Strong and weak coupling of eigenvalues of complex matrices

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Abstract— **The paper presents a general theory of coupling of eigenvalues of complex matrices of arbitrary dimension smoothly depending on real parameters. The cases of weak and strong coupling are distinguished and their geometric interpretation in two and three-dimensional spaces is given. General asymptotic formulae for eigenvalue surfaces near diabolic and exceptional points are presented demonstrating crossing and avoided crossing scenarios. Two numerical examples from crystal optics illustrate effectiveness and accuracy of the presented theory.**

I. Introduction

Behavior of eigenvalues of matrices dependent on parameters is a problem of general interest having many important applications in natural and engineering sciences. In modern physics, e.g. quantum mechanics, crystal optics, physical chemistry, acoustics and mechanics, multiple eigenvalues in matrix spectra associated with specific effects attract great interest of researchers since the papers [1], [2]. In recent papers, see e.g. [3]–[6], two important cases are distinguished: the diabolic points (DPs) and the exceptional points (EPs). From mathematical point of view DP is a point where the eigenvalues coalesce, while corresponding eigenvectors remain different; and EP is a point where both eigenvalues and eigenvectors merge forming a Jordan block. Both the DP and EP cases are interesting in applications and were observed in experiments [6], [7].

In this paper we present a general theory of coupling of eigenvalues of complex matrices of arbitrary dimension smoothly depending on multiple real parameters. Two essential cases of weak and strong coupling based on a Jordan form of the system matrix are distinguished. These two cases correspond to diabolic and exceptional points, respectively. We derive general formulae describing coupling and decoupling of eigenvalues, crossing and avoided crossing of eigenvalue surfaces. It is emphasized that the presented theory of coupling of eigenvalues of complex matrices gives not only qualitative, but also quantitative results on behavior of eigenvalues based only on the information taken at the singular points. The paper is based on the author's previous research on interaction of eigenvalues for matrices and differential operators depending on multiple parameters [8]–[11]; for more references see the recent book [12].

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II. Strong coupling of eigenvalues

Let us consider the eigenvalue problem

$$\mathbf{A}\mathbf{u} = \lambda\mathbf{u} \tag{1}$$

for a general $m \times m$ complex matrix $\mathbf{A}$ smoothly depending on a vector of n real parameters $\mathbf{p} = (p_1, \ldots, p_n)$. Assume that, at $\mathbf{p} = \mathbf{p}_0$, two eigenvalues coalesce, i.e., the matrix $\mathbf{A}_0 = \mathbf{A}(\mathbf{p}_0)$ has an eigenvalue λ_0 of algebraic multiplicity 2. This eigenvalue can have one or two linearly independent eigenvectors $\mathbf{u}$, which determine the geometric multiplicity.

Let us consider a double λ_0 possessing a single eigenvector $\mathbf{u}_0$. This case corresponds to the exceptional point. An associated vector $\mathbf{u}_1$ given by

$$\mathbf{A}_0\mathbf{u}_1 = \lambda_0\mathbf{u}_1 + \mathbf{u}_0 \tag{2}$$

is the second vector of the invariant subspace corresponding to λ_0. An eigenvector $\mathbf{v}_0$ and an associated vector $\mathbf{v}_1$ corresponding to the complex conjugate eigenvalue $\overline{\lambda}_0$ of the adjoint matrix (Hermitian transpose) $\mathbf{A}^*$ are determined by

$$\begin{gathered}\mathbf{A}_0^*\mathbf{v}_0 = \overline{\lambda}_0\mathbf{v}_0,\ \ \mathbf{A}_0^*\mathbf{v}_1 = \overline{\lambda}_0\mathbf{v}_1 + \mathbf{v}_0,\\ (\mathbf{u}_1, \mathbf{v}_0) = 1,\ \ (\mathbf{u}_1, \mathbf{v}_1) = 0,\end{gathered} \tag{3}$$

where $(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^n u_i \overline{v}_i$ denotes the Hermitian inner product. The last two equations in (3) are the normalization conditions determining $\mathbf{v}_0$ and $\mathbf{v}_1$ uniquely for a given $\mathbf{u}_1$.

Let us introduce real n-dimensional vectors $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$, $\mathbf{r}$ with the components

$$\begin{gathered}f_s = \mathrm{Re}\left(\frac{\partial\mathbf{A}}{\partial p_s}\mathbf{u}_0, \mathbf{v}_0\right),\ \ g_s = \mathrm{Im}\left(\frac{\partial\mathbf{A}}{\partial p_s}\mathbf{u}_0, \mathbf{v}_0\right),\\ h_s = \mathrm{Re}\left(\left(\frac{\partial\mathbf{A}}{\partial p_s}\mathbf{u}_0, \mathbf{v}_1\right) + \left(\frac{\partial\mathbf{A}}{\partial p_s}\mathbf{u}_1, \mathbf{v}_0\right)\right),\\ r_s = \mathrm{Im}\left(\left(\frac{\partial\mathbf{A}}{\partial p_s}\mathbf{u}_0, \mathbf{v}_1\right) + \left(\frac{\partial\mathbf{A}}{\partial p_s}\mathbf{u}_1, \mathbf{v}_0\right)\right),\\ s = 1, \ldots, n.\end{gathered} \tag{4}$$

