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STRONG AND Δ-CONVERGENCE OF A FASTER ITERATION PROCESS IN HYPERBOLIC SPACE

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Abstract. In this article, we first give metric version of an iteration scheme of Agarwal et al. [1] and approximate fixed points of two finite families of nonexpansive mappings in hyperbolic spaces through this iteration scheme which is independent of but faster than Mann and Ishikawa scheme. Also we consider case of three finite families of nonexpansive mappings. But, we need an extra condition to get convergence. Our convergence theorems generalize and refine many know results in the current literature.

1. Introduction

Throughout the article, $\mathbb{N}$ denotes the set of positive integers and I denotes the set of first N natural numbers. Let (X, d) be a metric space and K be a nonempty subset of X. A selfmap T on K is said to be nonexpansive if $d(Tx, Ty) \leq d(x, y)$. Denote by $F(T)$ the set of fixed points of T and by $F = \cap_{i=1}^{N} (F(T_i) \cap F(S_i))$ the set of common fixed points of two finite families of mappings $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$.

We know that Mann and Ishikawa iteration processes are defined for given x_1 in K (a subset of Banach space) respectively as:

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T x_n, \tag{1}$$

and

$$\begin{cases} x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T y_n, \\ y_n = (1 - \beta_n) x_n + \beta_n T x_n, \quad n \in \mathbb{N}, \end{cases} \tag{2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0, 1)$.

Recently, Agarwal et al. [1] introduced the following iteration process:

$$\begin{cases} x_{n+1} = (1 - \alpha_n) T x_n + \alpha_n T y_n, \\ y_n = (1 - \beta_n) x_n + \beta_n T x_n, \quad n \in \mathbb{N}. \end{cases} \tag{3}$$

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They showed that this process converges at a rate same as that of Picard iteration and faster than Mann and Ishikawa iterations for contractions.

Obviously the above process deals with one mapping only. The case of two mappings in iterative processes has also remained under study since Das and Debata [3] gave and studied a two mappings process. Also see, for example, [13] and [25]. Note that two mappings case, that is, approximating the common fixed points, has its own importance as it has a direct link with the minimization problem, see for example [24].

In [11], Khan et al. modified the iteration process (3) to the case of two mappings as follows.

$$\begin{cases} x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n S y_n, \\ y_n = (1-\beta_n)\, x_n + \beta_n T x_n, \quad n \in \mathbb{N}, \end{cases} \tag{4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$.

It is to be noted that (4) reduces to (3) when $S = T$ and (1) if T is identity mapping.

The purpose of this article is to investigate Δ-convergence as well as strong convergence of algorithm (4) for two finite families of nonexpansive maps in the more general setup of hyperbolic spaces. At first glance, it looks like the process will also work for three families of nonexpansive mappings without any difficulty. However, this is not the case. We must impose an extra condition on the mappings. Our results can be viewed as refinement and generalization of several well-known results in CAT(0) and uniformly convex Banach spaces.

A hyperbolic space [16] is a triple (X, d, W) where (X, d) is a metric space and $W : X^2 \times [0,1] \to X$ is such that

W1. $d(u, W(x,y,\alpha)) \le (1-\alpha)\, d(u,x) + \alpha d(u,y)$
W2. $d(W(x,y,\alpha), W(x,y,\beta)) = |\alpha - \beta|\, d(x,y)$
W3. $W(x,y,\alpha) = W(y,x,(1-\alpha))$
W4. $d(W(x,z,\alpha), W(y,w,\alpha)) \le (1-\alpha)\, d(x,y) + \alpha d(z,w)$

for all $x, y, z, w \in X$ and $\alpha, \beta \in [0,1]$. If a hyperbolic space (X, d, W) satisfies only (W1), then it coincides with the convex metric space introduced by Takahashi [23]. A subset K of a hyperbolic space X is convex if $W(x,y,\alpha) \in K$ for all $x, y \in K$ and $\alpha \in [0,1]$.

