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Strominger–Yau–Zaslow conjecture for Calabi–Yau hypersurfaces in the Fermat family

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1. Introduction

The Strominger–Yau–Zaslow (SYZ) conjecture [36] is the following: given a family of n-dimensional polarised Calabi–Yau (CY) manifolds $(X_s, g_s, J_s, \omega_s, \Omega_s)$ of holonomy $\mathrm{SU}(n)$ degenerating to the large complex structure limit, then

• After suitable scaling, the metric spaces (X_s, g_s) converge in the Gromov–Hausdorff sense to a *singular affine manifold* B homeomorphic to S^n. The limiting metric g_∞ is a *real Monge–Ampère metric* on the smooth locus. (This part is also known as the Kontsevich–Soibelman conjecture [29], [30].)

• Near the degenerating limit, the manifold X_s admits a *special Lagrangian T^n fibration* over the base B with some singular fibres. The diameters of the fibres are much smaller compared to $\mathrm{diam}(B)$. In the generic region on X_s, which covers most of the measure on X_s, the metric g_s is a small perturbation of a *semiflat metric*, meaning that the T^n fibres are almost flat.

• Mirror manifolds should be constructed as another T^n fibration over the same base B, by fibrewise replacing the T^n fibres with the dual tori.

An early achievement is Gross and Wilson's gluing construction [21] of degenerating CY metrics on K3 surfaces with elliptic fibrations, which becomes a special Lagrangian T^2-fibration after hyperkähler rotation. In this setting the metric is known semi-explicitly. The same period brought forth many insights concerning topological [18], combinatorial [23], [24], and differential geometric [43] aspects of the SYZ conjecture, until Joyce [27] discovered through his study of special Lagrangian singularities that the SYZ fibration map cannot be naïvely expected to be smooth, indicating the difficulty of the metric problem.

Later research on the SYZ conjecture gradually shifted focus from its metric geometric roots, in favour of softer approaches based on algebraic or symplectic methods, taking the original SYZ conjecture mainly as an inspiration. This has led to spectacular progress in the mathematical understanding of mirror symmetry, described in the excellent survey [19].

In the metric vein, the SYZ conjecture fits into the more general question of understanding how CY metrics degenerate as the complex and Kähler structures vary. The main dichotomy is whether the family of metrics are *non-collapsed*, meaning there is a uniform lower bound on the volume, once the diameter is normalised to 1. In the non-collapsing case much is known: for example, a polarised family of non-collapsed CY manifolds can only degenerate to normal CY varieties with klt singularities, and the notion of metric convergence agrees with the algebro-geometric notion of flat limit [14].

The collapsing case is widely open. Tosatti et al. made substantial progress on describing collapsing metrics associated with holomorphic fibrations [40], [20], in particular generalising much of [21] to hyperkähler manifolds with holomorphic Lagrangian fibrations. Recently there are many efforts to describe the degenerating CY metrics in special cases, notably for K3 surfaces [17], [26], [34], and higher-dimensional generalisations [37].

The metric SYZ conjecture resisted most attempts, because the large complex structure limit is a very severe degeneration mechanism. An interesting program of Boucksom et al. [3], [5] proposes that in the case of polarised algebraic degenerations the underlying Calabi–Yau manifolds converge naturally into a non-archimedean (NA) space, and the CY metrics should converge in a potential theoretic sense to their NA analogue. Their greatest achievements so far is to define and solve the NA Monge–Ampère (MA) equation, building on heavy machinery from birational geometry. To make contact with the SYZ conjecture, it would still remain to compare the non-archimedean MA equation with the real MA equation, prove the potential theoretic convergence, and improve it to the metric convergence. Notwithstanding these difficulties, this program has the promise to prove the SYZ conjecture in great generality.

The viewpoint of this paper is much more concrete. We focus on the Fermat family of projective hypersurfaces of any dimension n, approaching the large complex structure limit:

$$X_s=\left\{Z_0Z_1\dots Z_{n+1}+e^{-s}\sum_{i=0}^{n+1}Z_i^{n+2}=0\right\},\quad s\gg 1.$$

The most striking aspect of our results is the following.

THEOREM 1.1. (Cf. §5.4) *For the Fermat family, consider the Calabi–Yau metrics on X_s in the polarisation class $s^{-1}[\Delta]$, where $[\Delta]$ is a fixed Kähler class on $\mathbb{CP}^{n+1}$ restricted to X_s. Then, for a subsequence of X_s as $s\to\infty$, there exists a special Lagrangian T^n-fibration on the generic region $U_s\subset X_s$ such that*

$$\frac{\mathrm{Vol}(U_s)}{\mathrm{Vol}(X_s)}\to 1 \quad \text{as } s\to\infty.$$

We also summarize informally the other results in this paper:

- (Cf. §5.3) Any subsequential Gromov–Hausdorff limit of the CY metrics contains an open dense subset locally isometric to $\mathcal{R}\subset\partial\Delta_\lambda^\vee$ carrying a smooth real MA metric, where $\partial\Delta_\lambda^\vee$ denotes the boundary of a certain $(n+1)$-dimensional simplex $\Delta_\lambda^\vee$ in $\mathbb{R}^{n+1}$ arising naturally from tropical geometry, and $\partial\Delta_\lambda^\vee\setminus\mathcal{R}$ has zero $(n-1)$-Hausdorff measure.
- (Cf. Proposition 5.12) The diameters of the subsequence of CY metrics are uniformly bounded.
- (Cf. §5.2) In the generic region of X_s for $s\gg 1$, the CY metrics are C^∞_{loc} close to a sequence of semiflat metrics. In particular the sectional curvature in the generic region is uniformly bounded.

A basic feature of the complex geometry of CY hypersurfaces near the large complex structure limit, is that in generic regions the local structure is a large annulus region in $(\mathbb{C}^*)^n$, equipped with a holomorphic volume form which modulo a scale factor is very close to $d\log z_1\wedge \ldots d\log z_n$. An elementary observation is that *plurisubharmonic* (psh) *functions are intimately related to convex functions*:

- Let ϕ be psh on an annulus $\{1<|z_j|<\Lambda\}\subset(\mathbb{C}^*)^n$, then the fibrewise average function

$$\bar{\phi}(x_1,\ldots x_n)=\fint_{T^n}\phi(e^{x_1+i\theta_1},\ldots,e^{x_n+i\theta_n})\,d\theta_1\ldots d\theta_n$$

is convex.

- Let u be a convex function on $\{0<x_j<\log\Lambda\}$, then the pull-back of u to

$$\{1<|z_j|<\Lambda\}\subset(\mathbb{C}^*)^n$$

via the logarithm map is psh, and u solves the real MA equation $\det(D^2u)=\text{const}$ if and only if its pull-back solves the complex MA equation

$$\det\left(\frac{\partial^2 u}{\partial\log z_i\,\partial\overline{\log z_j}}\right)=\text{const}.$$

Our strategy is to show that in the highly collapsed regime $s \gg 1$, the local Kähler potentials are C^0-approximated by convex functions, whose regularity properties can be then transferred back to the local Kähler potentials at least in the generic region. In effect, this implies in the generic region the Calabi–Yau metrics are collapsing with uniformly bounded sectional curvature; then the existence of the special Lagrangian fibration in the generic region is a simple perturbation argument. Keeping in mind that the local complex structure is an annulus in $(\mathbb{C}^*)^n$, the special Lagrangian fibration is just a small C^∞-perturbation of the logarithm map

$$(\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n.$$

The essential problem is to obtain uniform estimates on the CY metrics as $s \to \infty$. Our techniques differ very significantly from Yau's proof of the Calabi conjecture. Our Kähler potential estimates are largely based on Kołodziej's method in pluripotential theory, which has the advantage of robustness even in collapsing settings. The technical core of our contribution is to produce a regularisation of the Calabi–Yau potential, and prove an improved version of the global Skoda inequality, which for large s forces the potential to be very close to its regularisation. As convexity is built into the construction of the regularisation, this furnishes a bridge between holomorphic and convex geometry, and one can start to transfer the a-priori much better regularity from the convex world into the holomorphic world near the collapsing limit $s \to \infty$. Our higher-order estimates exploit the local regularity theory of real MA equations, and a result of Savin from non-linear PDE theory.

The structure of the paper is as follows. We survey the rather extensive analytical backgrounds in §2. The complex geometry of the degenerating hypersurfaces is discussed in §3, with particular emphasis on its interplay with tropical geometry. We estimate the Calabi–Yau potentials in §4; in particular, we prove the Skoda type estimates, the uniform L^∞ bound, and the C^0-approximation by the convex regularisations. In §5, we use uniform Lipschitz bounds on the regularisation to extract a subsequential limit, and show that this defines a real MA metric. We then use the local regularity theory of real MA metrics to show the higher order estimates on the CY local potentials, and prove the existence of the special Lagrangian fibration.

We now discuss some directions of future research.

- It seems highly plausible that the SYZ conjecture on generic regions will hold also on many other degenerating CY manifolds, or at least CY hypersurfaces. In fact the only reason we restrict to the Fermat case is to utilize the large discrete symmetry group to give a relatively simple proof of a technical extension property for locally convex functions, which seems likely to generalize to other contexts.

• One would like to study the existence, uniqueness, and regularity of the real MA equation on compact polyhedral sets, which are covered by charts whose transition functions are only piecewise linear but not smooth in general; the SYZ conjecture predicts the solutions to such real MA equations should arise as possible limits of the collapsing CY metrics. This question may be parallel to the non-archimedean MA approach taken up in [4]. At present according to the author's knowledge, it is not clear how to define the real MA equation globally on such sets, and in fact we do not even have an established notion of local convexity.

Such questions on the real MA equations have direct bearings on improving our main theorem. For instance, if one can establish uniquenss, then there is no need to pass to subsequences in all of our results. If one can establish sufficient regularity, then it may be possible to prove the Gromov–Hausdorff limit is homeomorphic to

$$\partial\Delta_\lambda^\vee \simeq S^n.$$

The problem to set up the real MA equation is quite subtle. On a piecewise linear manifold the notion of a convex function is dependent on charts, and so does the real MA operator. To set up an invariant notion of the real MA equation, it is necessary to make branch cuts to charts. The location of such cuts seems to depend on some gradient condition on the convex function in question, and is hard to predict in the absence of symmetry. Thus the global real MA equation on polyhedral sets has the feature of a free boundary problem.

• The a-priori estimate approach in this paper says very little about the CY metrics in regions with high curvature concentration. In the case of CY 3-folds, the author [31] recently constructed the 3-dimensional analogues of the Ooguri–Vafa metric, which are conjectured to be the universal metric models for the neighbourhood of the most singular fibres in a generic SYZ fibration. A program to tackle the 3-fold case of the SYZ conjecture based on gluing ideas is outlined in [31], which has the ultimate aim to give a global description of the metric, and to produce a special Lagrangian fibration globally. This gluing approach requires very refined information on the singularities of the real MA equation, which is still far from what we can establish by a-priori estimate considerations.

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2. Analytic backgrounds

2.1. Skoda inequality

An upper semicontinuous L^1-function ϕ on a coordinate ball in $\mathbb{C}^n$ is called *plurisubharmonic* (psh) if it satisfies the sub mean value inequality when restricted to complex lines; this implies $\sqrt{-1}\,\partial\bar{\partial}\phi \geqslant 0$. The basic intuition is that regularity properties for psh functions in general dimensions are analogous to subharmonic functions on Riemann surfaces. This is captured by the basic version of the *Skoda inequality.*

THEOREM 2.1. (Cf. [41, Theorem 3.1]) *If ϕ is psh on $B_2 \subset \mathbb{C}^n$, with $\int_{B_2} |\phi| \omega_E^n \leqslant 1$ with respect to the standard Euclidean metric ω_E, then there are dimensional constants α and C such that*

$$\log \int_{B_1} e^{-\alpha\phi} \omega_E^n \leqslant C.$$

Remark 2.2. If instead $\int_{B_2} |\phi| \omega_E^n \leqslant C'$ for some constant C', then we can apply Theorem 2.1 to a scaling of ϕ, to get a Skoda inequality with modified α and C.

Remark 2.3. Assuming an L^1-bound on ϕ, then we can take a suitable cutoff function χ, and via integration by parts,

$$\begin{aligned}\int_{B_1} \sqrt{-1}\,\partial\bar{\partial}\phi \wedge \omega_E^{n-1} &\leqslant \int_{B_2} \chi \sqrt{-1}\,\partial\bar{\partial}\phi \wedge \omega_E^{n-1} \\ &= \int_{B_2} \phi \sqrt{-1}\,\partial\bar{\partial}\chi \wedge \omega_E^{n-1} \leqslant \|\chi\|_{C^2} \|\phi\|_{L^1} \leqslant C.\end{aligned}$$

This simple idea is a basic version of the *Chern–Levine inequality.*

The basic Skoda inequality immediately implies a global version. On a compact Kähler manifold (X, ω), we say an upper semicontinuous L^1-function $\phi \in \mathrm{PSH}(X, \omega)$ if its sum with the local potential of ω is psh, so that $\omega_\phi = \omega + \sqrt{-1}\,\partial\bar{\partial}\phi \geqslant 0$. This is the generalised notion of Kähler potentials.

THEOREM 2.4. *On a fixed (X, ω), there are positive constants α and C depending only on X and ω such that*

$$\int_X e^{-\alpha\phi} \omega_X^n \leqslant C \quad \textit{for all } \phi \in \mathrm{PSH}(X, \omega) \textit{ with } \sup \phi = 0.$$

Remark 2.5. Here, $\int_X |\phi| \omega_X^n$ is automatically bounded using the Harnack inequality, because $\Delta\phi \geqslant -n$ for $\phi \in \mathrm{PSH}(X, \omega)$.

Remark 2.6. The supremum of all such α is known as Tian's alpha invariant [38].

2.2. Kołodziej's estimate on pluripotentials

Here, we outline a method to estimate Kähler potentials, pioneered by Kołodziej [28], and further developed by [12] and [16], [22]. Our exposition largely adapts [16], [22], [15], with special attention to the dependence of constants. Unlike in [16], we do not impose a volume normalisation.

Given an n-dimensional Kähler manifold (X, ω), for $\phi \in \mathrm{PSH}(X, \omega) \cap L^\infty$, pluripotential theory allows one to make sense of the Monge–Ampère (MA) measure ω_ϕ^n, generalising the notion of volume forms. The basic problem is to estimate ϕ from a-priori bounds on ω_ϕ^n. A key concept is the *capacity* of subsets $K \subset X$:

$$Cap_\omega(K) = \sup\left\{ \int_K \omega_u^n | u \in \mathrm{PSH}(X, \omega), 0 \leqslant u \leqslant 1 \right\}.$$

We wish to sketch the main ideas behind a prototypical result.

Theorem 2.7. *Let (X, ω) be a compact Kähler manifold, and $\phi \in \mathrm{PSH}(X, \omega) \cap C^0$ be such that ω_ϕ^n is an absolutely continuous measure. Assume there are positive constants α and A such that the Skoda type estimate holds with respect to ω_ϕ^n:*

$$\int_X e^{-\alpha u} \frac{\omega_\phi^n}{\mathrm{Vol}(X)} \leqslant A \quad \text{for all } u \in \mathrm{PSH}(X, \omega) \text{ with } \sup_X u = 0. \tag{2.1}$$

(i) *For fixed n, α, and A, there is number $B(n, \alpha, A)$such that, if*

$$\frac{\int_{\phi \leqslant -t_0} \omega_\phi^n}{\mathrm{Vol}(X)} < (2B)^{-2n}$$

for some t_0, then

$$\min \phi \geqslant -t_0 - (4B+1) \left(\frac{\int_{\phi \leqslant -t_0} \omega_\phi^n}{\mathrm{Vol}(X)} \right)^{1/2n}.$$

(ii) *If $\sup_X \phi = 0$, then $\|\phi\|_{C^0} \leqslant C(n, \alpha, A)$.*

The first ingredient is the following.

Lemma 2.8. (Cf. [16, Lemma 2.3])) *The* MA *measure of sublevel sets controls the capacity of lower sublevel sets: for $\tau \geqslant 0$ and $0 \leqslant t \leqslant 1$,*

$$t^n \,\mathrm{Cap}_\omega(\phi < -\tau - t) \leqslant \int_{\phi < -\tau} \omega_\phi^n.$$

The second ingredient below contains the most substance.

LEMMA 2.9. (Volume-capacity estimate) *In the setting of Theorem 2.7, for any compact set $K\subset X$ we have*

$$\frac{\int_K \omega_\phi^n}{\mathrm{Vol}(X)} \leqslant Ae^\alpha \exp\left(-\alpha\left(\frac{\mathrm{Vol}(X)}{\mathrm{Cap}_\omega(K)}\right)^{1/n}\right). \tag{2.2}$$

In particular there is a constant $B=B(n,\alpha,A)$ verifying the power law bound

$$\frac{\int_K \omega_\phi^n}{\mathrm{Vol}(X)} \leqslant B^n\left(\frac{\mathrm{Cap}_\omega(K)}{\mathrm{Vol}(X)}\right)^2.$$

Sketch of proof. We may assume K is not pluripolar, for otherwise $\int_K \omega_\phi^n=0$ and $\mathrm{Cap}_\omega(K)=0$. We introduce the Siciak extremal function

$$V_{K,\omega}=\sup\{u\in \mathrm{PSH}(X,\omega): u\leqslant 0 \text{ on } K\},$$

whose upper semicontinuous regularisation $V_{K,\omega}^*\in\mathrm{PSH}(X,\omega)$. By the Alexander–Taylor comparison principle (cf. [22, Proposition 7.1]),

$$e^{-\sup_X V_{K,\omega}} \leqslant e\exp\left(-\left(\frac{\mathrm{Vol}(X)}{\mathrm{Cap}_\omega(K)}\right)^{1/n}\right).$$

By the Skoda integrability assumption (2.1), and the fact that $V_{K,\omega}=V_{k,\omega}^*$ a.e with respect to ω^n (so by absolute continuity also for ω_ϕ^n),

$$\int_X e^{\alpha(\sup_X V_{K,\omega}-V_{K,\omega})}\omega_\phi^n=\int_X e^{\alpha(\sup_X V_{K,\omega}-V_{K,\omega}^*)}\omega_\phi^n\leqslant A\,\mathrm{Vol}(X),$$

and hence

$$\int_K e^{-\alpha V_{K,\omega}}\omega_\phi^n\leqslant \int_X e^{-\alpha V_{K,\omega}}\omega_\phi^n\leqslant Ae^\alpha\,\mathrm{Vol}(X)\exp\left(-\alpha\left(\frac{\mathrm{Vol}(X)}{\mathrm{Cap}_\omega(K)}\right)^{1/n}\right).$$

The volume-capacity estimate (2.2) follows because $V_{K,\omega}\leqslant 0$ on K. □

The third ingredient is an elementary decay lemma:

LEMMA 2.10. (Cf. [16, Lemma 2.4 and Remark 2.5]) *Let $f:[t_0',\infty)\to[0,\infty)$ be a non-increasing right-continuous function such that*

$$\begin{cases} f(t_0')<\dfrac{1}{2B}, \\ tf(\tau+t)\leqslant Bf(\tau)^2 \quad \text{for all } \tau\geqslant 0 \text{ and } 0\leqslant t\leqslant 1, \\ \lim_{t\to\infty} f(t)=0. \end{cases}$$

Then, $f(t)=0$ for $t\geqslant t_0'+4Bf(t_0')$.