Then the bifurcation of λ_0 into a pair of simple eigenvalues λ_+ and λ_- under the perturbation of the parameter vector $\mathbf{p} = \mathbf{p}_0 + \Delta\mathbf{p}$ is described by the asymptotic formula [12]

$$\begin{aligned}\lambda_\pm &= \lambda_0 \pm \sqrt{\langle\mathbf{f}, \Delta\mathbf{p}\rangle + i\langle\mathbf{g}, \Delta\mathbf{p}\rangle}\\ &\quad + (\langle\mathbf{h}, \Delta\mathbf{p}\rangle + i\langle\mathbf{r}, \Delta\mathbf{p}\rangle)/2,\end{aligned} \tag{5}$$

where the angular brackets denote inner product of real vectors (terms of order $o(\|\Delta\mathbf{p}\|)$ are neglected inside and outside the square root). The corresponding eigenvectors are given by the asymptotic formula

$$\mathbf{u}_\pm = \mathbf{u}_0 \pm \sqrt{\langle\mathbf{f}, \Delta\mathbf{p}\rangle + i\langle\mathbf{g}, \Delta\mathbf{p}\rangle}\,\mathbf{u}_1. \tag{6}$$

One can see that both the eigenvalues and eigenvectors coalesce at the exceptional point. We call such a coupling of eigenvalues *strong*.

Expressing real and imaginary parts of the eigenvalues $\lambda_\pm$ from formula (5), we find

$$\begin{aligned}\mathrm{Re}\lambda_\pm &= \lambda_0 + \langle \mathbf{h}, \Delta\mathbf{p}\rangle/2 \\ &\pm \sqrt{\left(\sqrt{\langle \mathbf{f}, \Delta\mathbf{p}\rangle^2 + \langle \mathbf{g}, \Delta\mathbf{p}\rangle^2} + \langle \mathbf{f}, \Delta\mathbf{p}\rangle\right)/2},\end{aligned} \tag{7}$$

$$\begin{aligned}\mathrm{Im}\lambda_\pm &= \lambda_0 + \langle \mathbf{r}, \Delta\mathbf{p}\rangle/2 \\ &\pm \sqrt{\left(\sqrt{\langle \mathbf{f}, \Delta\mathbf{p}\rangle^2 + \langle \mathbf{g}, \Delta\mathbf{p}\rangle^2} - \langle \mathbf{f}, \Delta\mathbf{p}\rangle\right)/2}.\end{aligned} \tag{8}$$

According to equation (5), the eigenvalue remains double if $\langle \mathbf{f}, \Delta\mathbf{p}\rangle = \langle \mathbf{g}, \Delta\mathbf{p}\rangle = 0$. Thus, the double complex eigenvalue with a single eigenvector has codimension 2 and appears at points of the surface of dimension $n-2$ in the space of n parameters [13].

Let us study behavior of the eigenvalues λ_+ and λ_- depending on one parameter, say p_1, when the other parameters $p_2, \ldots, p_n$ are fixed in the neighborhood of $\mathbf{p}_0$. We assume that $f_1^2 + g_1^2 \neq 0$, which is the nondegeneracy condition. Separating real and imaginary parts in (5) and isolating the increment Δp_1 in one of the equations, we get

$$g_1(\mathrm{Re}\Delta\lambda)^2 - 2f_1 \mathrm{Re}\Delta\lambda\, \mathrm{Im}\Delta\lambda - g_1(\mathrm{Im}\Delta\lambda)^2 = \gamma, \tag{9}$$

where $\Delta\lambda = \lambda_\pm - \lambda_0$ and $\gamma = \sum_{s=2}^n (f_s g_1 - f_1 g_s)\Delta p_s$ is a small real constant.

If $\Delta p_j = 0$, $j = 2, \ldots, n$, or if they are nonzero but satisfy the equality $\gamma = 0$, then equation (9) yields two perpendicular lines intersecting at the point λ_0 of the complex plane. Due to variation of the parameter p_1 two eigenvalues $\lambda_\pm$ approach along one of the lines, merge to λ_0 at $\Delta p_1 = 0$, and then diverge along the other line; see Figure 1b, where the arrows show motion of eigenvalues with a monotonous change of p_1. The real and imaginary parts of the eigenvalues $\lambda_\pm$ cross at $p_1 = p_1^0$ forming the double cusps.

If $\gamma \neq 0$, then equation (9) defines hyperbolae in the complex plane. As Δp_1 changes monotonously, two eigenvalues λ_+ and λ_- moving along different branches of hyperbola come closer, turn and diverge; see Figure 1a,c. Note that for a small γ the eigenvalues $\lambda_\pm$ come arbitrarily close to each other without coupling that means *avoided crossing*. When γ changes the sign, the quadrants containing hyperbola branches are changed to the adjacent. Either real parts of the eigenvalues $\lambda_\pm$ cross due to variation of p_1 while the imaginary parts avoid crossing or vice-versa, as shown in Figure 1a,c. By using (7), (8) we find that the crossings occur at $p_1^\times = p_1^0 - \sum_{s=2}^n (g_s/g_1)\Delta p_s$ and

$$\mathrm{Re}\lambda_h = \mathrm{Re}\lambda_0 - \tfrac{1}{2g_1}\textstyle\sum_{s=2}^n (h_1 g_s - g_1 h_s)\Delta p_s,$$

$$\mathrm{Im}\lambda_r = \mathrm{Im}\lambda_0 - \tfrac{1}{2g_1}\textstyle\sum_{s=2}^n (r_1 g_s - g_1 r_s)\Delta p_s.$$

If the vector of parameters consists of only two components $\mathbf{p} = (p_1, p_2)$, then in the vicinity of the point $\mathbf{p}_0$, corresponding to the double eigenvalue λ_0, the eigenvalue surfaces (7) and (8) have the form of the well-known Whitney umbrella; see Figure 2.

