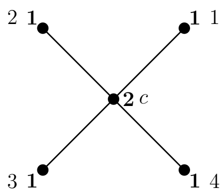
$n^{(0)}_{d_b=2,d^1_{\hat{A}_2},d^2_{\hat{A}_2},1}$

$d^1_{\hat{A}_2} \backslash d^2_{\hat{A}_2}$	0	1	2	3	4	5
0	0	0	0	27	286	1651
1	0	0	0	64	800	5184
2	0	0	25	266	1998	11473
3	27	64	266	1332	6260	26880
4	286	800	1998	6260	21070	70362
5	1651	5184	11473	26880	70362	191424

$n^{(0)}_{d_b=3,d^1_{\hat{A}_2},d^2_{\hat{A}_2},0}$

4.2.3 $\mathcal{O}(-4) \to \mathbb{P}^1$ geometry with $\hat{D}_4$ resolution

In this section we describe the elliptic Calabi-Yau which has base $B = \mathbb{F}_4$ and corresponds to the two-dimensional quiver studied in Section 3. Taking the local limit by sending the size of the $\mathbb{P}^1$ fiber of $\mathbb{F}^4$ to infinity one arrives at a local Calabi-Yau which has an affine $\hat{D}_4$ Kodaira singularity over the (-4) curve. The singularity in the elliptic fiber is resolved by sphere configurations with the affine D_4 intersection numbers and multiplicities as depicted below:



The toric data are given by a reflexive polyhedron Δ^* , whose points ν are the first entries in the table below.

$$
\begin{array}{c|cccc|ccccccc|ccccccc|}
D & & \nu_i^* & & & l^{(1)} & l^{(2)} & l^{(3)} & l^{(4)} & l^{(5)} & l^{(6)} & l^{(7)} & l_{T^2} & l^{(1)}_{\hat{D}_4} & l^{(2)}_{\hat{D}_4} & l^{(3)}_{\hat{D}_4} & l^{(4)}_{\hat{D}_4} & l^{(c)}_{\hat{D}_4} & l_{de} \\
D_0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & 0 & -2 & -2 & 0 & -1 & 0 \\
D_1 & -1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 2 & 0 & 0 & 0 & 0 \\
D_2 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 3 & 1 & 1 & 1 & 0 & 0 & 0 \\
D_3 & 1 & 1 & 0 & -1 & 1 & 0 & 2 & 0 & 0 & -2 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 \\
D_4 & 0 & 1 & 0 & -1 & 1 & 3 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & -2 & 2 & 0 & 0 & 0 \\
D_5 & 1 & 2 & 0 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & 0 & 1 & 0 \\
S' & 2 & 3 & 0 & -1 & 0 & 1 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 \\
K & 2 & 3 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -\frac{3}{2} \\
S'' & 2 & 3 & 0 & -2 & 0 & -2 & 0 & -2 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & -2 & \frac{1}{2} \\
F & 2 & 3 & -1 & -4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
S & 2 & 3 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
F & 2 & 3 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array} \; . \qquad (4.18)
$$

The Calabi-Yau hypersurface has $\chi(M_3) = -528$ and $h_{11} = 7$. The polyhedron Δ^* has 30 star triangulations and the $l^{(1)}, \ldots, l^{(7)}$ are generators of a simple geometrical Mori cone, which are needed to solve the topological string on the global model. Note that the evaluation of the Kähler classes against the second Chern class are:

$$\{\int c_2 J_i\} = \{620, 204, 140, 24, 72, 452, 616\}.$$

It turns out that J_4 appears only linearly in the intersection ring and $\int c_2 J_4 = 24$. Oguiso's criterion implies that M_3 is a $K3$ fibration whose base $\mathbb{P}^1$ is represented by $l^{(4)}$. This $\mathbb{P}^1$ is also the base of the local surface $B = \mathcal{O}(-4) \to \mathbb{P}^1$ and we will henceforth denote it by l_b. Since J_5 appears only quadratically in the intersection ring , $l^{(5)}$ represents the base of the elliptic fibration. Also, l_{T^2} represents the elliptic fiber class and the $l^{(i)}_{D_4}$ correspond to the simple roots of the affine $\hat{D}_4$. Finally, l_{de} is the class that one can take large to zoom in on the local surface geometry. The relation to the nef classes in the global model are

$$
\begin{aligned}
l_{T^2} &= 6l^{(1)} + 2l^{(2)} + l^{(3)} + 4l^{(6)} + 6l^{(7)}, l^{(1)}_{\hat{D}_4} = l^{(6)}, \;\; l^{(2)}_{\hat{D}_4} = 2l^{(1)} + l^{(6)} + 2l^{(7)}, \\
l^{(3)}_{\hat{D}_4} &= 2l^{(1)} + l^{(6)}, \;\; l^{(4)}_{\hat{D}_4} = l^{(3)} + l^{(6)}, \;\; l^{(c)}_{\hat{D}_4} = l^{(1)} + l^{(2)} + 2l^{(7)} \; .
\end{aligned} \qquad (4.19)
$$

A first check on these identifications is that the $\mathbb{P}^1$ curve classes called $l^{(i)}_{\hat{D}_4}$ add up to the class of the elliptic fiber with Coxeter labels a^i indicated in the affine Dynkin diagram, i.e.

$$l_{T^2} = \mathbf{1} l^{(1)}_{\hat{D}_4} + \mathbf{1} l^{(2)}_{\hat{D}_4} + \mathbf{1} l^{(3)}_{\hat{D}_4} + \mathbf{1} l^{(4)}_{\hat{D}_4} + \mathbf{2} l^{(c)}_{\hat{D}_4} \; , \qquad (4.20)$$

which is obviously the case. Another check is that after transforming to this basis the intersection form takes on a very simple appearance and is symmetric in $l^{(i)}_{\hat{D}_4}$,

$i = 1, \ldots, 4$:

$$\mathcal{R} = 9J_{T^2}^3 + \frac{3}{2}J_4 J_{T^2}^2 + 6J_{de}J_{T^2}^2 + 4J_{de}^2 J_{T^2} + J_4 J_{de} J_{T^2} - \sum_{i=1}^4 (J_{\hat{D}_4}^{(i)})^3 - \frac{1}{2}J_4 \sum_{i=1}^4 (J_{\hat{D}_4}^{(i)})^2. \quad (4.21)$$

The curve whose volume has to be scaled to infinity to decouple the $\mathcal{O}(-4) \to \mathbb{P}^1$ geometry from the compact manifold is the Kähler class dual to the Mori cone vector $l^{(5)}$. This decompactifies the base of the Cababi-Yau threefold by scaling the fiber of the Hirzebruch surface to infinity. The class can be further modified to l_{de} above to make the intersections even simpler.