Proof of Theorem 2.7. We consider the normalised capacity of sublevel sets

$$f(t)=\left(\frac{\mathrm{Cap}_\omega(\phi<-t)}{Vol(X)}\right)^{1/n}.$$

Combining the first two ingredients, we have

$$tf(t+\tau)\leqslant Bf(\tau)^2 \quad \text{for all } 0\leqslant t\leqslant 1 \text{ and } \tau\geqslant 0.$$

Using the first ingredient to control capacity by the volume,

$$f(t_0')\leqslant\left(\frac{\int_{\phi\leqslant -t_0}\omega_\phi^n}{\mathrm{Vol}(X)}\right)^{1/2n}<\frac{1}{2B}, \quad \text{with } t_0'=t_0+\left(\frac{\int_{\phi\leqslant -t_0}\omega_\phi^n}{\mathrm{Vol}(X)}\right)^{1/2n}.$$

Since $\phi\in C^0$ is bounded below, $\lim_{t\to\infty}f(t)=0$. Applying the decay lemma, for

$$t>t_0'+4Bf(t_0')$$

the sublevel set $\{\phi\leqslant -t\}$ has zero capacity, hence zero Lebesgue measure, so ϕ has the lower estimate as claimed in the first statement.

For the second statement, by (2.1) we have an a-priori exponential decay

$$\left(\frac{\int_{\phi\leqslant -t}\omega_\phi^n}{\mathrm{Vol}(X)}\right)^{1/2n}\leqslant A^{1/2n}e^{-\alpha t/2n}, \quad t\geqslant 0,$$

which allows us to find an appropriate t_0. □

Remark 2.11. Theorem 2.7 implies a famous result of Kołodziej stating that, if we fix (X,ω) and $p>1$, then ϕ has a C^0-bound depending only on

$$X, \quad \omega, \quad \text{and} \quad \left\|\frac{\omega_\phi^n}{\omega^n}\right\|_{L^p}$$

(cf. [16, Theorem A]). It is enough to check (2.1), which reduces by Hölder inequality to the standard Skoda inequality (cf. Theorem 2.4), with modified constants. The strength of Theorem 2.7 is that it still applies when the complex/Kähler structures are highly degenerate, as it distills the dependence on (X,ω) to only three constants n, α, and A.

Theorem 2.7 gives a criterion for two Kähler potentials to be close to each other.

Corollary 2.12. (Stability estimate) *Let (X,ω) be a compact Kähler manifold, and $\phi,\psi\in\mathrm{PSH}(X,\omega)\cap C^0$ be such that ω_ϕ^n is absolutely continuous. Assume $\|\psi\|_{C^0}\leqslant A'$ and the Skoda type estimate (2.1). Then, there is a number $B(n,A,A',\alpha)$, such that if*

$$\frac{\int_{\phi-\psi\leqslant -t_0}\omega_\phi^n}{\mathrm{Vol}(X)}<(2B)^{-2n}$$

for some t_0, then

$$\min(\phi-\psi)\geqslant -t_0-(4B+1)\left(\frac{\int_{\phi-\psi\leqslant -t_0}\omega_\phi^n}{\mathrm{Vol}(X)}\right)^{1/2n}.$$

.

Proof. If ψ is smooth, this follows from Theorem 2.7 by changing ω into ω_ψ, and changing ϕ into $\phi-\psi$, and checking the Skoda type estimate holds with modified constants. In general, one can approximate $\psi\in\mathrm{PSH}(X,\omega)$ by a decreasing sequence of functions in $\mathrm{PSH}(X,\omega)\cap C^\infty$ [2], and since $\psi\in C^0$, the convergence is uniform by Dini's theorem. $\square$

2.3. Algebraic metrics and asymptotes

This section is included for motivational purposes. On any compact complex manifold X with a positive line bundle L, any fixed Kähler metric ω in the class $2\pi c_1(L)$ is the curvature form of a Hermitian metric h on L. Consider the projective embedding $\iota_k\colon X\hookrightarrow\mathbb{P}(H^0(X,L^k)^*)$ for $k\gg1$. The L^2 norms on sections induce Euclidean metrics on the vector spaces $H^0(X,L^k)$, hence Fubini–Study metrics $\omega_{FS,k}$ on $\mathbb{P}(H^0(X,L^k)^*)$. A famous result of Tian says that ω is approximated by the algebraic metrics $k^{-1}\iota_k^*\omega_{FS,k}$ as $k\to\infty$; this idea has been much exploited in *regularisation theorems.*

This construction is particularly transparent in the toric case, as explained in [13]. Let (X,L) be an n-dimensional polarised toric manifold with moment polytope P, so a T^n-invariant basis $\{s_m\}$ of $H^0(X,L^k)$ corresponds to $kP\cap\mathbb{Z}^n$, or equivalently $P\cap k^{-1}\mathbb{Z}^n$ after rescaling. The L^2-metric on $H^0(X,L^k)$ is *diagonal* in the basis; i.e. the toric assumption reduces the unitary group acting on $H^0(X,L^k)$ to its maximal torus. Concretely, let ϕ denote the torus-invariant Kähler potential on $X\cap(\mathbb{C}^*)^n$, equivalently thought as some convex function of $\vec{t}\in\mathbb{R}^n$ via the logarithm map $(\mathbb{C}^*)^n\to\mathbb{R}^n$. Then,

$$I_m(k)=\|s_m\|_{L^2}^2=\int_X |s_m|^2 d\mathrm{Vol}=\mathrm{const}\int_{\mathbb{R}^n} e^{-k(\phi-\vec{t}\cdot m)}\,d\vec{t},\quad m\in P\cap k^{-1}\mathbb{Z}^n, \tag{2.3}$$

and the Fubini–Study potentials are

$$k^{-1}\iota_k^*\phi_{FS,k}=k^{-1}\log\Bigg(\sum_{m\in P\cap k^{-1}\mathbb{Z}^n} I_m(k)^{-1}|s_m|^2\Bigg). \tag{2.4}$$

Now, the right-hand side of (2.3) is a *Laplace-type integral*, and its dominant contribution comes from the neighbourhood of the point $\vec{t}_0$, where $\vec{t}\cdot m-\phi(\vec{t})$ is maximized among $\vec{t}\in\mathbb{R}^n$. The maximum is the value of the *Legendre transform* of ϕ:

$$u(m)=\sup_{\vec{t}}(\vec{t}\cdot m-\phi(t)).$$

The steepest descent method yields the asymptote

$$k^{-1}\log I_m(k)=u(m)+O(k^{-1}\log k),\quad k\to\infty.$$

In the 'continuum limit' $k\to\infty$, the discrete sum $\sum_{m\in P\cap k^{-1}\mathbb{Z}^n}$ is replaced by an integral. Now, the right-hand side of (2.4) is to leading order

$$k^{-1}\log\int_P e^{k(-u(m)+\vec{t}\cdot m)}dm, \quad \vec{t}\in\mathbb{R}^n.$$

This is another Laplace-type integral, and its limit as $k\to\infty$ is the Legendre transform of u, which gives back the function ϕ.

The moral is that *in the presence of toric symmetry, algebraic approximation of Kähler metrics is related to Legendre transforms.*

2.4. Extension of Kähler potentials

Extension theorems allow us to think extrinsically about Kähler currents on subvarieties in some ambient projective manifold.

THEOREM 2.13. ([10, Theorem B]) *Let (X,ω) be a projective manifold with a Kähler form representing an integral class, and Y be a smooth subvariety of X. Then, any $\phi\in\mathrm{PSH}(Y,\omega|_Y)$ extends to $\phi\in\mathrm{PSH}(X,\omega)$.*

2.5. Savin's small perturbation theorem

Savin [35] proved that for a large class of second-order elliptic equations satisfying certain structural conditions, any viscosity solution C^0-close to a given smooth solution has interior $C^{2,\gamma}$-bound. In particular, this applies to complex MA equation. Combined with the Schauder estimate,

THEOREM 2.14. *Fix $k\geqslant 2$ and $0<\gamma<1$. On the unit ball, let v be a given smooth solution to the complex Monge–Ampère equation $(\sqrt{-1}\,\partial\bar{\partial}v)^n=1$. Then, there are constants $0<\varepsilon\ll 1$ and C depending on n, k, γ, and $\|v\|_{C^{k,\gamma}}$, such that, if*

$$(\sqrt{-1}\,\partial\bar{\partial}(u+v))^n=1+f, \quad \|f\|_{C^{k-2,\gamma}}<\varepsilon, \quad \text{and} \quad \|u\|_{C^0}<\varepsilon,$$

then $\|u\|_{C^{k,\gamma}(B_{1/2})}\leqslant C\varepsilon$.

Savin's theorem has fully non-linear nature, because the perturbative machinery only applies once the solution has a-priori C^2 bound. His proof has two main parts: first he shows a Harnack inequality by a non-trivial application of Aleksandrov–Bakelman–Pucci estimates, and then uses a compactness argument to prove $C^{2,\gamma}$ estimate, similar to De Giorgi's almost flatness theorem for minimal surfaces [11].

2.6. Regularity theory for real Monge–Ampère

There is an extensive literature on the local regularity theory for the real Monge–Ampère equation, largely due to the Caffarelli school. The author thanks C. Mooney for bringing some of these results to his attention. All results surveyed here can be found in [33].

Any convex function on an open set $v:\Omega\subset\mathbb{R}^n\to\mathbb{R}$ has an associated Borel measure called the Monge–Ampère measure, defined by

$$\mathrm{MA}(v)(E)=|\partial v(E)|,$$

where $|\partial v(E)|$ denotes the Lebesgue measure of the image of the subgradient map on $E\subset\Omega$. Given a Borel measure μ, a solution to $\mathrm{MA}(v)=\mu$ is called an *Aleksandrov solution* to $\det(D^2v)=\mu$; if $v\in C^2$, this is the classical real Monge–Ampère equation. We shall assume a 2-sided density bound

$$\det(D^2v)=f \text{ in } B_1, \quad 0<\Lambda_1\leqslant f\leqslant\Lambda_2.$$

Let $B_1\setminus\Sigma$ be the set of strictly convex points of v, namely there is a supporting hyperplane touching the graph of v only at one point. Then Caffarelli [6]–[8] shows that

- If $f\in C^\gamma(B_1)$, then $v\in C^{2,\gamma}_{\mathrm{loc}}(B_1\setminus\Sigma)$. So, by Schauder theory, if f is smooth, then v is smooth in $B_1\setminus\Sigma$.
- If L is a supporting affine linear function to v such that the convex set $\{v=L\}$ is not a point, then $\{v=L\}$ has no extremal point in the interior of B_1.
- The above affine linear set $\{v=L\}$ has dimension $k<\frac{1}{2}n$.

Mooney [33] shows further that

- The singular set Σ has $(n-1)$-Hausdorff measure zero. Consequently $B_1\setminus\Sigma$ is path connected (because a generic path joining two given points does not intersect a subset of zero $(n-1)$-Hausdorff measure).
- The solution $v\in W^{2,1}_{\mathrm{loc}}(B_1)$, even if Σ is non-empty.

Remark 2.15. A classical counterexample of Pogorelov shows that, for $n=3$, the singular set Σ can contain a line segment. This is generalised by Caffarelli [8], who for any $k<\frac{1}{2}n$ constructs examples where f is smooth but Σ contains a k-plane. A surprising example of Mooney [33] shows that the Hausdorff dimension of Σ can be larger than $n-1-\varepsilon$ for any small ε. This means that the local regularity theory surveyed above is essentially optimal.

Remark 2.16. On a compact Hessian manifold, the real MA equation makes sense, and Viaclovsky and Caffarelli [9] show that the interior singularity cannot occur if the density f is smooth and positive.

2.7. Special Lagrangian fibration

A real n-dimensional submanifold L of a compact Calabi–Yau n-fold (X,ω,J,Ω) is called a *special Lagrangian* (SLag) with phase angle θ if

$$\omega|_L = 0 \quad \text{and} \quad \mathrm{Im}(e^{i\theta}\Omega)|_L = 0. \tag{2.5}$$

They are special cases of calibrated submanifolds introduced by Harvey and Lawson [25], and in particular are minimal submanifolds. The classical result of McLean says that the deformation theory of SLags with phase θ is unobstructed, and the first-order deformation space is isomorphic to $H^1(L,\mathbb{R})$. Thus, if L is diffeomorphic to T^n, then the deformation space is n-dimensional, compatible with the SYZ conjecture that X admits a SLag T^n-fibration. A sufficient condition to construct Slag fibrations, under the very strong hypothesis of collapsing metric with locally bounded sectional curvature, is obtained by Zhang [42, Theorem 1.1].

The essence of Zhang's result is a standard application of the implicit function theorem, and we shall summarize the key points (cf. [42, §4] for more details). Denote $Y_r = T^n \times B(0,r) \subset T^n_{x_i} \times \mathbb{R}^n_{y_i} \simeq T^*T^n$, where $r \gg 1$ is fixed. The trivial example of a SLag fibration is the following: the CY structure is the flat model

$$g = \sum (dx_i^2 + dy_i^2), \quad \omega = \sum dx_i \wedge dy_i, \quad \Omega = \bigwedge (dx_j + \sqrt{-1}\, dy_j),$$

and the Slag fibration is just the projection to the $\mathbb{R}^n_{y_i}$ factor, namely the tori $T^n \times \{y\}$ are SLags. Zhang considers a family of CY structures $(g_k, \omega_k, \Omega_k)$ converging to (g,ω,Ω) in the C^∞-sense on Y_{2r} (which follows from his bounded sectional curvature assumptions by elliptic bootstrap), such that $\omega_k \in [\omega] \in H^2(Y_{2r},\mathbb{R})$. Small deformations of the standard T^n fibres can be represented as graphs on T^n: for $y \in \mathbb{R}^n$ and a 1-form σ on T^n orthogonal to the harmonic 1-forms $dx_1, \ldots, dx_n$, write

$$L(y,\sigma) = \mathrm{Graph}(x \mapsto y + \sigma(x)) \subset T^*T^n.$$

The condition for $L(y,\sigma)$ to be a SLag with respect to $(g_k, \omega_k, \Omega_k)$ is

$$\omega_k|_{L(y,\sigma)} = 0 \quad \text{and} \quad \mathrm{Im}(e^{\sqrt{-1}\theta_k}\Omega_k)|_{L(y,\sigma)} = 0, \tag{2.6}$$

where θ_k are chosen so that

$$\int_{T^n} e^{\sqrt{-1}\theta_k}\Omega_k > 0.$$

Zhang shows by perturbation arguments that, for each $y \in B\left(0, \frac{3}{2}r\right)$ and $k \geqslant k_0 \gg 1$, there is a unique $\sigma = \sigma_{k,y}$ such that $L(y,\sigma_{k,y})$ solves (2.6) with small norm bound $\|\sigma_{k,y}\| < \delta \ll 1$. He then uses another implicit function argument to show that these SLags indeed define a local SLag T^n-fibration on some open subset of $Y_{3r/2}$ containing Y_r.

3. Degenerating Calabi–Yau hypersurfaces

We now set the scene for the main work: a particular class of Calabi–Yau hypersurfaces X_s inside $\mathbb{CP}^{n+1}$ near the large complex structure limit, polarised by the class $\mathcal{O}(n+2)|_{X_s}$ up to a rescaling factor. Special attention will be focused on the simplest case of the Fermat family (cf. Example 3.1). We freely borrow from Haase–Zharkov [23], [24], whose setting includes more general CY hypersurfaces in toric varieties. The key notion is that the degenerating complex structures are controlled by *piecewise linear* data, an idea studied extensively under the name of tropical geometry.

The philosophy is that every concept in Kähler geometry ought to have an analogue in the tropical world, and the combinatorial nature of the tropical version should simplify the original problem in Kähler geometry. However, it does not appear clear what is the tropical analogue of the notion of Kähler metrics; we devote §3.4 and S3.5 to investigate this question, and answer it in the Fermat case by utilizing the large discrete symmetry group.

3.1. Complex structure

Let $N\simeq\mathbb{Z}^{n+1}$, $M=\mathrm{Hom}(N,\mathbb{Z})$, and denote $N_{\mathbb{R}}=N\otimes\mathbb{R}$ and $M_{\mathbb{R}}=M\otimes\mathbb{R}$. We regard $\mathbb{CP}^{n+1}$ as a toric Fano manifold $\mathbb{P}_\Delta$, with moment polytope $\Delta\subset M_{\mathbb{R}}$ corresponding to the anticanonical class $\mathcal{O}(n+2)$. More explicitly, Δ is the $(n+1)$-simplex inside

$$M_{\mathbb{R}}\simeq\left\{\sum_0^{n+1} y_i=0\right\}\subset\mathbb{R}^{n+2}$$

spanned by the vertices

$$(n+1,-1,\dots-1),\quad(-1,n+1,-1,\dots,-1),\quad\dots,\quad(-1,\dots,-1,n+1).$$

In particular, Δ is a reflexive integral Delzant polytope, with dual polytope

$$\Delta^\vee=\{w\in N\otimes\mathbb{R}:\langle m,w\rangle\geqslant-1\text{for all } m\in\Delta\}\subset\mathbb{R}^{n+2}/\mathbb{R}(1,1,\dots1)$$

being the $(n+1)$-simplex spanned by the vertices $(1,0,\dots,0),\dots,(0,\dots,0,1)$. The integral points $m\in\Delta_{\mathbb{Z}}=\Delta\cap M$ parameterize monomials z^m in the anticanonical linear system $H^0(\mathbb{P}_\Delta,\mathcal{O}(n+2))$. We study the family of hypersurfaces

$$X_s=\left\{F_s(z)=\sum_{m\in\Delta_{\mathbb{Z}}} a_m e^{s\lambda(m)}z^m=0\right\}\subset\mathbb{P}_\Delta,\quad s\gg1. \tag{3.1}$$

Here, a_m are a fixed collection of coefficients, with $a_0=1$ corresponding to the unique interior integral point $0\in\Delta_{\mathbb{Z}}$. For any vertex m of Δ, we require $a_m\neq 0$. The function λ is defined for those $m\in\Delta_{\mathbb{Z}}$ for which $a_m\neq 0$; by assumption $\lambda(0)=0$, and $\lambda(m)<0$ otherwise. The natural piecewise linear extension of λ to $M_{\mathbb{R}}$ is assumed to be concave, whose domains of linearity are by assumption simplices, producing a triangulation of Δ. Using the adjunction formula, we can write down a holomorphic volume form Ω_s on X_s such that, along X_s,

$$dF_s\wedge\Omega_s = d\log z^1\wedge\ldots d\log z^{n+1}, \tag{3.2}$$

with $z^1, z^2, \ldots, z^{n+1}$ the standard coordinates on $(\mathbb{C}^*)^{n+1}\subset\mathbb{P}_\Delta$. We will always assume $s\gg 1$, and all the constants in the estimates are independent of s.