Different notions of "hyperbolic space" [6, 7, 14, 15] can be found in the literature. We work in the setting of hyperbolic spaces as introduced by Kohlenbach [16], which are slightly more restrictive than the spaces of hyperbolic type [6] by (W4), but more general then the concept of hyperbolic space from [19]. Spaces like CAT(0) and Banach are special cases of hyperbolic space. The class of hyperbolic spaces also contains Hadamard manifolds, Hilbert ball equipped with the hyperbolic metric [7], $\mathbb{R}$-trees and Cartesian products of Hilbert balls, as special cases.

A hyperbolic space (X, d, W) is said to be uniformly convex [22] if for all $u, x, y \in X$, $r > 0$ and $\varepsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that

$$\left.\begin{array}{r} d(x,u) \leq r \\ d(y,u) \leq r \\ d(x,y) \geq \varepsilon r \end{array}\right\} \Rightarrow d\left(W\left(x, y, \frac{1}{2}\right), u\right) \leq (1-\delta)\, r.$$

A map $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ which provides such a $\delta = \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0, 2]$, is called modulus of uniform convexity. We call η monotone if it decreases with r (for a fixed ε).

The concept of Δ-convergence in a metric space was introduced by Lim [18] and its analogue in CAT(0) spaces has been investigated by Dhompongsa and Panyanak [4]. In [9], Khan et al. continued the investigation of Δ-convergence in the general setup of hyperbolic spaces. Later on, some authors discussed the convergence of the iterative process in hyperbolic spaces (see, for example, [5, 20]).

Now, we collect some basic concepts.

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space X. For $x \in X$, define a continuous functional $r(\cdot, \{x_n\}) : X \to [0, \infty)$ by:

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $\rho = r(\{x_n\})$ of $\{x_n\}$ is given by:

$$\rho = \inf \{r(x, \{x_n\}) : x \in X\}.$$

The asymptotic center of a bounded sequence $\{x_n\}$ with respect to a subset K of X is defined as follows:

$$A_K(\{x_n\}) = \{x \in X : r(x, \{x_n\}) \leq r(y, \{x_n\}) \text{ for any } y \in K\}.$$

If the asymptotic center is taken with respect to X, then it is simply denoted by $A(\{x_n\})$. It is known that uniformly convex Banach spaces and even CAT(0) spaces enjoy the property that "bounded sequences have unique asymptotic centers with respect to closed convex subsets". The following lemma is due to Leustean [17] and ensures that this property also holds in a complete uniformly convex hyperbolic space.

Lemma 1.1 ([17]). *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset K of X.*

Recall that a sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_n x_n = x$ and call x as Δ-limit of $\{x_n\}$.

Lemma 1.2 ([9]). *Let (X, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{a_n\}$ be a sequence in $[b, c]$ for some $b, c \in (0, 1)$. If $\{x_n\}$and $\{y_n\}$ are sequences in X such that*

$\limsup_{n\to\infty} d(x_n, x) \le r$, $\limsup_{n\to\infty} d(y_n, x) \le r$ *and* $\lim_{n\to\infty} d(W(x_n, y_n, \alpha_n), x) = r$ *for some* $r \ge 0$, *then* $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Lemma 1.3 ([9]). *Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space and $\{x_n\}$ a bounded sequence in K such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in K such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = \rho$, then $\lim_{m\to\infty} y_m = y$.*

2. Main results

2.1. The case of two finite families of nonexpansive mappings

In this section, we establish Δ-convergence and strong convergence of the algorithm (5).

The two-step algorithm (4) can be defined for two finite families of nonexpansive self-maps in a hyperbolic space as:

$$\begin{aligned} x_{n+1} &= W(T_n x_n, S_n y_n, \alpha_n), \\ y_n &= W(x_n, T_n x_n, \beta_n), \ n \ge 1, \end{aligned} \tag{5}$$

where $T_n = T_{n(\text{mod } N)}$ and $S_n = S_{n(\text{mod } N)}$.