Let us next come to the evaluation of BPS numbers. We denote the charge associated to the class dual to $l^{(4)}$, which is the base, by n_b, the one dual to l_{de} by n_{de}, the ones dual to $l_{\hat{D}_4}^{(i)}$ by n_1, n_2, n_3, n_4, n_c and the one dual to l_{T^2} by n_e. From the viewpoint of the strings of the 6d SCFT, n_b denotes the string charge; $n_i, i = 1, \cdots, 4, n_c$ correspond to the flavor fugacity charges and n_e is the exponent of Q_τ in an expansion of the elliptic genus Z_{n_b}. We consider by definition of the local limit only $n_{de} = 0$. Due to the relation (4.20) the class l_{T^2} is not an independent class and therefore when labelling BPS states we can omit the dependence on n_e. The genus zero invariants are then given by $n^{(0)}_{n_b,n_1,n_2,n_3,n_4,n_c}$ and are fully symmetric in $n_1, \ldots, n_4$. Let us first focus on the BPS states associated to a single string (that is, those corresponding to $n_b = 1$). For $n_e = 0$ and $n_c = 1$ we get:

$n_1 \backslash n_2$	0	1	2	3	4	5	6
0	-4	-6	-6	-10	-14	-18	-22
1	-6	-8	-6	-10	-14	-18	-22
2	-6	-6	0	0	0	0	0
3	-10	-10	0	0	0	0	0
4	-14	-14	0	0	0	0	0
5	-18	-18	0	0	0	0	0
6	-22	-22	0	0	0	0	0

$n^{(0)}_{1,n_1,n_2,0,0,1}$

For $n_e = 0$ and $n_c = 2$ we get:

$n_1 \backslash n_2$	0	1	2	3	4	5	6
0	-6	-10	-12	-12	-18	-24	-30
1	-10	-16	-18	-16	-24	-32	-40
2	-12	-18	-18	-12	-18	-24	-30
3	-12	-16	-12	0	0	0	0
4	-18	-24	-18	0	0	0	0
5	-24	-32	-24	0	0	0	0
6	-30	-40	-30	0	0	0	0

$n^{(0)}_{1,n_1,n_2,0,0,2}$

For $n_e = 1$ one finds:

$n_1 \backslash n_2$	0	1	2	3	4
0	-80	-78	-96	144	-192
1	-78	-48	32	-	-
2	-96	-32	-	-	-
3	-144	-	-	-	-
4	-192	-	-	-	-

$n^{(0)}_{1,n_1+1,n_2+1,1,1,2}$

Let us now consider the two-string sector. For $n_b = 2$, $n_e = 0$ and $n_c = 1$ we get:

$n_1 \backslash n_2$	0	1	2	3	4	5	6
0	-6	-10	-12	-30	-98	-306	-814
1	-10	-16	-18	-40	-112	-324	-836
2	-12	-18	-18	-30	-42	-54	-66
3	-30	-40	-30	-50	-70	-90	-110
4	-98	-112	-42	-70	-98	-126	-154
5	-306	-324	-54	-90	-126	-162	-198
6	-814	-836	-66	-110	-154	-198	-242

$n^{(0)}_{2,n_1,n_2,0,0,1}$

For $n_b = 1, \ldots, 4$ and $n_c = 0$ we also find the following invariants:

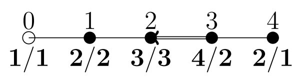
$n_b \backslash n_1$	0	1	2	3	4	5	6	7	8	9	10
1	-2	-2	-4	-6	-8	-10	-12	-14	-16	-18	-20
2	0	0	0	-6	-32	-110	-288	-644	-1280	-2340	4000
3	0	0	0	0	-8	-110	-756	-3556	-13072	-40338	-109120
4	0	0	0	0	0	-10	-288	-3556	-27264	-153324	-690400

$n^{(0)}_{n_b,n_1,0,0,0,0}$

Remarkably, and this is the main non-trivial test of the paper, all the invariants listed above can be reproduced from the elliptic genus computation in Section 3!

4.2.4 $\mathcal{O}(-5) \to \mathbb{P}^1$ geometry with $\hat{F}_4$ resolution

This elliptic singularity corresponds now to a non-simply laced Lie algebra. Unlike in the simply laced case, the Coxeter labels differ from the dual Coxeter labels. We indicate both Coxeter/dual Coxeter labels in the following diagram:



The toric polyhedron has 25 star triangulations; we present here the polyhedron together with the simplest choice of Mori vectors:

$$
\begin{array}{c|cccc|ccccccc|ccccc}
D & & \nu_i^* & & & l^{(1)} & l^{(2)} & l^{(3)} & l^{(4)} & l^{(5)} & l^{(6)} & l^{(7)} & l^{(4)}_{\hat{F}_4} & l^{(3)}_{\hat{F}_4} & l^{(3)}_{\hat{F}_4} & l^{(1)}_{\hat{F}_4} & l^{(0)}_{\hat{F}_4} \\
D_0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\
D_1 & -1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
D_2 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
D_3 & 0 & 1 & 0 & -1 & 0 & 3 & 0 & 0 & 0 & 1 & -2 & -2 & 1 & 0 & 0 & 0 \\
D_4 & 1 & 2 & 0 & -2 & 2 & 0 & 0 & 0 & 0 & -2 & 1 & 1 & -2 & 0 & 0 & 2 \\
S' & 2 & 3 & 0 & -1 & 0 & 1 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & -2 & 1 & 0 \\
S'' & 2 & 3 & 0 & -2 & 1 & -2 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\
S''' & 2 & 3 & 0 & -3 & -2 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & -2 \\
K & 2 & 3 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
F & 2 & 3 & -1 & -5 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
S & 2 & 3 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
F & 2 & 3 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array} \ . \tag{4.22}
$$

Evaluation of the second Chern class on the Kälher forms yields

$$\{\int c_2 J_i\} = \{336, 240, 164, 24, 84, 692, 708\} \ .$$

The intersection ring has the property that J_4 appears only linearly so that $l^{(4)}$ represents the base of a K3 fibration and the base of the local surface B, we therefore have $l^{(4)} = l_b$. J_5 appears only quadratically in the intersection ring and $l^{(5)}$ represents the base of an elliptic fibration. As before, it is the normal direction to the base of the local surface and gets decompactified.