III. Weak coupling of eigenvalues

Let us consider the case when λ_0 is a double eigenvalue of the matrix $\mathbf{A}_0 = \mathbf{A}(\mathbf{p}_0)$ with two eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$. This coupling point is known as a diabolic point. Since the eigenvectors do not coincide at $\mathbf{p}_0$, we call such a coupling of eigenvalues *weak*.

Let us denote by $\mathbf{v}_1$ and $\mathbf{v}_2$ two eigenvectors of the complex conjugate eigenvalue $\overline{\lambda}_0$ for the matrix $\mathbf{A}_0^*$ satisfying the normalization conditions $(\mathbf{u}_1, \mathbf{v}_1) = (\mathbf{u}_2, \mathbf{v}_2) = 1$, $(\mathbf{u}_1, \mathbf{v}_2) = (\mathbf{u}_2, \mathbf{v}_1) = 0$. These conditions define the unique vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ for given $\mathbf{u}_1$ and $\mathbf{u}_2$. Under perturbation of parameters $\mathbf{p} = \mathbf{p}_0 + \Delta\mathbf{p}$, the bifurcation of λ_0 into two simple eigenvalues λ_+ and λ_- and corresponding eigenvectors are described by the asymptotic formulae [12]

$$\lambda_\pm = \lambda_0 + \Delta\lambda_\pm, \quad \mathbf{u}_\pm = \alpha_\pm \mathbf{u}_1 + \beta_\pm \mathbf{u}_2. \tag{10}$$

The quantities $\Delta\lambda_\pm$, $\alpha_\pm$, and $\beta_\pm$ are found from the 2×2 eigenvalue problem

$$\begin{pmatrix} \langle \mathbf{g}_{11}, \Delta\mathbf{p}\rangle & \langle \mathbf{g}_{12}, \Delta\mathbf{p}\rangle \\ \langle \mathbf{g}_{21}, \Delta\mathbf{p}\rangle & \langle \mathbf{g}_{22}, \Delta\mathbf{p}\rangle \end{pmatrix} \begin{pmatrix} \alpha_\pm \\ \beta_\pm \end{pmatrix} = \Delta\lambda_\pm \begin{pmatrix} \alpha_\pm \\ \beta_\pm \end{pmatrix}, \tag{11}$$

where $\mathbf{d}_{ij} = (d_{ij}^1, \ldots, d_{ij}^n)$ is a complex vector with the components

$$d_{ij}^k = \left(\frac{\partial \mathbf{A}}{\partial p_k}\mathbf{u}_i, \mathbf{v}_j\right), \tag{12}$$

and $\langle \mathbf{d}_{ij}, \Delta\mathbf{p}\rangle = \langle \mathrm{Re}\,\mathbf{d}_{ij}, \Delta\mathbf{p}\rangle + i\langle \mathrm{Im}\,\mathbf{d}_{ij}, \Delta\mathbf{p}\rangle$.

By using (10), (11), we find the expressions for real and imaginary parts of the increments $\Delta\lambda_\pm$ as

$$\mathrm{Re}\,\Delta\lambda_\pm = \frac{\mathrm{Re}\,\langle \mathbf{d}_{11} + \mathbf{d}_{22}, \Delta\mathbf{p}\rangle}{2} \pm \sqrt{\frac{|c| + \mathrm{Re}\,c}{2}}, \tag{13}$$

$$\mathrm{Im}\,\Delta\lambda_\pm = \frac{\mathrm{Im}\,\langle \mathbf{d}_{11} + \mathbf{d}_{22}, \Delta\mathbf{p}\rangle}{2} \pm \sqrt{\frac{|c| - \mathrm{Re}\,c}{2}}, \tag{14}$$

where

$$c = \frac{\langle \mathbf{d}_{11} - \mathbf{d}_{22}, \Delta\mathbf{p}\rangle^2}{4} + \langle \mathbf{d}_{12}, \Delta\mathbf{p}\rangle\langle \mathbf{d}_{21}, \Delta\mathbf{p}\rangle. \tag{15}$$

The eigenvalue remains double under perturbation of parameters ($\lambda_+ = \lambda_-$) if $\mathrm{Re}\,c = \mathrm{Im}\,c = 0$. In general, the perturbed double eigenvalue $\lambda_+ = \lambda_-$ possesses a single eigenvector $\mathbf{u}_+ = \mathbf{u}_-$, i.e., the weak coupling becomes strong due to perturbation [12]. The perturbed double eigenvalue has two eigenvectors only when the matrix in the left-hand side of (11) is proportional to the identity matrix, i.e., $\langle \mathbf{d}_{11}, \Delta\mathbf{p}\rangle = \langle \mathbf{d}_{22}, \Delta\mathbf{p}\rangle$ and $\langle \mathbf{d}_{12}, \Delta\mathbf{p}\rangle = \langle \mathbf{d}_{21}, \Delta\mathbf{p}\rangle = 0$. These conditions represent six independent equations taken for real and imaginary parts. Thus, weak coupling of eigenvalues is a phenomenon of codimension 6 [13], [14]. We remark that some symmetries may decrease this codimension, e.g., the codimension is 3 for Hermitian matrices [1]. Another interesting case encountered in physical applications corresponds to a complex non-Hermitian perturbation of a symmetric two-parameter real matrix, when the eigenvalue surfaces have coffee-filter singularity [3], [4], [14]. A general theory of this phenomenon is given in [15].

