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STRONG APPROXIMATION OF MULTIPLE ITO AND STRATONOVICH STOCHASTIC INTEGRALS: MULTIPLE FOURIER SERIES APPROACH

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This book is the first monograph where the problem of strong (mean-square) approximation of multiple Ito and Stratonovich stochastic integrals is sistematically analyzed in the context of numerical integration of stochastic differential Ito equations.

This monograph for the first time successfully use the tool of multiple and iterative Fourier series, built in the space L_2 and poitwise, for the strong approximation of multiple stochastic integrals. The aforesaid means were not used in this academic field before.

We obtained a general result connected with expansion of multiple stochastic Ito integrals with any fixed multiplicity k, based on generalized multiple Fourier series converging in the space $L_2([t,T]^k)$. This result is adapted for multiple Stratonovich stochastic integrals of 1 – 3 multiplicity for Legendre polynomial system and system of trigonometric functions, as well as for other types of multiple stochastic integrals. The theorem on expansion of multiple Stratonovich stochastic integrals with any fixed multiplicity k, based on generalized Fourier series converging pointwise is verified. We obtained exact expressions for mean-square errors of approximation of multiple stochastic Ito integrals of 1 – 4 multiplicity. We provided a significant practical material devoted to expansion and approximation of specific multiple Ito and Stratonovich stochastic integrals of 1 – 5 multiplicity using the system of Legendre polynomials and the system of trigonometric functions. We compared the methods formulated in this book with existing methods of strong approximation of multiple stochastic integrals.

This monograph open a new direction in researching of multiple Ito and Stratonovich stochastic integrals. This book will be interesting for specialists dealing with the theory of stochastic processes, applied and computational mathematics, senior students and postgraduates of technical institutes and universities, as well as for computer experts.

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Preface

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The basis of this book is composed on chapters 5 and 6 of the monograph: Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. SPb: Publishing house of the Polytechnical University 2010, 816 p. 4th edition (in Russian).

It is well known, that Ito stochastic differential equations are adequate mathematical models of dynamic systems of various physical nature which are under the influence of random disturbances. We can meet the mathematical models built on the basis of Ito stochastic differential equations or systems of such equations in finances, medicine, geophysics, electrotechnics, seismology, chemical kinetics and other areas [7] - [21]. Importance of computational solution of Ito stochastic differential equations is arisen from this circumstance.

It is well known, that one of effective and perspective approaches to numerical integration of Ito stochastic differential equations is an approach based on stochastic analogues of Taylor formula for solution of this equations [24], [25], [48]. This approach uses finite discretization of temporary variable and performs numerical modeling of solution of Ito stochastic differential equation in discrete moments of time using stochastic analogue of Taylor formula.

The most important difference of such stochastic analogues of Taylor formula for solution of Ito stochastic differential equations is presence of so called multiple stochastic integrals in them in the forms of Ito or Stratonovich which are the complex functionals in relation to the components of vector Wiener process. These multiple stochastic integrals are subjects for study in this book. In one of the most common forms of record used in this monograph the mentioned multiple stochastic Ito and Stratonovich integrals are detected using the following formulas:

$$J[\psi^{(k)}]_{T,t}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\ \text{(Ito integrals)},$$

$$J^*[\psi^{(k)}]_{T,t}=\int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\ \text{(Stratonovich integrals)},$$

where $\psi_l(\tau);\ l=1,\ldots,k$ — are continuous functions at the interval $[t,T]$ (as a rule, in the applications they are identically equal to 1 or have polynomial shape; $\mathbf{w}_\tau$ — is a random vector with $m+1$ component: $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ when $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau$; $i_1,\ldots,i_k=0,\ 1,\ldots,m$; $\mathbf{f}_\tau^{(i)}$; $i=1,\ldots,m$ — are independent standard Wiener processes.

The given multiple stochastic integrals are the specific objects of the theory of stochastic processes. From one side, nonrandomness of weight functions $\psi_l(\tau);\ l=1,\ldots,k$ is the factor simplifying their structure. From the other side, nonscalarity of Wiener process $\mathbf{f}_\tau$ with independent components $\mathbf{f}_\tau^{(i)}$; $i=1,\ldots,m$, and the fact, that functions $\psi_l(\tau);\ l=1,\ldots,k$ are different for various $l;\ l=1,\ldots,k$ are essential complicating factors of structure of multiple stochastic integrals. Considering features mentioned above, the systems of multiple stochastic Ito and Stratonovich integrals play the extraordinary and perhaps the key role for solving the problems of numerical integration of stochastic Ito differential equations. We want to mention in short, that there are two main criteria of numerical methods convergence for stochastic Ito differential equations: a strong or mean-square criterion and a weak criterion where the subject of approximation is not the solution of stochastic Ito differential equation, simply stated, but the distribution of stochastic Ito differential equation solution. Both mentioned criteria are independent, i.e. in general it is impossible to state, that from execution of strong criterion follows execution of weak criterion and vice versa. Each of two convergence criteria is oriented on solution of specific classes of mathematical problems connected with stochastic differential equations.

Using the strong numerical methods, we may build sample pathes of stochastic Ito differential equation numerically. These methods require the combined mean-square approximation for collections of multiple stochastic Ito and Stratonovich integrals. Effective solution of this task composes the subject of this monograph. The strong numerical methods are using when building new mathematical models on the basis of stochastic Ito differential equations, when solving the task of numerical solution of filtering problem of signal under the influence of random disturbance in various arrangements, when solving the task connected with stochastic optimal control, and the task connected with testing procedures of evaluating parameters of stochastic systems and other tasks.

The problem of effective jointly numerical modeling (in terms of the mean-square convergence criterion) of multiple stochastic Ito or Stratonovich integrals is very important and difficult from theoretical and computing point of view.

Seems, that multiple stochastic integrals may be approximated by multiple integral sums. However, this approach implies partition of the interval of integration of multiple stochastic integrals (this interval is a small value, because it is a step of integration of numerical methods for stochastic differential equations, and according to numerical experiments this additional partition leads to significant calculating costs.

The problem of effective decreasing of mentioned costs (in several times or even in several orders) is very difficult and requires new complex investigations (the only exception is connected with a narrow particular case, when $i_1 = \ldots = i_k \neq 0$ and $\psi_1(s), \ldots, \psi_k(s) \equiv \psi(s)$. This case allows the investigation with using of Ito formula. In more general case, when not all numbers $i_1, \ldots, i_k$ are equal, the mentioned problem turns out to be more complex (it can't be solved using Ito formula and requires more deep and complex investigation). Note, that even for mentioned coincidence ($i_1 = \ldots = i_k \neq 0$), but for different functions $\psi_1(s), \ldots, \psi_k(s)$ the mentioned difficulties persist, and relatively simple families of multiple stochastic Ito integrals, which can be often met in the applications, cannot be expressed effectively in a finite form (for mean-square approximation) using the system of standard Gaussian random values. The Ito formula is also useless in this case and as a result we need to use more complex but effective expansions.

Why the problem of mean-square approximation of multiple stochastic integrals is so complex?

Firstly, the mentioned stochastic integrals (in case of fixed limits of integration) are the random values, whose density functions are unknown in the general case. Even the knowledge of these density functions would hardly be useful for mean-square approximation of multiple stochastic integrals.

Secondly, we need to approximate not only one stochastic integral, but several multiple stochastic integrals which are complexly depended in a probability meaning.

Often, the problem of combined mean-square approximation of multiple stochastic Ito and Stratonovich integrals occurs even in cases when the exact solution of stochastic differential Ito equation is known. It means, that even if you know the solution of stochastic differential Ito equation, you can't model it without enganging combained numerical modelling of multiple stochastic integrals.

Note, that for a number of special types of stochastic Ito differential equations the problem of approximation of multiple stochastic integrals may be simplified but can't be solved. The equation with additive vector noise, with scalar non-additive noise, scalar additive noise, equation with a small parameter is related to such types of equation. For the mentioned types of equations, simplifications are connected with the fact, that either some coefficient functions from stochastic analogues of Taylor formula identically equal to zero, or scalar noise has a strong effect, or due to presence of a small parameter we may neglect some members form the stochastic analogues of Taylor formula, which include difficult for approximation multiple stochastic integrals.

Furthermore, the problem of combined numerical modeling (proceeding from the mean-square convergence criterion) of multiple stochastic Ito and Stratonovich integrals is rather new.

One of the main and unexpected achievements of this book is successful usage of functional analysis methods (repeated and multiple generalized Fourier series (converging in $L_2([t,T]^k)$ and pointwise) through various systems of basis functions in this academic field.

The problem of combined numerical modeling (proceeding from the mean-square convergence criterion) of multiple stochastic Ito and Stratonovich integral systems was analyzed in the context of problem of numerical integration of stochastic differential Ito equations in the following monographs:

[I] Milstein G.N. Numerical integration of stochastic differential equations. Kluwer, 1995, 228 p. (translation from edition to Russian language, 1988);

[II] Kloeden P.E., Platen E. Numerical solution of stochastic differential equations. Berlin: Springer-Verlag, 1992. 632 p. (2nd edition 1995, 3rd edition 1999);

[III] Milstein G.N., Tretyakov M. V. Stochastic numerics for mathematical physics. Berlin: Springer-Verlag, 2004. 596 p.;

[IV] Kuznetsov D.F. Stochastic Differential Equations: Theory and Practice of Numerical Solution. SPb: Publishing house of the Polytechnical University, 2007. 777 c. (2nd edition 2007, 3rd edition 2009, 4th edition 2010; in Russian).

Books [I] and [III] analyze the problem of mean-square approximation only for two elementary multiple stochastic Ito integrals of first and second multiplicities ($k = 1$ and 2; $\psi_1(s)$ and $\psi_2(s) \equiv 1$) for the multivariable case: $i_1,\ i_2 = 0,\ 1, \ldots, m$. In addition, the main idea is based on the expansion of so called process of Brownian bridge into the Fourier series. This method is called in [I] and [III] as the method of Fourier.

In [II] using the method of Fourier, the attempt was made to perform mean-square approximation of elementary stochastic integrals of 1 – 3 multiplicity ($k = 1, \ldots, 3;\ \psi_1(s), \ldots, \psi_3(s) \equiv 1$) for the multivariate case: $i_1, \ldots, i_3 = 0,\ 1, \ldots, m$. However, as we can see in chapter 6, the results of monograph [II], related to the mean-square approximation of multiple stochastic integral of 3 multiplicity, cause a number of critical remarks.

The main purpose of this monograph is to detect, validate and adapt newer and more effective for applications methods (than presented in books [I] - [III]) of combined mean-square approximation of multiple stochastic Ito and Stratonovich integrals.

Talking about the history of solving the problem of combined mean-square approximation of multiple stochastic integrals, the idea to find bases of random values using which we may represent multiple stochastic integrals turned out to be useful. This idea was transformed several times in the course of time.

Attempts to approximate multiple stochastic integrals using various integral sums were made until 1980s, i.e. the interval of integration of stochastic integral was divided into n parts and the multiple stochastic integral was represented approximately by the multiple integral sum, included the system of independent standard Gaussian random variables, whose numerical modeling is not a problem.

However, as we noted before, it is obvious, that the interval of integration of multiple stochastic integrals is a step of numerical method of integration for the stochastic Ito differential equation which is already a rather small value even without additional splitting. Numerical experiments demonstrate, that such approach results in abrupt increasing of computational costs accompanied by the growth of multiplicity of stochastic integrals (beginning from 2nd and 3rd multiplicity), that is necessary for building more accurate numerical methods for stochastic Ito differential equations or in case of decrease of the numerical method integration step, and thereby it is almost useless for practice.

The new step for solution of the problem of combined mean-square approximation of stochastic integrals was made by G.N. Milstein in his monograph [I] (1988), who proposed to use converging in the mean-square sense trigonometric Fourier expansion of Wiener process, using which we may expand a multiple stochastic integral. In [I] using this method, the expansions of two simplest stochastic integrals of 1st and 2nd multiplicities into the products of standard Gaussian random values was obtained and their mean-square convergence was proved.

As we noted, the attempt to develop this idea was made in monograph [II] (1992), where it is obtained expansions of simplest multiple stochastic integrals of 1 – 3 multiplicity. However, due to the number of limitations and technical difficulties which are typical for method [I], in [II] and following publications this problem was not solved more completely. In addition, the author has reasonable doubts about handling of the method of series summation, given in [II], related to integrals of 3rd multiplicity (see section 6.1.4).

It is necessary to note, that the method [I] excelled in times or even in orders the method of integral sums considering computational costs in the sense of their diminishing.

Regardless of the method [I] positive features, the number of its limitation is also outlined: absence of obvious formula for calculation of expansion coefficients of the multiple stochastic integral; practical impossibility to make exact calculation of the mean-square error of approximation of stochastic integrals with the exception of simplest integrals of 1st and 2nd multiplicity (as a result, it is necessary to consider redundant terms of expansion and it results to the growth of computational costs and complication of numerical methods for stochastic Ito differential equations); there is a hard limit for a system of basis functions in the course of approximation — it may be only trigonometric functions; there are some technical problems if we use this method for stochastic integrals whose multiplicity is higher than 2nd (see 6.1.4). It is necessary to note, that the analyzed method is a concrete forward step in this academic field.

The author thinks, that the method presented by him in chapter 1 (hereafter, the method based on multiple Fourier series) is a breakthrough in solution of problem of combined mean-square approximation of multiple stochastic integrals.

The idea of this method is as follows: multiple stochastic Ito integral of multiplicity k is represented as a multiple stochastic integral from the certain non-random discontinuous function of k variables, detected on the hypercube $[t,T]^k$, where $[t,T]$ — is an interval of integration of multiple stochastic Ito integral. Then, the indicated nonrandom function is expanded in the hypercube into the generalized multiple Fourier series converging at the mean-square in the space $L_2([t,T]^k)$. After a number of nontrivial transformations we come (theorem 1) to the mean-square converging expansion of multiple stochastic Ito integral into the multiple series of products of standard Gaussian random values. The coefficients of this series are the coefficients of multiple Fourier series for the mentioned nonrandom function of several variables, which can be calculated using the explicit formula regardless of the multiplicity k of the multiple stochastic Ito integral.

As a result we obtain the following new possibilities and advantages in comparison with the method of Fourier [I].

1. There is an obvious formula for calculation of expansion coefficients of multiple stochastic Ito integral with any fixed multiplicity k. In other words, we can calculate (without any preliminary and additional work) the expansion coefficient with any fixed number in the expansion of multiple stochastic Ito integral of preset fixed multiplicity. At that, we don't need any knowledge about coefficients with other numbers or about other multiple stochastic Ito integrals, included in the analyzed collection.

2. We have new possibilities for explicit calculations of mean-square error of approximation of multiple stochastic Ito integrals. These possibilities are realized using exact formulas (see chapter 4) for mean-square errors of approximations of multiple stochastic Ito integrals. As a result, we won't need to consider redundant terms of expansion, that may complicate approximations of multiple stochastic integrals.

3. Since the used multiple Fourier series is a generalized in the sense, that it is built using various full orthonormal systems of functions in the space $L_2([t,T])$, we have new possibilities for approximation — we may use

not only trigonometric functions as in [I] but Legendre polynomials as well as function systems of Haar and Rademacher-Uolsh (see chapters 2 and 5).

4. As it turned out (see chapter 5), it is more convenient to work with Legendre polynomials for building approximations of multiple stochastic integrals — it is enough just to calculate coefficients of the multiple Fourier series, and approximations themselves appear to be simpler than for the case of the system of trigonometric functions. For the systems of Haar and Rademacher-Uolsh functions the expansions of multiple stochastic integrals become extremely complex and ineffective for practice (see chapter 2).

5. The question about what kind of functions (polynomial or trigonometric) is more convenient in the context of computational costs of approximation turns out to be nontrivial, since it is necessary to compare approximations made not for one integral but for several stochastic integrals at the same time. At the same time there is a possibility, that computational costs for some integrals will be smaller for the system of Legendre polynomials and for others — for the system of trigonometric functions. The author thinks, that (see bottom lines in tables 6.2 and 6.3) computational costs are 3 times less for the system of Legendre polynomials at least in case of approximation of special family of multiple stochastic integrals of $1-3$ multiplicity. In addition, the author supposes, that this effect will be more impressive when analyzing more complex families of multiple stochastic integrals. This supposition is based on the fact, that the polynomial system of functions has a significant advantage (in comparison with the trigonometric system) for approximation of multiple stochastic integrals for which not all weight functions are equal to 1 (compare formulas (5.4), (5.5), (5.7), (5.8) with formulas (5.43), (5.48), (5.47), (5.46) correspondently).

6. The Milstein method leads to repeated series (in contrast with multiple series taken from theorem 1 in this book) starting at least from the third multiplicity of multiple stochastic integral (we mean at least triple integration on Wiener processes). Multiple series are more preferential in terms of approximation than the repeated ones, since partial sums of multiple series converge in any possible case of joint converging to infinity of their upper limits of summation (lets define them as $p_1, \ldots, p_k$). For example, for more simple and convenient for practice case when $p_1 = \ldots = p_k = p \to \infty$. For repeated series it is obviously not the case. However, in [II] the authors unreasonably use the condition $p_1 = p_2 = p_3 = p \to \infty$ — within the frames of the Milstein method.

7. The convergence (see chapters 1 and 5) in the mean of degree $2n$, $n \in N$ of approximations from theorem 1 and convergence with probability 1 for some of these approximations is proven.

Let's deal with the content of this monograph according to chapters.

Chapter 1 is devoted to expansions of multiple Ito stochastic integrals. The new method of expansion of multiple stochastic Ito integrals based on the generalized multiple Fourier series and converging in the mean-square sense is formulated and proven. This method is generalized for the case of discontinuous full orthonormal systems of functions in the space $L_2([t,T])$. Using the example of multiple stochastic Ito integrals of 2nd and 3rd multiplicity it is demonstrated, that expansions from theorems 1 and 2 are similar for a particular case: $\psi_1(s), \psi_2(s), \psi_3(s) \equiv \psi(s)$; $i_1 = i_2 = i_3 = 1,\ldots,m$ with well–known representations of multiple Ito stochastic integrals based on Hermit polynomials. The convergence in the mean of degree $2n$, $n \in N$ of expansions from theorems 1 and 2 is proven.

Chapter 2 is devoted to expansions of multiple Stratonovich stochastic integrals. We adapt the results of theorems 1 and 2 for expansions of multiple Stratonovich stochastic integrals in the first part of this chapter. The theorem about expansion of multiple Stratonovich stochastic integrals of 2nd multiplicity (theorem 3) is proven for the case of twice continuously differentiated functions $\psi_1(s)$ и $\psi_2(s)$ $(i_1, i_2 = 1,\ldots,m)$. We obtained similar expansions for multiple Stratonovich stochastic integrals of 3rd multiplicity for the cases of system of Legendre polynomials and the system of trigonometric functions when $\psi_1(s), \psi_2(s), \psi_3(s) \equiv 1$ $(i_1, i_2, i_3 = 1,\ldots,m)$. The generalization of some of these results (theorem 4) for the system of Legendre polynomials and binomial expressions $\psi_j(s) \equiv (t-s)^{l_j}$ $(j = 1,\ 2,\ 3)$ are given in the following cases:

1. $i_1 \neq i_2,\ i_2 \neq i_3,\ i_1 \neq i_3$ and $l_1,\ l_2,\ l_3 = 0,\ 1,\ 2,\ldots$;
2. $i_1 = i_2 \neq i_3$; $l_1 = l_2 \neq l_3$ and $l_1,\ l_2,\ l_3 = 0,\ 1,\ 2,\ldots$;
3. $i_1 \neq i_2 = i_3$; $l_1 \neq l_2 = l_3$ and $l_1,\ l_2,\ l_3 = 0,\ 1,\ 2,\ldots$;
4. $i_1, i_2, i_3 = 1,\ldots,m$; $l_1 = l_2 = l_3 = l$ and $l = 0,\ 1,\ 2,\ldots$.

In the second part of chapter 2 we analyze another approach to expansion of multiple Stratonovich stochastic integrals of any fixed multiplicity k, based on the generalized repeated Fourier series converging pointwise. We analyze in detail the cases when $k = 1,\ 2,\ 3$ and propose generalization for the case of fixed k (theorem 5).

In chapter 3 we analyze versions of the theorem 1 for other types of multiple stochastic integrals. We formulated and proved analogues of theorem 1 for multiple stochastic integrals according to martingale Poisson measures (theorem 7) and for multiple stochastic integrals according to martingales (theorem 8).

Chapter 4 is devoted to obtainment of exact expressions for mean-square errors of approximation of multiple stochastic Ito integrals, created using theorem 1. We analyzed the case of any fixed k and pairwise various $i_1,\ldots,i_k=1,\ldots,m$, as well as the cases when $k=1,\ 2,\ 3,\ 4$ and any $i_1,\ldots,i_k=1,\ldots,m$. Here k — is a multiplicity of multiple stochastic Ito integral.