We find the following basis, which reflect the curves that represent the Cartan elements of the affine $\hat{F}_4$.

$$l^{(0)}_{\hat{F}_4} = l^{(3)}, \quad l^{(1)}_{\hat{F}_4} = 2l^{(7)} + l^{(2)} + l^{(6)}, \quad l^{(2)}_{\hat{F}_4} = l^{(1)} \quad l^{(3)}_{\hat{F}_4} = l^{(6)}, \quad l^{(4)}_{\hat{F}_4} = l^{(7)} \ . \tag{4.23}$$

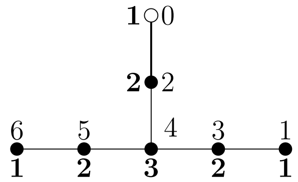
We see that in this basis

$$l_{T^2} = l^{(0)}_{\hat{F}_4} + 2l^{(1)}_{\hat{F}_4} + 3l^{(2)}_{\hat{F}_4} + 4l^{(3)}_{\hat{F}_4} + 2l^{(4)}_{\hat{F}_4}, \tag{4.24}$$

as expected (the vector corresponding to a given node is multiplied by the Coxeter label of that node). Note that the height in the last coordinate of ν_i^* is the dual Coxeter number of the Dynkin diagram of $\mathbb{F}_4$. This is a consequence of the F-theory realization of the G bundle moduli space of the heterotic string on an elliptically fibered K3 over the same $\mathbb{P}^1$ base as $\mathbb{P}(a_0, \ldots, a_r)$ [42] and will hold for all models below. From these data it is possible to calculate genus zero BPS invariants analogously to the $n \leq 4$ cases[6].

[6] We have calculated thes BPS invariants up to high multi-degree; these numbers are available on request.

4.2.5 $\mathcal{O}(-6) \to \mathbb{P}^1$ geometry with $\hat{E}_6$ resolution



In this hase the hypersurface CY has Euler number $\chi(M_3) = -624$ and $h_{11} = 9$. The polyhedron Δ^* has 199 star triangulations. Again we choose a simple one

D	ν_i^*				$l^{(1)}$	$l^{(2)}$	$l^{(de)}$	$l^{(4)}$	$l^{(b)}$	$l^{(6)}$	$l^{(7)}$	$l^{(8)}$	$l^{(9)}$
D_0	0	0	0	0	0	0	0	0	0	0	0	−1	0
D_1	−1	0	0	0	−2	0	0	0	0	0	0	0	1
D_2	0	−1	0	0	0	−1	0	0	0	0	0	1	0
D_3	0	0	0	−1	0	0	0	0	0	0	1	−1	−1
D_4	0	1	0	−1	3	0	0	0	0	0	−1	1	−1
D_5	1	1	0	−2	0	2	0	0	0	1	−1	0	1
D_6	1	2	0	−2	0	0	0	0	0	−2	1	0	0
F	2	3	−1	−6	0	0	0	0	1	0	0	0	0
S'	2	3	0	−3	0	−2	0	0	−2	1	0	0	0
S''	2	3	0	−2	−2	1	0	1	0	0	0	0	0
S'''	2	3	0	−1	1	0	1	−2	0	0	0	0	0
K	2	3	0	0	0	0	−2	1	0	0	0	0	0
S	2	3	0	1	0	0	1	0	0	0	0	0	0
F	2	3	1	0	0	0	0	0	1	0	0	0	0 ,

(4.25)

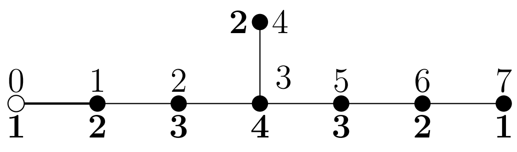
which has the property that

$$\{\int c_2 J_i\} = \{276, 360, 96, 188, 24, 764, 1524, 728, 800\}.$$

By Oguiso's criterion J_b represents the volume of the base of a K3 fibration while J_{de} respresents the volume of the base of an elliptic fibration. We have calculated the genus zero BPS invariants up to multi-degree 21.

4.2.6 The cases with $\hat{E}_7$ resolution

Here we discuss the cases $\mathcal{O}(-n) \to \mathbb{P}^1$ with $n = 7$ and $n = 8$. Let us start with $n = 8$, that is, the pure E_7 gauge theory case. The reflexive polyhedron Δ^* is the convex hull of the points ν_i^* listed below, together with the Mori cone that corresponds to a simple of a total of 420 star triangulations of Δ^*.



D	ν_i^*				$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(4)}$	$l^{(5)}$	$l^{(6)}$	$l^{(7)}$	$l^{(8)}$	$l^{(9)}$	$l^{(10)}$
D_0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1
D_1	−1	0	0	0	−1	0	0	0	0	0	0	0	0	1
D_2	0	−1	0	0	0	−1	0	0	0	0	0	0	1	0
D_3	0	0	0	−1	0	0	0	0	0	0	0	1	−2	0
D_4	0	1	0	−2	0	0	0	0	0	0	1	1	1	−1
D_5	1	1	0	−2	0	2	0	0	0	0	0	−1	0	1
D_6	1	2	0	−3	3	0	0	0	0	0	−2	−1	0	0
S'	2	3	0	−4	−3	0	−2	0	0	0	1	0	0	0
S''	2	3	0	−3	1	−2	0	0	0	1	0	0	0	0
S'''	2	3	0	−2	0	1	0	0	1	−2	0	0	0	0
S''''	2	3	0	−1	0	0	0	1	−2	1	0	0	0	0
F	2	3	−1	−8	0	0	1	0	0	0	0	0	0	0
K	2	3	0	0	0	0	0	−2	1	0	0	0	0	0
S	2	3	0	1	0	0	0	1	0	0	0	0	0	0
F	2	3	1	0	0	0	1	0	0	0	0	0	0	0

(4.26)

By standard toric methods we calculate $\chi(M_3) = -732$ and $h_{11} = 10$, i.e. $h_{21} = 376$. For this choice of Mori cone one has:

$$\{\int_{M_3} c_2 J_i\} = \{560, 456, 24, 120, 236, 348, 1724, 1764, 884, 848\}.$$

In the intersection ring J_3 appears only linearly, while J_4 appears only quadratically indicating a K3- and an elliptic fibration structure respectively.