Lemma 2.1. *Let K be a nonempty closed convex subset of a hyperbolic space X and let $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ be two finite families of nonexpansive selfmaps on K such that $F \neq \emptyset$. Then for the sequence $\{x_n\}$ defined in* (5)*, we have $\lim_{n\to\infty} d(x_n, p)$ exists for $p \in F$.*

Proof. For any $p \in F$, it follows from (5) that

$$\begin{aligned} d(x_{n+1}, p) &= d(W(T_n x_n, S_n y_n, \alpha_n), p) \\ &\le (1-\alpha_n) d(T_n x_n, p) + \alpha_n d(S_n y_n, p) \\ &\le (1-\alpha_n) d(x_n, p) + \alpha_n d(y_n, p) \\ &= (1-\alpha_n) d(x_n, p) + \alpha_n d(W(x_n, T_n x_n, \beta_n), p) \\ &\le (1-\alpha_n) d(x_n, p) + \alpha_n (1-\beta_n) d(x_n, p) + \alpha_n \beta_n d(T_n x_n, p) \\ &\le (1-\alpha_n) d(x_n, p) + \alpha_n (1-\beta_n) d(x_n, p) + \alpha_n \beta_n d(x_n, p) \\ &\le d(x_n, p). \end{aligned}$$

That is

$$d(x_{n+1}, p) \le d(x_n, p). \tag{6}$$

It follows from (6) that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$. Consequently, $\lim_{n\to\infty} d(x_n, p)$ exists. □

We give a key theorem for later use.

Lemma 2.2. *Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let*

$\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ be two finite families of nonexpansive selfmaps of K such that $F \neq \emptyset$. Then for the sequence $\{x_n\}$ defined in (5), we have

$$\lim_{n\to\infty} d(x_n, T_l x_n) = \lim_{n\to\infty} d(x_n, S_l x_n) = 0 \text{ for each } l = 1, 2, \ldots, N.$$

Proof. It follows from Lemma 2.1 that, $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$. Call it c. The case $c = 0$ is trivial. Next, we deal with the case $c > 0$. Now

$$\begin{aligned} d(y_n, p) &= d(W(x_n, T_n x_n, \beta_n), p) \\ &\leq (1 - \beta_n) d(x_n, p) +_n \beta_n d(T_n x_n, p) \\ &\leq d(x_n, p) \end{aligned}$$

implies that

$$\limsup_{n\to\infty} d(y_n, p) \leq c. \tag{7}$$

Also

$$d(T_n x_n, p) \leq d(x_n, p)$$

for all $n = 1, 2, \ldots$, so

$$\limsup_{n\to\infty} d(T_n x_n, p) \leq c. \tag{8}$$

Next,

$$d(S_n y_n, p) \leq d(y_n, p)$$

gives by (7) that

$$\limsup_{n\to\infty} d(S_n y_n, p) \leq c.$$

Moreover, $c = \lim_{n\to\infty} d(x_{n+1}, p) = \lim_{n\to\infty} d(W(T_n x_n, S_n y_n, \alpha_n), p)$ gives by Lemma 1.2,

$$\lim_{n\to\infty} d(T_n x_n, S_n y_n) = 0. \tag{9}$$

Now

$$\begin{aligned} d(x_{n+1}, p) &= d(W(T_n x_n, S_n y_n, \alpha_n), p) \\ &\leq (1 - \alpha_n) d(T_n x_n, p) + \alpha_n d(S_n y_n, p) \\ &\leq (1 - \alpha_n) d(T_n x_n, p) + \alpha_n d(S_n y_n, T_n x_n) + \alpha_n d(T_n x_n, p) \\ &\leq d(T_n x_n, p) + \alpha_n d(S_n y_n, T_n x_n) \end{aligned}$$

yields that

$$c \leq \liminf_{n\to\infty} d(T_n x_n, p)$$

so that (8) gives

$$\lim_{n\to\infty} d(T_n x_n, p) = c. \tag{10}$$

In turn,

$$\begin{aligned} d(T_n x_n, p) &\leq d(T_n x_n, S_n y_n) + d(S_n y_n, p) \\ &\leq d(T_n x_n, S_n y_n) + d(y_n, p) \end{aligned}$$

implies

$$c \le \liminf_{n\to\infty} d\left(y_n, p\right). \tag{11}$$

Thus $c = \lim_{n\to\infty} d\left(y_n, p\right) = \lim_{n\to\infty} d\left(W\left(x_n, T_n x_n, \beta_n\right), p\right)$ gives by Lemma 1.2 that