In chapter 5 we provide a significant practical material, based on the results of chapters 1 and 2. We got approximations of specific multiple stochastic Ito and Stratonovich integrals with multiplicities 1 – 5 using theorems 1 – 4 and the system of Legendre polynomials. For the case of trigonometric system of functions using theorem 1 and the results of chapter 2 we obtained approximations for specific multiple stochastic Ito and Stratonovich integrals with multiplicities 1 – 3. We obtained a big number of formulas for mean-square errors for developed approximations.

Chapter 6 is devoted to other methods of mean-square approximation of multiple stochastic Ito and Stratonovich integrals. We analyzed Milstein method and compared it with the method based on theorem 1. We also analyzed a combined method and a method of integral sums of mean-square approximation of multiple stochastic integrals. We represented multiple stochastic Ito integrals based on Hermit polynomials.

In chapter 7 we gathered a support material which may be used while reading this book. We provided concepts of stochastic Ito and Stratonovich integrals, of Ito formula, of stochastic Ito differential equation, of stochastic integrals according to Poisson random measures and martingales, of various variants of Taylor–Ito and Taylor–Stratonovich expansions for solution of stochastic Ito differential equation.

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Chapter 1

Expansions of multiple Ito stochastic integrals based on generalized multiple Fourier series, converging in the mean

This chapter is devoted to expansions of multiple Ito stochastic integrals, based on generalized multiple Fourier series converging in the mean. The method of generalized multiple Fourier series for expansions and mean-square approximations of multiple Ito stochastic integrals is derived here. In this chapter it is also obtained generalization of this method for discontinuous basis functions. And here is given a comparison of derived method with well-known expansions of multiple Ito stochastic integrals based on Ito formula and Hermit polynomials. As well as the proof of convergence in the mean of degree $2n$, $n \in \mathrm{N}$ of considered method is obtained.

1.1 Introduction

The results of this chapter are fundamental for following chapters of this monograph and perhaps for the book in whole. For the first time we use power tool of generalized multiple Fourier series converging in the mean in order to derive expansions of stochastic integrals.

The idea of representing of multiple Ito and Stratonovich stochastic integrals in the form of multiple stochastic integrals from specific nonrandom functions of several variables and following expansion of these functions using Fourier series in order to get effective mean-square approximations of mentioned stochastic integrals was represented in several works of the author. Specifically, this approach appeared for the first time in [49] (1994). In that work the mentioned idea is formulated more likely at the level of guess (without any satisfactory grounding), and as a result the work [49] contains rather fuzzy formulations and a number of incorrect conclusions. Nevertheless, even in [49] we can find, for example formulas (4.3), (4.21), (4.22). Note, that in [49] we used multiple Fourier series according to the trigonometric system of functions converging in the mean. It should be noted, that the results of work [49] are true for a sufficiently narrow particular case when numbers $i_1, ..., i_k$ are pairwise different; $i_1, \ldots, i_k = 1, \ldots, m$ (see formula (1.1)).

Usage of Fourier series according to the system of Legendre polynomials for approximation of multiple stochastic integrals took place for the first time in [32] (1998), [34], [35], [37] (1999), as well as in [36] (2000). In particular, you can find formulas (5.3) – (5.8), (5.18) in this works. Note, that the approach taken from work [49] was formulated, proved and generalized in its final variant by the author in [42] (2006) (theorem 1 in this book).

The question about what integrals (Ito or Stratonovich) are more suitable for expansions within the frames of distinguished direction of researches has turned out to be rather interesting and difficult.

On the one side, theorem 1 conclusively demonstrates, that the structure of multiple Ito stochastic integrals is rather convenient for expansions into multiple series according to the system of standard Gaussian random variables regardless of their multiplicity.

On the other side, the results of chapter 2 convincingly testify, that there is a doubtless relation between multiplier factor $\frac{1}{2}$, which is typical for stochastic Stratonovich integral and included into the sum, connecting stochastic Stratonovich and Ito integrals, and the fact, that in point of finite discontinuity of sectionally smooth function $f(x)$ its Fourier series converges to the value $\frac{1}{2}(f(x-0)+f(x+0))$. In addition, as it is demonstrated in chapter 2, final formulas for expansions of multiple stochastic Stratonovich integrals (of second multiplicity in the common case and of third and fourth multiplicity in some particular cases) are more compact than their analogues for stochastic Ito integrals. The expansion of multiple stochastic Stratonovich integrals of any fixed multiplicity k based on repeated Fourier series and obtained in chapter 2 [35] is also seems interesting.

And still, estimating the results of chapter 1 and 2 of this monograph, the author adhered to the judgment, that the structure of multiple stochastic Ito integrals is more suitable for expansion in multiple series according to the system of Gaussian random variables.

Actually, when proving theorem 1 for the case of any fixed multiplicity k of multiple stochastic Ito integral we used multiple Fourier series converging in the mean. The deduction of theorems 3 –8 for multiple Stratonovich stochastic integrals of 2nd, 3rd and 4th multiplicity in addition to the results of theorem 1 required also usage of the theory of repeated or multiple Fourier series converging pointwise, and resulted to more complex researches than those that were performed for proving of theorem 1, which nevertheless didn't provide common results (we analyzed the cases of multiple Stratonovich stochastic integrals of 2nd, 3rd and 4th multiplicity, where the results related to integrals of 3rd and 4th multiplicity have an individual pattern, although they are of vital importance for practice).

Expansions of multiple Stratonovich stochastic integrals of any fixed multiplicity k obtained at the end of chapter 2 are rather interesting but include repeated series, approximation of which is less convenient than approximation of multiple series.

1.2 Theorem on expansion of multiple Ito stochastic integrals of any multiplicity

In this section we will get the expansion of multiple Ito stochastic integrals of any fixed multiplicity k based on generalized multiple Fourier series converging in the mean in the space $L_2([t,T]^k)$.

Assume, that (Ω, F, P) — is a fixed probability space and $\{F_t,\ t\in[0,T]\}$ — is a non-decreasing collection of σ-algebras, defined at (Ω, F, P).

Assume, that $f(t,\omega)\stackrel{\mathsf{def}}{=} f_t,\ t\in[0,T]$ — is a standard Wiener process, which is F_t-measurable for all $t\in[0,T]$, and process $f_{t+\Delta}-f_{\Delta}$ for all $\Delta\ge 0,\ t>0$ is independent with the events of σ-algebra F_{Δ}.

Hereafter we call stochastic process $\xi:\ [0,T]\times\Omega\to\Re^1$ as non-anticipative when it is measurable according to the family of variables (t,ω) and function $\xi(t,\omega)\stackrel{\mathsf{def}}{=}\xi_t$ is F_t-measurable for all $t\in[0,T]$ and ξ_τ independent with increments $f_{t+\Delta}-f_{\Delta}$ for $\Delta\ge\tau,\ t>0$.

Let's examine the following multiple Ito and Stratonovich stochastic integrals:

$$J[\psi^{(k)}]_{T,t}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{1.1}$$

$$J^*[\psi^{(k)}]_{T,t}=\int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \tag{1.2}$$

where $\psi_l(\tau);\ l=1,\ldots,k$ — are continuous functions at the interval $[t,T]$; $\mathbf{w}_t^{(i)}=\mathbf{f}_t^{(i)}$ when $i=1,\ldots,m$; $\mathbf{w}_t^{(0)}=t$; $i_1,\ldots,i_k=0,\ 1,\ldots,m$; $\mathbf{f}_\tau^{(i)}$ $(i=1,\ldots,m)$ — are independent standard Wiener processes.

The problem of effective jointly numerical modeling (in terms of the mean-square convergence criterion) of multiple stochastic Ito integrals, as we mentioned before, is very important and complex from theoretical and computing point of view. The exception is a very marrow particular case, when $i_1 = \ldots = i_k \neq 0$ and $\psi_1(s), \ldots, \psi_k(s) \equiv \psi(s)$ (see this chapter and chapter 6). We can analyze this case using the Ito formula.

This problem, as we will see in this chapter, cannot be solved using the Ito formula and it requires deeper and more complex investigation for the case when not all numbers $i_1, \ldots, i_k$ coincide among themselves. Note, that even in case of such coincidence ($i_1 = \ldots = i_k \neq 0$), but with various $\psi_1(s), \ldots, \psi_k(s)$ the mentioned problem persists, and relatively simple families of multiple stochastic Ito integrals, which can be often met in the applications, cannot be expressed effectively in a finite form (for mean-square approximation) using the system of standard Gaussian random values. The Ito formula is useless in this case, and as a result we need to use more complex but effective expansions.

Assume, that $\{\phi_j(x)\}_{j=0}^{\infty}$ — is a full orthonormal system of functions in the space $L_2([t,T])$; $\psi_1(\tau), \ldots, \psi_k(\tau)$ — are continuous functions at the interval $[t,T]$.

Let's analyze the following function

$$K(t_1,\ldots,t_k) = \begin{cases} \psi_1(t_1)\ldots\psi_k(t_k), \ t_1 < \ldots < t_k \\ 0, \ \text{otherwise} \end{cases}; \ t_1,\ldots,t_k \in [t,T].$$

The function $K(t_1,\ldots,t_k)$ is sectionally continuous in the hypercube $[t,T]^k$, i.e. the hypercube may be cut in finite number of parts using the sectionally continuous surfaces in such manner, that the function $K(t_1,\ldots,t_k)$ is continuous in each part and has limits at the border of part, and it may have gaps along these cuts.

At this situation it is well known, that the multiple Fourier series of function $K(t_1,\ldots,t_k) \in L_2([t,T]^k)$ is converging to this function in the hypercube in the mean-square sense, i.e.

$$\lim_{p_1,\ldots,p_k\to\infty} \left\| K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l) \right\| = 0, \tag{1.3}$$

where $\|f\| = \left(\int\limits_{[t,T]^k} f^2(t_1,\ldots,t_k)dt_1\ldots dt_k \right)^{\frac{1}{2}}$ and we have Parseval equality

$$\int\limits_{[t,T]^k} K^2(t_1,\ldots,t_k)dt_1\ldots dt_k = \lim_{p_1,\ldots,p_k\to\infty} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C^2_{j_k\ldots j_1}. \tag{1.4}$$

Let's formulate the basic theorem:

Theorem 1. *Assume, that the following conditions are met:*

1. $\psi_i(\tau);\ i = 1,\ 2, \ldots, k$ — *are continuous functions at the interval* $[t, T]$.

2. $\{\phi_j(x)\}_{j=0}^{\infty}$ — *is a full orthonormal system of continuous functions in the space* $L_2([t, T])$.

Then the multiple stochastic Ito integral $J[\psi^{(k)}]_{T,t}$ *of the form* (1.1) *is expanded in the multiple series converging in the mean-square sense*

$$J[\psi^{(k)}]_{T,t} = \underset{p_1,\ldots,p_k\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k} \zeta_{j_l}^{(i_l)} -$$

$$-\underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k} \phi_{j_1}(\tau_{l_1})\Delta \mathbf{w}_{\tau_{l_1}}^{(i_1)} \ldots \phi_{j_k}(\tau_{l_k})\Delta \mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr), \tag{1.5}$$

where $\mathrm{G}_k = \mathrm{H}_k \backslash \mathrm{L}_k$; $\mathrm{H}_k = \{(l_1, \ldots, l_k):\ l_1, \ldots, l_k = 0,\ 1, \ldots, N-1\}$,

$$\mathrm{L}_k = \Bigl\{(l_1, \ldots, l_k):\ l_1, \ldots, l_k = 0,\ 1, \ldots, N-1;$$
$$l_g \ne l_r (g \ne r);\ g, r = 1, \ldots, k\Bigr\};$$

$\zeta_{j_l}^{(i_l)} = \int\limits_t^T \phi_{j_l}(s) d\mathbf{w}_s^{(i_l)}$ — *are independent standard Gaussian random variables for various* i_l *or* j_l (*if* $i_l \ne 0$);

$$C_{j_k\ldots j_1} = \int\limits_{[t,T]^k} K(t_1, \ldots, t_k) \prod_{l=1}^{k} \phi_{j_l}(t_l) dt_1 \ldots dt_k; \tag{1.6}$$

$$K(t_1, \ldots, t_k) = \begin{cases} \psi_1(t_1) \ldots \psi_k(t_k),\ t_1 < \ldots < t_k \\ 0,\ \text{otherwise} \end{cases};\ t_1, \ldots, t_k \in [t, T].$$

Proof. At first, let's prove preparatory lemmas.

Let's analyze the partition $\{\tau_j\}_{j=0}^{N}$ of interval $[t, T]$ for which

$$t = \tau_0 < \ldots < \tau_N = T,\ \Delta_N = \max_{0\le j\le N-1} \Delta\tau_j \to 0 \text{ with } N \to \infty, \tag{1.7}$$

where $\Delta\tau_j = \tau_{j+1} - \tau_j$.

Lemma 1. *Assume, that condition* 1 *of theorem* 1 *is met. Then*

$$J[\psi^{(k)}]_{T,t} = \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_k=0}^{N-1} \ldots \sum_{j_1=0}^{j_2-1} \prod_{l=1}^{k} \psi_l(\tau_{j_l}) \Delta \mathbf{w}_{\tau_{j_l}}^{(i_l)} \ \ w.\ p.\ 1, \tag{1.8}$$

where $\Delta \mathbf{w}_{\tau_{j_l}}^{(i_l)} = \mathbf{w}_{\tau_{j_l+1}}^{(i_l)} - \mathbf{w}_{\tau_{j_l}}^{(i_l)}$; $i_l = 0,\ 1, \ldots, m$; $\{\tau_{j_l}\}_{j_l=0}^{N-1}$ *— partition of interval* $[t, T]$, *satisfying the condition* (1.7); *hereinafter "w.p.1" means "with probability* 1*".*

Proof. Proving it is easy to notice, that using the property of stochastic integral additivity, we can write down:

$$J[\psi^{(k)}]_{T,t} = \sum_{j_k=0}^{N-1} \ldots \sum_{j_1=0}^{j_2-1} \prod_{l=1}^{k} J[\psi_l]_{\tau_{j_l+1},\tau_{j_l}} + \varepsilon_N \ \text{w p.1}, \tag{1.9}$$

where

$$\varepsilon_N = \sum_{j_k=0}^{N-1} \int\limits_{\tau_{j_k}}^{\tau_{j_k+1}} \psi_k(s) \int\limits_{\tau_{j_k}}^{s} \psi_{k-1}(\tau) J[\psi^{(k-2)}]_{\tau,t} d\mathbf{w}_{\tau}^{(i_{k-1})} d\mathbf{w}_{s}^{(i_k)} +$$

$$+ \sum_{r=1}^{k-3} G[\psi_{k-r+1}^{(k)}]_N \times$$

$$\times \sum_{j_{k-r}=0}^{j_{k-r+1}-1} \int\limits_{\tau_{j_{k-r}}}^{\tau_{j_{k-r}+1}} \psi_{k-r}(s) \int\limits_{\tau_{j_{k-r}}}^{s} \psi_{k-r-1}(\tau) J[\psi^{(k-r-2)}]_{\tau,t} d\mathbf{w}_{\tau}^{(i_{k-r-1})} d\mathbf{w}_{s}^{(i_{k-r})} +$$

$$+ G[\psi_3^{(k)}]_N \sum_{j_2=0}^{j_3-1} J[\psi^{(2)}]_{\tau_{j_2+1},\tau_{j_2}},$$

$$G[\psi_m^{(k)}]_N = \sum_{j_k=0}^{N-1} \sum_{j_{k-1}=0}^{j_k-1} \ldots \sum_{j_m=0}^{j_{m+1}-1} \prod_{l=m}^{k} J[\psi_l]_{\tau_{j_l+1},\tau_{j_l}},$$

$$J[\psi_l]_{s,\theta} = \int\limits_{\theta}^{s} \psi_l(\tau) d\mathbf{w}_{\tau}^{(i_l)},$$

$\psi_m^{(k)} \stackrel{\rm def}{=} (\psi_m, \psi_{m+1}, \ldots, \psi_k)$; $\psi_1^{(k)} \stackrel{\rm def}{=} \psi^{(k)} = (\psi_1, \ldots, \psi_k)$.

Using standard evaluations (7.3) and (7.4) for the moments of stochastic integrals, we obtain

$$\underset{N\to\infty}{\text{l.i.m.}}\ \varepsilon_N = 0. \tag{1.10}$$

Comparing (1.9) and (1.10) we get

$$J[\psi^{(k)}]_{T,t} = \sum_{j_k=0}^{N-1} \ldots \sum_{j_1=0}^{j_2-1} \prod_{l=1}^{k} J[\psi_l]_{\tau_{j_l+1},\tau_{j_l}} \ \text{w.p.1}. \tag{1.11}$$

Let's rewrite $J[\psi_l]_{\tau_{j_l+1},\tau_{j_l}}$ in the form

$$J[\psi_l]_{\tau_{j_l+1},\tau_{j_l}} = \psi_l(\tau_{j_l}) \Delta \mathbf{w}_{\tau_{j_l}}^{(i_l)} + \int\limits_{\tau_{j_l}}^{\tau_{j_l+1}} (\psi_l(\tau) - \psi_l(\tau_{j_l})) d\mathbf{w}_{\tau}^{(i_l)}$$

and put it in (1.11).

Then, due to moment properties of stochastic integrals, continuity (as a result uniform continuity) of functions $\psi_l(s)$ $(l = 1, \ldots, k)$ it is easy to see, that the prelimit expression in the right part of (1.11) is a sum of prelimit expression in the right part of (1.8) and of the value which goes to zero in the mean-square sense if $N \to \infty$. The lemma is proven. $\square$

Remark 1. *The result of lemma* 1 *may be generalized, i.e. the function* $\psi_l(s)$ *in* (1.8) *may be replaced with a stochastic process* ϕ_s *from the class* $M_2([0,T])$ (*see sect.* 7.1)

Remark 2. *It is easy to see, that if* $\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}$ *in* (1.8) *for some* $l \in \{1, \ldots, k\}$ *is replaced with* $\left(\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}\right)^p$; $p = 2$; $i_l \neq 0$, *then the differential* $d\mathbf{w}_{t_l}^{(i_l)}$ *in the integral* $J[\psi^{(k)}]_{T,t}$ *will be replaced with* dt_l. *If* $p = 3, \ 4, \ldots,$ *then the right part of the formula* (1.8) *with probubility* 1 *will become zero. If we replace* $\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}$ *in* (1.8) *for some* $l \in \{1, \ldots, k\}$ *with* $(\Delta\tau_{j_l})^p, \ p = 2, \ 3, \ldots,$ *then the right part of the formula* (1.8) *also with probubility* 1 *will be equal to zero.*

Let's define the following stochastic integral:

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{N-1} \Phi\left(\tau_{j_1}, \ldots, \tau_{j_k}\right) \prod_{l=1}^{k} \Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)} \stackrel{\text{def}}{=} J[\Phi]_{T,t}^{(k)}. \tag{1.12}$$

Assume, that $D_k = \{(t_1, \ldots, t_k): \ t \leq t_1 < \ldots < t_k \leq T\}$. We will write $\Phi(t_1, \ldots, t_k) \in C(D_k)$, if $\Phi(t_1, \ldots, t_k)$ is a continuous in the closed domain D_k nonrandom function of k variables.

Let's analyze the multiple stochastic integral of Ito type:

$$I[\Phi]_{T,t}^{(k)} \stackrel{\text{def}}{=} \int\limits_t^T \ldots \int\limits_t^{t_2} \Phi(t_1, \ldots, t_k) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)},$$

where $\Phi(t_1, \ldots, t_k)$ is a nonrandom function of k variables.