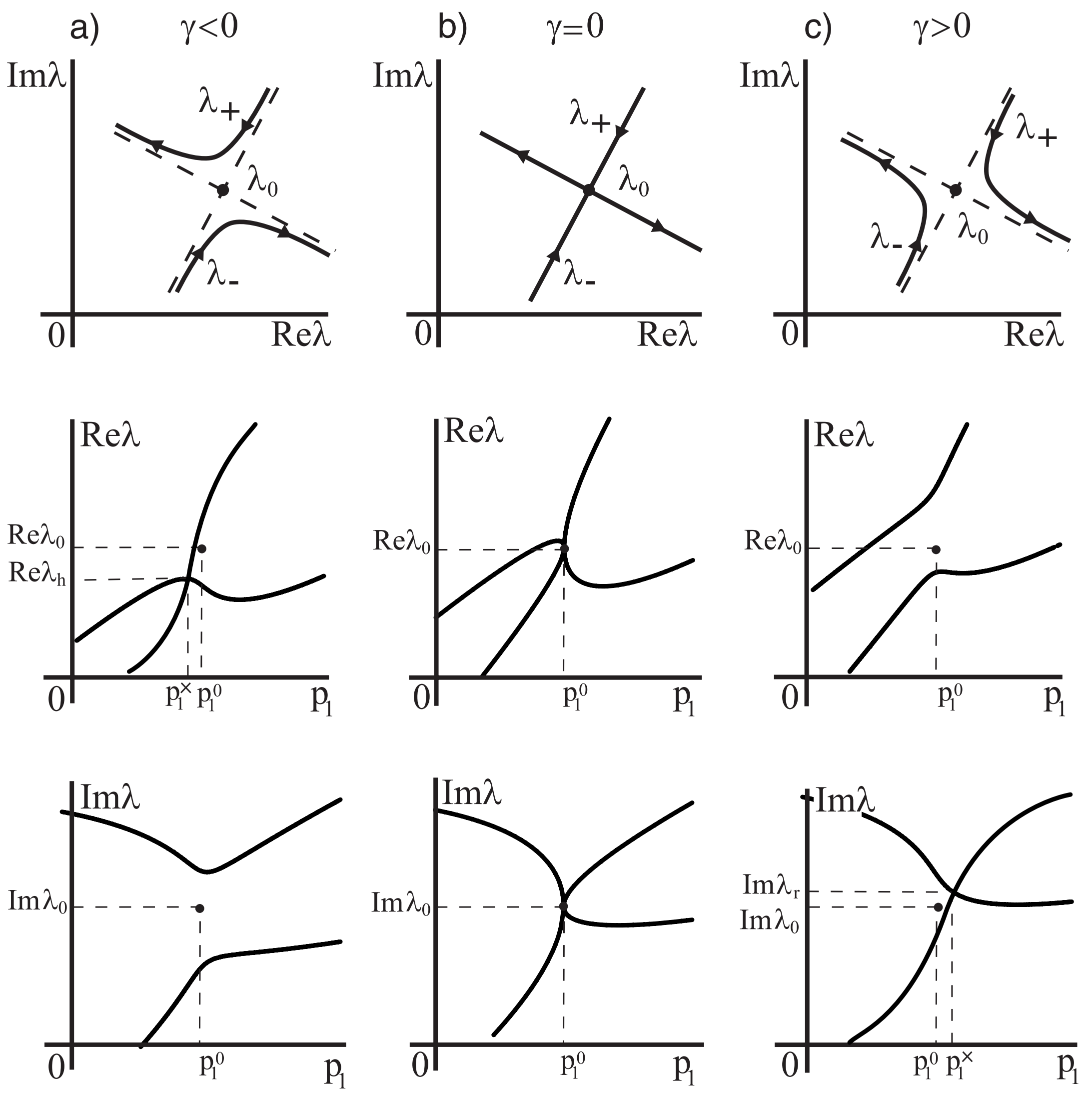

Fig. 1. Strong coupling of eigenvalues and avoided crossing.

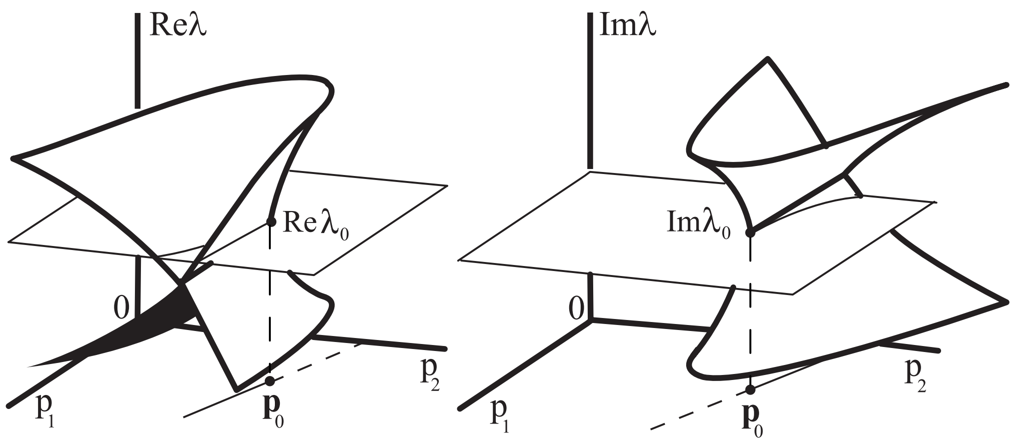

Fig. 2. Crossing of eigenvalue surfaces near a point of strong interaction.