The case with an additional $\frac{1}{2}\mathbf{56}$ hypermultiplet is obtained by replacing the point $\nu_F^* = (2, 3, -1, -8)$ with $\nu_F^* = (2, 3, -1, -7)$. The Hodge numbers change to $h_{21} = 348$, $h_{11} = 10$, and hence $\chi(M_3) = -676$. The only change in the Mori generators is the modified element

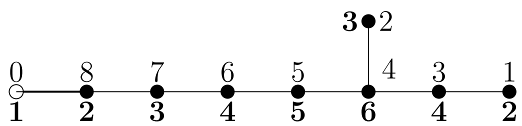
$$l^{(3)} = (0, 0, 0, 0, 0, 0, 0, -1, -1, 0, 0, 1, 0, 0, 1),$$

which leads to

$$\{\int_{M_3} c_2 J_i\} = \{524, 408, 24, 108, 212, 312, 1592, 1608, 806, 792\}.$$

While the intersection ring is modified, the fibration structure with respect to the classes J_3 and J_4 is maintained. For the $\hat{E}_7$ geometries, we have computed genus zero BPS invariants up to multi-degree 21.

4.2.7 The cases with $\hat{E}_8$ resolution



The Calabi-Yau hypersurface has $h_{11} = 11$ and $\chi(M_3) = -960$ and is the case with maximal absolute value of the Euler number within the class of toric hypesurfaces. From the 588 star triangulations of Δ^* we choose one leading to simple Mori cone with desired fibration structure:

D	ν_i^*				$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(4)}$	$l^{(5)}$	$l^{(6)}$	$l^{(7)}$	$l^{(8)}$	$l^{(9)}$	$l^{(10)}$	$l^{(11)}$
D_0	0	0	0	0	0	0	0	0	0	0	0	0	−1	0	0
D_1	−1	0	0	0	−2	0	0	0	0	0	0	0	0	0	1
D_2	0	−1	0	0	1	−1	0	0	0	0	0	0	0	0	0
D_3	0	1	0	−2	3	0	0	0	0	0	0	0	0	1	−2
D_4	1	1	0	−3	−2	2	0	0	0	0	0	0	1	0	0
D_5	1	2	0	−4	0	0	0	0	0	0	0	0	1	−2	1
F	2	3	−1	−12	0	0	0	0	0	0	0	1	0	0	0
S'	2	3	0	−6	0	0	0	0	0	0	0	−2	−2	1	0
S''	2	3	0	−5	0	−2	0	0	0	0	1	0	1	0	0
S'''	2	3	0	−4	0	1	0	0	0	1	−2	0	0	0	0
S''''	2	3	0	−3	0	0	0	0	1	−2	1	0	0	0	0
S'''''	2	3	0	−2	0	0	0	1	−2	1	0	0	0	0	0
S''''''	2	3	0	−1	0	0	1	−2	1	0	0	0	0	0	0
K	2	3	0	0	0	0	−2	1	0	0	0	0	0	0	0
S	2	3	0	1	0	0	1	0	0	0	0	0	0	0	0
F	2	3	1	0	0	0	0	0	0	0	0	1	0	0	0

(4.27)

The evaluations of the Kähler forms against the second Chern class are:

$$\{\int_{M_3} c_2 J_i\} = \{1496, 948, 168, 332, 492, 648, 800, 24, 1092, 2228, 3360\}.$$

We have a K3 fibration over the $\mathbb{P}^1$ represented by J_8 and an elliptic fibration over the base whose volume is given by J_3. As in all models with K3 fibrations, the BPS numbers depend only on the square of the curve classes in the K3, whose intersection form in the Picard lattice is given by coefficients of the ring that is linear in J_8, and can be obtained by a heterotic one-loop calculation. The genus zero BPS states are available to multi-degree 18.

The geometry can be modified by successively adding tensor multiplets corresponding to small instantons. This can be done by blow-ups of (-1) curves in the base. In the particular case of E_8 gauge symmetry this corresponds to the blowing up Hirzebruch surfaces $\mathbb{F}_n$, $n = 11, 10, 9$. We can construct explicitly models in which all divisors have toric representatives, by modifying Δ^* in the following way:

- $n = 11$ (one small instanton case): ν_F gets replaced by two points ($\nu_F \to \{\{2, 3, -1, -11\}, \{0, 0, 0, -1\}\}$). Thus $h_{11} = 12$, and in accord with the 6d anomaly condition we find $\chi(M_3) = -900$;

- $n = 10$: one must replace $\nu_F \to \{\{2,3,-1,-10\},\{0,0,0,-1\},\{2,3,1,-1\}\}$;
- $n = 9$: here, $\nu_F \to \{\{2,3,-1,-9\},\{0,0,0,-1\},\{2,3,1,-1\},\{2,3,1,-2\}\}$.

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A Geometry of Landau-Ginzburg models

In this part of the Appendix we describe a mirror construction that applies directly to the $(T^2 \times \mathbb{C}^2)/\mathbb{Z}_k$, $k = 3,4,6$ local orbifold using a Landau-Ginzburg description of the torus and the LG/CY correspondence. We focus on the case $k = 3$, which we compare in the next subsection with the constuction of the $\mathcal{O}(-3) \to \mathbb{P}^1$ geometry discussed in section 4.2.2.