It is easy to check, that this stochastic integral exists in the mean-square sense, if the following condition is met:

$$\int\limits_t^T \ldots \int\limits_t^{t_2} \Phi^2(t_1, \ldots, t_k) dt_1 \ldots dt_k < \infty.$$

Using the arguments which similar to the arguments used for proving of lemma 1 it is easy to demonstrate, that if $\Phi(t_1, \ldots, t_k) \in C(D_k)$, then the following equality is fulfilled:

$$I[\Phi]_{T,t}^{(k)} \stackrel{\text{def}}{=} \int\limits_t^T \ldots \int\limits_t^{t_2} \Phi(t_1, \ldots, t_k) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)} =$$

$$
= \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_k=0}^{N-1} \ldots \sum_{j_1=0}^{j_2-1} \Phi(\tau_{j_1},\ldots,\tau_{j_k}) \prod_{l=1}^{k} \Delta \mathbf{w}_{\tau_{j_l}}^{(i_l)} \ \text{w.p.1.} \qquad (1.13)
$$

In order to explain, let's check rightness of the equality (1.13) when $k=3$. For definiteness we will suggest, that $i_1, i_2, i_3 = 1, \ldots, m.$

We have

$$
I[\Phi]_{T,t}^{(3)} \stackrel{\text{def}}{=} \int_t^T \int_t^{t_3} \int_t^{t_2} \Phi(t_1,t_2,t_3) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} =
$$

$$
= \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1} \int_t^{\tau_{j_3}} \int_t^{t_2} \Phi(t_1,t_2,\tau_{j_3}) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)} =
$$

$$
= \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \int_{\tau_{j_2}}^{\tau_{j_2+1}} \int_t^{t_2} \Phi(t_1,t_2,\tau_{j_3}) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)} =
$$

$$
= \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \int_{\tau_{j_2}}^{\tau_{j_2+1}} \left(\int_t^{\tau_{j_2}} + \int_{\tau_{j_2}}^{t_2} \right) \Phi(t_1,t_2,\tau_{j_3}) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)} =
$$

$$
= \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \sum_{j_1=0}^{j_2-1} \int_{\tau_{j_2}}^{\tau_{j_2+1}} \int_{\tau_{j_1}}^{\tau_{j_1+1}} \Phi(t_1,t_2,\tau_{j_3}) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)} +
$$

$$
+ \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \int_{\tau_{j_2}}^{\tau_{j_2+1}} \int_{\tau_{j_2}}^{t_2} \Phi(t_1,t_2,\tau_{j_3}) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)}. \qquad (1.14)
$$

Let's demonstrate, that the second limit in the right part of (1.14) equals to zero.

Actually, the second moment of its prelimit expression equals to

$$
\sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \int_{\tau_{j_2}}^{\tau_{j_2+1}} \int_{\tau_{j_2}}^{t_2} \Phi^2(t_1,t_2,\tau_{j_3}) dt_1 dt_2 \Delta\tau_{j_3} \le M^2 \sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \frac{1}{2} \left(\Delta\tau_{j_2}\right)^2 \Delta\tau_{j_3} \to 0,
$$

when $N\to\infty$. Here M is a constant, which restricts the module of function $\Phi(t_1,t_2,t_3)$, because of its continuity; $\Delta\tau_j = \tau_{j+1} - \tau_j$.

Considering the obtained conclusions we have

$$
I[\Phi]_{T,t}^{(3)} \stackrel{\text{def}}{=} \int_t^T \int_t^{t_3} \int_t^{t_2} \Phi(t_1,t_2,t_3) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} d\mathbf{w}_{t_3}^{(i_3)} =
$$

$$
= \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1} \sum_{j_2=0}^{j_3-1} \sum_{j_1=0}^{j_2-1} \int_{\tau_{j_2}}^{\tau_{j_2+1}} \int_{\tau_{j_1}}^{\tau_{j_1+1}} \Phi(t_1,t_2,\tau_{j_3}) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)} =
$$

$$= \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \left(\Phi(t_1,t_2,\tau_{j_3}) - \Phi(t_1,\tau_{j_2},\tau_{j_3})\right) \times$$

$$\times d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)}$$

$$+ \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \left(\Phi(t_1,\tau_{j_2},\tau_{j_3}) - \Phi(\tau_{j_1},\tau_{j_2},\tau_{j_3})\right) \times$$

$$\times d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)}$$

$$+ \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1} \Phi(\tau_{j_1},\tau_{j_2},\tau_{j_3}) \Delta\mathbf{w}_{\tau_{j_1}}^{(i_1)} \Delta\mathbf{w}_{\tau_{j_2}}^{(i_2)} \Delta\mathbf{w}_{\tau_{j_3}}^{(i_3)}. \qquad (1.15)$$

In order to get the sought result, we just have to demonstrate, that the first two limits in the right part of (1.15) equal to zero. Let's prove, that the first one of them equals to zero (proving for the second limit is similar).

The second moment of prelimit expression of the first limit in the right part of (1.15) equals to the following expression:

$$\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \left(\Phi(t_1,t_2,\tau_{j_3}) - \Phi(t_1,\tau_{j_2},\tau_{j_3})\right)^2 dt_1 dt_2 \Delta\tau_{j_3}. \qquad (1.16)$$

Since the function $\Phi(t_1,t_2,t_3)$ is continuous in the closed bounded domain D_3, then it is uniformly continuous in this domain. Therefore, if the distance between two points in the domain D_3 is less than $\delta > 0$ ($\delta > 0$ and chosen for all $\varepsilon > 0$ and it doesn't depend on mentioned points), then the corresponding oscillation of function $\Phi(t_1,t_2,t_3)$ for these two points of domain D_3 is less than ε.

If we assume, that $\Delta\tau_j < \delta$ ($j = 0,\ 1,\ldots,N-1$), then the distance between points (t_1,t_2,τ_{j_3}), $(t_1,\tau_{j_2},\tau_{j_3})$ is obviously less than δ. In this case

$$\left|\Phi(t_1,t_2,\tau_{j_3}) - \Phi(t_1,\tau_{j_2},\tau_{j_3})\right| < \varepsilon.$$

Consequently, when $\Delta\tau_j < \delta$ ($j = 0,\ 1,\ldots,N-1$) the expression (1.16) is evaluated by the following value:

$$\varepsilon^2 \sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1} \Delta\tau_{j_1}\Delta\tau_{j_2}\Delta\tau_{j_3} < \varepsilon^2 \frac{(T-t)^3}{6}.$$

Because of this, the first limit in the right part of (1.15) equals to zero. Similarly we can prove equality to zero of the second limit in the right part of (1.15).

Consequently, the equality (1.13) is proven when $k = 3$. The cases when $k = 2$ and $k > 3$ are analyzed absolutely similarly.

It is necessary to note, that proving of formula (1.13) rightness is similar, when the nonrandom function $\Phi(t_1, \ldots, t_k)$ is continuous in the open domain D_k and bounded at its border.

Assume, that

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{j_1,\ldots,j_k=0\\ j_q\ne j_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1} \Phi\left(\tau_{j_1},\ldots,\tau_{j_k}\right)\prod_{l=1}^{k}\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)} \stackrel{\text{def}}{=} J'[\Phi]_{T,t}^{(k)}.$$

Then we will get according to (1.13)

$$J'[\Phi]_{T,t}^{(k)} = \int_t^T \ldots \int_t^{t_2} \sum_{(t_1,\ldots,t_k)} \left(\Phi(t_1,\ldots,t_k) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_k}^{(i_k)}\right), \qquad (1.17)$$

where summation according to derangements $(t_1, \ldots, t_k)$ is performed only in the expression, which is enclosed in parentheses, and the nonrandom function $\Phi(t_1, \ldots, t_k)$ is assumed to be continuous in the corresponding domains of integration.

Lemma 2. *Let us assume, that the following condition is met*

$$\int_t^T \ldots \int_t^{t_2} \Phi^2(t_1,\ldots,t_k) dt_1 \ldots dt_k < \infty,$$

where $\Phi(t_1, \ldots, t_k)$ — is a nonrandom function. Then

$$\mathsf{M}\left\{\left|I[\Phi]_{T,t}^{(k)}\right|^2\right\} \le C_k \int_t^T \ldots \int_t^{t_2} \Phi^2(t_1,\ldots,t_k) dt_1 \ldots dt_k,\ C_k < \infty.$$

Proof. Using standard properties and estimations of stochastic integrals (see sect. 7.1) for $\xi_\tau \in \mathrm{M}_2([t_0, t])$ (see also sect. 7.1) we have

$$\mathsf{M}\left\{\left|\int_{t_0}^{t} \xi_\tau df_\tau\right|^2\right\} = \int_{t_0}^{t} \mathsf{M}\left\{|\xi_\tau|^2\right\} d\tau,\ \mathsf{M}\left\{\left|\int_{t_0}^{t} \xi_\tau d\tau\right|^2\right\} \le (t-t_0)\int_{t_0}^{t} \mathsf{M}\left\{|\xi_\tau|^2\right\} d\tau. \qquad (1.18)$$

Let's denote

$$\xi[\Phi]_{t_{l+1},\ldots,t_k,t}^{(l)} = \int_t^{t_{l+1}} \ldots \int_t^{t_2} \Phi(t_1,\ldots,t_k) d\mathbf{w}_{t_1}^{(i_1)} \ldots d\mathbf{w}_{t_l}^{(i_l)},$$

where $l=1,\ldots,\ k-1$ and $\xi[\Phi]^{(0)}_{t_1,\ldots,t_k,t}\stackrel{\rm def}{=}\Phi(t_1,\ldots,t_k)$.

In accordance with induction it is easy to demonstrate, that $\xi[\Phi]^{(l)}_{t_{l+1},\ldots,t_k,t}\in$ $M_2([t,T])$ using the variable t_{l+1}.

Subsequently, using the estimations (1.18) repeatedly we can be led to confirmation of the lemma. □

Lemma 3. *Asume, that $\varphi_i(s);\ i=1,\ldots,k$ — are continuous functions at the interval $[t,T]$. Then*

$$\prod_{l=1}^{k}J[\varphi_l]_{T,t}=J[\Phi]^{(k)}_{T,t}\ w.p.1, \tag{1.19}$$

where $J[\varphi_l]_{T,t}=\int\limits_t^T\varphi_l(s)d{\bf w}_s^{(i_l)};\ \ \Phi(t_1,\ldots,t_k)=\prod\limits_{l=1}^{k}\varphi_l(t_l)$ *and the integral* $J[\Phi]^{(k)}_{T,t}$ *is defined by the equality* (1.12).

Proof. Let at first $i_l\ne 0;\ \ l=1,\ldots,k$. Let's denote

$$J[\varphi_l]_N\stackrel{\rm def}{=}\sum_{j=0}^{N-1}\varphi_l(\tau_j)\Delta{\bf w}_{\tau_j}^{(i_l)}.$$

Since

$$\prod_{l=1}^{k}J[\varphi_l]_N-\prod_{l=1}^{k}J[\varphi_l]_{T,t}=\sum_{l=1}^{k}\left(\prod_{g=1}^{l-1}J[\varphi_g]_{T,t}\right)\left(J[\varphi_l]_N-J[\varphi_l]_{T,t}\right)\left(\prod_{g=l+1}^{k}J[\varphi_g]_N\right),$$

then because of the Minkowsky inequality and inequality of Cauchy-Bunyakovsky

$$\left({\sf M}\left\{\left|\prod_{l=1}^{k}J[\varphi_l]_N-\prod_{l=1}^{k}J[\varphi_l]_{T,t}\right|^2\right\}\right)^{\frac{1}{2}}\le C_k\sum_{l=1}^{k}\left({\sf M}\left\{|J[\varphi_l]_N-J[\varphi_l]_{T,t}|^4\right\}\right)^{\frac{1}{4}}, \tag{1.20}$$

where $C_k<\infty$.

Note, that

$$J[\varphi_l]_N-J[\varphi_l]_{T,t}=\sum_{g=0}^{N-1}J[\Delta\varphi_l]_{\tau_{g+1},\tau_g},$$

where

$$J[\Delta\varphi_l]_{\tau_{g+1},\tau_g}=\int\limits_{\tau_g}^{\tau_{g+1}}\left(\varphi_l(\tau_g)-\varphi_l(s)\right)d{\bf w}_s^{(i_l)}.$$

Since $J[\Delta\varphi_l]_{\tau_{g+1},\tau_g}$ are independent for various g, then [27]

$${\sf M}\left\{\left|\sum_{j=0}^{N-1}J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\}=\sum_{j=0}^{N-1}{\sf M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\}+$$

$$+6\sum_{j=0}^{N-1}\mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^2\right\}\sum_{q=0}^{j-1}\mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{q+1},\tau_q}\right|^2\right\}. \qquad (1.21)$$

Because of gaussianity of $J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}$ we have

$$\mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^2\right\}=\int\limits_{\tau_j}^{\tau_{j+1}}(\varphi_l(\tau_j)-\varphi_l(s))^2ds,$$

$$\mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\}=3\Biggl(\int\limits_{\tau_j}^{\tau_{j+1}}(\varphi_l(\tau_j)-\varphi_l(s))^2ds\Biggr)^2.$$

Using this correlations and continuity and as a result the uniform continuity of functions $\varphi_i(s)$, we get the situation when the right part of (1.21) tends to zero when $N\to\infty$.

Considering this fact, as well as (1.20), we come to (1.19).

If for some $l\in\{1,\ldots,k\}:\mathbf{w}_{t_l}^{(i_l)}=t_l$, then proving of this lemma becomes obviously simpler and it is performed similarly. □

According to lemma 1 we have

$$J[\psi^{(k)}]_{T,t}=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_k=0}^{N-1}\ldots\sum_{l_1=0}^{l_2-1}\psi_1(\tau_{l_1})\ldots\psi_k(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_k=0}^{N-1}\ldots\sum_{l_1=0}^{l_2-1}K(\tau_{l_1},\ldots,\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_k=0}^{N-1}\ldots\sum_{l_1=0}^{N-1}K(\tau_{l_1},\ldots,\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=$$

$$=\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_1,\ldots,l_k=0\\ l_q\ne l_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1}K(\tau_{l_1},\ldots,\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}=$$

$$=\int\limits_t^T\ldots\int\limits_t^{t_2}\sum_{(t_1,\ldots,t_k)}\Bigl(K(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\Bigr), \qquad (1.22)$$

where derangements $(t_1,\ldots,t_k)$ for summing are executed only in the expression, enclosed in parentheses.

It is easy to see, that (1.22) may be rewritten in the form:

$$J[\psi^{(k)}]_{T,t}=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}K(t_1,\ldots,t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)},$$

where derangements $(t_1,\ldots,t_k)$ for summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$, at the same time the indexes near upper limits of integration in the multiple stochastic integrals are changed correspondently and if t_r changed places with t_q in the derangement $(t_1,\ldots,t_k)$, then i_r changes places with i_q in the derangement $(i_1,\ldots,i_k)$.

Note, that since integration of bounded function using the set of null measure for Riemann integrals gives zero result, then the following formula is reasonable for these integrals:

$$\int\limits_{[t,T]^k} G(t_1,\ldots,t_k)dt_1\ldots dt_k = \sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2} G(t_1,\ldots,t_k)dt_1\ldots dt_k,$$

where derangements $(t_1,\ldots,t_k)$ for summing are executed only in the values $dt_1,\ldots,dt_k$, at the same time the indexes near upper limits of integration are changed correspondently and the function $G(t_1,\ldots,t_k)$ is considered as integrated in hypercube $[t,T]^k$.

According to lemmas 1 – 3 and (1.17), (1.22) with probubility 1 we get the following representation

$$J[\psi^{(k)}]_{T,t} =$$

$$= \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\int\limits_t^T\ldots\int\limits_t^{t_2}\sum_{(t_1,\ldots,t_k)}\left(\phi_{j_1}(t_1)\ldots\phi_{j_k}(t_k)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}\right)+$$

$$+R_{T,t}^{p_1,\ldots,p_k} = \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\times$$

$$\times\underset{N\to\infty}{\text{l.i.m.}}\sum_{\substack{l_1,\ldots,l_k=0\\ l_q\ne l_r;\ q\ne r;\ q,r=1,\ldots,k}}^{N-1}\phi_{j_1}(\tau_{l_1})\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}+$$

$$+R_{T,t}^{p_1,\ldots,p_k} =$$

$$= \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\left(\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_1,\ldots,l_k=0}^{N-1}\phi_{j_1}(\tau_{l_1})\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}-\right.$$

$$\left.-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\right)+R_{T,t}^{p_1,\ldots,p_k} =$$

$$= \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\left(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}-\right.$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\phi_{j_1}(\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta\mathbf{w}_{\tau_{l_k}}^{(i_k)}\Biggr)+R_{T,t}^{p_1,\ldots,p_k},$$

where

$$R_{T,t}^{p_1,\ldots,p_k}=\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\Biggl(K(t_1,\ldots,t_k)-$$

$$-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\Biggr)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)},\qquad(1.23)$$

where derangements $(t_1,\ldots,t_k)$ for summing are performed only in the values $d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}$, at the same time the indexes near upper limits of integration in the multiple stochastic integrals are changed correspondently and if t_r changed places with t_q in the derangement $(t_1,\ldots,t_k)$, then i_r changes places with i_q in the derangement $(i_1,\ldots,i_k)$.

Let's evaluate the remainder $R_{T,t}^{p_1,\ldots,p_k}$ of the series.

According to lemma 2 we have

$$\mathsf{M}\left\{\left(R_{T,t}^{p_1,\ldots,p_k}\right)^2\right\}\le C_k\sum_{(t_1,\ldots,t_k)}\int\limits_t^T\ldots\int\limits_t^{t_2}\Biggl(K(t_1,\ldots,t_k)-$$

$$-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\Biggr)^2dt_1\ldots dt_k=$$

$$=C_k\int\limits_{[t,T]^k}\Biggl(K(t_1,\ldots,t_k)-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\Biggr)^2dt_1\ldots dt_k\to 0$$

if $p_1,\ldots,p_k\to\infty$, where the constant C_k depends only on the multiplicity k of multiple Ito stochastic integral. The theorem is proven. □

1.3 Expansion of multiple Ito stochastic integrals with multiplicities 1 − 6

In order to evaluate significance of theorem 1 for practice we will demonstrate its transformed particular cases for $k=1,\ldots,6$:

$$\int\limits_t^T\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}=\sum_{j_1=0}^{\infty}C_{j_1}\zeta_{j_1}^{(i_1)},\qquad(1.24)$$

$$\int\limits_t^T\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}=\sum_{j_1,j_2=0}^{\infty}C_{j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}-\mathbf{1}_{\{i_1=i_2\neq 0\}}\mathbf{1}_{\{j_1=j_2\}}\right),\qquad(1.25)$$

$$J[\psi^{(3)}]_{T,t}=\sum_{j_1,j_2,j_3=0}^{\infty}C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr), \tag{1.26}$$

$$J[\psi^{(4)}]_{T,t}=\sum_{j_1,\ldots,j_4=0}^{\infty}C_{j_4\ldots j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}^{(i_l)}-$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-$$

$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}-$$

$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \tag{1.27}$$

$$J[\psi^{(5)}]_{T,t}=\sum_{j_1,\ldots,j_5=0}^{\infty}C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)}-$$

$$-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-$$

$$-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}-$$

$$-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}+$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}+$$

$$+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}+$$

$$+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\Biggr), \tag{1.28}$$

$$J[\psi^{(6)}]_{T,t}=\sum_{j_1,\ldots,j_6=0}^{\infty}C_{j_6\ldots j_1}\Biggl(\prod_{l=1}^{6}\zeta_{j_l}^{(i_l)}-$$

$$-\mathbf{1}_{\{i_1=i_6\ne 0\}}\mathbf{1}_{\{j_1=j_6\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_2=i_6\ne 0\}}\mathbf{1}_{\{j_2=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}$$

$$-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}-\mathbf{1}_{\{i_4=i_6\ne 0\}}\mathbf{1}_{\{j_4=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}$$

$$-\mathbf{1}_{\{i_5=i_6\ne 0\}}\mathbf{1}_{\{j_5=j_6\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}-\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}$$

$$-\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}$$

$$-\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}$$

$$-\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}$$

$$-\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}-\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}$$

$$-\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_5}^{(i_5)}\zeta_{j_6}^{(i_6)}$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}$$
$$+\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_6}^{(i_6)}$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_6}^{(i_6)}$$
$$+\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_6}^{(i_6)}$$
$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}$$
$$+\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}$$
$$+\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_6}^{(i_6)}$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}$$
$$+\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}$$
$$+\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}$$
$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}$$
$$+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}$$

$$
\begin{aligned}
&+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}\\
&+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\\
&+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}\\
&+\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_4}^{(i_4)}\zeta_{j_5}^{(i_5)}\\
&+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\\
&+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\\
&+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_5}^{(i_5)}\\
&+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\\
&+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_5}^{(i_5)}\\
&+\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_5}^{(i_5)}\\
&+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\\
&+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_3)}\\
&+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_4)}\\
&+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\\
&+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_4)}\\
&+\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\\
&-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\\
&-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\\
&-\mathbf{1}_{\{i_6=i_1\ne 0\}}\mathbf{1}_{\{j_6=j_1\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\\
&-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}\\
&-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}\\
&-\mathbf{1}_{\{i_6=i_2\ne 0\}}\mathbf{1}_{\{j_6=j_2\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\\
&-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\\
&-\mathbf{1}_{\{i_6=i_3\ne 0\}}\mathbf{1}_{\{j_6=j_3\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}\\
&-\mathbf{1}_{\{i_3=i_6\ne 0\}}\mathbf{1}_{\{j_3=j_6\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_4=i_5\ne 0\}}\mathbf{1}_{\{j_4=j_5\}}\\
&-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_5\ne 0\}}\mathbf{1}_{\{j_1=j_5\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}
\end{aligned}
$$

$$-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_5\ne 0\}}\mathbf{1}_{\{j_2=j_5\}}$$
$$-\mathbf{1}_{\{i_6=i_4\ne 0\}}\mathbf{1}_{\{j_6=j_4\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_5\ne 0\}}\mathbf{1}_{\{j_3=j_5\}}$$
$$-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_4\ne 0\}}\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{i_2=i_3\ne 0\}}\mathbf{1}_{\{j_2=j_3\}}$$
$$-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{i_3=i_4\ne 0\}}\mathbf{1}_{\{j_3=j_4\}}$$
$$-\mathbf{1}_{\{i_6=i_5\ne 0\}}\mathbf{1}_{\{j_6=j_5\}}\mathbf{1}_{\{i_1=i_3\ne 0\}}\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{i_2=i_4\ne 0\}}\mathbf{1}_{\{j_2=j_4\}}\Biggr), \qquad (1.29)$$

where $\mathbf{1}_A$ — indicator of set A ($\mathbf{1}_A = 1$, if condition A is executed and $\mathbf{1}_A = 0$ otherwise).