$$\lim_{n\to\infty} d\left(x_n, T_n x_n\right) = 0. \tag{12}$$

Now

$$\begin{aligned} d\left(y_n, x_n\right) &= d\left(W\left(x_n, T_n x_n, \beta_n\right), x_n\right) \\ &\le \beta_n d\left(T_n x_n, x_n\right) \end{aligned}$$

implies by (12) that

$$\lim_{n\to\infty} d\left(y_n, x_n\right) = 0. \tag{13}$$

Using (9), (12) and (13), we have

$$\begin{aligned} d\left(x_n, S_n x_n\right) &\le d\left(x_n, T_n x_n\right) + d\left(T_n x_n, S_n y_n\right) + d\left(S_n y_n, S_n x_n\right) \\ &\le d\left(x_n, T_n x_n\right) + d\left(T_n x_n, S_n y_n\right) + d\left(y_n, x_n\right) \end{aligned}$$

and so

$$\lim_{n\to\infty} d\left(x_n, S_n x_n\right) = 0. \tag{14}$$

Next,

$$\begin{aligned} d\left(x_{n+1}, x_n\right) &= d\left(W\left(T_n x_n, S_n y_n, \alpha_n\right), x_n\right) \\ &\le \alpha_n d\left(T_n x_n, x_n\right) + \left(1-\alpha_n\right) d\left(S_n y_n, x_n\right) \\ &\le \alpha_n d\left(T_n x_n, x_n\right) + \left(1-\alpha_n\right)\left(d\left(S_n y_n, T_n x_n\right) + d\left(T_n x_n, x_n\right)\right) \end{aligned}$$

gives by (9) and (12) that

$$\lim_{n\to\infty} d\left(x_{n+1}, x_n\right) = 0.$$

This implies that

$$\lim_{n\to\infty} d\left(x_{n+l}, x_n\right) = 0 \text{ for each } l \in I.$$

Further, observe that

$$\begin{aligned} d\left(x_n, T_{n+l} x_n\right) &\le d\left(x_n, x_{n+l}\right) + d\left(x_{n+l}, T_{n+l} x_{n+l}\right) + d\left(T_{n+l} x_{n+l}, T_{n+l} x_n\right) \\ &\le d\left(x_n, x_{n+l}\right) + d\left(x_{n+l}, T_{n+l} x_{n+l}\right) + d\left(x_{n+l}, x_n\right) \\ &\le 2d\left(x_n, x_{n+l}\right) + d\left(x_{n+l}, T_{n+l} x_{n+l}\right). \end{aligned}$$

Taking lim on both sides of the above inequality, we have

$$\lim_{n\to\infty} d\left(x_n, T_{n+l} x_n\right) = 0 \text{ for each } l \in I.$$

Since for each $l \in I$, the sequence $\{d(x_n, T_l x_n)\}$ is a subsequence of $\bigcup_{i=1}^{N} \{d(x_n, T_{n+l}x_n)\}$ and $\lim_{n\to\infty} d(x_n, T_{n+l}x_n) = 0$ for each $l \in I$, therefore

$$\lim_{n\to\infty} d(x_n, T_l x_n) = 0 \text{ for each } l \in I.$$

Similarly, we have

$$\lim_{n\to\infty} d(x_n, S_{n+l}x_n) = 0 \text{ for each } l \in I,$$

and hence

$$\lim_{n\to\infty} d(x_n, S_l x_n) = 0 \text{ for each } l \in I. \qquad \square$$

Theorem 2.3. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ be two finite families of nonexpansive selfmaps on K such that $F \neq \emptyset$. Then the sequence $\{x_n\}$ defined in (5), Δ-converges to a common fixed point of $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$.*

Proof. It follows from Lemma 2.1 that $\{x_n\}$ is bounded. Therefore by Lemma 1.1, $\{x_n\}$ has a unique asymptotic center, that is, $A(\{x_n\}) = \{x\}$. Assume that $\{u_n\}$ is any subsequence of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. Then by Lemma 2.2, we have $\lim_{n\to\infty} d(u_n, T_l u_n) = \lim_{n\to\infty} d(u_n, S_l u_n) = 0$ for each $l = 1, 2, \ldots, N$. We claim that u is the common fixed point of $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$.