The weighted homogenous Landau-Ginzburg potential W is the relevant term determining the CFT that corresponds to the Calabi-Yau sigma model at the infrared fixed point. The building blocks of W can be labeled by a simply laced Lie algebra. The A_{k+1} case corresponds to the monomial x^{k+2}; $D_{\frac{k+2}{2}}$ (with k even) corresponds to $x_i^{\frac{k}{2}+1} + xy^2$; finally, one has the following exceptional cases: $E_6 : x^3 + y^4$ (with $k = 10$), $E_7 : x^3 + xy^3$ (with $k = 16$), and $E_8 : x^3 + y^5$ (with $k = 28$). Here the notation is so that $k+2$ is the Coxeter number of the Lie algebra and the contribution of each block to the central charge is

$$c = \frac{3k}{k+2}. \tag{A.1}$$

Note that quadratic terms do not contribute to the central charge. Each building block is identified with a *minimal* $(2,2)$ superconformal theory with this central charge and the label of the Lie algebra specifies the way the left and the right sectors of the theory are glued to form a modular invariant partition function, a problem that enjoys an ADE classification. When one identifies tensor products of LG models or minimal $(2,2)$ SCFT with the CY geometry one has to make a consistent projection on integral $U(1)_{L/R}$ charges and the correct spin structure. This involves orbifoldizations on the LG or minimal model side.

The $c = 3$ cases correspond to elliptic curves on the Calabi-Yau side. There are three cases with only A invariants,

$$W_3 = x_1^3 + x_2^3 + x_3^3 - 3\alpha x_1 x_2 x_3, \tag{A.2}$$
$$W_4 = x_1^4 + x_2^4 + x_3^2 - 4\alpha x_1 x_2 x_3, \tag{A.3}$$
$$W_6 = x_1^6 + x_2^3 + x_3^2 - 6\alpha x_1 x_2 x_3, \tag{A.4}$$

which have been identified with $T^2/\mathbb{Z}_3, T^2/\mathbb{Z}_4$ and $T^2/\mathbb{Z}_6$ orbifolds of the elliptic curve, see for example [43].

If one adds up three copies the above potential to a $c = 9$ LG potential and projects to integral charges one gets string vacua that correspond to the resolved $(T^2)^3/(\mathbb{Z}_k \times \mathbb{Z}_k)$ CY manifolds. To get the simpler $(T^2)^3/(\mathbb{Z}_k)$ orbifolds one has to mod out on the LG side by a further $\mathbb{Z}_k$. Aspects of the mirror description have been described in [44].

Let us consider for instance the $k = 3$ case. The LG potential is

$$W = \sum_{i=1}^{9} x_i^3 + \sum_{i \neq j \neq k} \alpha_{ijk} x_i x_j x_k, \tag{A.5}$$

where we listed the $\binom{9}{3} = 84$ independent complex structure deformations in the homogeneous degree 3 ring $\mathbb{C}[\underline{x}]/\{\partial_{x_i} W : i = 1, \ldots, 9\}$. The $(T^2)^3/(\mathbb{Z}_k \times \mathbb{Z}_k)$ orbifold has 84 Kähler deformations and the Landau-Ginzburg model can be viewed as its mirror. Some aspects of the B-model have been analyzed in [44]. In particular one has a $(5, 2)$-form in the sevenfold $W = 0$ in $\mathbb{P}^8$ given by

$$\Omega_{5,2} = \frac{1}{2\pi i} \oint_{S^1} \frac{\mu_8}{W^3} , \tag{A.6}$$

where

$$\mu_8 = \epsilon_{i_1,\ldots,i_9} x_{i_1} \mathrm{d}x_{i1} \ldots \mathrm{d}x_{i_9} . \tag{A.7}$$

is a eight form, well defined under scaling $x_i \to \lambda x_i$. Integrating over the S^1 around $W = 0$ makes it a $(5, 2)$ form.

The orbifold that relates the LG model to the $(T^2)^3/\mathbb{Z}_3$ geometry acts by

$$\begin{pmatrix} x_i, i \in I \\ x_j, j \in J \\ x_k, k \in K \end{pmatrix} \mapsto \begin{pmatrix} \alpha x_i, i \in I \\ \alpha^2 x_j, j \in J \\ x_k, k \in K \end{pmatrix}, \tag{A.8}$$

where the sets are $I = \{1, 2, 3\}$, $K = \{4, 5, 6\}$ and $J = \{7, 8, 9\}$ and α is a third root of unity. The invariant monomials are $m_1 = x_1 x_2 x_3$, $m_2 = x_4 x_5 x_6$ and $m_3 = x_7 x_8 x_9$ as well as the 27 monomials $m_{ijk} = x_i x_j x_k$, where indices i, j, k are in the sets I, J, K respectively. Hence, we get the invariant LG potential

$$W = \sum_{i=1}^{9} x_i^3 - 3 \sum_{i=1}^{3} \alpha_i m_i - 3 \sum_{i \in i, j \in J, k \in K} \alpha_{ijk} x_i x_j x_k. \tag{A.9}$$

On each T^2 of $(T^2)^3/\mathbb{Z}_3$ (which we parametrize by complex coordinates z_i, $i = 1,2,3$), the Z_3 action has three fixed points, corresponding to $z_i = 0, \frac{1}{\sqrt{3}}\beta$, and $\frac{2}{\sqrt{3}}\beta$, where $\beta^{12} = 1$, so that the global orbifold has $3^3 = 27$ $\mathbb{Z}_3$ fixed points. Locally it is given by $\mathbb{C}^3/\mathbb{Z}_3$, with the action of $\mathbb{Z}_3$ as in (2.3). The resolved geometry near each fixed point looks locally like the total space $\mathcal{O}(-3) \to \mathbb{P}^2$ of the anti-canonical bundle over $\mathbb{P}^2$. Here the $\mathbb{P}^2$ is the exceptional divisor E_i of the blow up. There are nine further Kähler classes in the invariant sector: the three invariant $(1,1)$ forms $h_i = dz_i \wedge d\bar{z}_i$, $i = 1,2,3$ and the six invariant $(1,1)$ forms $h_{ij} = dz_i \wedge d\bar{z}_j$, $i \neq j$, $i,j = 1,2,3$. If we denote the dual divisors H_i and H_{ij} respectively one has the non-vanishing intersections

$$\begin{array}{c} H_1 \cdot H_2 \cdot H_3 = \kappa, \quad H_{ij} \cdot H_{ki} \cdot H_{jk} = \kappa, \\ H_i \cdot H_{jk} \cdot H_{kj} = -\kappa, \quad \text{for } \; i \neq j, i \neq k \;, \quad E_i^3 = \kappa, \quad i = 1, \dots 27. \end{array} \tag{A.10}$$

Here we have taken a normalization $\kappa = 9$. Similarly, the intersection numbers of orbifolds with fixed tori have been calculated in [45].