Example 3.1. The *Fermat family* is given explicitly as

$$X_s = \left\{ Z_0 Z_1 \ldots Z_{n+1} + e^{-s}\sum_{i=0}^{n+1} Z_i^{n+2} = 0\right\}, \tag{3.3}$$

namely we choose $a_m=1$ for m corresponding to the monomials $Z_0\ldots Z_{n+1}$ and Z_i^{n+2}, and choose λ to be the piecewise linear function with value 0 at the origin and -1 at the vertices of Δ.

The key notion to describe the complex structure degeneration is a piecewise linear object called the *tropicalisation* of the hypersurfaces. Define the non-negative piecewise linear function L_λ on $N_{\mathbb{R}}$ by

$$L_\lambda(x) = \max_{\substack{m\in\Delta_{\mathbb{Z}}\\ a_m\neq 0}} \{\langle x, m\rangle + \lambda(m)\}.$$

The tropicalisation $\mathcal{A}_\lambda^\infty$ is defined as the non-smooth locus of L_λ, or equivalently the locus inside $N_{\mathbb{R}}$ where the maximum L_λ is achieved by at least two values of m. There is precisely one bounded component in the complement of $\mathcal{A}_\lambda^\infty$,

$$\Delta_\lambda^\vee = \{x : L_\lambda(x) = 0\}\subset N_{\mathbb{R}},$$

whose boundary $\partial\Delta_\lambda^\vee\subset\mathcal{A}_\lambda^\infty$. The relation between the hypersurfaces and the tropicalisation is furnished by the *rescaled log map*, that is

$$\begin{aligned}\mathrm{Log}_s\colon \mathbb{P}_\Delta\supset(\mathbb{C}^*)^{n+1} &\longrightarrow \mathbb{R}^{n+1}\simeq N_{\mathbb{R}},\\ z &\longmapsto \frac{1}{s}(\log|z_1|, \ldots, \log|z_{n+1}|).\end{aligned}$$

The image $\mathcal{A}_\lambda^s = \mathrm{Log}_s(X_s\cap(\mathbb{C}^*)^{n+1})$ is called the *amoeba*. The following proposition will be tacitly used frequently, as it allows us to think of regions on X_s efficiently in terms of the regions on $\mathcal{A}_\lambda^\infty$, up to a tiny amount of fuzziness.

Proposition 3.2. (Cf. [23, Proposition 3.2]) *The amoebas $\mathcal{A}_\lambda^s$ converge in the Hausdorff distance to $\mathcal{A}_\lambda^\infty$ as $s\to\infty$. In fact,*

$$\begin{cases} \operatorname{dist}_{\mathbb{R}^{n+1}}(x,\mathcal{A}_\lambda^\infty)\leqslant \dfrac{C}{s}, & \textit{for all } x\in \operatorname{Log}_s(X_s),\\ \operatorname{dist}_{\mathbb{R}^{n+1}}(x,\operatorname{Log}_s(X_s))\leqslant \dfrac{\mathbb{C}}{s}, & \textit{for all } x\in \mathcal{A}_\lambda^\infty. \end{cases}$$

Sketch of proof. Let $x=\operatorname{Log}_s(z)$ and let $m'\in\Delta_\mathbb{Z}$ saturate the maximum for $L_\lambda(x)$. Applying Log_s to the inequality

$$|e^{s\lambda(m')}z^{m'}|=\left|-\sum_{m\neq m'}\frac{a_m}{a_{m'}}e^{s\lambda(m)}z^m\right|\leqslant C\max_{m\neq m'}\{e^{s\lambda(m)}|z^m|\},$$

we see that

$$L_\lambda(x)=\langle x,m'\rangle+\lambda(m')\leqslant \max_{m\neq m'}\{\langle x,m\rangle+\lambda(m)\}+\frac{C}{s},$$

so $\operatorname{dist}_{\mathbb{R}^{n+1}}(x,\mathcal{A}_\lambda^\infty)\leqslant C/s$. The other inequality of the claim can be proved by constructing local models of X_s in regions whose Log_s-images are close to $x\in\mathcal{A}_\lambda^\infty$, and then use the implicit function theorem to show that X_s is a small perturbation of these local models. A good general survey on tropical geometry and the amoeba can be found in Mikhalkin [32]. □

Example 3.3. In the Fermat family example above, $\Delta_\lambda^\vee=-\Delta^\vee$ is the reflexion of $\Delta^\vee$.

The tropicalisation $\mathcal{A}_\lambda^\infty$ is naturally *stratified* according to the subset of $m\in\Delta_\mathbb{Z}$ saturating the maximum $L_\lambda(x)$. This induces a kind of quantitative stratification structure on $\mathcal{A}_\lambda^s$ for $s\gg 1$.

Lemma 3.4. *There is a fixed number $\delta_1>0$ such that, for any $x\in N_\mathbb{R}$, there is a simplex σ in the triangulation of Δ verifying $\langle x,m\rangle+\lambda(m)<L_\lambda(x)-\delta_1$ for $m\in\Delta_\mathbb{Z}\setminus\sigma$.*

Sketch of proof. For any fixed $x\in N_\mathbb{R}$, the function $\langle x,m\rangle+\lambda(m)$ is a concave function of $m\in M_\mathbb{R}$. By our assumptions, the set of $m\in\Delta_\mathbb{Z}$ saturating the maximum must be the set of vertices of some simplex σ in the triangulation of Δ. By a compactness argument, for any x in a given compact subset of $N_\mathbb{R}$, there is a simplex σ in the triangulation of Δ, verifying $\langle x,m\rangle+\lambda(m)<L_\lambda(x)-\delta_1$ for $m\in\Delta_\mathbb{Z}\setminus\sigma$.

To reduce to the compact case, we consider all possible subsets $S\subset\Delta_\mathbb{Z}$ with elements $m^{(1)},...,m^{(|S|)}$, and the possibly unbounded polytopes defined by

$$\begin{aligned}\langle m^{(1)},x\rangle+\lambda(m^{(1)})\geqslant\langle m^{(2)},x\rangle+\lambda(m^{(2)})\geqslant\langle m^{(|S|)},x\rangle+\lambda(m^{(|S|)})&\\ \geqslant\langle m^{(1)},x\rangle+\lambda(m^{(|S|)})-1\geqslant\langle m,x\rangle+\lambda(m)&\qquad\text{for all } m\in\Delta_\mathbb{Z}\setminus S.\end{aligned}$$

There are only finitely many choices of such polytopes, and together they cover $N_{\mathbb{R}}$. By construction, only $m\in S$ can allow $\langle x,m\rangle+\lambda(m)$ to come close to $L_\lambda(x)$. Any unbounded polytope is the sum of a bounded polytope (generated by the vertices) and a convex polyhedral cone (generated by the infinite rays). The polyhedron cone is contained in the subspace

$$\langle m^{(1)},x\rangle=\ldots=\langle m^{(|S|)},x\rangle.$$

We can use this to reduce an arbitrary x in the (unbounded) polytope to a bounded polytope, and thus extending the claim to all $x\in N_{\mathbb{R}}$. □

With any simplex $\sigma\subset\partial\Delta$ in the triangulation, we associate the subset

$$\mathcal{A}^\infty_{\lambda,\sigma}=\{x\in\mathcal{A}^\infty_\lambda : L_\lambda(x)=\langle x,m\rangle+\lambda(m) \text{ for all } m\in\sigma\}.$$

Clearly, if $\sigma\prec\sigma'$, then $\mathcal{A}^\infty_{\lambda,\sigma}\supset\mathcal{A}^\infty_{\lambda,\sigma'}$. The intuition is that larger σ correspond to more non-generic regions, and the complement of their neighbourhoods correspond to more generic regions.

Notation. We need a few terminologies to describe $\mathcal{A}^\infty_{\lambda,\sigma}$. The face of $\partial\Delta^\vee_\lambda$ dual to σ is $F^\vee_\sigma=\partial\Delta^\vee_\lambda\cap\mathcal{A}^\infty_{\lambda,\sigma}$. The outward normal cone to σ is

$$NC_\Delta(\sigma)=\{x\in N_{\mathbb{R}} : \langle m,x\rangle\leqslant\langle m',x\rangle \text{ for all } m\in\Delta_{\mathbb{Z}} \text{ and } m'\in\sigma\}.$$

By the Delzant polytope property, $NC_\Delta(\sigma)$ is isomorphic to $\mathbb{R}^l_{\geqslant 0}$, where $n+1-l$ is the dimension of the minimal face of $\partial\Delta$ containing σ. The Minkowski sum of two sets A and B means $A+B=\{a+b : a\in A \text{ and } b\in B\}$.

Lemma 3.5. (Cf. [23, Lemma 3.1]) *If* $\dim\sigma\geqslant 1$, *then* $\mathcal{A}^\infty_{\lambda,\sigma}=F^\vee_\sigma+NC_\Delta(\sigma)$.

Lemma 3.6. *We have*

$$\mathcal{A}^\infty_\lambda=\partial\Delta^\vee_\lambda\cup\bigcup_{\dim\sigma\geqslant 1}\mathcal{A}^\infty_{\lambda,\sigma}.$$

Proof. Let $x\in\mathcal{A}^\infty_\lambda$. If $m=0\in\Delta$ achieves the maximum $L_\lambda(x)$, then $x\in\partial\Delta^\vee_\lambda$. If not, then the maximum is achieved by at least two $m\in\partial\Delta$, so $x\in\mathcal{A}^\infty_{\lambda,\sigma}$ for some $\sigma\subset\partial\Delta$ with $\dim\sigma\geqslant 1$. □

Remark 3.7. The intuition is that a neighbourhood of $\partial\Delta^\vee_\lambda$ corresponds to a toric region, while $\mathcal{A}^\infty_{\lambda,\sigma}$ controls how X_s approaches the toric boundary of $\mathbb{P}_\Delta$, and the stratification is related to how the toric boundary components intersect.

Our next goal is to assign good holomorphic charts to X_s related to the stratification structure. We first consider the *toric region*, which shall be covered by $(\mathbb{C}^*)^n$-charts. Let $w\in N$ be the primitive integral outward normal vector to a facet

$$F(w)=\{m\in\Delta : \langle w,m\rangle=1\}$$

of Δ. The chart parameterised by w is contained inside the region

$$U_w^{s,o} = \{z \in X_s : e^{s\lambda(m)}|z^m| \ll 1 \text{ for all } m \in \Delta_{\mathbb{Z}} \setminus (F(w) \cup \{0\})\}. \tag{3.4}$$

Let $m_0 \in F(w) \cap \Delta_{\mathbb{Z}}$, and choose an integral basis $m_1, ..., m_n$ for $\{m \in M : \langle w, m \rangle = 0\}$. Then, the monomials $z^{m_1}, ..., z^{m_n}$ provide the local $(\mathbb{C}^*)^n$-coordinates on the chart, since, by the implicit function theorem, X_s is locally a graph $\{z^{m_0} = f(z^{m_1}, ..., z^{m_n})\}$. In fact, by the defining equation (3.1) of the hypersurface, we have

$$z^{-m_0} \approx - \sum_{m \in F(w)} a_m e^{s\lambda(m)} z^{m - m_0},$$

whence the holomorphic volume form is (cf. (3.2))

$$\Omega_s = \pm \frac{d\log z^{m_0} \wedge \dots d\log z^{m_n}}{dF_s} \approx d\log z^{m_1} \wedge \dots d\log z^{m_n}. \tag{3.5}$$

(Here, m_i are suitably oriented to take care of ± 1.) We regard the above region as an open subset of $(\mathbb{C}^*)^n$, and denote the chart U_w^s as the largest T^n-invariant subset, delineated by a collection of affine linear inequalities on the variables $\log |z^{m_i}|$.

In the tropical limit $s = \infty$, the region $\mathrm{Log}_s(U_w^{s,o})$ becomes

$$U_w^{\infty,o} = \{x \in \mathcal{A}_\lambda^\infty : \langle m, x \rangle + \lambda(m) < 0 \text{ for all } m \in \Delta_{\mathbb{Z}} \setminus (F(w) \cup \{0\})\}$$

Inside this, the limiting version of $\mathrm{Log}_s(U_w^s)$ is

$$U_w^\infty = \{(U_w^{\infty,o} \cap \partial \Delta_\lambda^\vee) + \mathbb{R}_{\geqslant 0} w\} \cap \mathcal{A}_\lambda^\infty.$$

Later we shall also need the slightly shrunken regions for $0 < \delta \ll \delta_1$: let

$$U_{w,\delta}^{s,o} = \{z \in X_s : e^{s\lambda(m)}|z^m| \ll e^{-s\delta} \text{ for all } m \in \Delta_{\mathbb{Z}} \setminus (F(w) \cup \{0\})\},$$

whose largest T^n-invariant subset is $U_{w,\delta}^s$. The tropical limit of $\mathrm{Log}_s(U_{w,\delta}^{s,o})$ is

$$U_{w,\delta}^{\infty,o} = \{x \in \mathcal{A}_\lambda^\infty : \langle m, x \rangle + \lambda(m) < -\delta \text{ for all } m \in \Delta_{\mathbb{Z}} \setminus (F(w) \cup \{0\})\},$$

containing the limiting version of $\mathrm{Log}_s(U_{w,\delta}^s)$, namely

$$U_{w,\delta}^\infty = \{(U_w^{\infty,o} \cap \partial \Delta_\lambda^\vee) + \mathbb{R}_{\geqslant 0} w\} \cap \mathcal{A}_\lambda^\infty.$$

As the choice of w varies, such regions $U_{w,\delta}^{\infty,o}$ cover a neighbourhood of $\partial \Delta_\lambda^\vee$ as a consequence of Lemma 3.4; so do $U_{w,\delta}^\infty$. This means that the charts of toric type already cover part of the neighbourhood of the toric boundary.

Example 3.8. In the $n=1$ case, X_s are elliptic curves, and the toric charts cover the entire X_s. In the $n=2$ case, X_s are quartic K3 surfaces, and the toric charts cover most parts of X_s including a large portion of the intersection of X_s with the toric boundary of $\mathbb{P}^3$, but do not cover a tiny neighbourhood of the 24 points located at the intersection of X_s with $\{Z_i=Z_j=0\}$.

We now consider the neighbourhood of the *toric boundary* near the stratum $\mathcal{A}^{\infty}_{\lambda,\sigma}$, but keeping away from higher strata and from $\partial\Delta^\vee_\lambda$. Here,

$$|a_{m'}e^{s\lambda(m')}z^{m'}|\ll|a_m e^{s\lambda(m)}z^m| \quad \text{for all } m'\in\Delta_{\mathbb{Z}}\backslash\sigma \text{ and } m\in\sigma.$$

Since most terms in the defining equation (3.1) are negligible in our region, the hypersurface is locally approximately

$$\sum_{m\in\sigma\cap\Delta_{\mathbb{Z}}} a_m e^{s\lambda(m)}z^m\approx 0.$$

We focus on the subregion where $m_0'\in\sigma$ achieves the maximal magnitude for

$$|a_m e^{s\lambda(m)}z^m|,$$

and $m_1'\in\sigma$ achieves the second largest magnitude. These two magnitudes must be of comparable size by the hypersurface equation. Choose an integral basis $w_1,\dots w_l$ for the outward normal cone $NC_\Delta(\sigma)$, so $\langle m,w_i\rangle=1$ for $m\in\sigma\cap\Delta_{\mathbb{Z}}$. Denote the vertices of σ as m_i', for $i=0,1,\dots,\dim\sigma$, and choose an integral basis $m_1,\dots,m_{\dim\sigma-1}$ of $\mathrm{span}_{\mathbb{Q}}\{m_2'-m_0',\dots,m'_{\dim\sigma}-m_0'\}\cap M$. Choose m_0 so that $m_0,\dots,m_{\dim\sigma-1}$ is an integral basis of $\mathrm{span}_{\mathbb{Q}}\{m_1'-m_0',\dots,m'_{\dim\sigma}-m_0'\}\cap M$, and complete this into an integral basis $\{m_0,\dots,m_{n-l}\}$ for $\mathrm{span}\{w_1,\dots w_l\}^\perp$, providing $(n+1-l)$ $\mathbb{C}^*$-variables $z^{m_0},\dots,z^{m_{n-l}}$. We then find $\mathfrak{m}_j$, for $j=1,2,\dots l$, with $\langle\mathfrak{m}_j,w_i\rangle=-\delta_{ij}$, and we can demand $\mathfrak{m}_1+\dots\mathfrak{m}_l=-m_0'$, since $\langle m_0',w_i\rangle=1$. These provide the $\mathbb{C}$-variables $z^{\mathfrak{m}_j}$, for $1\leqslant j\leqslant l$, which can vanish on the toric boundary. On this local piece of X_s, the variables $z^{\mathfrak{m}_j}$ and $z^{m_1},\dots z^{m_{n-l}}$ furnish a set of local coordinates, as the $\mathbb{C}^*$-variable z^{m_0} is expressible locally as a function of them.

The holomorphic volume form (3.2) is

$$\begin{aligned}\Omega_s&=\pm\frac{d\log z^{\mathfrak{m}_1}z^{\mathfrak{m}_1}\wedge\dots d\log z^{\mathfrak{m}_l}\wedge d\log z^{m_0}\wedge\dots d\log z^{m_{n-l}}}{dF_s}\\&=\pm\frac{dz^{\mathfrak{m}_1}\wedge\dots dz^{\mathfrak{m}_l}}{z^{-m_0'}}\bigwedge\frac{d\log z^{m_0}\wedge\dots d\log z^{m_{n-l}}}{dF_s}\\&\approx\pm(dz^{\mathfrak{m}_1}\wedge\dots dz^{\mathfrak{m}_l})\wedge(d\log z^{m_1}\wedge\dots d\log z^{m_{n-l}})\wedge\frac{d\log z^{m_0}}{a_{m_1'}e^{s\lambda(m_1')}dz^{m_1'-m_0'}}\\&=\frac{dz^{\mathfrak{m}_1}\wedge\dots dz^{\mathfrak{m}_l}}{a_{m_1'}e^{s\lambda(m_1')}z^{m_1'-m_0'}\mathfrak{d}}\wedge(d\log z^{m_1}\wedge\dots d\log z^{m_{n-l}}),\end{aligned}\tag{3.6}$$

up to choosing appropriate ordering of the coordinates. Here, $\mathfrak{d}$ is the divisibility of $m_1'-m_0'$ inside the group

$$\operatorname{span}_{\mathbb{Q}}\{m_1'-m_0', ..., m_{\dim\sigma}'-m_0'\}\cap M/\operatorname{span}_{\mathbb{Z}}\{m_1, ..., m_{\dim\sigma-1}\}\simeq\mathbb{Z}.$$

Notice that $a_{m_1'}e^{s\lambda(m_1')}z^{m_1'-m_0'}$ is uniformly equivalent to $a_{m_0'}e^{s\lambda(m_0')}$ in this region.