1.4 Expansion of multiple Ito integrals of any multiplicity k

Lets generalize formulas (1.24) – (1.29) for the case of any multiplicity of the multiple stochastic Ito integral. In order to do it we will introduce several denotations.

Let's examine the unregulated set $\{1, 2, \ldots, k\}$ and separate it up in two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one — of the remains $k - 2r$ numbers.

So, we have:

$$(\{\underbrace{\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}}_{\text{part 1}}\}, \{\underbrace{q_1, \ldots, q_{k-2r}}_{\text{part 2}}\}), \qquad (1.30)$$

where $\{g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r}\} = \{1, 2, \ldots, k\}$, curly braces mean irregularity of the set taken in them, and the round braces — regularity.

Let's call (1.30) as partition and examine the sum using all possible partitions:

$$\sum_{\substack{(\{\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}\}, \{q_1, \ldots, q_{k-2r}\}) \\ \{g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r}\} = \{1, 2, \ldots, k\}}} a_{g_1 g_2, \ldots, g_{2r-1} g_{2r}, q_1 \ldots q_{k-2r}}. \qquad (1.31)$$

We give an example of sums in the form (1.31):

$$\sum_{\substack{(\{g_1, g_2\}) \\ \{g_1, g_2\} = \{1, 2\}}} a_{g_1 g_2} = a_{12},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\})\\ \{g_1,g_2,g_3,g_4\}=\{1,2,3,4\}}} a_{g_1g_2g_3g_4} = a_{1234} + a_{1324} + a_{2314},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2\})\\ \{g_1,g_2,q_1,q_2\}=\{1,2,3,4\}}} a_{g_1g_2,q_1q_2} = a_{12,34} + a_{13,24} + a_{14,23} +$$

$$+ a_{23,14} + a_{24,13} + a_{34,12},$$

$$\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} a_{g_1g_2,q_1q_2q_3} = a_{12,345} + a_{13,245} + a_{14,235} +$$

$$+ a_{15,234} + a_{23,145} + a_{24,135} + a_{25,134} + a_{34,125} + a_{35,124} + a_{45,123},$$

$$\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} a_{g_1g_2,g_3g_4,q_1} = a_{12,34,5} + a_{13,24,5} + a_{14,23,5} +$$

$$+ a_{12,35,4} + a_{13,25,4} + a_{15,23,4} + a_{12,54,3} + a_{15,24,3} + a_{14,25,3} +$$

$$+ a_{15,34,2} + a_{13,54,2} + a_{14,53,2} + a_{52,34,1} + a_{53,24,1} + a_{54,23,1}.$$

Now we can formulate the basic result of theorem 1 (formula (1.5)) using alternative more comfortable form.

Theorem 2. *In conditions of the theorem* 1 *the following converging in mean-square sense expansion is valid:*

$$J[\psi^{(k)}]_{T,t} = \sum_{j_1,\ldots,j_k=0}^{\infty} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} +$$

$$+ \sum_{r=1}^{\left[\frac{k}{2}\right]}(-1)^r \cdot \sum_{\substack{(\{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}\},\{q_1,\ldots,q_{k-2r}\})\\ \{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}}} \prod_{s=1}^{r} \mathbf{1}_{\{i_{g_{2s-1}}=\ i_{g_{2s}}\neq 0\}}\times$$

$$\times \mathbf{1}_{\{j_{g_{2s-1}}=\ j_{g_{2s}}\}} \prod_{l=1}^{k-2r}\zeta_{j_{q_l}}^{(i_{q_l})}\Biggr). \qquad (1.32)$$

In particular from (1.32) if $k=5$ we obtain:

$$J[\psi^{(5)}]_{T,t} = \sum_{j_1,\ldots,j_5=0}^{\infty} C_{j_5\ldots j_1}\Biggl(\prod_{l=1}^{5}\zeta_{j_l}^{(i_l)} -$$

$$-\sum_{\substack{(\{g_1,g_2\},\{q_1,q_2,q_3\})\\ \{g_1,g_2,q_1,q_2,q_3\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}}\prod_{l=1}^{3}\zeta_{j_{q_l}}^{(i_{q_l})}+$$

$$+\sum_{\substack{(\{\{g_1,g_2\},\{g_3,g_4\}\},\{q_1\})\\ \{g_1,g_2,g_3,g_4,q_1\}=\{1,2,3,4,5\}}} \mathbf{1}_{\{i_{g_1}=\ i_{g_2}\neq 0\}}\mathbf{1}_{\{j_{g_1}=\ j_{g_2}\}}\times$$

$$\left.\times\mathbf{1}_{\{i_{g_3}=\ i_{g_4}\neq 0\}}\mathbf{1}_{\{j_{g_3}=\ j_{g_4}\}}\zeta_{j_{q_1}}^{(i_{q_1})}\right).$$

The last equality obviously agree with (1.28).

1.5 Comparison of theorem 2 with representations of multiple stochastic Ito integrals, based on Hermit polynomials

Note, that rightness of formulas (1.24) – (1.29) can be collaterally verified by the fact, that if $i_1=\ldots=i_6=i=1,\ldots,m$ and $\psi_1(s),\ldots,\psi_6(s)\equiv\psi(s)$, then we can deduce the following equalities which are right with probability 1:

$$J[\psi^{(1)}]_{T,t}=\frac{1}{1!}\delta_{T,t},$$

$$J[\psi^{(2)}]_{T,t}=\frac{1}{2!}\left(\delta_{T,t}^2-\Delta_{T,t}\right),$$

$$J[\psi^{(3)}]_{T,t}=\frac{1}{3!}\left(\delta_{T,t}^3-3\delta_{T,t}\Delta_{T,t}\right),$$

$$J[\psi^{(4)}]_{T,t}=\frac{1}{4!}\left(\delta_{T,t}^4-6\delta_{T,t}^2\Delta_{T,t}+3\Delta_{T,t}^2\right),$$

$$J[\psi^{(5)}]_{T,t}=\frac{1}{5!}\left(\delta_{T,t}^5-10\delta_{T,t}^3\Delta_{T,t}+15\delta_{T,t}\Delta_{T,t}^2\right),$$

$$J[\psi^{(6)}]_{T,t}=\frac{1}{6!}\left(\delta_{T,t}^6-15\delta_{T,t}^4\Delta_{T,t}+45\delta_{T,t}^2\Delta_{T,t}^2-15\Delta_{T,t}^3\right),$$

where $\delta_{T,t}=\int\limits_t^T\psi(s)d\mathbf{f}_s^{(i)}$, $\Delta_{T,t}=\int\limits_t^T\psi^2(s)ds$, which can be independently obtained using the Ito formula (see sect. 7.2).

When $k=1$ everything is evident. Let's examine the cases $k=2,\ 3$. When $k=2$ (we put $p_1=p_2=p$):

$$J[\psi^{(2)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\left(\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}-\sum_{j_1=0}^{p}C_{j_1j_1}\right)=$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \left(\sum_{j_1=0}^{p}\sum_{j_2=0}^{j_1-1}\left(C_{j_2j_1}+C_{j_1j_2}\right)\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}+\sum_{j_1=0}^{p}C_{j_1j_1}\left(\left(\zeta_{j_1}^{(i)}\right)^2-1\right)\right)=$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \left(\sum_{j_1=0}^{p}\sum_{j_2=0}^{j_1-1}C_{j_1}C_{j_2}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}+\frac{1}{2}\sum_{j_1=0}^{p}C_{j_1}^2\left(\left(\zeta_{j_1}^{(i)}\right)^2-1\right)\right)=$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \left(\frac{1}{2}\sum_{\substack{j_1,j_2=0\\ j_1\ne j_2}}^{p}C_{j_1}C_{j_2}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}+\frac{1}{2}\sum_{j_1=0}^{p}C_{j_1}^2\left(\left(\zeta_{j_1}^{(i)}\right)^2-1\right)\right)=$$

$$= \underset{p\to\infty}{\text{l.i.m.}} \left(\frac{1}{2}\left(\sum_{j_1=0}^{p}C_{j_1}\zeta_{j_1}^{(i)}\right)^2-\frac{1}{2}\sum_{j_1=0}^{p}C_{j_1}^2\right)=\frac{1}{2!}\left(\delta_{T,t}^2-\Delta_{T,t}\right). \qquad (1.33)$$

Let's explain the last step in (1.33). For stochastic Ito integrals the following estimation [5] is right:

$$\mathsf{M}\left\{\left|\int_t^T \xi_\tau df_\tau\right|^q\right\} \le K_q \mathsf{M}\left\{\left(\int_t^T |\xi_\tau|^2 d\tau\right)^{\frac{q}{2}}\right\}, \qquad (1.34)$$

where $q>0$ — is a fixed number; f_τ — is a scalar standard Wiener process; $\xi_\tau \in \mathrm{M}_2([t,T])$ (see sect. 7.1); K_q — is a constant, depending only on q;

$$\int_t^T |\xi_\tau|^2 d\tau < \infty \ w.p.1; \ \mathsf{M}\left\{\left(\int_t^T |\xi_\tau|^2 d\tau\right)^{\frac{q}{2}}\right\} < \infty.$$

Since

$$\delta_{T,t}-\sum_{j_1=0}^{p}C_{j_1}\zeta_{j_1}^{(i)}=\int_t^T\left(\psi(s)-\sum_{j_1=0}^{p}C_{j_1}\phi_{j_1}(s)\right)d\mathbf{f}_s^{(i)},$$

then using the estimation (1.34) to the right part of this expression and considering, that

$$\int_t^T\left(\psi(s)-\sum_{j_1=0}^{p}C_{j_1}\phi_{j_1}(s)\right)^2 ds \ \to 0$$

if $p\to\infty$ we obtain

$$\int_t^T \psi(s)d\mathbf{f}_s^{(i)} = q\ \underset{p\to\infty}{\text{-l.i.m.}}\ \sum_{j_1=0}^{p}C_{j_1}\zeta_{j_1}^{(i)},\ q>0. \qquad (1.35)$$

Hence, if $q=4$, then it is easy to conclude, that $\underset{p\to\infty}{\text{l.i.m.}}\left(\sum_{j_1=0}^{p}C_{j_1}\zeta_{j_1}^{(i)}\right)^2 = \delta_{T,t}^2$.

This equality was used in the last transition of the formula (1.33).
If $k=3$ (we put $p_1=p_2=p_3=p$):

$$J[\psi^{(3)}]_{T,t}=\underset{p\to\infty}{\text{l.i.m.}}\left(\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}\zeta_{j_3}^{(i)}-\sum_{j_1,j_3=0}^{p}C_{j_3j_1j_1}\zeta_{j_3}^{(i)}-\right.$$

$$\left.-\sum_{j_1,j_2=0}^{p}C_{j_2j_2j_1}\zeta_{j_1}^{(i)}-\sum_{j_1,j_2=0}^{p}C_{j_1j_2j_1}\zeta_{j_2}^{(i)}\right)=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\left(\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}\zeta_{j_3}^{(i)}-\sum_{j_1,j_3=0}^{p}\left(C_{j_3j_1j_1}+C_{j_1j_1j_3}+C_{j_1j_3j_1}\right)\zeta_{j_3}^{(i)}\right)$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\left(\sum_{j_1=0}^{p}\sum_{j_2=0}^{j_1-1}\sum_{j_3=0}^{j_2-1}\left(C_{j_3j_2j_1}+C_{j_3j_1j_2}+C_{j_2j_1j_3}+C_{j_2j_3j_1}+\right.\right.$$

$$\left.+C_{j_1j_2j_3}+C_{j_1j_3j_2}\right)\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}\zeta_{j_3}^{(i)}+$$

$$+\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}\left(C_{j_3j_1j_3}+C_{j_1j_3j_3}+C_{j_3j_3j_1}\right)\left(\zeta_{j_3}^{(i)}\right)^2\zeta_{j_1}^{(i)}+$$

$$+\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}\left(C_{j_3j_1j_1}+C_{j_1j_1j_3}+C_{j_1j_3j_1}\right)\left(\zeta_{j_1}^{(i)}\right)^2\zeta_{j_3}^{(i)}+\sum_{j_1=0}^{p}C_{j_1j_1j_1}\left(\zeta_{j_1}^{(i)}\right)^3-$$

$$\left.-\sum_{j_1,j_3=0}^{p}\left(C_{j_3j_1j_1}+C_{j_1j_1j_3}+C_{j_1j_3j_1}\right)\zeta_{j_3}^{(i)}\right)=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\left(\sum_{j_1=0}^{p}\sum_{j_2=0}^{j_1-1}\sum_{j_3=0}^{j_2-1}C_{j_1}C_{j_2}C_{j_3}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}\zeta_{j_3}^{(i)}+\right.$$

$$+\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_3}^2C_{j_1}\left(\zeta_{j_3}^{(i)}\right)^2\zeta_{j_1}^{(i)}+\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_1}^2C_{j_3}\left(\zeta_{j_1}^{(i)}\right)^2\zeta_{j_3}^{(i)}+$$

$$\left.+\frac{1}{6}\sum_{j_1=0}^{p}C_{j_1}^3\left(\zeta_{j_1}^{(i)}\right)^3-\frac{1}{2}\sum_{j_1,j_3=0}^{p}C_{j_1}^2C_{j_3}\zeta_{j_3}^{(i)}\right)=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\left(\frac{1}{6}\sum_{\substack{j_1,j_2,j_3=0\\ j_1\neq j_2,j_2\neq j_3,j_1\neq j_3}}^{p}C_{j_1}C_{j_2}C_{j_3}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}\zeta_{j_3}^{(i)}+\right.$$

$$+\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_3}^2C_{j_1}\left(\zeta_{j_3}^{(i)}\right)^2\zeta_{j_1}^{(i)}+\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_1}^2C_{j_3}\left(\zeta_{j_1}^{(i)}\right)^2\zeta_{j_3}^{(i)}+$$

$$\left.+\frac{1}{6}\sum_{j_1=0}^{p}C_{j_1}^3\left(\zeta_{j_1}^{(i)}\right)^3-\frac{1}{2}\sum_{j_1,j_3=0}^{p}C_{j_1}^2C_{j_3}\zeta_{j_3}^{(i)}\right)=\underset{p\to\infty}{\text{l.i.m.}}\left(\frac{1}{6}\sum_{j_1,j_2,j_3=0}^{p}C_{j_1}C_{j_2}C_{j_3}\zeta_{j_1}^{(i)}\zeta_{j_2}^{(i)}\zeta_{j_3}^{(i)}\right.$$

$$-\frac{1}{6}\Biggl(3\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_3}^2C_{j_1}\left(\zeta_{j_3}^{(i)}\right)^2\zeta_{j_1}^{(i)}+3\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_1}^2C_{j_3}\left(\zeta_{j_1}^{(i)}\right)^2\zeta_{j_3}^{(i)}+\sum_{j_1=0}^{p}C_{j_1}^3\left(\zeta_{j_1}^{(i)}\right)^3\Biggr)$$

$$+\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_3}^2C_{j_1}\left(\zeta_{j_3}^{(i)}\right)^2\zeta_{j_1}^{(i)}+\frac{1}{2}\sum_{j_1=0}^{p}\sum_{j_3=0}^{j_1-1}C_{j_1}^2C_{j_3}\left(\zeta_{j_1}^{(i)}\right)^2\zeta_{j_3}^{(i)}+$$

$$+\frac{1}{6}\sum_{j_1=0}^{p}C_{j_1}^3\left(\zeta_{j_1}^{(i)}\right)^3-\frac{1}{2}\sum_{j_1,j_3=0}^{p}C_{j_1}^2C_{j_3}\zeta_{j_3}^{(i)}\Biggr)=$$

$$=\underset{p\to\infty}{\text{l.i.m.}}\left(\frac{1}{6}\left(\sum_{j_1=0}^{p}C_{j_1}\zeta_{j_1}^{(i)}\right)^3-\frac{1}{2}\sum_{j_1=0}^{p}C_{j_1}^2\sum_{j_3=0}^{p}C_{j_3}\zeta_{j_3}^{(i)}\right)=\frac{1}{3!}\left(\delta_{T,t}^3-3\delta_{T,t}\Delta_{T,t}\right). \tag{1.36}$$

The last step in (1.36) is arisen from the equality $\underset{p\to\infty}{\text{l.i.m.}}\left(\sum_{j_1=0}^{p}C_{j_1}\zeta_{j_1}^{(i)}\right)^3=\delta_{T,t}^3$, which can be obtained easily when $q=8$ (see (1.35)).

In addition, we used the following correlations between the Fourier coefficients for the examined case: $C_{j_1j_2}+C_{j_2j_1}=C_{j_1}C_{j_2}$, $2C_{j_1j_1}=C_{j_1}^2$, $C_{j_1j_2j_3}+C_{j_1j_3j_2}+C_{j_2j_3j_1}+C_{j_2j_1j_3}+C_{j_3j_2j_1}+C_{j_3j_1j_2}=C_{j_1}C_{j_2}C_{j_3}$, $2\left(C_{j_1j_1j_3}+C_{j_1j_3j_1}+C_{j_3j_1j_1}\right)=C_{j_1}^2C_{j_3}$, $6C_{j_1j_1j_1}=C_{j_1}^3$ and the formula (2.162) if $k=2,\ 3$.

Cases $k=4,\ 5,\ 6$ can be analyzed similarly using the formula (2.162) when $k=4,\ 5,\ 6$.

1.6 On usage of full orthonormal discontinuous systems of functions in theorem 1

Analyzing the proof of theorem 1, we can ask a natural question: can we weaken the condition of continuity of functions $\phi_j(x)$; $j=1,\ 2,\ldots$?

We will tell, that the function $f(x):[t,T]\to\Re^1$ satisfies the condition $(\star)$, if it is continuouse at the interval $[t,T]$ except may be for the finite number of points of the finite discontinuity, as well as it is continuouse from the right at the interval $[t,T]$.

Afterwards, let's suppose, that $\{\phi_j(x)\}_{j=0}^{\infty}$ — is a full orthonormal system of functions in the space $L_2([t,T])$, moreover $\phi_j(x)$, $j<\infty$ satisfies the condition $(\star)$.

It is easy to see, that the continuity of function $\phi_j(x)$ was used substantially for proving of theorem 1 in two places: lemma 3 and formula (1.13). It's clear, that without damage to generality, partition $\{\tau_j\}_{j=0}^{N}$ of the interval $[t,T]$ in lemma 3 and formula (1.13) can be taken so "small", that among the points τ_j of this partition will be all points of jumps of functions $\varphi_1(\tau) = \phi_{j_1}(\tau), \ldots, \varphi_k(\tau) = \phi_{j_k}(\tau)$; $j_1, \ldots, j_k < \infty$ and among the points $(\tau_{j_1}, \ldots, \tau_{j_k})$; $0 \le j_1 < \ldots < j_k \le N-1$ there will be all points of jumps of function $\Phi(t_1, \ldots, t_k)$.

Let's demonstrate how to modify proofs of lemma 3 and formula (1.13) in the case when $\{\phi_j(x)\}_{j=0}^{\infty}$ — is a full orthonormal system of functions in the space $L_2([t,T])$, moreover $\phi_j(x)$, $j < \infty$ satisfies the condition $(\star)$.