We know from the mirror map of the individual $T^2/\mathbb{Z}_3$'s, given by the cubic constraint $W_3 = 0$ in (A.4), that $\frac{1}{2\pi i}\log(\alpha_i) \to i\infty$ corresponds to the large volume limit $\mathrm{Im}(\tau_i) \to \infty$ of the i'th T^2. Hence the limit in which $(T^2)^3/\mathbb{Z}_3$ becomes the $T^2 \times \mathbb{C}^2$ geometry involves taking, say, $\alpha_2, \alpha_3 \to \infty$, while keeping $\alpha \equiv \alpha_1$ finite. These limits can be taken individually and produce a term $\infty x_4 x_5 x_6$ or $\infty x_7 x_8 x_9$ in W, which requires taking, say, $x_5 \to 0$ and $x_8 \to 0$. The complex volumes of the 27 $\mathbb{P}^2$'s in the local $\mathcal{O}(-3) \to \mathbb{P}^2$ geometies are parametrized in the large volume limit of the $\mathbb{P}^2$'s by $t_a \sim \frac{1}{2\pi i}\log(\alpha_{ijk})$, $a = 1,\dots,27$. We want to keep three of these finite, while 24 of them should be scaled to infinite volume. Again this gives infinite terms in W, which can be eliminated by setting in addition $x_6 = x_8 = 0$; that is, we keep only x_1, x_2, x_3, x_4, x_7 finite. Then, $\frac{1}{2\pi i}\log(\alpha_{i,4,7}) \equiv \frac{1}{2\pi i}\log(\beta_i) \sim t_i$, $i = 1,2,3$ parametrize the three finite $\mathbb{P}^2$s. Hence, we end up with a potential

$$W = x_1^3 + x_2^2 + x_3^3 + x_4^3 + x_7^3 - 3\alpha x_1 x_2 x_3 - 3\beta_1 x_1 x_4 x_7 - 3\beta_2 x_2 x_4 x_7 - 3\beta_3 x_3 x_4 x_7. \tag{A.11}$$

Let us relabel coordinates and rewrite W as

$$W = a_0 x_1 x_2 x_3 + \sum_{i=1}^{5} a_i x_i^3 + a_6 x_1 x_4 x_5 + a_7 x_2 x_4 x_5 + a_8 x_3 x_4 x_5 \equiv \sum_{i=0}^{8} a_i Y_i. \tag{A.12}$$

We note that there are k relations among the Y_i given by

$$\prod_i Y_i^{l_i^{(r)}} = 1, \qquad r = 1,\dots,k, \tag{A.13}$$

where $k = 4$ and

$$\begin{array}{l} l^{(1)} = (-3; 1,1,1,0,0, \quad 0, \quad 0, \quad 0) \\ l^{(2)} = (\quad 0; 1,0,0,1,1,-3, \quad 0, \quad 0) \\ l^{(3)} = (\quad 0; 0,1,0,1,1, \quad 0,-3, \quad 0) \\ l^{(4)} = (\quad 0; 0,0,1,1,1, \quad 0, \quad 0,-3) \;. \end{array} \tag{A.14}$$

The Y_i, $i = 0, \ldots, 4$ are $\mathbb{C}$ variables while the Y_j, $j = 4, \ldots, 8$ are $\mathbb{C}^*$ variables. All Y_i are subject to a $\mathbb{C}^*$ action $Y_i \to \nu Y_i$ with $\nu \in \mathbb{C}^*$. This yields a $(9-4-1-1) = 3$-dimensional local Calabi-Yau manifold which is the mirror to the $(T^2 \times \mathbb{C}^2)/\mathbb{Z}_3$ manifold.

Its four complex deformations $z_r = (-1)^{l_0^{(r)}} \prod_i a_i^{l_i^{(r)}}$, $r = 1, \ldots, 4$, correspond respectively to the complexified volume of the T^2 and three lines in different exceptional $\mathbb{P}^2$'s. The local (3,0) form is given by

$$\Omega_{3,0} = \frac{1}{2\pi i} \oint \frac{\epsilon_{ijk} x_i dx_j dx_k}{W} \frac{dx_4}{x_4} \frac{dx_5}{x_5} \qquad i, j, k = 1, 2, 3 \, . \tag{A.15}$$

From local expression of the (3,0) form in (A.15) or a limit of (A.6) one can see that the periods $\tilde{\Pi}$ in the local limit fulfill

$$\prod_{l_i^{(r)}<0} \partial_{a_i}^{l_i^{(r)}} \tilde{\Pi} = \prod_{l_i^{(r)}>0} \partial_{a_i}^{l_i^{(r)}} \tilde{\Pi}. \tag{A.16}$$

Also, the periods have the scale invariances acting on the a_i that allow one to eliminate the a_i by the invariant combinations z_r. The periods Π are therefore annihilated by the differential operators

$$\begin{aligned} \mathcal{D}_1 &= (\theta_1+\theta_2)(\theta_1+\theta_3)(\theta_1+\theta_4) + 3\theta_1(3\theta_1-2)(3\theta_1-1)z_1 \\ \mathcal{D}_2 &= (\theta_1+\theta_2)(\theta_2+\theta_3+\theta_4)^2 + 3(\theta_2-1)(3\theta_2-1)(3\theta_2-1)z_2 \\ \mathcal{D}_3 &= (\theta_1+\theta_3)(\theta_2+\theta_3+\theta_4)^2 + 3(\theta_3-1)(3\theta_3-2)(3\theta_3-1)z_3 \\ \mathcal{D}_4 &= (\theta_1+\theta_4)(\theta_2+\theta_3+\theta_4)^2 + 3(\theta_4-1)(3\theta_4-2)(3\theta_4-1)z_4, \end{aligned} \tag{A.17}$$

where $\theta_i = z_i \frac{\mathrm{d}}{\mathrm{d}z_i}$. Note the following:

- We normalized the periods to $\Pi = a_0 \tilde{\Pi}$.
- The periods are not completely determined by the operators; that is, there are more functions annihilated by the $\mathcal{D}_i$ than periods.