Remark 3.9. The discussion above can be simplified if one assumes the triangulation of Δ is *maximal*, namely each simplex is $\mathbb{Z}$-isomorphic to the standard simplex. We choose not to do so because this stronger assumption would exclude the Fermat family.

Remark 3.10. A problem when we work with the coordinates $z^{m_1}, ..., z^{m_{n-l}}, z^{\mathfrak{m}_j}$ is the inequality constraint to keep $a_{m_0'}e^{s\lambda(m_0')}z^{m_0'}$ and $a_{m_1'}e^{s\lambda(m_1')}z^{m_1'}$ as the two dominant monomials. This means such a holomorphic chart is not quite as simple as the product of $D(1)^\ell$ with a long annulus in $(\mathbb{C}^*)^{n-l}$. In practice we will cover this region by lots of simpler charts which we call the *charts of boundary type*. Let P be any point in this region, such that $\max_m |a_m e^{s\lambda(m)}z^m|$ is large but still comparable to 1 (to guarantee the chart overlaps non-trivially with some toric-type chart). The associated chart uses the same coordinates $z^{m_1}, ..., z^{m_{n-l}}, z^{\mathfrak{m}_j}$ as above, but describes only a small region:

$$U_P=\{|z^{\mathfrak{m}_j}|\lesssim|z^{\mathfrak{m}_j}(P)| \text{ for all } j, \text{ and } |z^{m_i}-z^{m_i}(P)|<c|z^{m_i}(P)| \text{ for all } i\},$$

where $0<c\ll 1$ is a fixed-dimensional constant. These charts have an interpretation in terms of the strata $\mathcal{A}_{\lambda,\sigma}^\infty$ (cf. Lemma 3.5): the point P corresponds roughly to a point P' on the face $F_\sigma^\vee\subset\Delta_\lambda^\vee$, and allowing $|z^{\mathfrak{m}_j}|$ to decrease to zero corresponds to taking the Minkowski sum with the outward normal cone $NC_\Delta(\sigma)$, so the tropical analogue of our small chart is $\{P'\}+NC_\Delta(\Sigma)$.

Example 3.11. For generic quartic K3 surfaces, the following simple situation models a small neighbourhood of the 24 points on $\mathrm{K3}\cap\{Z_i=Z_j=0\}$. Locally the dominant monomials are $(z_1z_2)^{-1}$, $(z_1z_2)^{-1}z_0$, and 1, where z_1 and z_2 are $\mathbb{C}$-coodinates which vanish on toric boundaries, and z_0 is a $\mathbb{C}^*$-coordinate; together z_0, z_1, and z_2 are local coordinates on $\mathbb{P}^3$. The local model hypersurface is

$$\{-(z_1z_2)^{-1}+(z_1z_2)^{-1}z_0=1\}=\{z_0=1+z_1z_2\},$$

so z_1 and z_2 can be used as local coordinates on the hypersurface. The holomorphic volume form Ω on the hypersurface is (up to a normalising factor)

$$\Omega=\frac{d\log z_0\wedge d\log z_1\wedge d\log z_2}{d(-(z_1z_2)^{-1}+(z_1z_2)^{-1}z_0-1)}=z_0^{-1}\,dz_1\wedge dz_2.$$

This is the typical boundary-type behaviour. A significant part of the boundary-type region overlaps with the toric region. In this example, when $|z_1|$ is not too small, we can view z_1 as a $\mathbb{C}^*$-coordinates, so $\{z_1, z_0\}$ provides a toric-type chart, as we can express $z_2 = z_1^{-1}(z_0 - 1)$. In this chart,

$$\Omega = z_0^{-1} dz_1 \wedge dz_2 = d\log z_1 \wedge d\log z_0,$$

which agrees with the standard holomorphic volume form in toric-type charts. The same behaviour happens when $|z_2|$ is not too small. The problem mentioned in Remark 3.10 is due to the fact that this local model is only a valid approximate description of the K3 for z_0, z_1, and z_2 satisfying some inequality constraints. The prescription of charts of boundary type means that we are simultaneously using the charts $\{|z_1| \lesssim \nu, |z_2| \lesssim \nu^{-1}\}$ for many choices of parameters ν. Notice the scaling symmetry

$$z_1 \longmapsto \nu z_1, \quad z_2 \longmapsto \nu^{-1} z_2$$

means that there is no obviously preferred chart of boundary type. More concrete examples can be found in [31, §1.1.6].

Local charts of the toric type and the boundary type cover the entire hypersurface X_s for $s \gg 1$, and a substantial portion of any boundary-type chart is in fact already covered by toric charts. Almost all the measure is contained in the toric-type region.

3.2. Piecewise linear structure

PROPOSITION 3.12. *The polyhedral complex $\partial\Delta_\lambda^\vee$ is homeomorphic to S^n.*

Proof. This is because $\partial\Delta_\lambda^\vee$ is the boundary of a convex polyhedron $\Delta_\lambda^\vee$ with nontrivial interior. □

We now assign a collection of charts to $\partial\Delta_\lambda^\vee$, whose transition functions are piecewise linear. (Some authors prefer the terminology ‘piecewise affine’.) These are closely related to the holomorphic charts on X_s in §3.1.

Let $w \in N$ be the primitive integral outward normal vector to a facet $F(w)$ of Δ, and choose an integral basis $m_1, \dots m_n$ for $\{m \in M : \langle w, m\rangle = 0\}$, suitably oriented to be compatible with (3.5). On the open subset of $\partial\Delta_\lambda^\vee$,

$$\partial\Delta_\lambda^\vee \cap U_w^\infty = \{x \in \partial\Delta_\lambda^\vee : \langle m, x\rangle + \lambda(m) < 0 \text{ for all } m \in \Delta_{\mathbb{Z}} \setminus (F(w) \cup \{0\})\},$$

we regard $m_1, \dots, m_n$ as the affine linear coordinates, also written as $x^{m_1}, \dots x^{m_n}$. Such charts cover $\partial\Delta_\lambda^\vee$. We denote by $\widetilde{\text{Sing}}$ the subset of points on $\partial\Delta_\lambda^\vee$ which do not lie on the interior of the top-dimensional faces. It is easy to check that the transition functions on overlapping charts in $\partial\Delta_\lambda^\vee\setminus\widetilde{\text{Sing}}$ lie in $\mathrm{SL}(n,\mathbb{Z})\ltimes\mathbb{R}^n$, so the volume form $dx^{m_1}\wedge\ldots dx^{m_n}$ is defined independent of the choice of charts. We call the associated measure $d\mu_\infty$ the *Lebesgue measure* on $\partial\Delta_\lambda^\vee$, with respect to which $\widetilde{\text{Sing}}$ is a null set. The set $\widetilde{\text{Sing}}$ has real codimension 1, and the transition functions are in general only piecewise linear.

Remark 3.13. The affine structure on $\partial\Delta_\lambda^\vee\setminus\widetilde{\text{Sing}}$ can be often extended to a subset of $\partial\Delta_\lambda^\vee$ with codimension-2 complement. This in general involves a somewhat ad-hoc choice of the singular locus. In the Fermat family case, due to the discrete symmetry, the barycentric subdivision provides a canonical choice (cf. §3.5).

We now examine the *normalised canonical measure* on X_s

$$d\mu_s=\frac{1}{(4\pi s)^n}(\sqrt{-1})^{n^2}\Omega_s\wedge\overline{\Omega}_s. \tag{3.7}$$

PROPOSITION 3.14. *As $s\to\infty$, the push-forward measure $(\mathrm{Log}_s)_*d\mu_s$ converges to the Lebesgue measure $d\mu_\infty$ supported on $\partial\Delta_\lambda^\vee$. In particular,*

$$\int_{X_s} d\mu_s\to\mathrm{Vol}(\partial\Delta_\lambda^\vee)=\int_{\partial\Delta_\lambda^\vee} d\mu_\infty. \tag{3.8}$$

Morever, there is a uniform exponential measure decay estimate

$$d\mu_s(\{z\in X_s:\mathrm{dist}_{\mathbb{R}^{n+1}}(\mathrm{Log}_s(z),\partial\Delta_\lambda^\vee)>s^{-1}\Lambda\})\leqslant C'e^{-C\Lambda}\quad \textit{for all } \Lambda>0. \tag{3.9}$$

Sketch of proof. Using Lemma 3.5 and the holomorphic volume form formula (3.6), the neighbourhood of the toric boundary near $\mathcal{A}^\infty_{\lambda,\sigma}$ only contributes $O(s^{-l})$ to the normalised measure, where $l=\dim NC_\Delta(\sigma)$. The same lemmas imply (3.9) by summing over contributions from boundary-type regions. In the toric region corresponding to the neighbourhood of $\partial\Delta_\lambda^\vee$, the convergence of the normalised volume measure follows from Proposition 3.2 and formula (3.5). □

Remark 3.15. The measure convergence holds for much more general degenerating families, by the work of Boucksom et al. [5]. The fact that the measure is concentrated along $\partial\Delta_\lambda^\vee$ justifies why we focus on $\partial\Delta_\lambda^\vee$ rather than $\mathcal{A}_\lambda^\infty$.

3.3. Kählerian polarisation

We specify a polarisation class $[\Delta]$ on the toric manifold $\mathbb{CP}^{n+1}=\mathbb{P}_\Delta$. A standard background Kähler metric is (a suitable multiple of) the Fubini–Study metric:

$$\begin{aligned}\omega_{FS}&=\frac{\sqrt{-1}(n+2)}{2}\,\partial\bar\partial\log(|Z_0|^2+\ldots|Z_{n+1}|^2)\\&=\frac{\sqrt{-1}(n+2)}{2}\,\partial\bar\partial\log\Bigg(\sum_{m\in\mathrm{vertices}(\Delta)}e^{(2/(n+2))\langle m,\mathrm{Log}(z)\rangle}\Bigg).\end{aligned}$$

Our normalisation guarantees that the potential has the asymptotic behaviour

$$\sup_z \left| \frac{n+2}{2} \log\left(\sum_{m\in \mathrm{vertices}(\Delta)} e^{(2/(n+2))\langle m, \mathrm{Log}(z)\rangle} \right) - \max_{m\in\Delta} \langle m, \mathrm{Log}(z)\rangle \right| < \infty.$$

A general (singular) Kähler metric ω_u on $(\mathbb{P}_\Delta, [\Delta])$ is given by a relative potential $u\in \mathrm{PSH}(X, \omega_{FS})$. Alternatively, one thinks of ω_u as a collection of local absolute potentials:

$$\begin{cases} u_0 = u + \dfrac{n+2}{2} \log\left(\displaystyle\sum_{m'\in \mathrm{vertices}(\Delta)} e^{(2/(n+2))\langle m', \mathrm{Log}(z)\rangle} \right), \\ u_m = u + \dfrac{(n+2)}{2} \log\left(\displaystyle\sum_{m'\in \mathrm{vertices}(\Delta)} e^{(2/(n+2))\langle m', \mathrm{Log}(z)\rangle} \right) - \langle m, \mathrm{Log}(z)\rangle, \end{cases} \tag{3.10}$$

where u_0 is a local potential in a compact region, and u_m give the local potentials near the toric boundary.

We call a convex function u on $N_{\mathbb{R}} = \mathbb{R}^{n+1}$ *admissible* if it satisfies the asymptotic growth condition

$$\sup_x \left| u(x) - \max_{m\in\Delta} \langle m, x\rangle \right| < \infty, \tag{3.11}$$

which captures the information of the Kähler class.

PROPOSITION 3.16. *A convex function u is admissible if and only if the Kähler current defined by the psh function $u\circ\mathrm{Log}$ on $(\mathbb{C}^*)^{n+1}$ extends to a torus-invariant Kähler current on $(\mathbb{P}_\Delta, [\Delta])$ with continuous local potentials.*

Sketch of proof. Convex functions on $(\mathbb{C}^*)^n$ correspond to torus-invariant psh functions via the log map (cf. Lemma 4.3 below). The asymptotic condition is clearly necessary by the local boundedness of u_m near the toric boundary pieces.

Conversely, if u is admissible, then near the toric boundary the appropriate local potential u_m is bounded. Using also the convexity of u_m, we claim $u_m\circ\mathrm{Log}$ extends continuously over the boundary region. This amounts a local problem: if v is a bounded convex function on $(-\infty, 0)^n$, hence necessarily an increasing function in each coordinate, then $v(\log|z_1|, ..., \log|z_n|)$ extends continuously over the $z_i = 0$ planes. Such an extension can be done inductively on dimension, and the continuity uses Dini's theorem. Morever, since the bounded function $u_m\circ\mathrm{Log}$ is psh away from the boundary divisor, it must remain psh after the extension, so gives rise to a continuous local potential. □

A general (singular) Kähler metric ω_φ on X_s in the polarisation class $s^{-1}[\Delta]$ is given by a potential $\varphi\in \mathrm{PSH}(X_s, s^{-1}\omega_{FS})$. The normalising factor s^{-1} is aimed at extracting non-trivial limits as $s\to\infty$. We can completely analogous define the local potentials:

$$\begin{cases} \varphi_0 = \varphi + \dfrac{n+2}{2s} \log\Bigg(\displaystyle\sum_{m'\in \text{vertices}\,(\Delta)} e^{(2/(n+2))\langle m', \mathrm{Log}(z)\rangle} \Bigg), \\ \varphi_m = \varphi + \dfrac{n+2}{2s} \log\Bigg(\displaystyle\sum_{m'\in \text{vertices}\,(\Delta)} e^{(2/(n+2))\langle m', \mathrm{Log}(z)\rangle} \Bigg) - \langle m, \mathrm{Log}_s(z)\rangle, \end{cases} \tag{3.12}$$

which are by definition psh on respective regions.

In particular, we can represent the Calabi–Yau metric $\omega_{CY,s}$ on X_s by a potential $\varphi_{CY,s}$. The Calabi–Yau condition is

$$\omega^n_{CY,s} = a_s s^{-n}\, d\mu_s, \tag{3.13}$$

where the normalising constant

$$a_s = \frac{\int_{X_s} [\Delta]^n}{\int_{X_s} d\mu_s} \to a_\infty = \frac{\int_{X_s} [\Delta]^n}{\mathrm{Vol}(\partial\Delta^\vee_\lambda)} \tag{3.14}$$

as $s\to\infty$ (cf. Proposition 3.14).

3.4. Extension property and locally convex functions

We now discuss the issue of finding a tropical notion analogous to Kähler metrics. The concept of a Kähler metric is formulated in terms of a collection of local psh functions ϕ_j on overlapping complex charts, whose differences $\{\phi_i - \phi_j\}$ represent a given cocycle of local pluriharmonic function. Intuitively, the analogue should be a collection of local convex functions u_j whose differences $\{u_i - u_j\}$ represent a given cocycle of local affine functions.

To the author's awareness there is no definitive formulation of local convexity on polyhedral sets. In the case of interest, we need to define a class of 'locally convex functions' on $\partial\Delta^\vee_\lambda$. The problem is that on $\widetilde{\mathrm{Sing}}\subset\partial\Delta^\vee_\lambda$, the transition functions between different charts are only piecewise linear, so convexity is not invariantly defined. This problem also prevents us from setting up a general global notion of real MA equation on $\partial\Delta^\vee_\lambda$, which is an essential ingredient in the SYZ conjecture in general. We will attempt to give a special definition in the Fermat case (cf. §3.5).

However, the extension Theorem 2.13 provides an alternative viewpoint: (1,1)-type Kähler currents can be defined extrinsically. By analogy, we propose that the correct notion should be equivalent to the following.

Definition 3.17. A continuous function u on $\partial\Delta^\vee_\lambda$ satisfies the *extension property* if it extends to an admissible convex function on $N_{\mathbb{R}} = \mathbb{R}^{n+1}$ defined in §3.3.

Example 3.18. The zero function extends to L_λ, which is admissible and convex.

The problem is to make this definition both intrinsic to $\partial\Delta_\lambda^\vee$, and local in nature. We do not fully succeed but shall make some partial progress.

PROPOSITION 3.19. *A continuous function u on $\partial\Delta_\lambda^\vee$ satisfies the extension property if and only if, for every $x\in\partial\Delta_\lambda^\vee$, there exists $p\in\Delta$ such that, for any $y\in\partial\Delta_\lambda^\vee$, one has*

$$u(y)\geqslant u(x)+\langle p, y-x\rangle.$$

Proof. The 'if' direction is because the asymptotic growth condition (3.11) implies the gradient of u must be contained in Δ.

For the 'only if' direction, we apply the Legendre transform:

$$u^*(p)=\sup_{x\in\partial\Delta_\lambda^\vee}\{\langle x,p\rangle-u(x)\},\quad p\in\Delta,$$

and consider a version of the double Legendre transform:

$$u^{**}(x)=\sup_{p\in\Delta}\{\langle x,p\rangle-u^*(p)\}.$$

Clearly u^{**} is convex, and admissible by the boundedness of u^*, and $u^{**}(x)\leqslant u(x)$ on $\partial\Delta_\lambda^\vee$ because

$$\langle x,p\rangle-u^*(p)\leqslant u(x)\quad \text{for all } p\in\Delta.$$

Our characterisation precisely ensures that $u^{**}(x)\geqslant u(x)$ on $\partial\Delta_\lambda^\vee$. Then u^{**} provides the canonical extension. □

Remark 3.20. The above characterisation is not completely intrinsic because it uses the extrinsic pairing $\langle\cdot,\cdot\rangle\colon M_\mathbb{R}\times N_\mathbb{R}\to\mathbb{R}$. On the positive side it uses only the value of u on $\partial\Delta_\lambda^\vee$.

Remark 3.21. At $x\in\partial\Delta_\lambda^\vee$, the vector $p\in\Delta$ in the hypothesis is a subgradient of the canonical extension u^{**}, namely $u^{**}(y)-u(x)\geqslant\langle p,y-x\rangle$.

We now introduce a local notion. The function u below will be analogous to ϕ_0 in (3.12). Recall the charts $\partial\Delta_\lambda^\vee\cap U_w^\infty$ associated with outward normal vectors w introduced in §3.2, with local coordinates $x^{m_1},\dots,x^{m_n}$.

Definition 3.22. Let u be a continuous function on $\partial\Delta_\lambda^\vee$, with which we associate a collection of local functions $\{u_m\}_{m\in\Delta_\mathbb{Z}}$ by the rule $u_m=u-\langle x,m\rangle$. We regard u_m as a function on the charts $\partial\Delta_\lambda^\vee\cap U_w^\infty$, with $\langle w,m\rangle=1$. We say that u is a *locally convex* function if all of the u_m's are convex on their corresponding charts.