First, let us study behavior of the eigenvalues λ_+ and λ_- depending on one parameter, say p_1, when the other parameters $p_2, \dots, p_n$ are fixed in the neighborhood of $\mathbf{p}_0$. In case $\Delta p_2 = \cdots = \Delta p_n = 0$, expressions (10) and (11) yield

$$\Delta\lambda_\pm = \Big((d_{11}^1 + d_{22}^1)/2 \pm\sqrt{(d_{11}^1 - d_{22}^1)^2/4 + d_{12}^1 d_{21}^1}\Big)\Delta p_1. \tag{16}$$

The eigenvalues λ_+ and λ_- are smooth functions of one parameter at the coupling point $\Delta p_1 = 0$, see Figure 3a. The corresponding eigenvectors $\mathbf{u}_+$ and $\mathbf{u}_-$ remain different (linearly independent) at all values of ε including their limits at the point $\mathbf{p}_0$.

If the perturbations $\Delta p_2, \dots, \Delta p_n$ are nonzero, the avoided crossing of the eigenvalues $\lambda_\pm$ with a change of p_1 is a typical scenario. We can distinguish different cases by checking intersections of real and imaginary parts of λ_+ and λ_-. By using (13), we find that $\mathrm{Re}\,\lambda_+ = \mathrm{Re}\,\lambda_-$ if $\mathrm{Im}\,c = 0$, $\mathrm{Re}\,c < 0$. Analogously, from (14) it follows that $\mathrm{Im}\,\lambda_+ = \mathrm{Im}\,\lambda_-$ if $\mathrm{Im}\,c = 0$, $\mathrm{Re}\,c > 0$. Let us write expression (15) in the form $c = c_0 + c_1\Delta p_1 + c_2(\Delta p_1)^2$, where the coefficients c_0, c_1, and c_2 are expressed in terms of components of the vectors $\mathbf{d}_{ij}$ and the fixed perturbations $\Delta p_2, \dots, \Delta p_n$. If the discriminant $D = (\mathrm{Im}\,c_1)^2 - 4\mathrm{Im}\,c_0 \mathrm{Im}\,c_2 > 0$, the equation $\mathrm{Im}\,c = 0$ yields two solutions

$$\Delta p_1^a = \frac{-\mathrm{Im}\,c_1 - \sqrt{D}}{2\mathrm{Im}\,c_2}, \quad \Delta p_1^b = \frac{-\mathrm{Im}\,c_1 + \sqrt{D}}{2\mathrm{Im}\,c_2}. \tag{17}$$

There are no real solutions if $D < 0$, and the single solution corresponds to the degenerate case $D = 0$. At the points $p_1^a = p_1^0 + \Delta p_1^a$ and $p_1^b = p_1^0 + \Delta p_1^b$ the values of c are real, and we denote them by c_a and c_b, respectively. The sign of $c_{a,b}$ determines whether the real or imaginary parts of $\lambda_\pm$ coincide at $p_1^{a,b}$.

In the nondegenerate case $D \neq 0$, there are four types of avoided crossing shown in Figure 3b–e. These cases are classified according to the number of intersection for real and imaginary parts of the eigenvalues $\lambda_\pm$, and are distinguished by the signs of the quantities D, c_a, and c_b.

Consider a system depending on two parameters p_1 and p_2. Let us write expression (15) in the form $c = c_{11}(\Delta p_1)^2 + c_{12}\Delta p_1 \Delta p_2 + c_{22}(\Delta p_2)^2$, where

$$\begin{aligned} c_{11} &= (d_{11}^1 - d_{22}^1)^2/4 + d_{12}^1 d_{21}^1, \\ c_{12} &= (d_{11}^1 - d_{22}^1)(d_{11}^2 - d_{22}^2)/2 + d_{12}^1 d_{21}^2 + d_{12}^2 d_{21}^1, \\ c_{22} &= (d_{11}^2 - d_{22}^2)^2/4 + d_{12}^2 d_{21}^2. \end{aligned}$$

If the discriminant $D' = (\mathrm{Im}\,c_{12})^2 - 4\mathrm{Im}\,c_{11}\mathrm{Im}\,c_{22} > 0$, the equation $\mathrm{Im}\,c = 0$ yields the two crossing lines

$$\begin{aligned} l_a: &\quad 2\mathrm{Im}\,c_{11}\Delta p_1 + (\mathrm{Im}\,c_{12} + \sqrt{D'})\Delta p_2 = 0, \\ l_b: &\quad 2\mathrm{Im}\,c_{11}\Delta p_1 + (\mathrm{Im}\,c_{12} - \sqrt{D'})\Delta p_2 = 0. \end{aligned} \tag{18}$$

There are no real solutions if $D' < 0$, and the lines l_a and l_b coincide in the degenerate case $D' = 0$. At points of the lines $l_{a,b}$ the values of c are real numbers of the same sign; we denote $\gamma_a = \mathrm{sign}\,c$ for the line l_a, and $\gamma_b = \mathrm{sign}\,c$ for the line l_b.