Now, we define a sequence $\{v_m\}$ in K by $v_m = T_m u$ where $T_m = T_{m(\text{mod N})}$. On the other hand,

$$\begin{aligned} d(v_m, u_n) &\leq d(T_m u, T_m u_n) + d(T_m u_n, T_{m-1}u_n) + \cdots + d(T u_n, u_n) \\ &\leq d(u, u_n) + \sum_{i=1}^{m-1} d(u_n, T_i u_n). \end{aligned}$$

Therefore, we have

$$r(v_m, \{u_n\}) = \limsup_{n\to\infty} d(v_m, u_n) \leq \limsup_{n\to\infty} d(u, u_n) = r(u, \{u_n\}).$$

This implies that $|r(v_m, \{u_n\}) - r(u, \{u_n\})| \to 0$ as $m \to \infty$. By Lemma 1.3, we get $T_{m(\text{mod}N)} u = u$. Thus u is the common fixed point of $\{T_i : i \in I\}$. By the same argument, we can show that u is the common fixed point of $\{S_i : i \in I\}$. Therefore u is the common fixed point of $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$. Moreover, $\lim_{n\to\infty} d(x_n, u)$ exists by Lemma 2.1.

Assume $x \neq u$.By the uniqueness of asymptotic centers,

$$\begin{aligned} \limsup_{n\to\infty} d(u_n, u) &< \limsup_{n\to\infty} d(u_n, x) \\ &\leq \limsup_{n\to\infty} d(x_n, x) \\ &< \limsup_{n\to\infty} d(x_n, u) \\ &= \limsup_{n\to\infty} d(u_n, u), \end{aligned}$$

a contradiction. Thus $x = u$. Since $\{u_n\}$ is an arbitrary subsequence of $\{x_n\}$, therefore $A(\{u_n\}) = \{u\}$ for all subsequences $\{u_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$ Δ-converges to a common fixed point of $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$. □

We have the following corollaries.

Corollary 2.4. *Let X, K and $\{T_i : i \in I\}$ be taken as above theorem and $\{x_n\}$ be defined as*

$$x_{n+1} = W(T_n x_n, T_n y_n, \alpha_n), \quad y_n = W(x_n, T_n x_n, \beta_n), \ n \geq 1 \tag{15}$$

Δ-converges to a common fixed point of $\{T_i : i \in I\}$.

Corollary 2.5. *Let X and K be taken as above theorem and S and T be two nonexpansive mappings. Let $\{x_n\}$ be defined as*

$$x_{n+1} = W(Tx_n, Sy_n, \alpha_n), \quad y_n = W(x_n, Tx_n, \beta_n), \ n \geq 1 \tag{16}$$

Δ-converges to a common fixed point of $\{T_i : i \in I\}$.

Recall that a sequence $\{x_n\}$ in a metric space X is said to be Fejér monotone with respect to K (a subset of X) if $d(x_{n+1}, p) \leq d(x_n, p)$ for all $p \in K$ and for all $n \geq 1$. A map $T : K \to K$ is semi-compact if any bounded sequence $\{x_n\}$ satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$, has a convergent subsequence.

Two mappings $T, S : K \to K$ are said to satisfy condition (A') [12] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0, \infty)$ such that

$$\frac{1}{2}(d(x, Tx) + d(x, Sx)) \geq f(d(x, F))$$

for all $x \in K$, where $d(x, F) = \inf\{d(x, p) : p \in F := F(T) \cap F(S)\}$.

We can modify this definition for two finite families of mappings as follows. Let $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ be two finite families of nonexpansive selfmaps on K with $F \neq \emptyset$. Then the two families are said to satisfy condition (B) on K if

$$\max_{1 \leq l \leq N} \left\{ \frac{1}{2}(d(x, T_l x) + d(x, S_l x)) \right\} \geq f(d(x, F)) \text{ for all } x \in K.$$

For further development, we need the following technical result.