Remark 3.23. One can reconstruct u from the local functions $\{u_m\}$, as long as their mutual differences define a correct cocycle $\{m-m'\}$. Thus, this definition has the intrinsic local feature we desire, in analogy with the notion of Kähler potentials.

Remark 3.24. For fixed w, and m and m' satisfying $\langle m,w\rangle=\langle m',w\rangle=1$, the properties of u_m and $u_{m'}$ being convex on the w-chart are equivalent, because $m-m'$ is an affine function. However, on the overlap of the w-chart and the w'-chart, if u_m is convex in one chart, it is not automatically convex in the other.

Remark 3.25. If a convex function is not sufficiently regular, there can be a null set of points at which the subgradient is not unique. Later we will abuse language to use the word gradient to refer to any choice of subgradient.

PROPOSITION 3.26. *If u satisfies the extension property, then u is locally convex.*

Proof. Let $\langle m,w\rangle=1$, and consider the function u_m on the chart $\partial\Delta_\lambda^\vee\cap U_w^\infty$. Given x in the chart, we need to find $\vec{p}$ such that

$$u_m(y)-u_m(x)\geqslant \vec{p}\cdot(y-x)_w,$$

where $\vec{p}$ is a covector, and $(y-x)_w$ refers to the representation of $y-x$ in the local coordinates $x^{m_1},\ldots x^{m_n}$; after identifying $x^{m_1},\ldots x^{m_n}$ as coordinates on the plane

$$m^\perp=\{\langle m,x'\rangle=0\},$$

we may regard $(y-x)_w$ as an element of $m^\perp$, and according to the decomposition

$$N_\mathbb{R}=m^\perp\oplus\mathbb{R}w,$$

we have

$$y-x=(y-x)_w+\langle y-x,m\rangle w.$$

Since the properties of u_m and $u_{m'}$ being convex in the w-chart are equivalent if $\langle m,w\rangle=\langle m,w\rangle=1$, we may assume that $L_\lambda(x)$ is attained by $\langle m,x\rangle+\lambda(m)$. By the extension property and Proposition 3.19, there is some $p\in\Delta$ such that

$$u(y)-u(x)\geqslant\langle p,y-x\rangle,$$

and hence

$$u_m(y)-u_m(x)\geqslant\langle p-m,y-x\rangle=\langle p-m,(y-x)_w\rangle+\langle y-x,m\rangle\langle p-m,w\rangle.$$

As $p\in\Delta$, we have $\langle p,w\rangle\leqslant 1=\langle m,w\rangle$. Since $L_\lambda(x)$ is attained by $\langle m,x\rangle+\lambda(m)$, and the polytope $\Delta_\lambda^\vee$ lies in the half-space $\{\langle m,\rangle+\lambda(m)\leqslant 0\}$, we have

$$\langle m,y\rangle+\lambda(m)\leqslant 0=\langle m,x\rangle+\lambda(m).$$

Combining the above, we have

$$u_m(x)-u_m(y)\geqslant\langle p-m,(y-x)_w\rangle+\langle y-x,m\rangle\langle p-m,w\rangle\geqslant\langle p-m,(y-x)_w\rangle,$$

so we have produced $\vec{p}$ as required. $\square$

3.5. Extension property: the Fermat case

We do not know the equivalence between the extension property and the local convexity property. However, in the case of the Fermat family Example 3.1, the polyhedral set $\partial\Delta_\lambda^\vee = -\partial\Delta^\vee$ has a *discrete symmetry* by the permutation group of the vertices of Δ, corresponding to the permutations of the monomials $Z_0^{n+2}, ..., Z_{n+1}^{n+2}$. This can be used to our advantage.

Notation. Denote the vertices of $\partial\Delta_\lambda^\vee$ as $w_0, ..., w_{n+1}$, which coincide with the outward normal vectors , because $\partial\Delta_\lambda^\vee = -\partial\Delta^\vee$. Denote the vertices of Δ as $m^0, ..., m^{n+1}$, so that

$$\langle w_i, m^j\rangle = \begin{cases} 1, & \text{if } i \neq j, \\ -(n+1), & \text{if } i = j. \end{cases}$$

Let $\mathrm{Star}(w_i)$ be the star of w_i in the barycentric subdivision of $\partial\Delta_\lambda^\vee$. Let $\mathrm{Sing} \subset \widetilde{\mathrm{Sing}}$ be the subset of points not contained in the interior of any of these stars. The affine structure on $\partial\Delta_\lambda^\vee \setminus \widetilde{\mathrm{Sing}}$ extends to $\partial\Delta_\lambda^\vee \setminus \mathrm{Sing}$, by decreeing that, on the interior of $\mathrm{Star}(w_i)$, we use the coordinates for the chart $U_w^\infty \cap \partial\Delta_\lambda^\vee$. As Sing has codimension 2 inside $\partial\Delta_\lambda^\vee$, this makes $\partial\Delta_\lambda^\vee$ into a *singular affine manifold.*

Proposition 3.27. *In the Fermat case, if u is a locally convex function on $\partial\Delta_\lambda^\vee$, which is invariant under the permutation group, then u satisfies the extension property.*

Proof. We need to prove the characterisation in Proposition 3.19. Without loss of generality, $L_\lambda(x)$ is achieved by $\langle m^0, x\rangle + \lambda(m^0)$. We need to find $p \in \Delta$, such that $u(y) - u(x) \geqslant \langle p, y - x\rangle$. For this we study the gradient of the function u_{m^0} on the various w-charts.

First, notice that, for x' and y' on the face $\{L_\lambda = \langle m^0, \rangle + \lambda(m^0)\}$, namely the convex hull of $w_1, ..., w_{n+1}$, the vector $y' - x'$ is parallel to the face, and by convexity of u_{m^0} the directional derivative $\nabla u_{m^0} \cdot (y' - x')$ is monotone along the path from x' to y', so must be maximized at y'. In particular, we consider such line segments on the face parallel to $w_i - w_j$ for $i, j \geqslant 1$. By the discrete symmetry, $\nabla u_{m^0} \cdot (w_j - w_i)$ must be zero on the plane of reflection bisecting the face. Thus, for $i, j \geqslant 1$, $i \neq j$, the subset of the face

$$\{\nabla u_{m^0} \cdot w_i \geqslant \nabla u_{m^0} \cdot w_j\} \cap \{L_\lambda = \langle m^0, \rangle + \lambda(m^0)\}$$

agrees exactly with the half of the face containing w_i. Therefore, the subset of the face

$$\{\nabla u_{m^0} \cdot w_i \geqslant \nabla u_{m^0} \cdot w_j \text{ for all } j \geqslant 1\}$$

is exactly the intersection of $\mathrm{Star}(w_i)$ with the face. Without loss of generality, x lies in $\mathrm{Star}(w_1)$.

We follow the notation in the proof of Proposition 3.26. In the w_1-chart, denote the gradient of u_{m^0} as $\vec{p}$, so that, for y in the w_1-chart, one has

$$u_{m^0}(y)-u_{m^0}(x)\geqslant \vec{p}\cdot(y-x)_{w_1}.$$

A priori $\vec{p}$ lives in $M_{\mathbb{R}}/\mathbb{R}m^0$. We lift $\vec{p}$ to $M_{\mathbb{R}}$ by demanding $\langle\vec{p},w_1\rangle=0$, so, by the above discussion, $\langle\vec{p},w_i\rangle\leqslant 0$ for $i\geqslant 1$. Define $p=\vec{p}+m_0$, then $\langle p,w_i\rangle\leqslant 1$ for all $i\geqslant 1$. We regard $p\in M_{\mathbb{R}}$ as the gradient of u at x, and write $p=\nabla u$ as a function of x. This construction can be made on other faces as well, and on the intersection of two faces the definitions are compatible.

We claim that $p\in\Delta$: it suffices to show that $\langle p,w_0\rangle\leqslant 1$. Notice that

$$w_0=-\sum_{1}^{n+1}w_i=\sum_{i=2}^{n+1}(w_1-w_i)-(n+1)w_1.$$

Consider the line segment in the face joining x to the boundary of the face in the direction $\sum_{i=2}^{n+1}(w_1-w_i)$, which stays inside Star(w_1), and along which $\nabla u\cdot\sum_{i=2}^{n+1}(w_1-w_i)$ increases, or equivalently $\langle\nabla u,w_0\rangle$ increases. But the boundary of the face

$$\{L_\lambda=\langle m^0,\rangle+\lambda(m^0)\}$$

lies also on a different face, and we can use the information from this new face to deduce that $\langle\nabla u,w_0\rangle\leqslant 1$ there.

By construction for y in the w_1-chart,

$$u(y)-u(x)\geqslant\langle\vec{p}(x),(y-x)_{w_1}\rangle+\langle m_0,y-x\rangle=\langle\nabla u(x),y-x\rangle.$$

We claim that, in fact,

$$u(y)-u(x)\geqslant\langle\nabla u(x),y-x\rangle$$

holds for all $y\in\partial\Delta_\lambda^\vee$. We are left to check the claim for y on the face $\{L_\lambda=\langle m^1,\rangle+\lambda(m^1)\}$, namely the complement of the w_1-chart. Consider the w_i-chart for $i>1$. We can write, according to the decomposition $N_{\mathbb{R}}=(m^1)^\perp\oplus\mathbb{R}w_i$, that

$$y-x=(y-x)_{w_i,m^1}+\langle m^1,y-x\rangle w_i.$$

By local convexity, in the w_i-chart u_{m^1} is convex, so there is some $\vec{p}'$ such that, for any y' in the w_i-chart, one has

$$u_{m^1}(y')-u_{m^1}(x)\geqslant\vec{p}'\cdot(y'-x)_{w_i,m^1}.$$

But a gradient vector of u_{m^1} at x is $\nabla u(x)-m^1$, so we may take $\vec{p}'=\nabla u(x)-m^1$. Thus,

$$u(y)-u(x)\geqslant \vec{p}'\cdot(y-x)_{w_i,m^1}+\langle m^1,y-x\rangle=\langle\nabla u(x),y-x\rangle-\langle\vec{p}',w_i\rangle\langle m^1,y-x\rangle.$$

Now, $\langle m^1,y-x\rangle\geqslant 0$ as in the proof of Proposition 3.26, and $\langle\vec{p}',w_i\rangle\leqslant 0$ by $\nabla u\in\Delta$. This implies that $u(y)-u(x)\geqslant\langle\nabla u(x),y-x\rangle$, as required.

We have verified the characterisation in Proposition 3.19, and hence the extension property. □

The proof above contains some additional information about the gradients.

COROLLARY 3.28. *In the region* $\mathrm{Star}(w_i)+\mathbb{R}_{\geqslant 0}w_i\subset N_{\mathbb{R}}$, *the directional derivative of the canonical extension* $u=u^{**}$ *satisfies* $\langle w_i,\nabla u\rangle=1$. *In particular, in this region, for any* m *with* $\langle m,w_i\rangle=1$, *the function* $u_m=u-m$ *is constant upon translation in the* w_i*-direction.*

Proof. By Remark 3.21, the ∇u introduced in the above proof is actually the gradient of the extension u over $N_{\mathbb{R}}$. By the proof above, we know that $\langle\nabla u,w_i\rangle=1$ on $\mathrm{Star}(w_i)\subset\partial\Delta_\lambda^\vee$. This directional derivative can only increase as $x\in N_{\mathbb{R}}$ moves in the w_1-direction. But $\nabla u\in\Delta$ on $N_{\mathbb{R}}$, since the extension is admissible, so $\langle\nabla u,w_i\rangle\leqslant 1$ everywhere, and hence the claim is proved. □

For later use, we define the notion of real MA equation in the Fermat case.

Definition 3.29. Let u be a locally convex function on $\partial\Delta_\lambda^\vee$ invariant under the discrete symmetry. Then, u is called an Aleksandrov solution of the *real* MA *equation* on $\partial\Delta_\lambda^\vee\setminus\mathrm{Sing}$ if th following conditions hold.

- On the interior of any top-dimensional face of $\partial\Delta_\lambda^\vee$, in a set of standard local affine coordinates $x^{m_1},\dots,x^{m_n}$ with $dx^{m_1}\wedge dx^{m_2}\dots dx^{m_n}$ equal to the standard volume form $d\mu_\infty$, the function u satisfies $MA(u)=d\mu_\infty$ in the Aleksandrov sense.
- On $\mathrm{Star}(w)\subset U_w^\infty\cap\partial\Delta_\lambda^\vee$, we use the standard affine coordinates $x^{m_1},\dots,x^{m_n}$ associated with the $U_w^\infty\cap\partial\Delta_\lambda^\vee$ chart. We demand, for any vertex m of Δ with $\langle m,w\rangle$, that the local function $u_m=u-m$ satisfies $\mathrm{MA}(u_m)=d\mu_\infty$ in the Aleksandrov sense.

Schematically we write $\mathrm{MA}(u)=d\mu_\infty$.

Remark 3.30. Notice that the definition is compatible on overlapping charts, because the transition functions lie in $\mathrm{SL}(n,\mathbb{Z})\ltimes\mathbb{R}^n$. On the locus $\mathrm{Sing}\subset\partial\Delta_\lambda^\vee$, we make no definition.

4. Estimates on the Kähler potential

This section is concerned with estimating the Kähler potential on the degenerating hypersurfaces X_s in the Fermat family. The expectation that the potentials converge in the $s\to\infty$ limit to a solution of a real MA equation, motivates us to produce local convex functions by taking average of local Kähler potentials. Convex functions have better a-priori regularity than psh functions: a Lipschitz bound is automatic. These arguments work for general Kähler potentials, without using the complex MA equation. The main difficulty is then to show that, for the Calabi–Yau metric, the local potentials are C^0-close to their averaging convex functions, at least in the generic region; equivalently, the local potentials have small local oscillations. This part relies on the method of Kołodziej, as outlined in §2.2, and a key ingredient is an improved uniform Skoda inequality.

Most arguments apply to more general contexts, and the only reason we restrict to the Fermat family of hypersurfaces is to use the extension property, which enables us to patch up the local convex functions into a global regularisation of the original Kähler potential.

4.1. Harnack inequality

Consider a general possibly singular Kähler potential $\varphi\in\mathrm{PSH}(X_s, s^{-1}\omega_{FS})$ on X_s, normalised to $\sup_{X_s}\varphi=0$. We think of φ equivalently as a collection of local potentials $\{\varphi_0, \varphi_m\}$ as in §3.3. In the region $U^s_w\subset X_s$, we can find $m\in\Delta_{\mathbb{Z}}$ with $\langle m, w\rangle=1$ and $\mathbb{C}^*$-coordinates $z^{m_1}, ..., z^{m_n}$ as in §3.1. Recall that $d\mu_s$ is the normalised canonical measure induced by the holomorphic volume form.

Notation. Denote X_s^{toric} as the union of all the toric regions $U^s_{w,\delta}$ for various choices of m and w. It is tacitly understood that slightly shrunken domains correspond to a slightly larger choice of δ, and we shall abusively use the same notation for shrunken domains.

Proposition 4.1. (Harnack-type inequality) *Suppose $\varphi\in\mathrm{PSH}(X_s, s^{-1}\omega_{FS})$ with $\sup_{X_s}\varphi=0$. Then the average integral*

$$\fint_{X_s^{\mathrm{toric}}} |\varphi|\, d\mu_s \leqslant C.$$

Proof. (Cf. [1, proof of Proposition 3.1]) Consider the local potentials $\phi=\varphi_m$ on various coordinate charts in §3.1, both of the toric type and of the boundary type. The charts can be chosen so that the Lebesgue measures thereof are uniformly equivalent to $d\mu_s$ up to a scaling factor. We have $|\phi-\varphi|\leqslant C$ uniformly on charts. Suppose that a coordinate ball $B(p,3R)$ is contained in (the universal cover of) the local chart. Since ϕ is psh and $\phi - C \leqslant 0$, for $z \in B(p,R)$,

$$\phi(y) - C \leqslant \fint_{B(y,2R)} (\phi - C) \lesssim \fint_{B(p,R)} (\phi - C),$$

hence

$$\fint_{B(p,R)} |\varphi| \lesssim 1 + \inf_{B(p,R)} (-\varphi).$$

To deduce the global version of the Harnack-type inequality, we need a transitivity property, namely we can connect the chart containing the maximum point of φ to any of the toric charts in X_s^{toric} via a chain of $O(1)$ number of charts, such that $\inf_{B(p,R)} |\varphi|$ on charts increase by only $O(1)$ in each step. This last fact is because we can choose the chains of successive charts $B(p_i, 5R_i)$ such that the measure of the overlap occupies a non-trivial portion of the previous chart:

$$|B(p_i, R_i) \cap B(p_{i+1}, R_{i+1})| \gtrsim \tfrac{1}{10} |B(p_i, R_i)|,$$

which would force

$$\begin{aligned} \inf_{B(p_{i+1}, R_{i+1})} |\varphi| &\leqslant \inf_{B(p_{i+1}, R_{i+1}) \cap B(p_i, R_i)} |\varphi| \leqslant \fint_{B(p_{i+1}, R_{i+1}) \cap B(p_i, R_i)} |\varphi| \\ &\lesssim \fint_{B(p_i, R_i)} |\varphi| \lesssim 1 + \inf_{B(p_i, R_i)} |\varphi|. \end{aligned}$$

□

Remark 4.2. Notice this transitivity argument allows us to move from boundary-type charts into toric charts, but not conversely, because the measure is much larger on toric charts.

4.2. Local potentials: convexity

We continue with a general $\varphi \in \mathrm{PSH}(X_s, s^{-1}\omega_{FS})$ normalised to $\sup_{X_s} \varphi = 0$, whose local potentials are $\{\varphi_0, \varphi_m\}$. A simple observation is the following.

LEMMA 4.3. *Let Φ be any psh function on the open subset of*

$$\{1 < |\zeta_i| < \Lambda \text{ for all } i = 1, \dots n\} \subset (\mathbb{C}^*)^n.$$

Then, the T^n-invariant function

$$\bar{\Phi}(\log|\zeta_1|, \dots, \log|\zeta_n|) = \frac{1}{(2\pi)^n} \int_{T^n} \Phi(|\zeta_1| e^{i\theta_1}, \dots, |\zeta_n| e^{i\theta_n})\, d\theta_1 \dots d\theta_n$$

is a convex function in the variables $x_1 = \log|\zeta_1|, \dots, x_n = \log|\zeta_n|$.