At first, appeal to lemma 3. Proving this lemma we got the following relations:

$$\mathsf{M}\left\{\left|\sum_{j=0}^{N-1} J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\} = \sum_{j=0}^{N-1} \mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\} +$$

$$+6\sum_{j=0}^{N-1} \mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^2\right\}\sum_{q=0}^{j-1} \mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{q+1},\tau_q}\right|^2\right\}, \qquad (1.37)$$

$$\mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^2\right\} = \int\limits_{\tau_j}^{\tau_{j+1}} (\varphi_l(\tau_j) - \varphi_l(s))^2 ds, \qquad (1.38)$$

$$\mathsf{M}\left\{\left|J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\} = 3\left(\int\limits_{\tau_j}^{\tau_{j+1}} (\varphi_l(\tau_j) - \varphi_l(s))^2 ds\right)^2. \qquad (1.39)$$

Propose, that functions $\varphi_l(s)$; $l = 1, \ldots, k$ satisfy the condition $(\star)$, and the partition $\{\tau_j\}_{j=0}^{N-1}$ includes all points of jumps of functions $\varphi_l(s)$; $l = 1, \ldots, k$. It means, that, for the integral $\int\limits_{\tau_j}^{\tau_{j+1}} (\varphi_l(\tau_j) - \varphi_l(s))^2 ds$ the subintegral function is continuous at the interval $[\tau_j,\ \tau_{j+1})$ and possibly it has finite discontinuity in the point τ_{j+1}.

Let $\mu \in (0,\ \Delta\tau_j)$ is fixed, then, because of continuity which means uniform continuity of the functions $\varphi_l(s)$; $l = 1, \ldots, k$ at the interval $[\tau_j,\ \tau_{j+1} - \mu]$ we have:

$$\int\limits_{\tau_j}^{\tau_{j+1}} (\varphi_l(\tau_j) - \varphi_l(s))^2 ds = \int\limits_{\tau_j}^{\tau_{j+1}-\mu} (\varphi_l(\tau_j) - \varphi_l(s))^2 ds +$$

$$+ \int\limits_{\tau_{j+1}-\mu}^{\tau_{j+1}} (\varphi_l(\tau_j) - \varphi_l(s))^2 ds < \varepsilon^2(\Delta\tau_j - \mu) + M^2\mu. \qquad (1.40)$$

Obtaining the inequality (1.40) we proposed, that $\Delta\tau_j < \delta$; $j = 0,\ 1, \ldots, N-1$ ($\delta > 0$ is exist for all $\varepsilon > 0$ and it doesn't depend on s); $|\varphi_l(\tau_j) - \varphi_l(s)| < \varepsilon$ if $s \in [\tau_{j+1} - \mu,\ \tau_{j+1}]$ (because of uniform continuity of functions $\varphi_l(s)$; $l = 1, \ldots, k$); $|\varphi_l(\tau_j) - \varphi_l(s)| < M$, M — is a constant;

potential point of discontinuity of function $\varphi_l(s)$ is supposed in the point τ_{j+1}.

Performing the passage to the limit in the inequality (1.40) when $\mu \to +0$, we get

$$\int\limits_{\tau_j}^{\tau_{j+1}} (\varphi_l(\tau_j) - \varphi_l(s))^2 ds \le \varepsilon^2 \Delta\tau_j.$$

Using this estimation for evaluation of the right part (1.37) we get

$$\mathsf{M}\left\{\left|\sum_{j=0}^{N-1} J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\} \le \varepsilon^4\Biggl(3\sum_{j=0}^{N-1}(\Delta\tau_j)^2 + 6\sum_{j=0}^{N-1}\Delta\tau_j\sum_{q=0}^{j-1}\Delta\tau_q\Biggr) <$$

$$< 3\varepsilon^4\left(\delta(T-t) + (T-t)^2\right). \qquad (1.41)$$

This implies, that $\mathsf{M}\left\{\left|\sum\limits_{j=0}^{N-1} J[\Delta\varphi_l]_{\tau_{j+1},\tau_j}\right|^4\right\} \to 0$ when $N \to \infty$ and lemma 3 remains reasonable.

Now, let's present explanations concerning the rightness of the formula (1.13), when $\{\phi_j(x)\}_{j=0}^{\infty}$ — is a full orthonormal system of functions in the space $L_2([t,T])$, moreover $\phi_j(x)$, $j < \infty$ satisfies the condition $(\star)$.

Let's examine the case $k = 3$ and representation (1.15). We can demonstrate, that in the studied case the first limit in the right part of (1.15) equals to zero (similarly we demonstrate, that the second limit in the right part of (1.15) equals to zero; proving of the second limit equality to zero in the right part of the formula (1.14) is the same as for the case of continuous functions $\phi_j(x)$; $j = 0,\ 1,\ldots$).

The second moment of prelimit expression of the first limit in the right part of (1.15) looks as follows:

$$\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1}\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}}\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \left(\Phi(t_1,t_2,\tau_{j_3}) - \Phi(t_1,\tau_{j_2},\tau_{j_3})\right)^2 dt_1 dt_2 \Delta\tau_{j_3}.$$

Further, for the fixed $\mu \in (0,\ \Delta\tau_{j_2})$ and $\rho \in (0,\ \Delta\tau_{j_1})$ we have

$$\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}}\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \left(\Phi(t_1,t_2,\tau_{j_3}) - \Phi(t_1,\tau_{j_2},\tau_{j_3})\right)^2 dt_1 dt_2 =$$

$$= \Biggl(\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}-\mu} + \int\limits_{\tau_{j_2+1}-\mu}^{\tau_{j_2+1}}\Biggr)\Biggl(\int\limits_{\tau_{j_1}}^{\tau_{j_1+1}-\rho} + \int\limits_{\tau_{j_1+1}-\rho}^{\tau_{j_1+1}}\Biggr) \left(\Phi(t_1,t_2,\tau_{j_3}) - \Phi(t_1,\tau_{j_2},\tau_{j_3})\right)^2 dt_1 dt_2$$

$$
= \left(\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}-\mu} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}-\rho} + \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}-\mu} \int\limits_{\tau_{j_1+1}-\rho}^{\tau_{j_1+1}} + \int\limits_{\tau_{j_2+1}-\mu}^{\tau_{j_2+1}} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}-\rho} + \int\limits_{\tau_{j_2+1}-\mu}^{\tau_{j_2+1}} \int\limits_{\tau_{j_1+1}-\rho}^{\tau_{j_1+1}} \right) \times
$$
$$
\times \left(\Phi(t_1,t_2,\tau_{j_3}) - \Phi(t_1,\tau_{j_2},\tau_{j_3})\right)^2 dt_1 dt_2 <
$$
$$
< \varepsilon^2 \left(\Delta\tau_{j_2}-\mu\right)\left(\Delta\tau_{j_1}-\rho\right) + M^2\rho\left(\Delta\tau_{j_2}-\mu\right) + M^2\mu\left(\Delta\tau_{j_1}-\rho\right) + M^2\mu\rho, \tag{1.42}
$$

where M — is a constant; $\Delta\tau_j < \delta$; $j = 0,\ 1,\ldots,N-1$ ($\delta > 0$ is exists for all $\varepsilon > 0$ and it doesn't depend on points (t_1,t_2,τ_{j_3}), $(t_1,\tau_{j_2},\tau_{j_3})$); we also propose that, the partition $\{\tau_j\}_{j=0}^{N-1}$ contains all points of discontinuity of the function $\Phi(t_1,t_2,t_3)$ as points τ_j (for every variable).

When obtaining of (1.42) we also suppose, that potential points of discontinuity of this function (for every variable) are in points $\tau_{j_1+1}, \tau_{j_2+1}, \tau_{j_3+1}$.

Let's explain in details how we obtained the inequality (1.42). Since the function $\Phi(t_1,t_2,t_3)$ is continuous at the closed bounded set $Q_3 = \{(t_1,t_2,t_3) : t_1 \in [\tau_{j_1},\tau_{j_1+1}-\rho], t_2 \in [\tau_{j_2},\tau_{j_2+1}-\mu], t_3 \in [\tau_{j_3},\tau_{j_3+1}-\nu],\}$, where ρ,μ,ν — are fixed small positive numbers ($\nu \in (0,\ \Delta\tau_{j_3})$, $\mu \in (0,\ \Delta\tau_{j_2})$, $\rho \in (0,\ \Delta\tau_{j_1})$), then this function is also uniformly continous at this set and bounded at closed set D_3 (see sect. 1.2).

Since the distance between points (t_1,t_2,τ_{j_3}), $(t_1,\tau_{j_2},\tau_{j_3}) \in Q_3$ is obviously less than δ ($\Delta\tau_j < \delta$; $j = 0,\ 1,\ldots,N-1$), then $|\Phi(t_1,t_2,\tau_{j_3}) - \Phi(t_1,\tau_{j_2},\tau_{j_3})| < \varepsilon$. This inequality was used during estimation of the first double integral in (1.42). Estimating of three remaining double integrals we used the feature of limitation of function $\Phi(t_1,t_2,t_3)$ in form of inequality $|\Phi(t_1,t_2,\tau_{j_3}) - \Phi(t_1,\tau_{j_2},\tau_{j_3})| < M$.

Performing the passage to the limit in the inequality (1.42) when $\mu,\rho \to +0$ we obtain the estimation

$$
\int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \left(\Phi(t_1,t_2,\tau_{j_3}) - \Phi(t_1,\tau_{j_2},\tau_{j_3})\right)^2 dt_1 dt_2 \le \varepsilon^2 \Delta\tau_{j_2}\Delta\tau_{j_1}.
$$

Usage of this estimation provides

$$
\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \left(\Phi(t_1,t_2,\tau_{j_3}) - \Phi(t_1,\tau_{j_2},\tau_{j_3})\right)^2 dt_1 dt_2 \Delta\tau_{j_3} \le
$$
$$
\le \varepsilon^2 \sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1} \Delta\tau_{j_1}\Delta\tau_{j_2}\Delta\tau_{j_3} < \varepsilon^2 \frac{(T-t)^3}{6}.
$$

The last evaluation means, that in the considered case the first limit in the right part of (1.15) equals to zero (similarly we may demonstrate, that the second limit in the right part of (1.15) equals to zero).

Consequently, formula (1.13) is reasonable when $k = 3$ in the analyzed case. Similarly, we perform argumentation for the case when $k = 2$ and $k > 3$.

Consequently, in theorem 1 we can use full orthonormal systems of functions $\{\phi_j(x)\}_{j=0}^{\infty}$ in the space $L_2([t, T])$, for which $\phi_j(x)$, $j < \infty$ satisfies the condition $(\star)$.

The example of such system of functions may serve as a full orthonormal system of Haar functions in the space $L_2([t, T])$:

$$\phi_0(x) = \frac{1}{\sqrt{T-t}},\ \phi_{nj}(x) = \frac{1}{\sqrt{T-t}}\varphi_{nj}\left(\frac{x-t}{T-t}\right),$$

where $n = 0,\ 1, \ldots;\ j = 1,\ 2, \ldots, 2^n$ and functions $\varphi_{nj}(x)$ has the following form:

$$\varphi_{nj}(x) = \begin{cases} 2^{\frac{n}{2}},\ x \in [\frac{j-1}{2^n},\ \frac{j-1}{2^n} + \frac{1}{2^{n+1}}) \\ -2^{\frac{n}{2}},\ x \in [\frac{j-1}{2^n} + \frac{1}{2^{n+1}},\ \frac{j}{2^n}) \\ 0 \text{ otherwise} \end{cases};$$

$n = 0,\ 1, \ldots;\ j = 1,\ 2, \ldots,\ 2^n$ (we choose the values of Haar functions in the points of discontinuity in order they will be continuous at the right).

The other example of similar system of functions is a full orthonormal Rademacher-Walsh system of functions in the space $L_2([t, T])$:

$$\phi_0(x) = \frac{1}{\sqrt{T-t}},\ \phi_{m_1 \ldots m_k}(x) = \frac{1}{\sqrt{T-t}}\varphi_{m_1}\left(\frac{x-t}{T-t}\right) \ldots \varphi_{m_k}\left(\frac{x-t}{T-t}\right),$$

where $0 < m_1 < \ldots < m_k;\ m_1, \ldots, m_k = 1,\ 2, \ldots;\ k = 1,\ 2, \ldots;\ \varphi_m(x) = (-1)^{[2^m x]};\ x \in [0, 1];\ m = 1,\ 2, \ldots;\ [y]$ — integer part of y.

1.7 Remarks about usage of full orthonormal systems in theorem 1

Note, that actually functions $\phi_j(s)$ of the full orthonormal system of functions $\{\phi_j(s)\}_{j=0}^{\infty}$ in the space $L_2([t, T])$ depend not only on s, but on $t,\ T$.

For example, full orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t, T])$ have the following form:

$$\phi_j(s, t, T) = \sqrt{\frac{2j+1}{T-t}}P_j\left(\left(s - \frac{T+t}{2}\right)\frac{2}{T-t}\right),$$

$P_j(s)$ — Legendre polynomials;

$$\phi_j(s,t,T)=\frac{1}{\sqrt{T-t}}\begin{cases}1 & \text{when } j=0\\ \sqrt{2}\sin\frac{2\pi r(s-t)}{T-t} & \text{when } j=2r-1\\ \sqrt{2}\cos\frac{2\pi r(s-t)}{T-t} & \text{when } j=2r\end{cases};$$

$r=1,\ 2,\ldots.$

Note, that the specified systems of functions will be used in the context of realizing of numerical methods for stochastic differential Ito equations for the sequences of time intervals: $[T_0,T_1]$, $[T_1,T_2]$, $[T_2,T_3],\ldots,$ and spaces $L_2([T_0,T_1])$, $L_2([T_1,T_2])$, $L_2([T_2,T_3]),\ldots.$

We can explain, that the dependence of functions $\phi_j(s,t,T)$ from $t,\ T$ (hereinafter these constants will mean fixed moments of time) will not affect the main characteristics of independence of random variables. $\zeta_{(j_l)T,t}^{(i_l)}=\int\limits_t^T\phi_{j_l}(s,t,T)d\mathbf{w}_s^{(i_l)};\ i_l\ne 0;\ l=1,\ldots,k.$

Indeed, for fixed $t,\ T$ due to orthonormality of mentioned systems of functions, we have:

$$\mathsf{M}\left\{\zeta_{(j_l)T,t}^{(i_l)}\zeta_{(j_r)T,t}^{(i_r)}\right\}=\mathbf{1}_{\{i_l=i_r\ne 0\}}\mathbf{1}_{\{j_l=j_r\}},$$

where $\zeta_{(j_l)T,t}^{(i_l)}=\int\limits_t^T\phi_{j_l}(s,t,T)d\mathbf{w}_s^{(i_l)};\ i_l\ne 0;\ l,\ r=1,\ldots,k.$

On the other side random variables $\zeta_{(j_l)T_1,t_1}^{(i_l)}=\int\limits_{t_1}^{T_1}\phi_{j_l}(s,t_1,T_1)d\mathbf{w}_s^{(i_l)}$ and $\zeta_{(j_l)T_2,t_2}^{(i_l)}=\int\limits_{t_2}^{T_2}\phi_{j_l}(s,t_2,T_2)d\mathbf{w}_s^{(i_l)}$ are independent if $[t_1,T_1]\cap[t_2,T_2]=\emptyset$ (the case $T_1=t_2$ is possible) according to property of stochastic Ito integrals.

Therefore, two important characteristics of random variables $\zeta_{(j_l)T,t}^{(i_l)}$ which are the basic motive of their usage are stored.

In the future, as it was before, instead of $\phi_j(s,t,T)$ we will write $\phi_j(s)$ and instead of $\zeta_{(j)T,t}^{(i)}$ we will write $\zeta_j^{(i)}$ for brevity sake.

1.8 Convergence in the mean of degree $2n$ of expansion of multiple stochastic Ito integrals from theorem 1

Creating expansions of stochastic Ito integrals from theorem 1 we stored all information about these integrals, that is why it is natural to expect, that the mentioned expansions will be converged not only in the mean-square sense but in the stronger probability meanings.

We will obtain the common evaluation which proves convergence in the mean of degree $2n$, $n \in N$ of approximations from theorem 1.

According to notations of theorem 1:

$$R_{T,t}^{p_1,\ldots,p_k} = \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} R_{p_1\ldots p_k}(t_1,\ldots,t_k) d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)}, \qquad (1.43)$$

$$R_{p_1\ldots p_k}(t_1,\ldots,t_k) \stackrel{\text{def}}{=} K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l).$$

For definiteness we will consider, that $i_1,\ldots,i_k = 1,\ldots,m$ (it is obviously quite enough for unified Taylor-Ito expansion (see sect. 7.9) and we can see decoding of other notations used in this section at the text of proving of theorem 1.

Note, that proving of theorem 1 we obtained, that

$$\mathsf{M}\{(R_{T,t}^{p_1,\ldots,p_k})^2\} \le C_k \sum_{(t_1,\ldots,t_k)} \int\limits_t^T \ldots \int\limits_t^{t_2} R_{p_1\ldots p_k}^2(t_1,\ldots,t_k) dt_1 \ldots dt_k =$$

$$= C_k \int\limits_{[t,T]^k} R_{p_1\ldots p_k}^2(t_1,\ldots,t_k) dt_1 \ldots dt_k, \ C_k < \infty.$$

Assume, that

$$\eta_{t_l,t}^{(l-1)} \stackrel{\text{def}}{=} \int\limits_t^{t_l} \ldots \int\limits_t^{t_2} R_{p_1\ldots p_k}(t_1,\ldots,t_k) d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_{l-1}}^{(i_{l-1})}; \ l = 2,\ 3,\ldots,k+1,$$

$$\eta_{t_{k+1},t}^{(k)} \stackrel{\text{def}}{=} \eta_{T,t}^{(k)}; \ \eta_{T,t}^{(k)} \stackrel{\text{def}}{=} \int\limits_t^T \ldots \int\limits_t^{t_2} R_{p_1\ldots p_k}(t_1,\ldots,t_k) d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)}.$$

Using Ito formula (see sect. 7.3) it is easy to demonstrate, that

$$\mathsf{M}\left\{\left(\int\limits_{t_0}^t \xi_\tau df_\tau\right)^{2n}\right\} = n(2n-1)\int\limits_{t_0}^t \mathsf{M}\left\{\left(\int\limits_{t_0}^s \xi_u df_u\right)^{2n-2} \xi_s^2\right\} ds.$$

Using the Holder inequality in the right part under the sign of integration if $p = n/(n-1)$, $q = n$ and using the increase of value $\mathsf{M}\left\{\left(\int\limits_{t_0}^t \xi_\tau df_\tau\right)^{2n}\right\}$ with the growth t, we get:

$$\mathsf{M}\left\{\left(\int\limits_{t_0}^t \xi_\tau df_\tau\right)^{2n}\right\} \le n(2n-1)\left(\mathsf{M}\left\{\left(\int\limits_{t_0}^t \xi_\tau df_\tau\right)^{2n}\right\}\right)^{\frac{n-1}{n}} \int\limits_{t_0}^t (\mathsf{M}\{\xi_s^{2n}\})^{\frac{1}{n}} ds.$$

Raising to power n the obtained inequality and dividing it on

$$\left(\mathsf{M}\left\{\left(\int\limits_{t_0}^{t}\xi_\tau df_\tau\right)^{2n}\right\}\right)^{n-1}$$

we get the following estimation

$$\mathsf{M}\left\{\left(\int\limits_{t_0}^{t}\xi_\tau df_\tau\right)^{2n}\right\}\le (n(2n-1))^n\left(\int\limits_{t_0}^{t}\left(\mathsf{M}\{\xi_s^{2n}\}\right)^{\frac{1}{n}}ds\right)^n. \tag{1.44}$$

Using estimation (1.44) we have

$$\mathsf{M}\{(\eta_{T,t}^{(k)})^{2n}\}\le (n(2n-1))^n\left[\int\limits_{t}^{T}\left(\mathsf{M}\{(\eta_{t_k,t}^{(k-1)})^{2n}\}\right)^{\frac{1}{n}}dt_k\right]^n\le$$

$$\le (n(2n-1))^n\left[\int\limits_{t}^{T}\left((n(2n-1))^n\left[\int\limits_{t}^{t_k}\left(\mathsf{M}\{(\eta_{t_{k-1},t}^{(k-2)})^{2n}\}\right)^{\frac{1}{n}}dt_{k-1}\right]^n\right)^{\frac{1}{n}}dt_k\right]^n=$$

$$=(n(2n-1))^{2n}\left[\int\limits_{t}^{T}\int\limits_{t}^{t_k}\left(\mathsf{M}\{(\eta_{t_{k-1},t}^{(k-2)})^{2n}\}\right)^{\frac{1}{n}}dt_{k-1}dt_k\right]^n\le\ldots$$

$$\ldots\le (n(2n-1))^{n(k-1)}\left[\int\limits_{t}^{T}\int\limits_{t}^{t_k}\ldots\int\limits_{t}^{t_3}\left(\mathsf{M}\{(\eta_{t_2,t}^{(1)})^{2n}\}\right)^{\frac{1}{n}}dt_3\ldots dt_{k-1}dt_k\right]^n=$$

$$=(n(2n-1))^{n(k-1)}(2n-1)!!\left[\int\limits_{t}^{T}\ldots\int\limits_{t}^{t_2}R_{p_1\ldots p_k}^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right]^n\le$$

$$\le (n(2n-1))^{n(k-1)}(2n-1)!!\left[\int\limits_{[t,T]^k}R_{p_1\ldots p_k}^2(t_1,\ldots,t_k)dt_1\ldots dt_k\right]^n.$$

The next to last step was obtained using the formula

$$\mathsf{M}\{(\eta_{t_2,t}^{(1)})^{2n}\}=(2n-1)!!\left[\int\limits_{t}^{t_2}R_{p_1\ldots p_k}^2(t_1,\ldots,t_k)dt_1\right]^n,$$

which follows from gaussianity of $\eta_{t_2,t}^{(1)}=\int\limits_{t}^{t_2}R_{p_1\ldots p_k}(t_1,\ldots,t_k)d\mathbf{f}_{t_1}^{(i_1)}$.