Nevertheless, we can identify the relevant solutions by the Frobenius method. In this method, we define a $\underline{z} = (z_1, \ldots, k)$ and $\underline{\rho} = (\rho_1, \ldots, \rho_k)$ dependent function as

$$\omega(\underline{z}, \underline{\rho}) = \sum_{\underline{n}} \left(\frac{\Gamma(\sum_a l_0^{(a)}(n_a+\rho_a)+1)}{\prod_{i>0} \Gamma(\sum_a l_i^{(a)}(n_a+\rho_a)+1)} \right) z^{\underline{n}+\underline{\rho}} \, . \tag{A.18}$$

Using the fact that $[\partial_{\rho_a}, \mathcal{D}_k] \sim 0$ and $\omega(\underline{z}, \underline{\rho}=0)$ is a solution, we get more solutions by taking derivatives with respect to the various ρ_i:

$$\begin{aligned} X^0 &= \omega(\underline{z}, \underline{\rho})|_{\underline{\rho}=0} \, , \\ X^r &= \partial_{\rho_r} \omega(\underline{z}, \underline{\rho})|_{\underline{\rho}=0} \, , \\ \tilde{F}_r &= \tfrac{1}{2} c_{rij} \partial_{\rho_i} \partial_{\rho_j} \omega(\underline{z}, \underline{\rho})|_{\underline{\rho}=0} \, , \\ \tilde{F}_0 &= \tfrac{1}{6} c_{ijk} \partial_{\rho_i} \partial_{\rho_j} \partial_{\rho_k} \omega(\underline{z}, \underline{\rho})|_{\underline{\rho}=0} \, . \end{aligned} \tag{A.19}$$

Here $r = 1, \ldots, h_{11}(M_3) = h_{21}(W_3)$, and the c_{ijk} are classical intersection numbers. Note that each derivative w.r.t. ρ_i gives a $\log(z_i)$ term and that there is only a finite number $b_3(W_3)$ of solutions[7], which are given in (A.19).

The single logarithmic solutions define the mirror maps as

$$t^k = \frac{1}{2\pi i} \frac{X^k}{X^0} \ . \tag{A.20}$$

Mirror symmetry and special geometry implies that an integral sympletic basis of periods is given by

$$\Pi = X^0 \begin{pmatrix} 1 \\ t^a \\ \partial_{t^a} F_0 \\ 2F_0 - t^a \partial_{t^a} F_0 \end{pmatrix} \ , \tag{A.21}$$

where the prepotential F_0 is determined in terms of the C.T.C Wall Data as

$$F_0 = X_0^2 \left[-\frac{1}{6} c_{ijk} t^i t^j t^k + \frac{1}{2} A_{ij} t^i t^j + c_i t^i - i\chi \frac{\zeta(3)}{2(2\pi)^3} + \sum_{\underline{n}} n_{\underline{n}}^{(0)} \mathrm{Li}_3(\underline{Q}^{\underline{n}}) \right] \ , \tag{A.22}$$

with $\mathrm{Li}_k(x) = \sum_{n=1} \frac{x^n}{n^k}$, $\underline{Q}^{\underline{n}} = \prod_{i=1}^k \exp(2\pi i t_i n_i)$, $c_i = \frac{1}{24} \int_X c_2 \, J_i$ and χ is the Euler number of M_3. In particular, we can read off the genus 0 GV invariants $n_{\underline{n}}^{(0)}$ already from the double logarithmic solutions

$$X^0 \partial_{t^r} F_0 \ . \tag{A.23}$$

In the example under consideration, one gets in particular the period X^0 and, by taking single derivatives with respect to ρ_i, one gets k further periods, which are to low orders in z given by

$$\begin{aligned} X^0(z_1) &= 1 + 6z_1 + 90z_1^2 + 1680z_1^3 + 34650z_1^4 + \mathcal{O}(z_1^5) \\ X^1(z_1) &= X^0 \log(z_1) + 15z_1 + (513z_1^2)/2 + 5018z_1^3 + \mathcal{O}(z_1^4) \\ X^k(z_1, z_k) &= X^0 \log(z_k) - 6(z_1 + z_k) + (45z_k^2 - 135z_1^2 - 18z_1 z_k) \\ &\quad + (90z_1 z_k^2 - 3080z_1^3 - 180z_1^2 z_k - 560z_k^3) + \mathcal{O}(z^4), \quad k = 2, 3, 4. \end{aligned} \tag{A.24}$$

In the local limit, only the intersections $E_i^3 \sim \kappa_{lim}$ contribute, with an appropriate normalization. We have calculated the genus zero BPS numbers using the description above and checked that they agree with the ones calculated in the decompatification limit of the Calabi-Yau threefold discussed in section 4.2.2 in the basis discussed in the next section.

[7] In fact the notation $[\partial_{\rho_a}, \mathcal{D}_k] \sim 0$ means that arbitrary derivatives are not all annihilated by the differential operators $\mathcal{D}_k$; rather, the result will be in general proportional to $\log(z_i)$ terms.

Toric Geometry realization of the $\mathcal{O}(-3) \to \mathbb{P}^1$ geometry

Here we want to obtain the geometry discussed in the last section from the toric polyhedron specified in Table 4.14. It has 10 star triangulations. We list only one Mori cone with the intersections

$$\begin{aligned}\mathcal{R} = {} & 1791J_1^3 + 957J_1^2J_2 + 511J_1J_2^2 + 272J_2^3 + 180J_1^2J_3 + \\ & 96J_1J_2J_3 + 51J_2^2J_3 + 18J_1J_3^2 + 9J_2J_3^2 + 360J_1^2J_4 + \\ & 192J_1J_2J_4 + 102J_2^2J_4 + 36J_1J_3J_4 + 19J_2J_3J_4 + 3J_3^2J_4 + 72J_1J_4^2 + \\ & 38J_2J_4^2 + 7J_3J_4^2 + 14J_4^3 + 900J_1^2J_5 + 480J_1J_2J_5 + 256J_2^2J_5 + 90J_1J_3J_5 + \\ & 48J_2J_3J_5 + 9J_3^2J_5 + 180J_1J_4J_5 + 96J_2J_4J_5 + 18J_3J_4J_5 + 36J_4^2J_5 + 450J_1J_5^2 + \\ & 240J_2J_5^2 + 45J_3J_5^2 + 90J_4J_5^2 + 225J_5^3.\end{aligned} \tag{A.25}$$

The evaluation of the second Chern class is given by $\int c_2 J_i = \{570, 308, 60, 116, 282\}$, and the model has no particular fibration structure except for the elliptic one.