Proof. Since the T^n-action on $(\mathbb{C}^*)^n$ is holomorphic, $\Phi(\zeta_1 e^{i\theta_1}, ..., \zeta_n e^{i\theta_n})$ is psh in ζ for any choice of θ_i, so the average function $\bar{\Phi}$ is also psh. Any T^n-invariant psh function must be convex in the log coordinates, because of the formula

$$\sqrt{-1}\,\partial\bar{\partial}\bar{\Phi} = \frac{1}{4}\sum \frac{\partial^2\bar{\Phi}}{\partial x_i \partial x_j}\sqrt{-1}\,d\log\zeta_i \wedge d\,\overline{\log\zeta_j} \geqslant 0. \qquad \square$$

In the region $U_w^s \subset X_s$, we can find $m \in \Delta_{\mathbb{Z}}$, with $\langle m, w\rangle = 1$, and $\mathbb{C}^*$-coordinates

$$z^{m_1}, ..., z^{m_n}$$

as in §3.1, and consider the local potential $\phi = \varphi_m$. Set

$$x^{m_i} = \frac{\log|z^{m_i}|}{s}.$$

We produce the local average function

$$\bar{\phi}(x^{m_1}, ... x^{m_n}) = \frac{1}{(2\pi)^n}\int_{T^n} \phi(|z^{m_1}|e^{i\theta_1}, ..., |z^{m_n}|e^{i\theta_n})\,d\theta_1 ... d\theta_n. \tag{4.1}$$

Proposition 4.4. *In the chart U_w^s the average function $\bar{\phi}$ is convex, and on the shrunken chart $U_{w,\delta}^s$ it has a Lipschitz bound*:

$$|\bar{\phi}| \leqslant C, \quad |\bar{\phi}(x) - \bar{\phi}(x')| \leqslant C|x - x'|. \tag{4.2}$$

Proof. By Lemma 4.3, $\bar{\phi}$ is convex, and by Proposition 4.1 it has an L^1 bound in the x^{m_i} coordinates:

$$\int |\bar{\phi}| dx^{m_1} ... dx^{m_n} \leqslant C.$$

Clearly, $\bar{\phi}$ is also bounded above, so for the argument we may pretend $\bar{\phi} \leqslant 0$ upon shifting by a bounded constant.

We claim that $\bar{\phi}(x)$ is bounded from below for x in a shrunken interior region. The ball $B(x, 2r)$ is contained in the coordinate chart, with r bounded below by a positive constant. For y in the annulus $B(x, 2r) \backslash B(x, r)$, we have

$$2\bar{\phi}\left(\frac{x+y}{2}\right) \leqslant \bar{\phi}(x) + \bar{\phi}(y),$$

so, upon integration,

$$\int |\bar{\phi}| \gtrsim \int 2\left|\bar{\phi}\left(\frac{x+y}{2}\right)\right| dy \geqslant \int (|\bar{\phi}(x)| + |\bar{\phi}(y)|)\,dy,$$

which bounds $|\bar{\phi}(x)|$. Thus, on a slightly shrunken x-domain, the oscillation is bounded:

$$\operatorname{osc}\bar{\phi} = (\sup - \inf)\bar{\phi} \leqslant C,$$

and the Lipschitz bound follows again by convexity. $\square$

Remark 4.5. We discuss some intuition about *log scales*. Let $P\in X_s$ lie in $U^s_{w,\delta}$, then a log scale $|z^{m_i}|\sim|z^{m_i}(P)|$ around P refers to the subregion

$$\left\{\tfrac{1}{2}|z^{m_i}(P)|\lesssim|z^{m_i}|\lesssim 2|z^{m_i}(P)| \text{ for all } 1\leqslant i\leqslant n\right\}.$$

Now, $\log|z^{m_i}|$ vary by order $O(s)$ within $U^s_{w,\delta}$, so there are an enormous number of log scales. The long range behaviour of X_s is similar to $(\mathbb{C}^*)^n$, with half of the dimensions compactified into T^n. On the other hand, over one log scale X_s behaves qualitatively like the unit disc in $\mathbb{C}^n$. The concept of *local oscillation* of a function refers to the oscillation within one log scale. In particular the Lipschitz bound (4.2) implies a local oscillation bound

$$\operatorname{osc}_{|z^{m_i}|\sim|z^{m_i}(P)|}\bar{\phi}\leqslant Cs^{-1}.$$

4.3. Local potentials: plurisubharmonicity

The following lemma is a special case of the principle that for a subharmonic function, the standard mean value inequality has interesting strengthenings if there is more information about microscopic averages.

Lemma 4.6. *Let Φ be a subharmonic function on $B_2^n\times T^k=B_2\times\mathbb{R}^k/\varepsilon\mathbb{Z}^k$ equipped with the Euclidean metric*

$$g=\sum_1^n dx_i^2+\sum_1^k dy_j^2,$$

where $0<\varepsilon\ll 1$. Let v be the averaging function of Φ over the T^k fibres. Assume $⨍|\Phi|\lesssim 1$ and a Lipschitz bound $\operatorname{Lip}(v)\lesssim 1$, then on $B_1\times T^k$ we have $\Phi\leqslant v+C\varepsilon^{1/2}$.

Proof. (Courtesy of W. Feldman) By passing to the universal cover $B_2\times\mathbb{R}^k$, the standard mean value inequality implies that

$$\sup_{B_{3/2}\times T^k}\Phi\lesssim ⨍|\Phi|\lesssim 1.$$

Let $p\in B_1\times T^k$, which lifts to a point p in $B_1\times\mathbb{R}^k$. Consider the Euclidean ball

$$B_g(p,\varepsilon R)\subset B_{3/2}\times\mathbb{R}^k,$$

where $R\gg 1$ is a parameter to be chosen. Then, by the mean value inequality,

$$\Phi(p)\leqslant ⨍_{B_g(p,\varepsilon R)}\Phi.$$

Define the subset $E\subset B_g(\varepsilon R)$ as the union of all interior lattice cubes, then

$$B_g(p,\varepsilon R)\setminus E\subset B_g(p,\varepsilon R)\setminus B_g(p,\varepsilon(R-C)),$$

and, by the lattice periodicity of Φ, we have $\int_E \Phi=\int_E v$. By partitioning the integral $\int_{B_g(p,\varepsilon)}\Phi$ into the contributions from E and $B_g(p,\varepsilon R)\setminus E$,

$$\fint_{B_g(p,\varepsilon R)}\Phi\leqslant\fint_{B_g(p,\varepsilon R)}v+CR^{-1}\sup_{B_g(p,\varepsilon R)}(\Phi-v)\leqslant\fint_{B_g(p,\varepsilon R)}v+CR^{-1}.$$

By the Lipschitz bound of v, the right-hand side is bounded above by

$$v(p)+\mathrm{Lip}(v)\varepsilon R+CR^{-1}\leqslant v(p)+C(\varepsilon R+R^{-1}).$$

Choosing $R=\varepsilon^{-1/2}$ gives $\Phi(p)\leqslant v(p)+C\varepsilon^{1/2}$. $\square$

Back to the setting of Proposition 4.4,

Corollary 4.7. (Local potential upper bound) *On* $U^s_{w,\delta}$, *then* $\phi-\bar\phi\leqslant Cs^{-1/2}$.

Proof. The psh property of ϕ implies subharmonicity. By the Harnack inequality in Proposition 4.1 the average L^1-integral is bounded, and by Proposition 4.4 there is a Lipschitz bound on the local average function $\bar\phi$. $\square$

Corollary 4.8. (Local L^1-oscillation bound) *Over one log scale inside* $U^s_{w,\delta}$,

$$\fint_{|z^{m_i}|\sim|z^{m_i}(P)|}|\phi-\bar\phi|\,d\mu_s\leqslant Cs^{-1/2}.$$

Proof. Recall the local oscillation of $\bar\phi$ in one log scale is $O(s^{-1})$. Since the local sup of ϕ differs from the local average of ϕ by $O(s^{-1/2})$, the local L^1-oscillation is likewise bounded by $O(s^{-1/2})$. $\square$

Remark 4.9. The s-dependence is probably not optimal.

We now seek a local L^1-oscillation bound on the charts of boundary type U_P (cf. Remark 3.10). The idea is that any chart of boundary-type overlaps with some chart of toric type in an annulus region, where the L^1-oscillation bound is already known. It would be enough to transfer the L^1-oscillation bound from the annulus to the deep interior of the chart.

Lemma 4.10. *Let* Φ *be a psh function on the* $\{|z_i|\leqslant 4$ *for all* $i\}\subset\mathbb{C}^n$. *Then,*

$$\fint_{B_1}|\Phi|\lesssim\fint_{\{1<|z_i|<4\ for\ all\ i\}}|\Phi|.$$

Proof. We induct on dimension. For $n=1$, the unit ball is already enclosed by an annulus, so $\sup_{B(1)} \Phi$ is bounded above, and the mean value property applied to all balls $B(p,2)$ with $1<|p|\leqslant 2$ gives a lower bound on $\fint_{B(1)} \Phi$. Thus, the L^1-bound in $B(1)$ is clear.

For general n, notice by induction that we can bound, for each $i\leqslant n$,

$$\fint_{\{1<|z_i|<4 \text{ and } |z_j|<4 \text{ for all } j\neq i\}} |\Phi| \lesssim \fint_{\{1<|z_j|<4 \text{ for all } j\}} |\Phi|,$$

so Φ is controlled in L^1 on an annulus enclosing $B(1)$, and we can bound $\fint_{B(1)} |\Phi|$ similar to the $n=1$ case. □

COROLLARY 4.11. (Local L^1-oscillation bound II) *In the chart of boundary type U_P, the local potential ϕ satisfies*

$$\fint_{U_P} \left| \phi - \fint_{U_P} \phi \right| d\mu_s \leqslant C s^{-1/2}.$$

4.4. Locally convex function

In §4.2 we produced a collection of local average functions $\bar\phi = \bar\phi_{m,w}$ on U_w^s corresponding to various choices of w and m with $\langle m, w\rangle = 1$. But the local coordinates $x^{m_1}, ..., x^{m_n}$ are naturally interpreted also as coordinates on $\partial\Delta_\lambda^\vee$ (cf. §3.2), so $\bar\phi_{m,w}$ can be alternatively viewed as a collection of convex functions on the charts $U_w^\infty \cap \partial\Delta_\lambda^\vee$ of $\partial\Delta_\lambda^\vee$. (Notice that these local functions are defined without the need to shrink the domain to $U_{w,\delta}^\infty$).

The intuition is that up to C^0-small error, the differences of these local functions agree with the cocycle $\{m-m'\}$, or equivalently, up to some C^0-small fuzziness

$$\bar\phi_{m,w} + \langle m, x\rangle$$

glue to a *locally convex function* on $\partial\Delta_\lambda^\vee$, in the sense of Definition 3.22. The more precise statement is the following.

LEMMA 4.12. *On overlapping charts of $\partial\Delta_\lambda^\vee$,*

$$|\bar\phi_{m,w} - \bar\phi_{m',w'} + (m-m')| \leqslant C s^{-1/2}.$$

Proof. Since we know that the local L^1-oscillation estimate holds in every local region, in a log scale in U_w^s, not necessarily in the shrunken region $U_{w,\delta}^s$,

$$\fint_{|z^{m_i}|\sim |z^{m_i}(P)|} \left| \varphi_m - \fint \varphi_m \right| d\mu_s \leqslant C s^{-1/2}.$$

Since $\bar{\phi}_{m,w}$ is convex, a local L^1-bound implies a local L^∞-bound in a slightly shrunken region, so, in the log scale,

$$\left|\bar{\phi}_{m,w} - \fint \varphi_m\right| \leqslant C s^{-1/2}.$$

Likewise for $\bar{\phi}_{m',w'}$. By definition, the local potentials differ by

$$\varphi_m - \varphi_{m'} = \langle m' - m, \operatorname{Log}_s(z)\rangle.$$

Notice that, for a given point P on $\partial\Delta_\lambda^\vee$, the log scales on U_w^s and $U_{w'}^s$ around P have a non-trivial percentage of overlapping measure. Thus,

$$\begin{aligned}|\bar{\phi}_{m,w} - \bar{\phi}_{m',w'} + (m-m')| &\lesssim s^{-1/2} + \left|\fint \varphi_m - \fint \varphi_{m'} + (m-m')\right| \\ &\lesssim s^{-1/2} + \fint_{\text{overlap}} |\varphi_m - \varphi_{m'} + (m-m')| \lesssim s^{-1/2}. \qquad \square\end{aligned}$$

Remark 4.13. The tropical version U_w^∞ of U_w^s is in general larger than $U_w^\infty \cap \partial\Delta_\lambda^\vee$; it typically contains also some subset stretching to infinity along the w-direction. If we regard $\bar{\phi}_{m,w}$ as local functions on $\mathcal{A}_\lambda^\infty$ instead of $\partial\Delta_\lambda^\vee$, then there is a delicate issue. The lemma above *does not* imply that $\bar{\phi}_{m,w} + \langle m, x\rangle$ for various choices of m and w glue approximately on overlapping regions far from $\partial\Delta_\lambda^\vee$. The problem is that such overlapping regions have too small measure, which breaks down the proof.

4.5. Legendre transform, extension, regularisation

We restrict to the Fermat case, and consider a general $\varphi \in \mathrm{PSH}(X_s, s^{-1}\omega_{FS})$ with

$$\sup_{X_s} \varphi = 0,$$

invariant under the symmetric group permuting the monomials $Z_0^{n+2}, \dots, Z_{n+1}^{n+2}$. The goal of this section is to canonically patch together the local convex functions in §4.4 approximately to produce a convex admissible function on $N_\mathbb{R} = \mathbb{R}^{n+1}$. We will then induce a potential $\psi \in \mathrm{PSH}(X_s, s^{-1}\omega_{FS}) \cap C^0$ which is a *regularisation* of φ in the sense that it enjoys better a-priori bounds than φ.

Proposition 4.14. *There is an admissible convex function u on $N_\mathbb{R}$ such that, on $U_w^\infty \cap \partial\Delta_\lambda^\vee$,*

$$|u - (\bar{\phi}_{m,w} + m)| \leqslant C s^{-1/2}. \tag{4.3}$$

Proof. The idea is to regard $\bar{\phi}_{m,w}+\langle m,x\rangle$ as approximately defining a locally convex function on $\partial\Delta_\lambda^\vee$ in the sense of Definition 3.22, and then the problem is essentially to prove an effective version of the extension property (cf. Proposition 3.27). We will outline the main modifications.

We will produce u by mimicking the Legendre duality construction in Proposition 3.19. For $p\in\Delta$, define

$$u^*(p)=\sup_{x\in\partial\Delta_\lambda^\vee}\{\langle x,p\rangle-(\bar{\phi}_{m,w}+\langle m,x\rangle)\},$$

where it is tacitly understood that $\bar{\phi}_{m,w}+\langle m,x\rangle$ is defined only over $\partial\Delta_\lambda^\vee\cap U_w^\infty$, and the sup is taken over all choices of m and w, whenever $\bar{\phi}_{m,w}$ is defined. Since $\bar{\phi}_{m,w}$ are uniformly bounded on $\partial\Delta_\lambda^\vee$, we see that $\|u^*\|_{C^0(\Delta)}\leqslant C$. We then define a convex function u on $N_{\mathbb{R}}$ by another Legendre transform

$$u(x)=\sup_{p\in\Delta}\{\langle p,x\rangle-u^*(p)\},$$

which is admissible, because u^* is bounded. By the same reasoning in Proposition 3.19, on $\partial\Delta_\lambda^\vee\cap U_w^\infty$,

$$u(x)\leqslant\bar{\phi}_{m,w}+\langle m,x\rangle+Cs^{-1/2}.$$

We are only left to show that

$$u(x)\geqslant\bar{\phi}_{m,w}+\langle m,x\rangle-Cs^{-1/2},$$

which amounts to showing that there exists $p\in\Delta$ such that, for any $y\in\partial\Delta_\lambda^\vee$,

$$\bar{\phi}_{m',w'}(y)+\langle m',y\rangle\geqslant\bar{\phi}_{m,w}(x)+\langle m,x\rangle+\langle p,y-x\rangle-Cs^{-1/2}.$$

Notice that our setting enjoys the discrete symmetry. This last step is the effective version of Proposition 3.27, and the proof is basically the same. □

By construction, u has a number of additional properties.

COROLLARY 4.15. *The canonical extension u satisfies an a-priori Lipschitz bound*

$$\begin{cases} |u-\max_m\langle m,x\rangle|\leqslant C, & \text{for all } x\in N_{\mathbb{R}},\\ |u(x)-u(x')|\leqslant C|x-x'|, & \text{for all } x,x'\in N_{\mathbb{R}}. \end{cases}\tag{4.4}$$

Morever, in the region $\mathrm{Star}(w)+\mathbb{R}_{\geqslant 0}w\subset N_{\mathbb{R}}$, *for any m with $\langle m,w\rangle=1$, the function $u_m=u-m$ is constant upon translation in the w-direction.*

Proof. The first inequality is because the Legendre transform $u^*(p)$ is bounded on Δ as in the above proof, and the second is because $\nabla u \in \Delta$. The morever statement is essentially identical to Corollary 3.28. □

By a small variant of Proposition 3.16, when we pull back the admissible convex functions u via Log_s, we obtain a torus-invariant Kähler current on $(\mathbb{P}_\Delta, s^{-1}[\Delta])$ with continuous local potentials. In details, we write $\psi_0 = u \circ \mathrm{Log}_s$, and define

$$\begin{cases} \psi = \psi_0 - \dfrac{n+2}{2s} \log\left(\displaystyle\sum_{m' \in \mathrm{vertices}(\Delta)} \exp\left(\frac{2}{n+2} \langle m', \mathrm{Log}(z) \rangle \right) \right), \\ \psi_m = \psi_0 - \langle m, \mathrm{Log}_s(z) \rangle = u_m \circ \mathrm{Log}_s\,. \end{cases} \tag{4.5}$$

By construction, $\psi \in \mathrm{PSH}(\mathbb{P}_\Delta, s^{-1}\omega_{FS}) \cap C^0$, and ψ_0 and ψ_m are the local potentials of ψ (cf. (3.12)). By Corollary 4.15, $\|\psi\|_{C^0} \leqslant C$, and ψ inherits the Lipschitz bound from u. By a slight abuse of notation, the restriction to X_s will still be denoted as

$$\psi \in \mathrm{PSH}(X_s, s^{-1}\omega_{FS}) \cap C^0.$$

We think of ψ as a regularisation of φ.

Remark 4.16. As explained in §2.3, on toric manifolds the Legendre transform arises from a limiting version of approximation by algebraic metrics, which in turn is a more standard way to regularize an arbitrary Kähler potential. Now X_s is not a toric manifold, but the toric symmetry holds approximately in generic regions, which motivates us to take the Legendre transform as a replacement of algebraic regularisation.

We now specify some subregions on X_s^{toric} with coordinate descriptions. These are intimately related to $\partial\Delta_\lambda^\vee \setminus \mathrm{Sing}$, which is covered by the stars of the vertices and the interior of the top-dimensional faces (cf. §3.5).

Notation. (Star-type regions on X_s) On the region $U_w^s \subset X_s$, recall the coodinates z^{m_i}, and regard $x^{m_i} = s^{-1} \log |z^{m_i}|$ as local coordinates also on $U_w^\infty \cap \partial\Delta_\lambda^\vee$. Let $U_w^{s,*} \subset U_{w,\delta}^s$ be the subset where the x^{m_i} coordinates correspond to points in $\mathrm{Star}(w) \subset \partial\Delta_\lambda^\vee$. The tropical analogue of $U_w^{s,*}$ is $(\mathrm{Star}(w) + \mathbb{R}_{\geqslant 0} w) \cap A_\lambda^\infty$.