Similarly we estimate each summand in the right part of (1.43). Then, from (1.43) using Minkowsky inequality we finally get

$$\mathsf{M}\{(R_{T,t}^{p_1,\ldots,p_k})^{2n}\}\le$$

$$\leq \Biggl(k!\Biggl((n(2n-1))^{n(k-1)}(2n-1)!!\Biggl[\int\limits_{[t,T]^k} R^2_{p_1\ldots p_k}(t_1,\ldots,t_k)dt_1\ldots dt_k\Biggr]^n\Biggr)^{\frac{1}{2n}}\Biggr)^{2n}$$

$$= (k!)^{2n}(n(2n-1))^{n(k-1)}(2n-1)!!\Biggl[\int\limits_{[t,T]^k} R^2_{p_1\ldots p_k}(t_1,\ldots,t_k)dt_1\ldots dt_k\Biggr]^n. \tag{1.45}$$

The inequality (1.45) means, that approximations of multiple stochastic Ito integrals, obtained using theorem 1, converge in the mean of degree $2n$, $n\in N$, as according to this theorem $\int\limits_{[t,T]^k} R^2_{p_1\ldots p_k}(t_1,\ldots,t_k)dt_1\ldots dt_k\to 0$ when $p_1,\ldots,p_k\to\infty$.

Chapter 2

Expansions of multiple stochastic Stratonovich integrals, based on generalized multiple and repeated Fourier series

This chapter is devoted to expansions of multiple Startonovich stochastic integrals. We adapted the results of chapter 1 (theorem 1) for multiple Stratonovich stochastic integrals of multiplicity $1-3$. Also, we consider other approach to expansion of multiple Stratonovich stochastic integrals of any fixed multiplicity k, based on repeated generalized Fourier series converging pointwise.

2.1 Expansions of multiple stochastic Stratonovich integrals of 1st and 2nd multiplicity

Assume, that $\psi_1(\tau),\ \psi_2(\tau)$ — are continuously differentiated functions at the interval $[t,T]$. For the case $k=1$ we obviously have

$$\int\limits_t^{*T}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}=\sum_{j_1=0}^{\infty}C_{j_1}\zeta_{j_1}^{(i_1)},$$

where the series converges in the mean-square sense;

$$C_{j_1}=\int\limits_t^T\psi_1(t_1)\phi_{j_1}(t_1)dt_1;\ \zeta_{j_1}^{(i_1)}=\int\limits_t^T\phi_{j_1}(t_1)d\mathbf{f}_{t_1}^{(i_1)};\ i_1=1,\ldots,m.$$

According to the standard connection of stochastic Stratonovich and Ito integrals with probability 1 we have

$$J^*[\psi^{(2)}]_{T,t}=J[\psi^{(2)}]_{T,t}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\neq 0\}}\int\limits_t^T\psi_1(t_1)\psi_2(t_1)dt_1. \qquad (2.1)$$

On the other side, according to theorem 1

$$\int\limits_t^T \psi_2(t_2)\int\limits_t^{t_2} \psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)} = \sum_{j_1,j_2=0}^{\infty} C_{j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\ne 0\}}\mathbf{1}_{\{j_1=j_2\}}\right) =$$

$$= \sum_{j_1,j_2=0}^{\infty} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2\ne 0\}}\sum_{j_1=0}^{\infty} C_{j_1j_1}.$$

The following natural questions take place: is it legal the partition of limit in two limits in the last formula and is the following equality reasonable (it proves the possibility of such partition):

$$\frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)dt_1 = \sum_{j_1=0}^{\infty} C_{j_1j_1}. \tag{2.2}$$

Since if $\psi_1(s)\equiv\psi_2(s)$ the equality $C_{j_1j_1}=\frac{1}{2}C_{j_1}^2$ is realized, then in this case the equality (2.2) is a conclusion of Parseval equality.

In order to check the equality (2.2) in the general case it is enought to demonstrate, that

$$\sum_{j_1,j_2=0}^{\infty} C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1) = \frac{1}{2}\psi_1(t_1)\psi_2(t_1), \tag{2.3}$$

where the multiple series converges uniformly according to variable t_1 at the interval $[t,T]$.

Evidently upon integrating the equality (2.3) and using the orthonormality of functions $\phi_j(\tau)$ we get the equality (2.2), that in its turn will mean that

$$J^*[\psi^{(2)}]_{T,t} = \sum_{j_1,j_2=0}^{\infty} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}, \tag{2.4}$$

where the series converge in the mean-square sense.

Let's prove (2.2) in more general case.

In order to prove (2.3) we should refer to the facts taken from the theory of multiple Fourier series, summarized in accordance with Princeheim.

For each $\delta>0$ *let's call the exact upper edge of difference* $|f(\mathbf{t}')-f(\mathbf{t}'')|$ *in the set of all points* $\mathbf{t}'$, $\mathbf{t}''$ *which belong to the domain* D*, as the module of continuity of function* $f(\mathbf{t})$ $(\mathbf{t}=(t_1,\ldots,t_k))$ *in the* k*-dimentional domain* D $(k\ge 1)$*, moreover distance satisfies the formula* $\rho(\mathbf{t}',\mathbf{t}'')<\delta$.

We will declare, that the function of k $(k\ge 1)$ *variables* $f(\mathbf{t})$ $(\mathbf{t}=(t_1,\ldots,t_k))$ *belongs to the Holder class with the parameter* 1 $(f(\mathbf{t})\in C^1(D))$ *in domain* D *if the module of continuity of function* $f(\mathbf{t})$ $(\mathbf{t}=(t_1,\ldots,t_k))$ *in the domain* D *has the order* $O(\delta)$.

Let's analyze the full full orthonormal system of functions $\{\phi_j(x)\}_{j=0}^{\infty}$ in the space $L_2([t,T])$.

We will declare that systen satisfies the condition $(\star\star)$, is the system $\{\phi_j(x)\}_{j=0}^{\infty}$ is such that for any function of k variables, which is continuous at the hypercube $[t,T]^k$ and belongs to the class $C^1([t,T]^k)$ its multiple Fourier series (summarized using the method of rectangular sums) at the $[t,T]^k$ built according to the system of functions $\{\phi_j(x)\}_{j=0}^{\infty}$ converge uniformly to this function within $[t,T]^k$ and it converge at the border of the hypercube $[t,T]^k$.

In 1967, L.V. Zhizhiashvili proved, that the rectangular sums of multiple trigonometric Fourier series in the hypercube $[t,T]^k$ of the function of k variables converge uniformly in the hypercube to this function, if it belongs to $C^{\alpha}([t,T]^k)$; $\alpha>0$ (definition of Holder class with the index $\alpha>0$ may be found in the well-known mathematical analysis manuals).

It is also well known, that for rightness of the similar statement for Fourier-Haar series, at least for a two-dimensional case, it is enough to have only continuity of function of two variables in the square $[t,T]^2$.

The author thinks, that for double Fourier-Legendre series the similar formulation will be true, if the function of two variables belongs to $C^1([t,T]^2)$. If this condition is not enough, then at least the result will be correct if the function is constant in $[t,T]^2$ (it corresponds to $\psi_i(\tau)\equiv 1$; $i=1,2,3$ in the following arguments).

Note that, according to (5.36), (5.37) the formula (2.2) is right, at least if $\psi_1(\tau)\equiv t-\tau$, $\psi_2(\tau)\equiv 1$; $\psi_1(\tau)\equiv 1$, $\psi_2(\tau)\equiv t-\tau$; $\psi_1(\tau)$, $\psi_2(\tau)\equiv t-\tau$; $\psi_1(\tau)\equiv (t-\tau)^2$, $\psi_2(\tau)\equiv 1$; $\psi_1(\tau)\equiv 1$, $\psi_2(\tau)\equiv (t-\tau)^2$; $\tau\in[t,T]$.

Let's analyze the auxiliary function:

$$K'(t_1,t_2)=\begin{cases}\psi_2(t_1)\psi_1(t_2), & t_1\ge t_2\\ \psi_1(t_1)\psi_2(t_2), & t_1\le t_2\end{cases},\quad t_1,t_2\in[t,T]$$

and demonstrate, that it belongs to $C^1([t,T]^2)$.

Let's analyze the increment: $\Delta K'=K'(t_1,t_2)-K'(t_1^*,t_2^*)$, where

$$\sqrt{(t_1-t_1^*)^2+(t_2-t_2^*)^2}<\delta,\ (t_1,t_2),\ (t_1^*,t_2^*)\in[t,T]^2.$$

Using the Lagrange formula for $\psi_1(t_1^*)$, $\psi_2(t_1^*)$ at $[\min\{t_1,t_1^*\},\max\{t_1,t_1^*\}]$ and for $\psi_1(t_2^*)$, $\psi_2(t_2^*)$ at $[\min\{t_2,t_2^*\},\max\{t_2,t_2^*\}]$ we will come to the representation

$$\Delta K'=\begin{cases}\psi_2(t_1)\psi_1(t_2), & t_1\ge t_2\\ \psi_1(t_1)\psi_2(t_2), & t_1\le t_2\end{cases}-\begin{cases}\psi_2(t_1)\psi_1(t_2), & t_1^*\ge t_2^*\\ \psi_1(t_1)\psi_2(t_2), & t_1^*\le t_2^*\end{cases}+O(\delta).$$

Hereafter, it is clear, that the difference staying in the right part of the last equality is different from zero and equals to

$$\pm(\psi_1(t_1)\psi_2(t_2)-\psi_1(t_2)\psi_2(t_1))+O(\delta) \qquad (2.5)$$

on the set: $\text{M}=\{(t_1,t_2): \min\{t_1,t_1+\varepsilon\}\le t_2\le \max\{t_1,t_1+\varepsilon\};\ t_1\in[t,T]\}$, where $\varepsilon=(t_1^*-t_1)-(t_2^*-t_2)=O(\delta)$.

Since we have $|t_2-t_1|=O(\delta)$ on the set M, then using the Lagrange formula to $\psi_2(t_2)$, $\psi_1(t_2)$ at the interval $[\min\{t_1,t_2\},\ \max\{t_1,t_2\}]$ and substituting the result into (2.5), we will get, that $K'(t_1,t_2)\in C^1([t,T]^2)$.

Let's expand the function $K'(t_1,t_2)$ in the square $[t,T]^2$ into the multiple Fourier series, summarized using the method of rectangular sums, i.e.

$$K'(t_1,t_2)=\lim_{n_1,n_2\to\infty}\sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2}\int\limits_t^T\int\limits_t^T K'(t_1,t_2)\phi_{j_1}(t_1)\phi_{j_2}(t_2)dt_1dt_2\cdot\phi_{j_1}(t_1)\phi_{j_2}(t_2)=$$

$$=\lim_{n_1,n_2\to\infty}\sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2}\Biggl(\int\limits_t^T\psi_2(t_2)\phi_{j_2}(t_2)\Biggl(\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\Biggr)dt_2+$$

$$+\int\limits_t^T\psi_1(t_2)\phi_{j_2}(t_2)\Biggl(\int\limits_{t_2}^T\psi_2(t_1)\phi_{j_1}(t_1)dt_1\Biggr)dt_2\Biggr)\phi_{j_1}(t_1)\phi_{j_2}(t_2)=$$

$$=\lim_{n_1,n_2\to\infty}\sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2}\left(C_{j_2j_1}+C_{j_1j_2}\right)\phi_{j_1}(t_1)\phi_{j_2}(t_2). \qquad (2.6)$$

Obtaining (2.6) we replaced the order of integration in the second repeated Riemann integral.

It is easy to see, that putting $t_1=t_2$ into (2.6), separating the limit in the right part in two limits and renaming j_1 by j_2, j_2 by j_1, n_1 by n_2 and n_2 by n_1 in the second limit, we will obtain

$$\psi_1(t_1)\psi_2(t_1)=2\lim_{n_1,n_2\to\infty}\sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1).$$

The required equality is obtained.

So, the following theorem is reasonable.

Theorem 3. *Assume, that the following conditions are met:*

1. $\psi_1(\tau)$, $\psi_2(\tau)$ — *are functions continuously differentiated at the interval* $[t,T]$.

2. $\{\phi_j(x)\}_{j=0}^{\infty}$ — *is a full orthonormal system of functions in the space* $L_2([t,T])$ *which satisfies if* $k=2$ *to condition* $(\star\star)$ *(see p.50), moreover its each function in the case of finite* j *satisfies the condition* $(\star)$ *(see p.39).*

Then, the multiple Stratonovich stochastic integral of the second multiplicity

$$J^*[\psi^{(2)}]_{T,t} = \int\limits_t^{*T} \psi_2(t_2) \int\limits_t^{*t_2} \psi_1(t_1) d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} \ \ (i_1, i_2 = 1, \ldots, m)$$

is expanded into the converging in the mean-square sense multiple series

$$J^*[\psi^{(2)}]_{T,t} = \underset{p_1,p_2\to\infty}{\text{l.i.m.}} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)},$$

where the meaning of notations introduced in the formulations of theorems 1 *and* 2 *is remained.*

Let's demonstrate expansions of multiple stochastic Stratonovich integrals of second multiplicity using Legendre polynomials, trigonometric functions, Haar functions and Rademacher-Wolsh functions.

Using Legendre polynomials:

$$\int\limits_t^{*T} d\mathbf{f}_{\tau}^{(i_1)} = \sqrt{T-t}\zeta_0^{(i_1)},$$

$$\int\limits_t^{*T} (t-\tau) d\mathbf{f}_{\tau}^{(i_1)} = -\frac{(T-t)^{3/2}}{2} \left(\zeta_0^{(i_1)} + \frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right),$$

$$\int\limits_t^{*T} \int\limits_t^{*s} d\mathbf{f}_{\tau}^{(i_1)} d\mathbf{f}_{s}^{(i_2)} = \frac{T-t}{2}\left[\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=1}^{\infty} \frac{1}{\sqrt{4i^2-1}}\left\{\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)} - \zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right\}\right].$$

For system of trigonometric functions:

$$\int\limits_t^{*T} d\mathbf{f}_{\tau}^{(i_1)} = \sqrt{T-t}\zeta_0^{(i_1)},$$

$$\int\limits_t^{*T} (t-\tau) d\mathbf{f}_{\tau}^{(i_1)} = -\frac{(T-t)^{\frac{3}{2}}}{2}\left[\zeta_0^{(i_1)} - \frac{\sqrt{2}}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_1)}\right],$$

$$\int\limits_t^{*T} \int\limits_t^{*s} d\mathbf{f}_{\tau}^{(i_1)} d\mathbf{f}_{s}^{(i_2)} = \frac{1}{2}(T-t)\left[\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \frac{1}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\left\{\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)} - \zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)} + \right.\right.$$

$$\left.\left. + \sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)} - \zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\right\}\right].$$

Using the system of Haar functions:

$$\int\limits_t^{*T} d\mathbf{f}_\tau^{(i_1)} = \sqrt{T-t}\zeta_0^{(i_1)},$$

$$\int\limits_t^{*T} (t-\tau) d\mathbf{f}_\tau^{(i_1)} = -\frac{(T-t)^{\frac{3}{2}}}{2}\Biggl(\zeta_0^{(i_1)} + \sum_{n=0}^{\infty}\sum_{j=1}^{2^n} \bar{C}_{nj}\zeta_{nj}^{(i_1)}\Biggr),$$

$$\int\limits_t^{*T}\int\limits_t^{*s} d\mathbf{f}_\tau^{(i_1)} d\mathbf{f}_s^{(i_2)} = \frac{T-t}{2}\Biggl(\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{n=0}^{\infty}\sum_{j=1}^{2^n} \bar{C}_{nj}(\zeta_{nj}^{(i_2)}\zeta_0^{(i_1)} - \zeta_{nj}^{(i_1)}\zeta_0^{(i_2)}) +$$

$$+ \sum_{n_1,n_2=0}^{\infty}\sum_{j_1=1}^{2^{n_1}}\sum_{j_2=1}^{2^{n_2}} \bar{C}_{n_2j_2,n_1j_1}\zeta_{n_2j_2}^{(i_2)}\zeta_{n_1j_1}^{(i_1)}\Biggr),$$

where

$$\bar{C}_{nj} = 2^{\frac{n}{2}}\Biggl(2\Bigl(\frac{j-1}{2^n} + \frac{1}{2^{n+1}}\Bigr)^2 - \Bigl(\frac{j-1}{2^n}\Bigr)^2 - \Bigl(\frac{j}{2^n}\Bigr)^2\Biggr),$$

$$\bar{C}_{n_2j_2,n_1j_1} = 2^{\frac{n_1+n_2}{2}}\Biggl(\Biggl(\Biggl(\min\Bigl\{\frac{j_2-1}{2^{n_2}} + \frac{1}{2^{n_2+1}},\ \frac{j_1-1}{2^{n_1}} + \frac{1}{2^{n_1+1}}\Bigr\} - \frac{j_1-1}{2^{n_1}}\Biggr)^2$$

$$-\Biggl(\max\Bigl\{\frac{j_2-1}{2^{n_2}},\ \frac{j_1-1}{2^{n_1}}\Bigr\} - \frac{j_1-1}{2^{n_1}}\Biggr)^2\Biggr)\times$$

$$\times\mathbf{1}_{\bigl\{\max\bigl\{\frac{j_2-1}{2^{n_2}},\ \frac{j_1-1}{2^{n_1}}\bigr\}<\min\bigl\{\frac{j_2-1}{2^{n_2}}+\frac{1}{2^{n_2+1}},\ \frac{j_1-1}{2^{n_1}}+\frac{1}{2^{n_1+1}}\bigr\}\bigr\}} -$$

$$-\Biggl(\Biggl(\min\Bigl\{\frac{j_2}{2^{n_2}},\ \frac{j_1-1}{2^{n_1}} + \frac{1}{2^{n_1+1}}\Bigr\} - \frac{j_1-1}{2^{n_1}}\Biggr)^2 -$$

$$-\Biggl(\max\Bigl\{\frac{j_2-1}{2^{n_2}} + \frac{1}{2^{n_2+1}},\ \frac{j_1-1}{2^{n_1}}\Bigr\} - \frac{j_1-1}{2^{n_1}}\Biggr)^2\Biggr)\times$$

$$\times\mathbf{1}_{\bigl\{\max\bigl\{\frac{j_2-1}{2^{n_2}}+\frac{1}{2^{n_2+1}},\ \frac{j_1-1}{2^{n_1}}\bigr\}<\min\bigl\{\frac{j_2}{2^{n_2}},\ \frac{j_1-1}{2^{n_1}}+\frac{1}{2^{n_1+1}}\bigr\}\bigr\}} +$$

$$+\Biggl(\Biggl(\min\Bigl\{\frac{j_2}{2^{n_2}},\ \frac{j_1}{2^{n_1}}\Bigr\} - \frac{j_1}{2^{n_1}}\Biggr)^2 -$$

$$-\Biggl(\max\Bigl\{\frac{j_2-1}{2^{n_2}} + \frac{1}{2^{n_2+1}},\ \frac{j_1-1}{2^{n_1}} + \frac{1}{2^{n_1+1}}\Bigr\} - \frac{j_1}{2^{n_1}}\Biggr)^2\Biggr)\times$$

$$\times\mathbf{1}_{\bigl\{\max\bigl\{\frac{j_2-1}{2^{n_2}}+\frac{1}{2^{n_2+1}},\ \frac{j_1-1}{2^{n_1}}+\frac{1}{2^{n_1+1}}\bigr\}<\min\bigl\{\frac{j_2}{2^{n_2}},\ \frac{j_1}{2^{n_1}}\bigr\}\bigr\}} -$$

$$-\Biggl(\Biggl(\min\Bigl\{\frac{j_2-1}{2^{n_2}} + \frac{1}{2^{n_2+1}},\ \frac{j_1}{2^{n_1}}\Bigr\} - \frac{j_1}{2^{n_1}}\Biggr)^2 -$$

$$-\Biggl(\max\Bigl\{\frac{j_2-1}{2^{n_2}},\ \frac{j_1-1}{2^{n_1}} + \frac{1}{2^{n_1+1}}\Bigr\} - \frac{j_1}{2^{n_1}}\Biggr)^2\Biggr)\times$$

$$\times\mathbf{1}_{\left\{\max\left\{\frac{j_2-1}{2^{n_2}},\ \frac{j_1-1}{2^{n_1}}+\frac{1}{2^{n_1+1}}\right\}<\min\left\{\frac{j_2-1}{2^{n_2}}+\frac{1}{2^{n_2+1}},\ \frac{j_1}{2^{n_1}}\right\}\right\}}\Biggr);$$