It follows from (13) and (14) that the real or imaginary parts of $\lambda_\pm$ coincide at $l_{a,b}$ for negative or positive $\gamma_{a,b}$, respectively. According to the signs of D', γ_a, and γ_b, we distinguish four types of the graphs for $\mathrm{Re}\,\lambda_\pm(p_1, p_2)$ and $\mathrm{Im}\,\lambda_\pm(p_1, p_2)$ as shown in Figure 4. The singularities of these surfaces include cones, intersecting surfaces, and "clusters of shells".

IV. Example

Consider propagation of light in a homogeneous non-magnetic crystal in the general case when the crystal possesses natural optical activity (chirality) and dichroism (absorption) in addition to biaxial birefringence. The optical properties of the crystal are characterized by the inverse dielectric tensor $\boldsymbol{\eta}$. The vectors of electric field $\mathbf{E}$ and displacement $\mathbf{D}$ are related as $\mathbf{E} = \boldsymbol{\eta}\mathbf{D}$. A monochromatic plane wave of frequency ω that propagates in a direction specified by a real unit vector $\mathbf{s} = (s_1, s_2, s_3)$ has the form

$$\begin{aligned} \mathbf{D}(\mathbf{r}, t) &= \mathbf{D}(\mathbf{s}) \exp i\omega\Big(\tfrac{n(\mathbf{s})}{c}\mathbf{s}^T\mathbf{r} - t\Big), \\ \mathbf{H}(\mathbf{r}, t) &= \mathbf{H}(\mathbf{s}) \exp i\omega\Big(\tfrac{n(\mathbf{s})}{c}\mathbf{s}^T\mathbf{r} - t\Big), \end{aligned} \tag{19}$$

where $n(\mathbf{s})$ is a refractive index, and $\mathbf{r} = (x_1, x_2, x_3)$ is the real vector of spatial coordinates. Substituting the wave (19) into Maxwell's equations, we find [3], [16]

$$\boldsymbol{\eta}\mathbf{D}(\mathbf{s}) - \mathbf{s}(\mathbf{s}^T\boldsymbol{\eta}\mathbf{D}(\mathbf{s})) = \frac{1}{n^2(\mathbf{s})}\mathbf{D}(\mathbf{s}). \tag{20}$$

Equation (20) can be written in the form of an eigenvalue problem for the complex non-Hermitian matrix $\mathbf{A}(\mathbf{s}) = (\mathbf{I} - \mathbf{s}\mathbf{s}^T)\boldsymbol{\eta}(\mathbf{s})$ dependent on the vector of parameters $\mathbf{s} = (s_1, s_2, s_3)$, where $\lambda = n^{-2}$, $\mathbf{u} = \mathbf{D}$, and $\mathbf{I}$ is the identity matrix. Notice that one of the eigenvalues of the matrix $\mathbf{A}$ is always zero.

As a numerical example, we choose the inverse dielectric tensor in the form

$$\boldsymbol{\eta} = \begin{pmatrix} 3 & i & 2i \\ i & 1 & 0 \\ 2i & 0 & 2 \end{pmatrix} + i\begin{pmatrix} 0 & -s_1 & 0 \\ s_1 & 0 & -s_3 \\ 0 & s_3 & 0 \end{pmatrix}, \tag{21}$$

where $s_3 = \sqrt{1 - s_1^2 - s_2^2}$. The first matrix in the right-hand side of (21) constitutes an anisotropy tensor, and the second matrix describes chirality of the crystal. When $s_1 = s_2 = 0$, the matrix $\mathbf{A}$ has the double eigenvalue $\lambda_0 = 2$ with the single eigenvector $\mathbf{u}_0 = (i, -1, 0)^T$ and associated vector $\mathbf{u}_1 = (0, 1, 0)^T$. The eigenvector $\mathbf{v}_0 = (i, 1, 1 + i/2)^T$ and associated vector $\mathbf{v}_1 = (i, 0, 1/2 - i/4)^T$ correspond to the double eigenvalue $\lambda_0 = 2$ of the adjoint matrix $\mathbf{A}^*$. Calculating the derivatives of the matrix $\mathbf{A}(s_1, s_2)$ at the point $\mathbf{s}_0 = (0, 0, 1)$ and using formulae (4), we obtain

$$\mathbf{f} = (0, 4), \;\; \mathbf{g} = (-4, 0), \;\; \mathbf{h} = (0, 0), \;\; \mathbf{r} = (-4, 0).$$

The eigensurfaces $\mathrm{Re}\lambda(s_1, s_2)$ and $\mathrm{Im}\lambda(s_1, s_2)$ in the vicinity of the point $\mathbf{s}_0 = (0, 0, 1)$ are given by the asymptotic