Notation. (Face-type regions on X_s) Consider a slightly shrunken subset of the interior of a given top-dimensional face of $\partial\Delta_\lambda^\vee$. This can be regarded as a subset of $U_{w,\delta}^\infty \cap \partial\Delta_\lambda^\vee$, where we regard $x^{m_i} = s^{-1} \log |z^{m_i}|$ as local affine coordinates. Let

$$U_w^{s,\mathrm{face}} \subset U_w^s$$

be the subset where the x^{m_i} coordinates correspond to points in this shrunken face. The tropical analogue of $U_w^{s,\mathrm{face}} \subset U_w^s$ is the shrunken face.

The intuition is that when $z \in X_s$ has Log_s image close to $\partial \Delta_\lambda^\vee$, or if this image approaches infinity in specific directions, then $\varphi - \psi$ is bounded above by a very small number.

PROPOSITION 4.17. (Local potential upper bound)

- *Inside $U_w^{s,*} \subset X_s$, for $\langle m, w\rangle = 1$, the local potentials satisfy*

$$\varphi_m - \psi_m \leqslant C s^{-1/2}, \quad \textit{or equivalently} \quad \varphi - \psi \leqslant C s^{-1/2}.$$

- *Inside $U_w^{s,\mathrm{face}}$, the local potentials satisfies*

$$\varphi_0 - \psi_0 \leqslant C s^{-1/2}, \quad \textit{or equivalently} \quad \varphi - \psi \leqslant C s^{-1/2}.$$

Proof. In the star-type-region case, by Corollary 4.7, we have the upper bound

$$\varphi_m - \bar{\phi}_{m,w} \leqslant C s^{-1/2}.$$

By Proposition 4.14 and Corollary 4.15, in $U_w^{s,*}$ we can replace $\bar{\phi}_{m,w}$ by ψ_m up to an error bounded by $Cs^{-1/2}$, hence the claim. The face-type-region case follows the same argument, without the translational invariance statement of Corollary 4.15. □

4.6. Improved Skoda inequality

Recall the local L^1-oscillation bounds in both toric- and boundary-type regions, from Corollaries 4.8 and 4.11. Consequently, we have the following.

LEMMA 4.18. (Local Skoda estimate) *Consider any*

$$\varphi \in \mathrm{PSH}(X_s, s^{-1}\omega_{FS})$$

normalised to $\sup_{X_s} \varphi = 0$. There are uniform positive constants α and C such that the local potentials ϕ satisfy the following conditions:

- *In a log scale in the toric region,*

$$\fint_{|z^{m_i}| \sim |z^{m_i}(P)|} e^{-\alpha\sqrt{s}(\phi - \fint \phi)}\, d\mu_s \leqslant C.$$

- *In a boundary-type chart,*

$$\fint_{U_P} e^{-\alpha\sqrt{s}(\phi - \fint \phi)}\, d\mu_s \leqslant C.$$

Proof. Apply the standard Skoda inequality (cf. Theorem 2.1) to the rescaled function $s^{1/2}(\phi - \fint \phi)$. □

Remark 4.19. The local average $\fint \phi$ can be replaced by the local supremum using the mean value inequality.

COROLLARY 4.20. (Global Skoda estimate) *Consider any* $\varphi \in \mathrm{PSH}(X_s, s^{-1}\omega_{FS})$ *normalised to* $\sup_{X_s} \varphi = 0$. *There are uniform positive constants* α *and* C *such that*

$$\fint_{X_s} e^{-\alpha\varphi}\, d\mu_s \leqslant C. \tag{4.6}$$

Proof. By the local Skoda estimate and the Remark above, for both a log scale in the toric region, and a boundary-type chart, the local average

$$\fint e^{\alpha\sqrt{s}(-\varphi + \sup_{\mathrm{loc}} \varphi)}\, d\mu_s \leqslant C, \tag{4.7}$$

so in particular $\fint e^{\alpha(-\varphi + \sup_{\mathrm{loc}} \varphi)} \leqslant C$. But we have already achieved a C^0-bound on local average functions, and in particular a lower bound on local suprema. Thus

$$\fint e^{-\alpha\varphi}\, d\mu_s \leqslant C e^{-\alpha \sup_{\mathrm{loc}} \varphi} \leqslant C,$$

or equivalently $\int_{\mathrm{loc}} e^{-\alpha\varphi}\, d\mu_s \leqslant C \int_{\mathrm{loc}} d\mu_s$ for local integrals. To pass from this to the global Skoda estimate, we need to take a large collection of log scales and boundary-type charts and sum over the estimates:

$$\int_{X_s} e^{-\alpha\varphi}\, d\mu_s \leqslant C \sum \int_{\mathrm{loc}} d\mu_s.$$

The only problem is to ensure that the local charts can be chosen without substantially overcounting the measure. For points on X_s whose Log_s image is at $O(s^{-1})$ Euclidean distance to $\partial\Delta_\lambda^\vee$, it is easy to choose the charts so that each point is contained in $O(1)$ number of charts. Away from $\partial\Delta_\lambda^\vee$, the points deep inside the boundary-type charts in general do not have this local finiteness property, but this is compensated by the fact that the measure $d\mu_s$ decays exponentially away from $\partial\Delta_\lambda^\vee$ (cf. (3.9)). The conclusion is that

$$\sum \int_{\mathrm{loc}} d\mu_s \leqslant C \int_{X_s} d\mu_s,$$

whence the global Skoda estimate. □

We now specialize to the Fermat case, and consider $\varphi \in \mathrm{PSH}(X, s^{-1}\omega_{FS})$ normalised to $\sup_{X_s} \varphi = 0$ with discrete symmetry, as in §4.5. The regularisation of φ produced via Legendre transform is denoted as ψ.

THEOREM 4.21. (Improved Skoda estimate) *In the Fermat case above, there are uniform constants α and C such that*

$$\fint_{X_s} e^{-\alpha\sqrt{s}(\varphi-\psi)}\, d\mu_s \leqslant C. \tag{4.8}$$

Proof. On either a log scale in the toric region, or a boundary-type chart, we have by the local L^1-oscillation estimate and the mean value inequality that

$$\left|\sup_{\text{loc}} \varphi_m - \fint_{\text{loc}} \varphi_m\right| \leqslant C s^{-1/2} \quad \text{and} \quad \left|\sup_{\text{loc}} \psi_m - \fint_{\text{loc}} \psi_m\right| \leqslant C s^{-1/2}.$$

Notice also the local averages of φ_m and ψ_m differ by $O(s^{-1/2})$, so

$$\left|\sup_{\text{loc}} \varphi_m - \sup_{\text{loc}} \psi_m\right| \leqslant C s^{-1/2} \quad \text{and} \quad \left|\sup_{\text{loc}} \varphi - \sup_{\text{loc}} \psi\right| \leqslant C s^{-1/2}.$$

Combined with (4.7),

$$\fint_{\text{loc}} e^{-\alpha\sqrt{s}(\varphi-\psi)}\, d\mu_s \leqslant C.$$

The summation argument as in the global Skoda estimate proves the claim. □

Remark 4.22. This means $\phi-\psi$ can only fail to be bounded below by $Cs^{-1/2}$ on a set with exponentially small probability measure. Notice that we have not yet used the complex MA equation.

4.7. L^∞ and stability estimates for CY potentials

We finally impose the Calabi–Yau condition, and consider the CY potential $\varphi = \varphi_{CY,s}$ normalised to $\sup_{X_s} \varphi = 0$, solving (3.13):

$$\omega_{CY,s}^n = (s^{-1}\omega_{FS} + \sqrt{-1}\,\partial\bar{\partial}\varphi)^n = a_s s^{-n}\, d\mu_s.$$

THEOREM 4.23. (L^∞-estimate) *The Calabi–Yau potential $\varphi_{CY,s}$ satisfies the uniform L^∞-estimate $\|\varphi_{CY,s}\|_{L^\infty} \leqslant C$.*

Proof. We apply Kołodziej's estimate in Theorem 2.7. The Skoda-type inequality (2.1) is verified in Corollary 4.20, and hence the L^∞ estimate. □

We now specialize to the Fermat case. Clearly φ is invariant under the discrete symmetry of the hypersurface. Recall the regularisation is denoted as $\psi = \psi_{CY,s}$, coming from the double Legendre transform construction $u = u_{CY,s}$ (cf. §4.5). The local potentials of $\varphi_{CY,s}$ and $\psi_{CY,s}$ are denoted $\varphi_m = \varphi_{CY,s,m}$ and $\psi_m = \psi_{CY,s,m}$ according to the same convention as (3.12).

Theorem 4.24. *In the Fermat case, there is a uniform stability estimate*

$$\varphi_{CY,s}-\psi_{CY,s}\geqslant -Cs^{-1/2}\log s. \tag{4.9}$$

Proof. We apply Corollary 2.12. The Skoda estimate is verified in Corollary 4.20. The improved Skoda estimate theorem (Theorem 4.21) implies an exponential volume decay:

$$\frac{1}{\mathrm{Vol}(X_s)}\int_{\varphi-\psi\leqslant -t}\omega_\phi^n\leqslant Ce^{-\alpha t\sqrt{s}},$$

hence there exists $c\gg 1$, such that for $t_0=cs^{-1/2}\log s$,

$$\left(\frac{1}{\mathrm{Vol}(X_s)}\int_{\varphi-\psi\leqslant -t_0}\omega_\phi^n\right)^{1/2n}\leqslant Ce^{-\alpha t_0\sqrt{s}/2n}=Ce^{-\alpha c\log s/2n}\leqslant Cs^{-1/2}.$$

Theorem 2.7 then implies $\varphi-\psi\geqslant -Cs^{-1/2}\log s$ as required. □

Remark 4.25. In the theorems above only an upper bound on the volume measure is actually needed. The intuition is that the Skoda inequality is already so close to an L^∞ estimate, that a very tiny amount of extra assumptions are needed to conclude L^∞-estimate.

Combining this with the upper bound from Proposition 4.17 yields the following.

Corollary 4.26. *In the Fermat case, there is a uniform C^0-stability estimate*:

- *Inside $U_w^{s,*}\subset X_s$, for $\langle m,w\rangle=1$, the local potentials satisfy*

$$|\varphi_{CY,s,m}-\psi_{CY,s,m}|\leqslant Cs^{-1/2}\log s,$$

or equivalently

$$|\varphi_{CY,s}-\psi_{CY,s}|\leqslant Cs^{-1/2}\log s.$$

- *Inside $U_w^{s,\mathrm{face}}$, the local potentials satisfy*

$$|\varphi_{CY,s,0}-\psi_{CY,s,0}|\leqslant Cs^{-1/2}\log s,$$

or equivalently

$$|\varphi_{CY,s}-\psi_{CY,s}|\leqslant Cs^{-1/2}\log s.$$

The point is that in the generic region of X_s *the Calabi–Yau local potentials are C^0-approximated by their regularisations*, which build in convexity by construction, and therefore have a-priori Lipschitz bounds.

5. Fermat case: Metric convergence and SYZ fibration

We focus on the Fermat family case. We will produce a solution of the real MA equation on $\partial\Delta_\lambda^\vee$ by a subsequential limit, which induces a real MA metric on the regular locus (cf.$\tilde{\S}$5.1). Then we show the Calabi–Yau metrics on the degenerating hypersurfaces converge to the real MA metric, both in a C^∞_{loc}-sense (cf. §5.2) and in the global Gromov–Hausdorff sense (cf. §5.3). The strong regularity estimates will in particular imply that in the generic region of X_s the CY metrics are collapsing with bounded curvature, which by a result of Zhang [42] allows one to produce a special Lagrangian fibration in the generic region of X_s (cf. §5.4).

5.1. Limiting real MA metric

We work in the context of §4.7, and use the notations therein. We shall extract some subsequential limit of local potentials for the CY metric $\omega_{CY,s}$, and check that up to a constant it solves the real MA equation on $\partial\Delta_\lambda^\vee\setminus\text{Sing}$ according to Definition 3.29 (cf. also §2.6).

Since the convex functions $u_{CY,s}$ on $N_\mathbb{R}$ produced by double Legendre transform have uniform Lipschitz bounds (4.4), by the Arzela–Ascoli theorem we can take a *subsequential limit* as $s\to\infty$ such that $u_{CY,s}\to u_\infty$ in C^0_{loc}-topology. Later we will sometimes suppress mentioning the subsequence for brevity. In particular, u_∞ is convex and admissible. We can also pass Corollary 4.15 to the limit, to see that in the region $\text{Star}(w)+\mathbb{R}_{\geqslant 0}w\subset N_\mathbb{R}$, for any m with $\langle m,w\rangle=1$, the function $u_{\infty,m}=u_\infty-m$ is constant upon translation in the w-direction. In particular, in such regions the C^0_{loc} convergence improves to

$$\|u_{CY,s,m}-u_{\infty,m}\|_{C^0}\to 0.$$

By construction $\psi_{CY,s,m}=u_{CY,s,m}\circ\text{Log}_s$, and $u_{CY,s,0}=u_{CY,s}\circ\text{Log}_s$. Thus the stability estimate Corollary 4.26 implies that

- Inside $U_w^{s,*}\subset X_s$, for $\langle m,w\rangle=1$, the local potentials satisfy

$$|\varphi_{CY,s,m}-u_{\infty,m}\circ\text{Log}_s|\to 0.$$

- Inside $U_w^{s,\text{face}}$, the local potentials satisfy

$$|\varphi_{CY,s,0}-u_\infty\circ\text{Log}_s|\to 0.$$

The rest of this section is devoted to proving the following result.

THEOREM 5.1. *On $\partial\Delta_\lambda^\vee\setminus\mathrm{Sing}$, the locally convex function u_∞ solves the real* MA *equation in the sense of Definition 3.29 up to a scaling constant*:

$$\mathrm{MA}(u_\infty)=\frac{a_\infty}{\pi^n n!}\,d\mu_\infty, \tag{5.1}$$

where $d\mu_\infty$ is the Lebesgue measure on $\partial\Delta_\lambda^\vee$, and the constant a_∞ is defined by (3.14).

The intuitive idea is to pass the complex MA equation to some weak limit. The main problem is that the sequence $\varphi_{CY,s}$ live on different manifolds, so we need more effective estimates to pass to the limit.

LEMMA 5.2. *Let u be a bounded convex function on the square $\{|x_i|<1\}\subset\mathbb{R}^n$. Via the rescaled log map $s^{-1}\,\mathrm{Log}\colon(\mathbb{C}^*)^n\to\mathbb{R}^n$, the function u pulls back to a psh function on $\{|\log|z_i||<s\}$. Then, the real* MA *measure of u is related to the push-forward of the complex* MA *measure of $u\circ s^{-1}\,\mathrm{Log}$ by*

$$\mathrm{MA}(u)=\frac{s^n}{\pi^n n!}(s^{-1}\,\mathrm{Log})_*(\sqrt{-1}\,\partial\bar{\partial}(u\circ s^{-1}\,\mathrm{Log}))^n.$$

Proof. If u is smooth, then

$$\mathrm{MA}(u)(K)=\int_K\det(D^2u)\,dx_1\ldots dx_n=\frac{s^n}{\pi^n n!}\int_{(s^{-1}\,\mathrm{Log})^{-1}(K)}(\sqrt{-1}\,\partial\bar{\partial}(u\circ s^{-1}\,\mathrm{Log}))^n.$$

Since $u\in C^0$, and both the real and complex MA operators are weakly continuous with respect to C^0-limits, this equality passes to general u. □

LEMMA 5.3. (Chern–Levine type estimate) *Let u be a psh function on the annulus region $U=\{|\log|z_i||<s, \text{for all } i\}\subset(\mathbb{C}^*)^n$, with $\|u\|_{L^\infty}\lesssim 1$. Then, the following statements hold.*

- *On the shrunken set $E=\left\{\left|\log|z_i|\right|<\frac{1}{2}s\right\}$, the measure*

$$\int_E(\sqrt{-1}\,\partial\bar{\partial}u)^n\leqslant Cs^{-n}.$$

- *Let $u+v$ is another psh function, with $\|v\|_{L^\infty}\ll 1$. Let f be any compactly supported function on the square $\{|x_i|<1\}\subset\mathbb{R}^n$. Then,*

$$\int f\{(\sqrt{-1}\,\partial\bar{\partial}(u+v))^n-(\sqrt{-1}\,\partial\bar{\partial}u)^n\}\leqslant Cs^{-n}\,\|f\|_{C^2}\,\|v\|_{L^\infty}.$$

Proof. Let χ be a compactly supported non-negative smooth function on the square $\{|x_i|<1\}\subset\mathbb{R}^n$, equal to 1 on $\left\{|x_i|\leqslant\frac{1}{2}\right\}$. We identify χ with $\chi\circ s^{-1}\,\mathrm{Log}$, and denote

$$\omega_{\mathrm{std}}=\sqrt{-1}\sum d\log z_i\wedge d\,\overline{\log z_i}.$$

Then,

$$-Cs^{-2}\omega_{\mathrm{std}} \leqslant \sqrt{-1}\,\partial\bar{\partial}\chi \leqslant Cs^{-2}\omega_{\mathrm{std}}.$$

The basic obervation is that if T is a positive current of bidegree $(n-1,n-1)$, then, by integration by part,

$$\int_E \sqrt{-1}\,\partial\bar{\partial}u\wedge T \leqslant \int_{\mathrm{supp}(\chi)} \chi\sqrt{-1}\,\partial\bar{\partial}u\wedge T$$
$$= \int_{\mathrm{supp}(\chi)} u\sqrt{-1}\,\partial\bar{\partial}\chi\wedge T \leqslant Cs^{-2}\int_{\mathrm{supp}(\chi)} \omega_{\mathrm{std}}\wedge T.$$

Iterating this argument to lower the power of $\sqrt{-1}\,\partial\bar{\partial}u$,

$$\int_E (\sqrt{-1}\,\partial\bar{\partial}u)^n \leqslant Cs^{-2n}\int_U \omega_{\mathrm{std}}^n \leqslant Cs^{-n}.$$

The second statement is proved similarly by removing $\sqrt{-1}\,\partial\bar{\partial}v$ factors iteratively. □

Proof of Theorem 5.1. There are two subcases: the interior of the top-dimensional faces of $\partial\Delta_\chi^\vee$, and the star of the vertices $\mathrm{Star}(w)$. Since the arguments are almost the same, we focus on the latter.