$\zeta_0^{(i)}=\int\limits_t^T\phi_0(\tau)d\mathbf{f}_\tau^{(i)},\ \zeta_{nj}^{(l)}=\int\limits_t^T\phi_{nj}(\tau)d\mathbf{f}_\tau^{(l)};\ n=0,\ 1,\ldots;\ j=1,\ 2,\ldots,\ 2^n$ — are independent as a whole according to lower indexes or if $i\ne l$ $(i,\ l=1,\ldots,m)$ standard Gaussian's random variables; $i_1,\ i_2=1,\ldots,m.$

For the system of Rademacher-Wolsh functions:

$$\int\limits_t^{*T}d\mathbf{f}_\tau^{(i_1)}=\sqrt{T-t}\zeta_0^{(i_1)},$$

$$\int\limits_t^{*T}(t-\tau)d\mathbf{f}_\tau^{(i_1)}=-\frac{(T-t)^{\frac{3}{2}}}{2}\Biggl(\zeta_0^{(i_1)}+\sum_{\substack{1\le m_1<\ldots<m_k\le\infty\\1\le k\le\infty}}\bar C_{m_1\ldots m_k}\zeta_{m_1\ldots m_k}^{(i_1)}\Biggr),$$

$$\int\limits_t^{*T}\int\limits_t^{*s}d\mathbf{f}_\tau^{(i_1)}d\mathbf{f}_s^{(i_2)}=\frac{T-t}{2}\Biggl(\zeta_0^{(i_1)}\zeta_0^{(i_2)}+$$

$$+\sum_{\substack{1\le m_1<\ldots<m_k\le\infty\\1\le k\le\infty}}\bar C_{m_1\ldots m_k}(\zeta_{m_1\ldots m_k}^{(i_2)}\zeta_0^{(i_1)}-\zeta_{m_1\ldots m_k}^{(i_1)}\zeta_0^{(i_2)})+$$

$$+\sum_{\substack{1\le n_1<\ldots<n_{k_1}\le\infty\\1\le m_1<\ldots<m_{k_2}\le\infty\\1\le k_1,k_2\le\infty}}\bar C_{n_1\ldots n_{k_2},m_1\ldots m_{k_1}}\zeta_{n_1\ldots n_{k_2}}^{(i_2)}\zeta_{m_1\ldots m_{k_1}}^{(i_1)}\Biggr),$$

where

$$\bar C_{m_1\ldots m_k}=\sum_{s_1=0}^{2^{m_1}-1}(-1)^{s_1}\ldots\sum_{s_k=0}^{2^{m_k}-1}(-1)^{s_k}\times$$

$$\times\mathbf{1}_{\left\{\max\left\{\frac{s_1}{2^{m_1}},\ldots,\frac{s_k}{2^{m_k}}\right\}<\min\left\{\frac{s_1+1}{2^{m_1}},\ldots,\frac{s_k+1}{2^{m_k}}\right\}\right\}}\times$$

$$\times\left(\left(\min\left\{\frac{s_1+1}{2^{m_1}},\ldots,\frac{s_k+1}{2^{m_k}}\right\}\right)^2-\left(\max\left\{\frac{s_1}{2^{m_1}},\ldots,\frac{s_k}{2^{m_k}}\right\}\right)^2\right),$$

$$\bar C_{n_1\ldots n_{k_2},m_1\ldots m_{k_1}}=\sum_{s_1=0}^{2^{m_1}-1}(-1)^{s_1}\ldots\sum_{s_{k_1}=0}^{2^{m_{k_1}}-1}(-1)^{s_{k_1}}\sum_{q_1=0}^{2^{n_1}-1}(-1)^{q_1}\ldots\sum_{q_{k_2}=0}^{2^{n_{k_2}}-1}(-1)^{q_{k_2}}$$

$$\times\mathbf{1}_{\left\{\max\left\{\frac{s_1}{2^{m_1}},\ldots,\frac{s_{k_1}}{2^{m_{k_1}}}\right\}<\min\left\{\frac{s_1+1}{2^{m_1}},\ldots,\frac{s_{k_1}+1}{2^{m_{k_1}}}\right\}\right\}}\mathbf{1}_{\left\{\max\left\{\frac{q_1}{2^{n_1}},\ldots,\frac{q_{k_2}}{2^{n_{k_2}}}\right\}<\min\left\{\frac{q_1+1}{2^{n_1}},\ldots,\frac{q_{k_2}+1}{2^{n_{k_2}}}\right\}\right\}}$$

$$\times\Biggl[\mathbf{1}_{\left\{\max\left\{\max\left\{\frac{q_1}{2^{n_1}},\ldots,\frac{q_{k_2}}{2^{n_{k_2}}}\right\},\max\left\{\frac{s_1}{2^{m_1}},\ldots,\frac{s_{k_1}}{2^{m_{k_1}}}\right\}\right\}<\min\left\{\min\left\{\frac{q_1+1}{2^{n_1}},\ldots,\frac{q_{k_2}+1}{2^{n_{k_2}}}\right\},\min\left\{\frac{s_1+1}{2^{m_1}},\ldots,\frac{s_{k_1}+1}{2^{m_{k_1}}}\right\}\right\}\right\}}$$

$$\times\Biggl(\Biggl(\min\Biggl\{\min\Biggl\{\frac{q_1+1}{2^{n_1}},\ldots,\frac{q_{k_2}+1}{2^{n_{k_2}}}\Biggr\},\min\Biggl\{\frac{s_1+1}{2^{m_1}},\ldots,\frac{s_{k_1}+1}{2^{m_{k_1}}}\Biggr\}\Biggr\}-$$

$$-\max\Biggl\{\frac{s_1}{2^{m_1}},\ldots,\frac{s_{k_1}}{2^{m_{k_1}}}\Biggr\}\Biggr)^2-$$

$$-\Biggl(\max\Biggl\{\max\Biggl\{\frac{q_1}{2^{n_1}},\ldots,\frac{q_{k_2}}{2^{n_{k_2}}}\Biggr\},\max\Biggl\{\frac{s_1}{2^{m_1}},\ldots,\frac{s_{k_1}}{2^{m_{k_1}}}\Biggr\}\Biggr\}-$$

$$-\max\Biggl\{\frac{s_1}{2^{m_1}},\ldots,\frac{s_{k_1}}{2^{m_{k_1}}}\Biggr\}\Biggr)^2\Biggr)+$$

$$+2\cdot\mathbf{1}_{\left\{\max\left\{\max\left\{\frac{q_1}{2^{n_1}},\ldots,\frac{q_{k_2}}{2^{n_{k_2}}}\right\},\min\left\{\frac{s_1+1}{2^{m_1}},\ldots,\frac{s_{k_1}+1}{2^{m_{k_1}}}\right\}\right\}<\min\left\{\min\left\{\frac{q_1+1}{2^{n_1}},\ldots,\frac{q_{k_2}+1}{2^{n_{k_2}}}\right\},\ 1\right\}\right\}}$$

$$\times\Biggl(\min\Biggl\{\frac{s_1+1}{2^{m_1}},\ldots,\frac{s_{k_1}+1}{2^{m_{k_1}}}\Biggr\}-\max\Biggl\{\frac{s_1}{2^{m_1}},\ldots,\frac{s_{k_1}}{2^{m_{k_1}}}\Biggr\}\Biggr)\times$$

$$\times\Biggl(\min\Biggl\{\min\Biggl\{\frac{q_1+1}{2^{n_1}},\ldots,\frac{q_{k_2}+1}{2^{n_{k_2}}}\Biggr\},\ 1\Biggr\}-$$

$$-\max\Biggl\{\max\Biggl\{\frac{q_1}{2^{n_1}},\ldots,\frac{q_{k_2}}{2^{n_{k_2}}}\Biggr\},\min\Biggl\{\frac{s_1+1}{2^{m_1}},\ldots,\frac{s_{k_1}+1}{2^{m_{k_1}}}\Biggr\}\Biggr\}\Biggr)\Biggr];$$

$\zeta_0^{(p)}=\int\limits_t^T\phi_0(\tau)d\mathbf{f}_\tau^{(p)}$, $\zeta_{m_1\ldots m_k}^{(g)}=\int\limits_t^T\phi_{m_1\ldots m_k}(\tau)d\mathbf{f}_\tau^{(g)}$; $0<m_1<\ldots<m_k$; $k,\ m_1,\ldots,\ m_k=1,\ 2,\ldots$ — are independent as a whole according to lower indexes or if $p\ne g$ $(p,\ g=1,\ldots,m)$ standard Gaussian random variables; $i_1,\ i_2=1,\ldots,m$.

' Apparently, due to its complexity (in comparison with expansions according to Legendre polynomials and trigonometric functions), the given expansions performed using Haar and Rademacher-Wolsh systems represent more theoretic interest than practical one.

Let's provide some additional remarks in the context of analyzed problem.

Note, that the following statement is reasonable.

Assume, that $\xi_{n,m},\ \mu_m,\ \rho_n;\ n,\ m=0,\ 1,\ 2,\ldots$ — *are sequences of random values, moreover*

$$\mathop{\text{l.i.m.}}_{n,m\to\infty}\ \xi_{n,m}=\zeta,\ \ \mathop{\text{l.i.m.}}_{n\to\infty}\ \xi_{n,m}=\mu_m,\ \ \mathop{\text{l.i.m.}}_{m\to\infty}\ \xi_{n,m}=\rho_n,$$

where ζ — *is a random value. Then*

$$\lim_{n\to\infty}\lim_{m\to\infty}\mathsf{M}\{(\xi_{n,m}-\zeta)^2\}=\lim_{m\to\infty}\lim_{n\to\infty}\mathsf{M}\{(\xi_{n,m}-\zeta)^2\}=0.$$

We prove this fact as in deterministic case using the inequality:

$$\mathsf{M}\{(x-y)^2\}\le 2\mathsf{M}\{(x-z)^2\}+2\mathsf{M}\{(z-y)^2\}$$

instead of the inequality: $|x-y| \le |x-z| + |z-y|$.

Assume, that

$$\xi_{n,m} = \sum_{j_1=0}^{n} \sum_{j_2=0}^{m} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)}, \ \zeta = J^*[\psi^{(2)}]_{T,t}.$$

Let's take for μ_m and ρ_n the following formulas

$$\sum_{j_2=0}^{m} \left(\sum_{j_1=0}^{\infty} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \right) \zeta_{j_2}^{(i_2)} \ \text{and} \ \sum_{j_1=0}^{n} \left(\sum_{j_2=0}^{\infty} C_{j_2 j_1} \zeta_{j_2}^{(i_2)} \right) \zeta_{j_1}^{(i_1)}$$

correspondingly.

Actually, since

$$\mathsf{M}\{(\xi_{n,m} - \mu_m)^2\} = \sum_{j_2=0}^{m} \sum_{j_1=n+1}^{\infty} C_{j_2 j_1}^2; \ \mathsf{M}\{(\xi_{n,m} - \rho_n)^2\} = \sum_{j_1=0}^{n} \sum_{j_2=m+1}^{\infty} C_{j_2 j_1}^2;$$

$$\sum_{j_1=0}^{n} C_{j_2 j_1}^2 \le \sum_{j_1, j_2=0}^{\infty} C_{j_2 j_1}^2; \ \sum_{j_2=0}^{m} C_{j_2 j_1}^2 \le \sum_{j_1, j_2=0}^{\infty} C_{j_2 j_1}^2;$$

$$\sum_{j_1, j_2=0}^{\infty} C_{j_2 j_1}^2 = \int\limits_{[t,T]^2} K^2(t_1, t_2) dt_1 dt_2 < \infty,$$

then

$$\underset{n\to\infty}{\text{l.i.m.}}\ \xi_{n,m} = \mu_m, \ \underset{m\to\infty}{\text{l.i.m.}}\ \xi_{n,m} = \rho_n.$$

Then, using the statement given before, we will obtain

$$J^*[\psi^{(2)}]_{T,t} = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)}, \ J^*[\psi^{(2)}]_{T,t} = \sum_{j_2=0}^{\infty} \sum_{j_1=0}^{\infty} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)},$$

where the series converges in the mean-square sense, i.e. for example for the first case

$$\lim_{n\to\infty} \lim_{m\to\infty} \mathsf{M}\left\{ \left(J^*[\psi^{(2)}]_{T,t} - \sum_{j_1=0}^{n} \sum_{j_2=0}^{m} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \right)^2 \right\} = 0.$$

The possibility to generalize theorem 3 to the case of multiple stochastic Stratonovich integral of any fixed multiplicity k seems quite natural. But this problem as we will demonstrate in the next section turned out to be rather difficult.

2.2 About the expansion of multiple stochastic Stratonovich integrals of 3rd multiplicity. Some relations for the case of weight functions of general form

Investigating the problem connected with a possibility to generalize theorem 3 for the case of multiple stochastic Stratonovich integrals of 3rd multiplicity, the author didn't obtain general results. However, he has noticed some interesting and useful practical facts.

In particular, we will show in this chapter, that in the case $\psi_1(\tau)$, $\psi_2(\tau)$, $\psi_3(\tau) \equiv 1$; $i_1, i_2, i_3 = 1, \ldots, m$ and the system of Legendre polynomials or the system of trigonometric function, generalization of theorem 3 for stochastic Stratonovich integrals is correct.

In addition in this chapter, we will show, that for some combinations of indexes i_1, i_2, i_3 and the functions $\psi_1(\tau)$, $\psi_2(\tau)$, $\psi_3(\tau)$ of polynomial type or even for some more general smooth functions, generalization of theorem 3 (case of Legendre polynomials) for stochastic Stratonovich integrals of 3rd multiplicity is also correct.

We will also analyze the more general situation for which using the formulas, obtained in this section, we may formulate sufficient conditions of correctness for generalization of theorem 3 for stochastic Stratonovich integrals of 3rd multiplicity in more general case using the terms of number series convergence.

In the following sections of this chapter we will denote full orthonormal systems of Legendre polynomials or trigonometric functions in the space $L_2([t, T])$ as $\{\phi_j(x)\}_{j=0}^{\infty}$.

In the mentioned sections we will also pay attention on the following well-known facts about these two systems of functions.

Assume, that $f(x)$ *— is a bounded at the interval* $[t, T]$ *and sectionally smooth function at the open interval* (t, T). *Then the Fourier series* $\sum\limits_{j=0}^{\infty} C_j\phi_j(x)$; $C_j = \int\limits_t^T f(x)\phi_j(x)dx$ *converges at any internal point* x *of the interval* $[t, T]$ *to the value* $\frac{1}{2}(f(x-0)+f(x+0))$ *and converges uniformly to* $f(x)$ *in any closed interval of continuity of the function* $f(x)$, *laying inside* $[t, T]$. *At the same time the Fourier series obtained using Legendre polinomials converges if* $x = t$ *and* $x = T$ *to* $f(t+0)$ *and* $f(T-0)$ *correspondently, and the trigonometric Fourier series converges if* $x = t$ *and* $x = T$ *to* $\frac{1}{2}(f(t+0)+f(T-0))$ *in case of periodic continuation of function.*

So, let's try to develop the approach described in the previous section for multiple stochastic Stratonovich integrals of 3rd multiplicity.

Let's write down the relation connecting the stochastic Stratonovich and Ito integrals of 3rd multiplicity:

$$J^*[\psi^{(3)}]_{T,t}=J[\psi^{(3)}]_{T,t}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\int\limits_t^T\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\psi_1(t_2)dt_2d\mathbf{f}_{t_3}^{(i_3)}+$$

$$+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\}}\int\limits_t^T\psi_3(t_3)\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}dt_3, \tag{2.7}$$

which is correct with probubility 1 and $\psi_1(\tau)$, $\psi_2(\tau)$, $\psi_3(\tau)$ — are continuously differentiated functions at the interval $[t,T]$.

From here we see, that there are the following particular cases:
1. i_1,i_2,i_3 are pairwise different;
2. $i_1=i_2\ne i_3$; 3. $i_1\ne i_2=i_3$; 4. $i_1=i_3\ne i_2$; 5. $i_1=i_2=i_3$.

Here we propose, that $i_1,i_2,i_3=1,\ldots,m$.

It is clear, that in the first case multiple stochastic Stratonovich and Ito integrals are simply the same. It also relates to the any multiplicity k, therefore we may use theorem 1 for these integrals.

Let's analyze the second particular case.

From theorem 1 if $i_1=i_2\ne i_3$ follows, that

$$J[\psi^{(3)}]_{T,t}=\sum_{j_1,j_2,j_3=0}^{\infty}C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}\zeta_{j_3}^{(i_3)}-\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\Biggr).$$

If we could rewrite the last equality in the form

$$J[\psi^{(3)}]_{T,t}=\sum_{j_1,j_2,j_3=0}^{\infty}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}\zeta_{j_3}^{(i_3)}-\sum_{j_1,j_3=0}^{\infty}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}$$

and could demonstrate, that in the mean-square sense

$$\sum_{j_1,j_3=0}^{\infty}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}=\frac{1}{2}\int\limits_t^T\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_1)\psi_1(t_1)dt_1d\mathbf{f}_{t_3}^{(i_3)}, \tag{2.8}$$

then we will obtain

$$J^*[\psi^{(3)}]_{T,t}=\sum_{j_1,j_2,j_3=0}^{\infty}C_{j_3j_2j_1}\prod_{l=1}^{3}\zeta_{j_l}^{(i_l)}\ \ (i_1=i_2\ne i_3), \tag{2.9}$$

where the series converges in the mean-square sense.

The author doesn't have a proof of equality (2.8) in the general case (leaping ahead we can note, that this equality is true in some practically important cases: theorems 4 – 6). We will only demonstrate here, that

$$\sum_{j_3=0}^{\infty}\sum_{j_1=0}^{\infty} C_{j_3 j_1 j_1}\zeta_{j_3}^{(i_3)} = \frac{1}{2}\int_t^T \psi_3(t_3)\int_t^{t_3}\psi_2(t_1)\psi_1(t_1)dt_1 d\mathbf{f}_{t_3}^{(i_3)}, \tag{2.10}$$

where the series $\sum\limits_{j_3=0}^{\infty}$ converges in the mean-square sense, and the series $\sum\limits_{j_1=0}^{\infty}$ converges in the common sense.

In accordance with the Ito formula, the last equality may be rewritten in the following form

$$\sum_{j_3=0}^{\infty}\sum_{j_1=0}^{\infty} C_{j_3 j_1 j_1}\zeta_{j_3}^{(i_3)} = \frac{1}{2}\int_t^T \psi_1(t_1)\psi_2(t_1)\int_{t_1}^{T}\psi_3(t_3)d\mathbf{f}_{t_3}^{(i_3)}dt_1.$$

Let's show, that

$$K_2(t_1,t_3) = \begin{cases} \frac{1}{2}\psi_3(t_3)\psi_1(t_1)\psi_2(t_1), & t_1 < t_3 \\ 0, & t_1 > t_3 \\ \frac{1}{6}\psi_3(t_1)\psi_1(t_1)\psi_2(t_1), & t_1 = t_3 \end{cases} =$$

$$= \sum_{j_3=0}^{\infty}\sum_{j_2,j_1=0}^{\infty} C_{j_3 j_2 j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1)\phi_{j_3}(t_3), \tag{2.11}$$

where $t_1, t_3 \in [t,T]$ and the convergence of series according to t_1 and t_3 is uniform at the intervals of continuity of expanded functions.