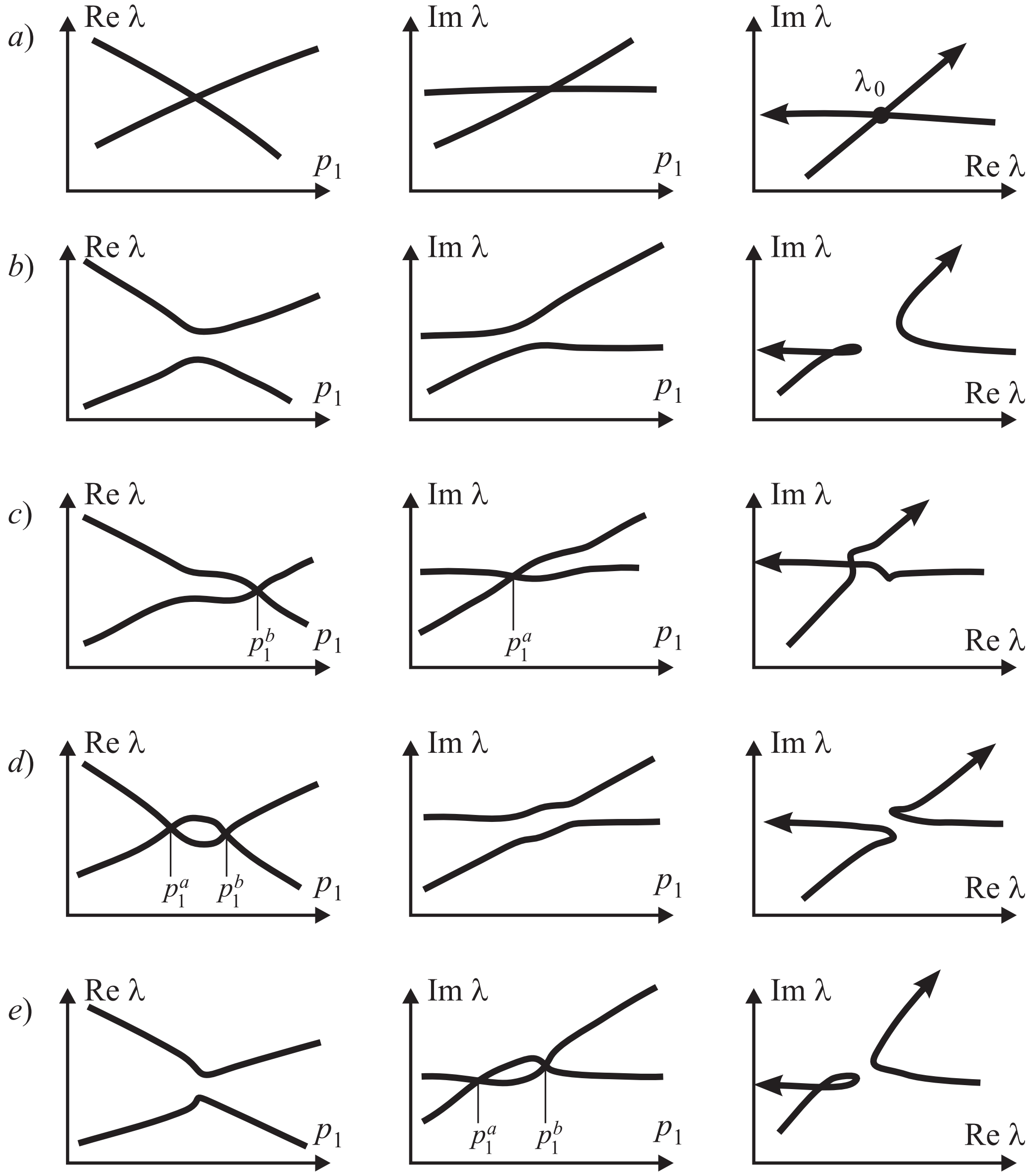

Fig. 3. Weak coupling of eigenvalues and avoided crossing: a) $\Delta p_2 = \cdots = \Delta p_n = 0$, b) $D < 0$, c) $D > 0$ and $c_a c_b < 0$, d) $D > 0$ and $c_{a,b} < 0$, e) $D > 0$ and $c_{a,b} > 0$.

expressions (7), (8) as

$$\begin{aligned} \mathrm{Re}\lambda_\pm &= 2 \pm \sqrt{2s_2 + 2\sqrt{s_1^2 + s_2^2}}, \\ \mathrm{Im}\lambda_\pm &= -2s_1 \pm \sqrt{-2s_2 + 2\sqrt{s_1^2 + s_2^2}}. \end{aligned} \tag{22}$$

These surfaces have the Whitney umbrella singularity at $s_1 = s_2 = 0$, see Figure 2.

As a second numerical example, consider

$$\begin{aligned} \boldsymbol{\eta} = &\begin{pmatrix} 1+5i & 0 & 1+4i \\ 0 & 1+5i & 2i \\ 1+4i & 2i & 4 \end{pmatrix} \\ &+ 4i \begin{pmatrix} 0 & -s_1 - is_2 & is_3 \\ s_1 + is_2 & 0 & -s_3 \\ -is_3 & s_3 & 0 \end{pmatrix}. \end{aligned} \tag{23}$$

At $\mathbf{s}_0 = (0, 0, 1)$, the matrix $\mathbf{A}$ has the double eigenvalue $\lambda_0 = 1 + 5i$ with two eigenvectors $\mathbf{u}_1 = (1, 0, 0)^T$ and $\mathbf{u}_2 = (0, 1, 0)^T$. The eigenvectors of $\overline{\lambda}_0$ for the adjoint matrix $\mathbf{A}^*$ are $\mathbf{v}_1 = (1, 0, \frac{-3-4i}{1-5i})^T$ and $\mathbf{v}_2 = (0, 1, \frac{2i}{1-5i})^T$. Taking derivatives of the matrix $\mathbf{A}$ with respect to parameters s_1 and s_2 and using formula (12), we obtain

$$\begin{aligned} \mathbf{d}_{11} = (-2 - 8i, 0),\ \mathbf{d}_{12} = (6i, -9 - 4i), \\ \mathbf{d}_{21} = (-10i, 7 - 4i),\ \mathbf{d}_{22} = (0, -4i). \end{aligned} \tag{24}$$