On the interior of $\mathrm{Star}(w)$, we have local affine coordinates $x^{m_1},\dots,x^{m_n}$, related to the holomorphic $\mathbb{C}^*$-coordinates $z^{m_1},\dots z^{m_n}$ by $x^{m_i}=s^{-1}\log|z^{m_i}|$. The star-type region $U_w^{s,*}\subset X_s$ can be viewed as a subset of $(\mathbb{C}^*)^n$, so we use the rescaled map

$$s^{-1}\,\mathrm{Log}\colon (\mathbb{C}^*)^n_{z^{m_i}} \longrightarrow \mathbb{R}^n_{x^{m_i}}$$

to pull back the function $u_{\infty,m}$ on $\mathrm{Star}(w)$. On the other hand, $X_s\cap(\mathbb{C}^*)^{n+1}$ maps into $N_{\mathbb{R}}$ via Log_s, so we can also pull back $u_{\infty,m}$ via Log_s. These two pull-backs differ by at most Cs^{-1}, using Corollary 4.15. We also write $\phi=\varphi_{CY,s,m}$.

Take a local test function $f\in C_c^2$ supported in the interior of $\mathrm{Star}(w)$, then f is identified as a local function on $U_w^{s,*}\subset X_s$ via $s^{-1}\,\mathrm{Log}$. By the Chern–Levine-type estimate above,

$$s^n\int f\{(\sqrt{-1}\,\partial\bar{\partial}\phi)^n-(\sqrt{-1}\,\partial\bar{\partial}(u_{\infty,m}\circ s^{-1}\,\mathrm{Log}))^n\}$$
$$\leqslant C\|\phi-u_{\infty,m}\circ s^{-1}\,\mathrm{Log}\,\|_{L^\infty}\,\|f\|_{C^2}\to 0,$$

as $s\to\infty$. By the Calabi–Yau condition (3.13) and Proposition 3.14,

$$s^n(\sqrt{-1}\,\partial\bar{\partial}\phi)^n = a_s d\mu_s = a_\infty d\mu_s(1+o(1)) \quad \text{as } s\to\infty.$$

Pushing forward via $s^{-1}\,\mathrm{Log}$, and applying Lemma 5.2,

$$\pi^n n!\int f\,\mathrm{MA}(u_\infty) = \lim_{s\to\infty}\int f a_s(s^{-1}\,\mathrm{Log})_*\,d\mu_s = a_\infty\int f\,d\mu_\infty.$$

Since this holds for every $f\in C_c^2$, on the interior of this top-dimensional face we obtain the measure equality (5.1). □

5.2. Higher regularity in the generic region

Once we know the subsequential limit u_∞ satisfies the real MA equation, then by the local regularity theory surveyed in §2.6, we have the following.

Corollary 5.4. (Regularity of real MA solution) *Inside* $\partial\Delta_\lambda^\vee\setminus\mathrm{Sing}$, *let* $\mathcal{R}$ *be the set of strictly convex points of* u_∞. *Then,* $u_\infty\in C^\infty_{\mathrm{loc}}(\mathcal{R})$, *and the complement of* $\mathcal{R}$ *is a closed subset of Hausdorff* $(n-1)$*-measure zero. In particular,* $\mathcal{R}$ *is path connected, and is open and dense in* $\partial\Delta_\lambda^\vee\setminus\mathrm{Sing}$.

Remark 5.5. In dimension 2, the local regularity theory implies that $\mathcal{R}=\partial\Delta_\lambda^\vee\setminus\mathrm{Sing}$, namely the real MA solution is smooth wherever the affine structure is defined. The same might hold in any higher dimension, although this cannot be concluded by local regularity results alone (cf. Remark 2.15).

We now proceed to a very explicit coordinate version of higher-order estimates for the local CY potentials, by transferring regularity from the real MA equation to the complex MA equation.

Let $x\in\mathcal{R}$, then u_∞ (resp. the appropriate $u_{\infty,m}$) has $C^{k,\gamma}$-bound on some coordinate ball $B(x,2r(x))\subset\mathcal{R}$ contained in a shrunken face (resp. $\mathrm{Star}(w)$). For clarity, we focus on the face case. The radius $r(x)$ and the $C^{k,\gamma}$-bound depend on the choice of x, but are uniform for x in any fixed compact subset of $\mathcal{R}$. We identify u_∞ with its pull-back to

$$(s^{-1}\,\mathrm{Log})^{-1}(B(x,r(x)))\subset U_w^{s,\mathrm{face}}\subset X_s.$$

The local CY potential $\varphi_{CY,s,0}$ on $(s^{-1}\,\mathrm{Log})^{-1}(B(x,2r(x)))$ satisfies

$$\left\|\varphi_{CY,s,0}-u_\infty\circ s^{-1}\,\mathrm{Log}\right\|_{C^0}\to 0,\qquad s\to\infty$$

along the subsequence. We may regard

$$(s^{-1}\,\mathrm{Log})^{-1}(B(x,2r(x)))$$

as an open subset of $(\mathbb{C}^*)^n$. On the universal cover of $(\mathbb{C}^*)^n$, we use the natural coordinates $s^{-1}\log z^{m_i}$ for $i=1,\dots n$.

Now, $\varphi_{CY,s,0}$ satisfies the complex MA equation (cf. (3.13) and (3.7))

$$(\sqrt{-1}\,\partial\bar\partial\varphi_{CY,s,0})^n=a_s s^{-n}\,d\mu_s=\frac{a_s}{(4\pi s^2)^n}\sqrt{-1}^{n^2}\,\Omega_s\wedge\bar\Omega_s.$$

By the holomorphic volume form formula (3.5),

$$(\sqrt{-1}\,\partial\bar\partial\varphi_{CY,s,0})^n=\frac{a_s}{(4\pi)^n}(1+o(1))\prod_i\sqrt{-1}s^{-1}d\log z^{m_i}\wedge s^{-1}d\,\overline{\log z^{m_i}},$$

where the $o(1)$ term in fact has exponentially small C^∞ bounds in $s^{-1}\log z^{m_i}$ coordinates; the higher-order bound uses that Ω_s is holomorphic. On the other hand, by the calculation in §5.1, the pull-back of u_∞ satisfies

$$(\sqrt{-1}\,\partial\bar{\partial}(u_\infty \circ s^{-1}\operatorname{Log}))^n = \frac{a_\infty}{(4\pi)^n}\prod_i s^{-1}\sqrt{-1}d\log z^{m_i}\wedge s^{-1}d\,\overline{\log z^{m_i}}.$$

To summarize, the deviation of the right-hand side is negligible and the deviation between $\varphi_{CY,s,0}$ and $u_\infty \circ s^{-1}\operatorname{Log}$ is small in C^0-norm. Applying Savin's theorem (Theorem 2.14), we get the following.

Theorem 5.6. (Smooth convergence in generic regions) *As $s\to\infty$ along the subsequence, assume the coordinate ball $B(x, 2r(x))\subset\mathcal{R}$. Then, on the region*

$$(s^{-1}\operatorname{Log})^{-1}(B(x,r(x)))\subset X_s,$$

we have the following higher-regularity estimates with respect to the $C^{k,\gamma}$-norm in the $s^{-1}\log z^{m_i}$ coordinates.

- *In the face-type-region-$U_w^{s,\mathrm{face}}$ case*

$$\|\varphi_{CY,s,0} - u_\infty \circ s^{-1}\operatorname{Log}\|_{C^{k,\gamma}((s^{-1}\operatorname{Log})^{-1}(B(x,r(x))))} \to 0.$$

- *In the star-type-region-$U_w^{s,*}$ case, for $\langle m, w\rangle = 1$,*

$$\|\varphi_{CY,s,m} - u_{\infty,m} \circ s^{-1}\operatorname{Log}\|_{C^{k,\gamma}((s^{-1}\operatorname{Log})^{-1}(B(x,r(x))))} \to 0.$$

The convergence rate is uniform for x on any fixed compact subset of $\mathcal{R}$.

The intuition is that in the generic regular locus in the toric part of X_s, the local CY potentials converge in some C^∞_{loc} sense.

Notation. For every compact $K\subset\mathcal{R}$, let $U_{s,K}$ denote the union of the regions

$$(s^{-1}\operatorname{Log})^{-1}(B(x,r(x))) \quad \text{for } x\in K;$$

the convergence rates will be uniform on $U_{s,K}$. Notice that

$$\limsup_{s\to\infty}\frac{\operatorname{Vol}(U_{s,K})}{\operatorname{Vol}(X_s)} \geqslant \frac{\int_K d\mu_\infty}{\int_{\partial\Delta_X^\vee} d\mu_\infty},$$

so by taking a compact exhaustion of $\mathcal{R}$, we may assume that $U_{s,K}$ occupies a percentage of the total measure arbitrarily close to 1.

Remark 5.7. If one can show that the limiting real MA metric is unique, then there will be no need to pass to a subsequence.

Next, we discuss CY metrics in $(s^{-1}\operatorname{Log})^{-1}(B(x,r(x)))\subset U_{s,K}$.

- In the face-type-region case, up to C^∞-small error in the $s^{-1}\log z^{m_i}$ coordinates,

$$\begin{aligned}\omega_{CY,s} &= \sqrt{-1}\,\partial\bar\partial\varphi_{CY,s,0}\\ &\approx \sqrt{-1}\,\partial\bar\partial u_\infty\circ s^{-1}\operatorname{Log} = \frac{1}{4}\frac{\partial^2 u_\infty}{\partial x^{m_i}\partial x^{m_j}}\sqrt{-1}s^{-1}d\log z^{m_i}\wedge s^{-1}\,d\,\overline{\log z^{m_j}},\end{aligned}$$

and hence the CY metrics $g_{CY,s}$ is, up to C^∞-small error,

$$g_{CY,s}\approx \operatorname{Re}\left\{\frac{1}{2}\frac{\partial^2 u_\infty}{\partial x^{m_i}\partial x^{m_j}}s^{-1}d\log z^{m_i}\otimes s^{-1}\,d\,\overline{\log z^{m_j}}\right\}. \tag{5.2}$$

- Likewise in the star-type-region case, up to C^∞ small error in the $s^{-1}\log z^{m_i}$ coordinates,

$$\begin{cases}\omega_{CY,s}\approx \dfrac{1}{4}\dfrac{\partial^2 u_{\infty,m}}{\partial x^{m_i}\partial x^{m_j}}\sqrt{-1}s^{-1}d\log z^{m_i}\wedge s^{-1}\,d\,\overline{\log z^{m_j}},\\ g_{CY,s}\approx \operatorname{Re}\left\{\dfrac{1}{2}\dfrac{\partial^2 u_{\infty,m}}{\partial x^{m_i}\partial x^{m_j}}s^{-1}d\log z^{m_i}\otimes s^{-1}\,d\,\overline{\log z^{m_j}}\right\}.\end{cases} \tag{5.3}$$

Notice that, in such local $(\mathbb{C}^*)^n$ coordinates, the rescaled log map $s^{-1}\operatorname{Log}$ gives a local T^n-fibration. The metric associated with $\sqrt{-1}\,\partial\bar\partial(u_\infty\circ s^{-1}\operatorname{Log})$ is a *semiflat metric*, namely a T^n-invariant metric which is flat when restricted to any T^n-fibre. Thus, (5.2) and (5.3) assert that *the Calabi–Yau metrics $g_{CY,s}$ are C^∞-approximated by semiflat metrics in the regular regions.*

Corollary 5.8. *On $U_{s,K}\subset X_s$ the sectional curvature has a uniform bound*

$$|\operatorname{Riem}(g_{CY,s})|\leqslant C,$$

and the injectivity radius satisfies

$$C^{-1}s^{-1}\leqslant \operatorname{inj}\leqslant Cs^{-1},$$

with constants depending on $K\subset\mathcal{R}$.

5.3. Gromov–Hausdorff convergence

On the regular locus $\mathcal{R}\subset\partial\Delta_\lambda^\vee$, we have a well-defined real MA metric

$$g_\infty=\begin{cases}\dfrac{1}{2}\displaystyle\sum_{i,j}\frac{\partial^2 u_\infty}{\partial x_i\partial x_j}\,dx_i\,dx_j, & \text{on the face regions,}\\ \dfrac{1}{2}\displaystyle\sum_{i,j}\frac{\partial^2 u_{\infty,m}}{\partial x_i\partial x_j}\,dx_i\,dx_j, & \text{on the star regions.}\end{cases} \tag{5.4}$$

Notice the definitions are compatible on overlapping regions. The metric asymptotes (5.2)(5.3) say that in some C^∞_{loc} sense the collapsing CY metrics $g_{CY,s}$ converge to the metric g_∞ on $\mathcal{R}$, and we know $\mathcal{R}$ is path connected because its complement has zero $\mathcal{H}^{n-1}$-measure.

Remark 5.9. We do not know if the metric completion of $\mathcal{R}$ is homeomorphic to $\partial\Delta^\vee_\lambda \simeq S^n$, as the regularity theory of the real MA equation on a singular affine manifold is not yet developed, and we know little about what can happen near singularities.

The goal of this section is to show the following result.

THEOREM 5.10. *Any subsequential Gromov–Hausdorff limit of the collapsing* CY *metrics* $(X_s, g_{CY,s})$ *contains an open dense subset locally isometric to* $\mathcal{R}$.

Remark 5.11. Here, 'local isometry' means a diffeomorphism Ψ between two open sets U and U', such that around any point $x\in U$, the map Ψ preserves the distance $d(\Psi(x), \Psi(y)) = d(x,y)$ if $d(x,y) < r_x$ is sufficiently small depending on x. There is no claim about isometry for points separated by large distances.

PROPOSITION 5.12. *There is a uniform diameter bound*

$$\operatorname{diam}(X_s, g_{CY,s}) \leqslant C.$$

Proof. This argument is essentially the same as [39, Theorem 3.1]. We quote [39, Lemma 3.2].

LEMMA 5.13. *Let* (M^{2n}, g) *be a closed Riemannian manifold with* $\mathrm{Ric}(g)\geqslant 0$, *let* $p\in M$ *and* $1 < R \leqslant \operatorname{diam}(X,g)$. *Then,*

$$\frac{R-1}{4n} \leqslant \frac{\mathrm{Vol}(B(p, 2(R+1)))}{\mathrm{Vol}(B(p,1))}.$$

Using Theorem 5.6, we can find inside the regular region of X_s some geodesic ball $B_{g_{CY,s}}(p,r)$ of radius $r<1$, occupying a non-trivial portion of the total volume:

$$\frac{\mathrm{Vol}(B_{g_{CY,s}}(p,r)))}{\mathrm{Vol}(X_s)} \geqslant \varepsilon > 0,$$

with ε independent of s. Now, applying the lemma to the rescaled CY metric $r^{-2}g_{CY,s}$,

$$\frac{\operatorname{diam}(X_s)-r}{4nr} \leqslant \frac{\mathrm{Vol}(B_{g_{CY,s}}(p, 2(\operatorname{diam}(X_s)+r)))}{\mathrm{Vol}(B_{g_{CY,s}}(p,r))} \leqslant \frac{\mathrm{Vol}(X_s)}{\mathrm{Vol}(B_{g_{CY,s}}(p,r))} \leqslant \varepsilon^{-1},$$

so $\operatorname{diam}(X_s) \leqslant Cr \leqslant C$ as required. $\square$

Proof of Theorem 5.10. By Theorem 5.6, we already know the C^∞ metric convergence over any properly contained open subset of $\mathcal{R}$, which corresponds to a region $U_s \subset X_s$, with nearly the full measure

$$\mathrm{Vol}(U_s) > (1-\varepsilon)\,\mathrm{Vol}(X_s),$$

where ε can be chosen arbitrarily small. This gives rise to an open subset U in any Gromov–Hausdorff subsequential limit, locally isometric to $\mathcal{R}$. It now suffices to show any point $p \in X_s \setminus U_s$ is close to U_s; this would imply the density of U inside the Gromov–Hausdorff limit.

For any $r>0$ such that the geodesic ball $B_{g_{CY,s}}(p,r) \subset X_s \setminus U_s$, the Bishop–Gromov inequality implies

$$\left(\frac{r}{\mathrm{diam}(X_s)}\right)^{2n} \leqslant \frac{\mathrm{Vol}(B_{g_{CY,s}}(p,r))}{\mathrm{Vol}(X_s)} \leqslant \frac{\mathrm{Vol}(X_s \setminus U_s)}{\mathrm{Vol}(X_s)} < \varepsilon.$$

Taking the sup of all such r,

$$\mathrm{dist}_{g_{CY,s}}(p, U_s) \leqslant \varepsilon^{1/2n}\,\mathrm{diam}(X_s) \leqslant C\varepsilon^{1/2n},$$

which can be made arbitrarily small. $\square$

5.4. Special Lagrangian fibration in the generic region

In the setting of §5.2, the very strong regularity bounds in the generic region leads to the existence of special Lagrangian T^n-fibrations thereon.

THEOREM 5.14. *For any fixed compact $K \subset \mathcal{R}$, for $s \gg 1$ depending on K, there is a special Lagrangian* (SLag) *T^n-fibration on an open subset of X_s containing $U_{s,K}$.*

Remark 5.15. By considering a compact exhaustion of $\mathcal{R}$, we can choose K so that the region $U_{s,K}$ occupies a percentage of the total measure on X_s arbitrarily close to 1.

Proof. As K is a compact subset in the open set $\mathcal{R}$, we can find an open set $\mathcal{U} \subset K$ properly contained in $\mathcal{R}$. This ensures that the smooth convergence in Theorem 5.6 happens uniformly on a slightly larger set $U_{s,K'}$ than $U_{s,K}$. We assume $s \gg 1$ as ususal.

Consider a coordinate region $(s^{-1}\,\mathrm{Log})^{-1}(B(x,r(x))$ contained in this larger set, which is topologically $T^n \times B(x,r(x))$. Here, the T^n is well defined as a homology cycle independent of the coordinates. We define the phase angles θ_s by requiring

$$\int_{T^n} e^{\sqrt{-1}\theta_s}\Omega > 0.$$

We consider the rescaled CY metrics $(s^2 g_{CY,s}, s^2\omega_{CY,s})$, so the diameter of T^n fibres are now of order $O(1)$ by (5.2) and (5.3). Within any log scale, these rescaled CY structures are C^∞-close to the standard flat structures in §2.7 up to constant factors. By construction the Kähler forms are exact in these coordinate charts. Thus, by Zhang's result surveyed in §2.7, within any log scale, we can construct a SLag T^n-fibration with phase θ_s, whose fibres are very small C^∞-perturbations of the fibres of the map

$$\begin{aligned}\mathrm{Log}\colon (\mathbb{C}^*)^n &\longrightarrow \mathbb{R}^n,\\ (z^{m_1}, ..., z^{m_n}) &\longmapsto (\log|z^{m_1}|, ..., \log|z^{m_n}|).\end{aligned}$$

Observe that on overlapping charts, the Log-fibres with respect to one chart are very small C^∞-perturbations of the Log-fibres of the other chart. Then the uniqueness part of Zhang's argument shows that on overlapping charts the SLag T^n-fibrations are in fact defined independent of charts. (It is the local universal family of SLags within the perturbative regime.) Thus, the local constructions glue to a SLag fibration on a subset of X_s containing $U_{s,K}$, as required. □

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