Lemma 2.6 ([2]). *Let K be a nonempty closed subset of a complete metric space (X, d) and $\{x_n\}$ be Fejér monotone with respect to K. Then $\{x_n\}$ converges to some $p \in K$ if and only if $\lim_{n\to\infty} d(x_n, K) = 0$.*

Now we prove our strong convergence theorems as follows:

Theorem 2.7. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ be two finite families of nonexpansive selfmaps on K such that $F \neq \emptyset$. Suppose that $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ satisfy condition (B). Then the sequence $\{x_n\}$ defined in* (5) *converges strongly to $p \in F$.*

Proof. By Lemma 2.1, $\lim_{n\to\infty} d(x_n, F)$ exists for all $p \in F$. Also, by Lemma 2.2, $\lim_{n\to\infty} d(x_n, T_l x_n) = \lim_{n\to\infty} d(x_n, S_l x_n) = 0$ for each $l \in I$. It follows from condition (B) that $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is nondecreasing with $f(0) = 0$, it follows that $\lim_{n\to\infty} d(x_n, F) = 0$. By (6), the sequence $\{x_n\}$ is Fejér monotone with respect to F. Therefore, Lemma 2.6 implies that $\{x_n\}$ converges strongly to a point p in F. □

Note that the Condition (B) is weaker than both the compactness of K and the semicompactness of the nonexpansive mappings $\{T_i : i\in I\}$ and $\{S_i : i\in I\}$, (see Senter and Dotson [21]) therefore we already have the following result.

Theorem 2.8. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ be two finite families of nonexpansive selfmaps on K such that $F \neq \emptyset$. Suppose that either K is compact or one of the map in $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ is semi-compact. Then the sequence $\{x_n\}$ defined in* (5) *converges strongly to $p \in F$.*

2.2. The case of three finite families of nonexpansive mappings

We can generalize (5) to the case of three finite families of nonexpansive mappings as follows.

$$x_{n+1} = W(T_n x_n, S_n y_n, \alpha_n),$$
$$y_n = W(x_n, Q_n x_n, \beta_n), \ n \geq 1,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$.This iteration is reduced to (5) (and hence (3), (4), (15), (16)). Moreover, it is reduced to the Ishikawa iteration process.

We can prove all the theorems of this paper with this process using an extra condition as : $d(x_n, S_i x_n) \leq d(T_i x_n, S_i x_n)$. The mapping Q have an important role because we have to have $\lim_{n\to\infty} d(x_n, Q_i x_n)$ to reach $d(T_i x_n, S_i x_n) = 0$, and, in turn, $\lim_{n\to\infty} d(x_n, T_i x_n) = \lim_{n\to\infty} d(x_n, S_i x_n) = 0$. Above condition is satisfied by the nonexpansive mappings $S, T : \mathbb{R} \to \mathbb{R}$ defined as $Sx = \frac{2x+1}{4}$, $Tx = 1 - x$ for all $x \in \mathbb{R}$.

Remark 2.9. Our results generalize the corresponding results Khan and Abbas [10] in two ways: (i) from one nonexpansive mapping to two finite families of nonexpansive mappings. (ii) from CAT(0) spaces to general setup of hyperbolic spaces.

Remark 2.10. Since our iteration process is faster than Mann iteration process, our result better than correspond results of Gunduz and Akbulut [8].

Remark 2.11. Theorems of this paper can also be proved with error terms using following proces:

$$x_{n+1} = W\left(T_n x_n, W\left(S_n y_n, u_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right),$$
$$y_n = W\left(x_n, W\left(T_n x_n, v_n \frac{\hat{\beta}_n}{1-\hat{\alpha}_n}\right) \hat{\alpha}_n\right),\ n \geq 1,$$

where $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in X and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\hat{\alpha}_n\}$, $\{\hat{\beta}_n\}$ are four sequences in $[0,1]$ such that $\alpha_n + \beta_n = 1 = \hat{\alpha}_n + \hat{\beta}_n$ for $n \in \mathbb{N}$.

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