Using (24) in (13)–(15), we find approximations for real and imaginary parts of the nonzero eigenvalues $\lambda_\pm$ near $\mathbf{s}_0$ as

$$\begin{aligned} \mathrm{Re}\,\lambda_\pm &= 1 - s_1 \pm \sqrt{(|c| + \mathrm{Re}\,c)/2}, \\ \mathrm{Im}\,\lambda_\pm &= 5 - 4s_1 - 2s_2 \pm \sqrt{(|c| - \mathrm{Re}\,c)/2}, \end{aligned} \tag{25}$$

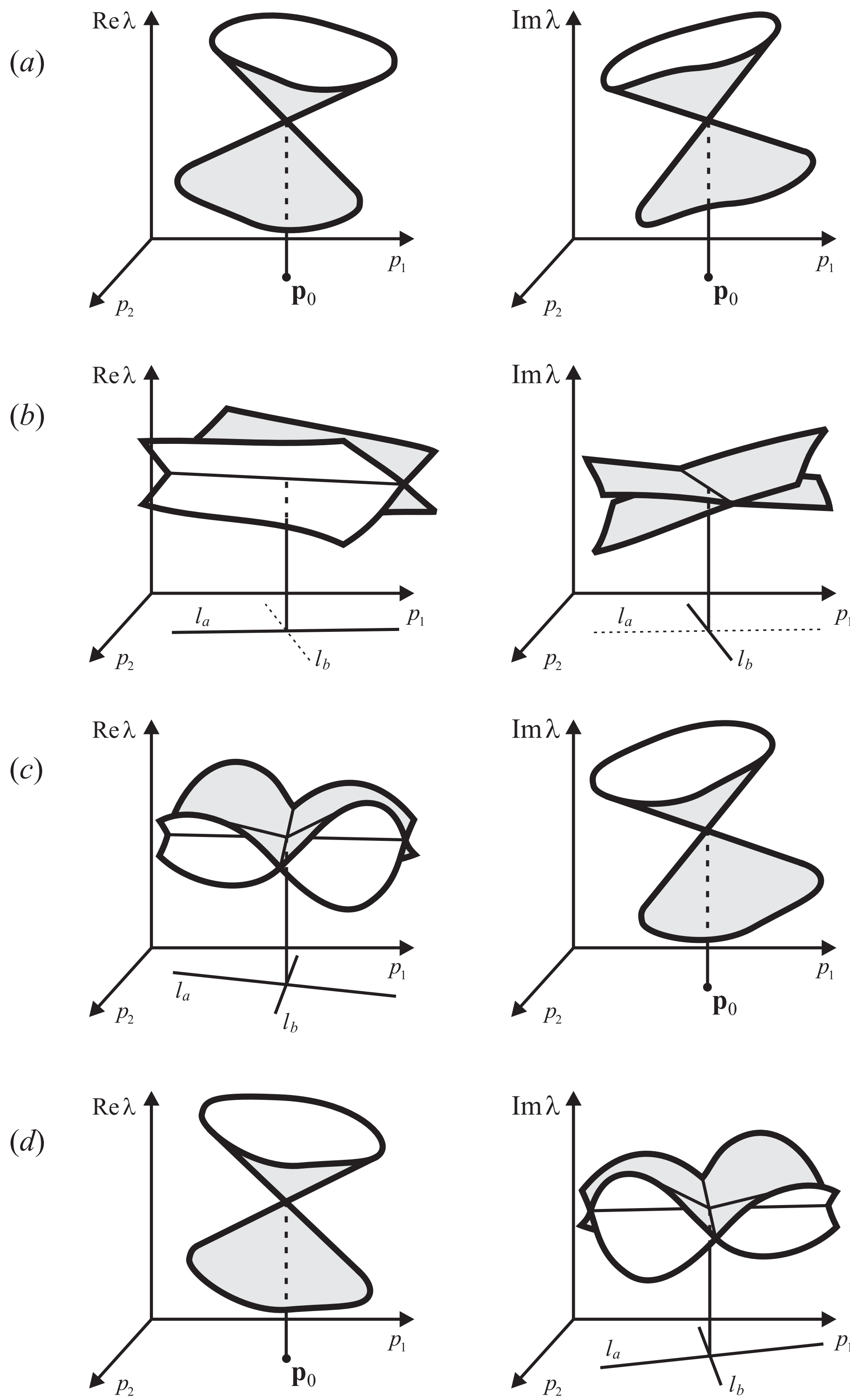

Fig. 4. Eigenvalue surfaces near a point of weak coupling: a) $D' < 0$, b) $D' > 0$ and $\gamma_a\gamma_b < 0$, c) $D' > 0$ and $\gamma_{a,b} < 0$, d) $D' > 0$ and $\gamma_{a,b} > 0$.

where $c = (45 + 8i)s_1^2 + 128is_1s_2 + (-83 + 8i)s_2^2$. The graphs for both $\mathrm{Re}\lambda(s_1, s_2)$ and $\mathrm{Im}\lambda(s_1, s_2)$ are given by two surfaces intersecting at $s_1 = s_2 = 0$ as shown in Figure 4b.

V. Conclusion

A general theory of coupling of eigenvalues of complex matrices smoothly depending on multiple real parameters has been presented. This theory gives a clear and complete picture of crossing and avoided crossing of eigenvalues with a change of parameters, providing qualitative and quantitative description of eigenvalue surfaces based only on the information at the diabolic and exceptional points. This information includes eigenvalues, eigenvectors and associated vectors with derivatives of the system matrix taken at the singular points.

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