In relation to the Landau-Ginzburg formulation of the Z orbifold, the following classes in the given base,

$$\begin{array}{l} l_{T^2} = 6l^{(1)} + 3l^{(2)} + l^{(4)} + 3l^{(5)}, \quad l^{(1)}_{\mathbb{P}^2} = 3l^{(1)} + l^{(2)} \\ l^{(2)}_{\mathbb{P}^2} = l^{(2)}, \quad l^{(3)}_{\mathbb{P}^2} = l^{(2)} + l^{(4)}, \quad l_{de} = l^{(3)} + \frac{1}{3}(l^{(2)} + l^{(5)}), \end{array} \tag{A.26}$$

are of particular interest, because for them the intersection form becomes symmetric in the three classes of the $\mathbb{P}^2$'s discussed in Section A:

$$\mathcal{R} = \frac{25}{3} J_{T^2}^3 - \frac{1}{3}(J^3_{\mathbb{P}^2_1} + J^3_{\mathbb{P}^2_2} + J^3_{\mathbb{P}^2_3}) + 5J_{T^2}^2 J_{de} + 3J_{T^2}J_{de}^2 \ . \tag{A.27}$$

Here J_{de} is the class that needs to be decompactified to obatain a non-compact geometry.

The curve classes, which correspond to the Mori cone vector $l^{(i)}_{\mathbb{P}^2}$ generate all the BPS states of $\mathcal{O}(-3) \to \mathbb{P}^2$. For example, at genus zero

$$\{n^{(0)}_{3k,k,0,0,0}\} = \{n^{(0)}_{0,k,0,0,0}\} = \{n^{(0)}_{0,k,0,k,0}\} = \{3, -6, 27, -192, 1695, -17064, \ldots\}. \tag{A.28}$$

These curves lie in the divisor classes D_3, D_4, D_5, which consist of an exceptional curve over the -3 curve in the base S' with $(S')^2 = -3$, a section of the Hirzebruch surface $\mathbb{F}_3$ fiber. The class l_{T^2} is the class of the elliptic fiber. We find that in this direction all BPS numbers are $\{n^{(0)}_{6k,3k,0,k,3k}\} = \{492, 492, \ldots\}$. Note that 492 is the Euler number of the Calabi-Yau and this is the first modular direction. This follows from the representation of the Eisenstein series E_4 as

$$E_4(q) = 1 + 240 \sum_{k=1}^{\infty} \frac{k^3 q^k}{1-q^k} \tag{A.29}$$

and from the multicovering formula for $g = 0$, which reads

$$F_0 = F_0^{classical} + \sum_{d=1}^{\infty} n^{(0)}_d \mathrm{Li}_3(q^d), \tag{A.30}$$

with $\mathrm{Li}_k(x)\sum_{n=1}^{\infty}\frac{x^d}{k^d}$. From this, one gets

$$\partial_t^3 F_0 = c + \sum_{d=1}^{\infty} \frac{n_d d^3 q^d}{1-q^d} = \frac{240c-\chi}{240} + \frac{\chi}{240} E_4(q) \tag{A.31}$$

Let us denote by d_{T^2} the degree of the elliptic curve, and by d_1, d_2, d_3 the degrees with respect to the three $\mathcal{O}(-3) \to \mathbb{P}^2$ classes L_i, $i = 1, 2, 3$, and denote now the BPS invariants by $n^{(0)}_{d_{T^2},d_1,d_2,d_3}$. We have the obvious property that $n^{(0)}_{d_{T^2},d_1,d_2,d_3}$ depends symmetrically on the d_i. We can study the mixing of the local $\mathbb{P}^2$ with the fiber. At $d_{T^2} = 0$ we have no mixing between the L_i classes:

$d_1\backslash d_2$	0	1	2	3	4
0	$\frac{i492\zeta(3)}{(2\pi)^3}$	3	-6	27	-192
1	3	0	0	0	0
2	-6	0	0	0	0
3	27	0	0	0	0
4	-192	0	0	0	0

$n_{d_{T^2}=0,d_1,d_2,0}$

In general $n_{0,d_1,d_2,d_3} = 0$ if more then two d_i are non zero. This is of obvious from the geometry, since the blow up points sit at distiguished points of the fiber and are uncorrelated as long there is no curve wrapping the fiber.

From $d_{T^2} > 1$ the mixing starts. In particular, one finds:

$d_1\backslash d_2$	0	1	2	3	4
0	492	36	-360	4752	-70560
1	36	-216	2052	-26082	376704
2	360	2052	-17760	211140	-2912544
3	4752,	-26082	211140	-2378484	31525200
4	-70560	376704	-2912544	31525200	-405029376

$n^{(0)}_{1,0,d_1,d_2}$

$d_1\backslash d_2$	1	2	3	4
1	1458	-12654	150903	-2087856
2	-12654	103536	-1177686	15735024
3	150903	-1177686	12859560	-1664394480
4	-2087856	15735024	-166439448	2099613312

$n^{(0)}_{1,1,d_1,d_2}$

$d_1\backslash d_2$	2	3	4
2	-812808	8923104	-115996032
3	8923104	-94862502	1201724208
4	-115996032	1201724208	-14901588864

$n^{(0)}_{1,2,d_1,d_2}$

$d_1\backslash d_2$	0	1	2	3
0	492	288	-10656	346356
1	288	-8604	225234	-5852520
2	-10656	225234	-4648248	102706623
3	346356	-5852520	102706623	-2009199816

$n^{(0)}_{2,0,d_1,d_2}$

$d_1\backslash d_2$	0	1	2
0	492	1788	-197568
1	1788	-76724	-
2	-197568	-	-

$n^{(0)}_{3,0,d_1,d_2}$

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