Let's analyze the auxiliary function

$$K_2'(t_1,t_2,t_3) = \begin{cases} \psi_3(t_3)\psi_2(t_2)\psi_1(t_1), & t_1 \le t_2 < t_3 \\ \psi_3(t_3)\psi_1(t_2)\psi_2(t_1), & t_2 \le t_1 < t_3 \\ 0, & \text{otherwise} \end{cases}, \quad t_1,t_2,t_3 \in [t,T].$$

Let's fix t_1, t_2 and expand the function $K_2'(t_1,t_2,t_3)$ using the variable t_3 at the interval $[t,T]$ into the Fourier series:

$$K_2'(t_1,t_2,t_3) = \sum_{j_3=0}^{\infty}\Biggl(\psi_1(t_1)\psi_2(t_2)\int_{t_2}^{T}\psi_3(t_3)\phi_{j_3}(t_3)dt_3\mathbf{1}_{\{t_1<t_2\}} +$$

$$+\psi_1(t_1)\psi_2(t_2)\int_{t_2}^{T}\psi_3(t_3)\phi_{j_3}(t_3)dt_3\mathbf{1}_{\{t_1=t_2\}} +$$

$$+\psi_1(t_2)\psi_2(t_1)\int\limits_{t_1}^{T}\psi_3(t_3)\phi_{j_3}(t_3)dt_3\mathbf{1}_{\{t_2<t_1\}}\Biggr)\phi_{j_3}(t_3)\ \ (t_3\ne t_1,\ t_2). \qquad (2.12)$$

It is easy to see, that the function staying in the parentheses looks as follows

$$\tilde{K}_{j_3}(t_1,t_2)=\begin{cases}\psi_1(t_1)\Psi_{j_3}(t_2),\ t_1\le t_2\\ \psi_1(t_2)\Psi_{j_3}(t_1),\ t_2\le t_1\end{cases};\ \Psi_{j_3}(s)=\psi_2(s)\int\limits_{s}^{T}\psi_3(t_3)\phi_{j_3}(t_3)dt_3.$$

Therefore, this function belongs to the Holder class $C^1([t,T]^2)$ (see the previous section). Let's expand it in the square $[t,T]^2$ into the multiple Fourier series, summarized according to Princeheim and substitute the result into (2.12):

$$K_2'(t_1,t_2,t_3)=\sum_{j_3=0}^{\infty}\lim_{p_1,p_2\to\infty}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\left(C_{j_3j_2j_1}+C_{j_3j_1j_2}\right)\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3).$$

Thinking $t_1=t_2$ in this equality, which is correct if $t_3\ne t_1,\ t_2$, we get (see the previous section):

$$\frac{1}{2}K_2'(t_1,t_1,t_3)=\sum_{j_3=0}^{\infty}\sum_{j_2,j_1=0}^{\infty}C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1)\phi_{j_3}(t_3)\ \ (t_3\ne t_1).$$

Let's analyze the auxiliary function

$$K_4(t_1,t_2,t_3)=\begin{cases}\psi_3(t_3)\psi_2(t_2)\psi_1(t_1),\ t_1\le t_2\le t_3\\ \psi_3(t_3)\psi_1(t_2)\psi_2(t_1),\ t_2\le t_1\le t_3\\ \psi_1(t_3)\psi_3(t_2)\psi_2(t_1),\ t_3\le t_1\le t_2\\ \psi_2(t_3)\psi_3(t_2)\psi_1(t_1),\ t_1\le t_3\le t_2\\ \psi_1(t_3)\psi_2(t_2)\psi_3(t_1),\ t_3\le t_2\le t_1\\ \psi_2(t_3)\psi_1(t_2)\psi_3(t_1),\ t_2\le t_3\le t_1\end{cases},\ t_1,t_2,t_3\in[t,T].$$

Let's expand this function in the cube $[t,T]^3$ into the multiple Fourier series, summarized according to Princeheim

$$K_4(t_1,t_2,t_3)=\lim_{n_1,n_2,n_3\to\infty}\sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2}\sum_{j_3=0}^{n_3}C_{j_3j_2j_1}^{(1)}\prod_{l=1}^{3}\phi_{j_l}(t_l), \qquad (2.13)$$

where

$$C_{j_3j_2j_1}^{(1)}=\int\limits_{[t,T]^3}K_4(t_1,t_2,t_3)\prod_{l=1}^{3}\phi_{j_l}(t_l)dt_1dt_2dt_3.$$

The function $K_4(t_1,t_2,t_3)$ is selected in such manner, that after using the property of additivity of Riemann integrals and usage of integration order replacement in these integrals we could get the equality:

$$C^{(1)}_{j_3j_2j_1} = C_{j_3j_2j_1} + C_{j_3j_1j_2} + C_{j_2j_1j_3} + C_{j_2j_3j_1} + C_{j_1j_2j_3} + C_{j_1j_3j_2}. \qquad (2.14)$$

Substituting (2.14) into (2.13), proposing in the obtained equality, that $t_1 = t_2 = t_3$ and separating the limit in the right part of obtained equality into 6 limits we get:

$$\frac{1}{6}\psi_1(t_1)\psi_2(t_1)\psi_3(t_1) = \sum_{j_1,j_2,j_3=0}^{\infty} C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1)\phi_{j_3}(t_1).$$

Since

$$\sum_{j_1,j_2=0}^{\infty} C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1) = \frac{1}{2}\tilde{K}_{j_3}(t_1,t_1) = \frac{1}{2}\psi_1(t_1)\psi_2(t_1)\int\limits_{t_1}^{T}\psi_3(s)\phi_{j_3}(s)ds$$

and because of the well-known statement about reducing the limit to the repeated one we come to (2.11). The equality (2.11) is proven.

Let's analyze

$$\mathsf{M}\left\{\left(\frac{1}{2}\psi_1(t_1)\psi_2(t_1)\int\limits_{t_1}^{T}\psi_3(t_3)d\mathbf{f}^{(i_3)}_{t_3} - \sum_{j_3=0}^{n}\zeta^{(i_3)}_{j_3}\sum_{j_1,j_2=0}^{\infty} C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1)\right)^2\right\} =$$

$$= \mathsf{M}\left\{\left(\int\limits_{t}^{T}\left(K_2(t_1,t_3) - \sum_{j_3=0}^{n}\phi_{j_3}(t_3)\sum_{j_1,j_2=0}^{\infty} C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1)\right)d\mathbf{f}^{(i_3)}_{t_3}\right)^2\right\} =$$

$$= \int\limits_{t}^{T}\left(K_2(t_1,t_3) - \sum_{j_3=0}^{n}\phi_{j_3}(t_3)\sum_{j_1,j_2=0}^{\infty} C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1)\right)^2 dt_3.$$

The right part of the last equality converges to zero if $n\to\infty$ because of uniform convergence of the series according to $t_3\in(t,T)$, $t_3\neq t_1$ (t_1 is fixed).

So, in the mean-square sense

$$\sum_{j_3=0}^{\infty}\zeta^{(i_3)}_{j_3}\sum_{j_1,j_2=0}^{\infty} C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1) = \frac{1}{2}\psi_1(t_1)\psi_2(t_1)\int\limits_{t_1}^{T}\psi_3(t_3)d\mathbf{f}^{(i_3)}_{t_3}.$$

Considering Parseval equality we have

$$\mathsf{M}\left\{\left(\int\limits_{t}^{T}\sum_{j_3=0}^{\infty}\zeta^{(i_3)}_{j_3}\sum_{j_1,j_2=0}^{\infty} C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1 - \right.\right.$$

$$
-\int\limits_t^T \sum_{j_3=0}^{n} \zeta_{j_3}^{(i_3)} \sum_{j_1,j_2=0}^{\infty} C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1\Biggr)^2\Biggr\} \le
$$

$$
\le L\int\limits_t^T \sum_{j_3=n+1}^{\infty}\Biggl(\sum_{j_1,j_2=0}^{\infty} C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1)\Biggr)^2 dt_1 =
$$

$$
= L\int\limits_t^T\Biggl(\frac{1}{4}\psi_1^2(t_1)\psi_2^2(t_1)\int\limits_{t_1}^T \psi_3^2(t_3)dt_3 - \sum_{j_3=0}^{n}\Biggl(\sum_{j_1,j_2=0}^{\infty} C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1)\Biggr)^2\Biggr)dt_1,
$$

where L — is a constant.

Because of continuity (here $\phi_j(\tau)$ are assumed to be continuous) and nondecreasing of members of functional sequence

$$
u_n(t_1) = \sum_{j_3=0}^{n}\Biggl(\sum_{j_1,j_2=0}^{\infty} C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1)\Biggr)^2,
$$

and because of the property of continuity of the limit function

$$
u(t_1) = \frac{1}{4}\psi_1^2(t_1)\psi_2^2(t_1)\int\limits_{t_1}^T \psi_3^2(t_3)dt_3
$$

according to Dini test we have a uniform convergence $u_n(t_1)$ to $u(t_1)$ at the interval $[t,T]$ $(t_1 \ne t_3,\ t_3 is fixed)$.

That is why, performing the passage to the limit under the sign of integration in the last equation we obtain in the mean-square sense

$$
\sum_{j_3=0}^{\infty}\zeta_{j_3}^{(i_3)}\int\limits_t^T \sum_{j_1,j_2=0}^{\infty} C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_1)dt_1 = \frac{1}{2}\int\limits_t^T \psi_1(t_1)\psi_2(t_1)\int\limits_{t_1}^T \psi_3(t_3)d\mathbf{f}_{t_3}^{(i_3)}dt_1.
$$

Replacing the sign of integration in the left part of this equality and the sign of the right series (that is possible due to uniform convergence of the last one according to t_1 at the interval $[t,T]$, $t_1 \ne t_3,\ t_3 is fixed)$. and taking into account orthonormality of functions $\phi_j(\tau)$, we come to (2.10).

Let's analyze the third particular case.

From theorem 1 if $i_1 \ne i_2 = i_3$ follows, that

$$
J[\psi^{(3)}]_{T,t} = \sum_{j_1,j_2,j_3=0}^{\infty} C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\Biggr).
$$

Again, if we could write down

$$
J[\psi^{(3)}]_{T,t} = \sum_{j_1,j_2,j_3=0}^{\infty} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}\zeta_{j_3}^{(i_3)} - \sum_{j_1,j_3=0}^{\infty} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}
$$

and could demonstrate, that in the mean-square sense

$$\sum_{j_1,j_3=0}^{\infty} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)} = \frac{1}{2}\int\limits_t^T \psi_3(t_3)\psi_2(t_3)\int\limits_t^{t_3} \psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}dt_3, \qquad (2.15)$$

then, because of connection between stochastic Ito and Stratonovich integrals we could get:

$$J^*[\psi^{(3)}]_{T,t} = \sum_{j_1,j_2,j_3=0}^{\infty} C_{j_3j_2j_1}\prod_{l=1}^{3}\zeta_{j_l}^{(i_l)} \ (i_1 \neq i_2 = i_3),$$

where the series converges in the mean-square sense.

Let's only demonstrate here, that in the mean-square sense

$$\sum_{j_1=0}^{\infty}\sum_{j_3=0}^{\infty} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)} = \frac{1}{2}\int\limits_t^T \psi_3(t_3)\psi_2(t_3)\int\limits_t^{t_3} \psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}dt_3, \qquad (2.16)$$

where the series $\sum\limits_{j_1=0}^{\infty}$ converges in the mean-square sense, and the series $\sum\limits_{j_3=0}^{\infty}$ converges in the common sense.

Let's demonstrate, that

$$K_3(t_1,t_3) = \begin{cases} \frac{1}{2}\psi_3(t_3)\psi_2(t_3)\psi_1(t_1), & t_1 < t_3 \\ 0, & t_1 > t_3 \\ \frac{1}{6}\psi_3(t_1)\psi_1(t_1)\psi_2(t_1), & t_1 = t_3 \end{cases} =$$

$$= \sum_{j_1=0}^{\infty}\sum_{j_2,j_3=0}^{\infty} C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_3)\phi_{j_3}(t_3), \qquad (2.17)$$

where $t_1, t_3 \in [t,T]$ and the series converges uniformly according to t_1 and t_3 in the intervals of continuity of expanded functions.

Let's analyze the auxiliary function

$$K_3'(t_1,t_2,t_3) = \begin{cases} \psi_3(t_3)\psi_2(t_2)\psi_1(t_1), & t_1 < t_2 \leq t_3 \\ \psi_3(t_2)\psi_2(t_3)\psi_1(t_1), & t_1 < t_3 \leq t_2 \\ 0, & \text{otherwise} \end{cases}, \ t_1,t_2,t_3 \in [t,T].$$

Let's fix t_2, t_3 and expand the function $K_3'(t_1,t_2,t_3)$ using the variable t_1 at the interval $[t,T]$ into the Fourier series.

$$K_3'(t_1,t_2,t_3) = \sum_{j_1=0}^{\infty}\Biggl(\psi_2(t_2)\psi_3(t_3)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\mathbf{1}_{\{t_2<t_3\}} +$$

$$+\psi_2(t_2)\psi_3(t_3)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\mathbf{1}_{\{t_2=t_3\}}+$$

$$+\psi_3(t_2)\psi_2(t_3)\int\limits_t^{t_3}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\mathbf{1}_{\{t_3<t_2\}}\Biggr)\phi_{j_1}(t_1)\ \ (t_1\ne t_2,\ t_3). \qquad (2.18)$$

It is easy to see, that the function staying in the parentheses looks as follows

$$\begin{cases}\psi_3(t_3)\Psi_{j_1}(t_2),\ t_2\le t_3\\ \psi_3(t_2)\Psi_{j_1}(t_3),\ t_3\le t_2\end{cases};\ \Psi_{j_1}(s)=\psi_2(s)\int\limits_t^{s}\psi_1(t_1)\phi_{j_1}(t_1)dt_1.$$

Therefore, this function is related belongs to the Holder class $C^1([t,T]^2)$ (see the previous section). Let's expand it in the square $[t,T]^2$ into the multiple Fourier series, summarized according to Princeheim and substitute the result into (2.18):

$$K_3'(t_1,t_2,t_3)=\sum_{j_1=0}^{\infty}\lim_{p_2,p_3\to\infty}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3}\left(C_{j_3j_2j_1}+C_{j_2j_3j_1}\right)\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3).$$

Taking $t_2=t_3$ in this equality, which is correct if $t_1\ne t_2,\ t_3$, we get (see the previous section):

$$\frac{1}{2}K_3'(t_1,t_3,t_3)=\sum_{j_1=0}^{\infty}\sum_{j_2,j_3=0}^{\infty}C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_3)\phi_{j_3}(t_3)\ \ (t_1\ne t_3).$$

The equality (2.17) is proven. The following proving of relation (2.16) is similar to the case which was investigated earlier.

In the fourth particular case the considered stochastic Ito and Stratonovich integrals with probability 1 will be the same, but as it follows from the theorem 1 the series

$$\sum_{j_3,j_2,j_1=0}^{\infty}C_{j_3\ldots j_1}\prod_{l=1}^{3}\zeta_{j_l}^{(i_l)},$$

generally speaking, may not converges to stochastic Stratonovich integral $J^*[\psi^{(3)}]_{T,t}$ when $i_1=i_3\ne i_2$.

In this case let's use the theorem 1 and formula (2.7) if $i_1=i_3\ne i_2$.

Nevertheless, close connection of formulas (2.8) and (2.10), as well as formulas (2.15) and (2.16) is non-random. In particular, in the following sections we will demonstrate, that for the case $\psi_1(\tau),\ \psi_2(\tau),\ \psi_3(\tau)\equiv 1$ and the system of Legendre polynomials or the system of trigonometric functions the formulas (2.8) and (2.15) passes to formulas (2.10) and (2.16) correspondently.

Besides, let's demonstrate, that within the frames of the mentioned case the generalization of theorem 3 for multiple stochastic Stratonovich integrals of 3rd multiplicity is correct.

2.3 Expansions of multiple stochastic Stratonovich integrals of 3rd multiplicity, based on theorem 1. Case of Legendre polynomials

2.3.1 The case $\psi_1(\tau),\ \psi_2(\tau),\ \psi_3(\tau)\equiv 1;\ i_1,i_2,i_3=1,\ldots,m$

Assume, that $\psi_1(\tau),\ \psi_2(\tau),\ \psi_3(\tau)\equiv 1$ and $\{\phi_j(x)\}_{j=0}^{\infty}$ — is a full orthonormal system of Legendre polynomials at the interval $[t,T]$.

In this section we will prove the following expansion for multiple stochastic Stratonovich integral of 3rd multiplicity:

$$\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2} d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}=\sum_{j_1,j_2,j_3=0}^{\infty} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}, \tag{2.19}$$

where the series converges in the mean-square sense, its coefficients has the following form:

$$C_{j_3j_2j_1}=\int\limits_t^{T}\phi_{j_3}(s)\int\limits_t^{s}\phi_{j_2}(s_1)\int\limits_t^{s_1}\phi_{j_1}(s_2)ds_2ds_1ds\ (i_1,i_2,i_3=1,\ldots,m).$$

If we prove the following formulas:

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\stackrel{\text{def}}{=}\sum_{j_1,j_3=0}^{\infty} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}=\frac{1}{4}(T-t)^{\frac{3}{2}}\left(\zeta_0^{(i_3)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_3)}\right), \tag{2.20}$$

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\stackrel{\text{def}}{=}\sum_{j_1,j_3=0}^{\infty} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=\frac{1}{4}(T-t)^{\frac{3}{2}}\left(\zeta_0^{(i_1)}-\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right), \tag{2.21}$$

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}\stackrel{\text{def}}{=}\sum_{j_1,j_3=0}^{\infty} C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}=0, \tag{2.22}$$

then in accordance with theorem 1, formulas (2.20) – (2.22), standard relations between multiple stochastic Stratonovich and Ito integrals, as well as in accordance with formulas (they also follows from theorem 1):

$$\frac{1}{2}\int\limits_t^{T}\int\limits_t^{\tau} ds d\mathbf{f}_{\tau}^{(i_3)}=\frac{1}{4}(T-t)^{\frac{3}{2}}\left(\zeta_0^{(i_3)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_3)}\right)\ \text{w. p. 1},$$

$$\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} d\mathbf{f}_s^{(i_1)}d\tau=\frac{1}{4}(T-t)^{\frac{3}{2}}\left(\zeta_0^{(i_1)}-\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right)\ \text{w. p. 1}$$

we will have

$$\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2} d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}=\sum\limits_{j_1,j_2,j_3=0}^{\infty}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}-$$

$$-\mathbf{1}_{\{i_1=i_2\}}\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} ds d\mathbf{f}_{\tau}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\}}\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} d\mathbf{f}_s^{(i_1)}d\tau.$$

It means, that the expansion (2.19) will be proven.

At first, note that the following relations result from formulas (2.10), (2.16)

$$\sum\limits_{j_3=0}^{\infty}\sum\limits_{j_1=0}^{\infty}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}=\frac{1}{2}\int\limits_t^T(\tau-t)d\mathbf{f}_{\tau}^{(i_3)}=\frac{1}{4}(T-t)^{\frac{3}{2}}\left(\zeta_0^{(i_3)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_3)}\right), \quad (2.23)$$

$$\sum\limits_{j_1=0}^{\infty}\sum\limits_{j_3=0}^{\infty}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau} d\mathbf{f}_s^{(i_1)}d\tau=\frac{1}{2}\int\limits_t^T(T-s)d\mathbf{f}_s^{(i_1)}=$$

$$=\frac{1}{2}\left((T-t)\int\limits_t^T d\mathbf{f}_s^{(i_1)}+\int\limits_t^T(t-s)d\mathbf{f}_s^{(i_1)}\right)=\frac{1}{2}(T-t)^{\frac{3}{2}}\left(\zeta_0^{(i_1)}-\frac{1}{2}\zeta_0^{(i_1)}-\frac{1}{2\sqrt{3}}\zeta_1^{(i_1)}\right)=$$

$$=\frac{1}{4}(T-t)^{\frac{3}{2}}\left(\zeta_0^{(i_1)}-\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right). \quad (2.24)$$

The series $\sum\limits_{j_3=0}^{\infty}$ in the left part of the formula (2.23) and the series $\sum\limits_{j_1=0}^{\infty}$ in the left part of the formula (2.24) converges in the mean-square sense. The number series $\sum\limits_{j_1=0}^{\infty}$ in the left part of the formula (2.23) and the number series $\sum\limits_{j_3=0}^{\infty}$ in the left part of the formula (2.24) converges in the common sense.

Let's examine (2.23). It follows from (2.23), that

$$\sum\limits_{j_1=0}^{\infty}C_{0j_1j_1}=\frac{1}{4}(T-t)^{\frac{3}{2}}, \quad (2.25)$$

$$\sum\limits_{j_1=0}^{\infty}C_{1j_1j_1}=\frac{1}{4\sqrt{3}}(T-t)^{\frac{3}{2}}, \quad (2.26)$$

$$\sum_{j_1=0}^{\infty} C_{j_3 j_1 j_1} = 0, \ \ j_3 \geq 2. \tag{2.27}$$

Let's check formulas (2.25) – (2.27) by direct calculation. Let's examine (2.25). We have

$$C_{000} = \frac{(T-t)^{\frac{3}{2}}}{6};$$

$$C_{0j_1j_1} = \int_t^T \phi_0(s) \int_t^s \phi_{j_1}(s_1) \int_t^{s_1} \phi_{j_1}(s_2) ds_2 ds_1 ds =$$

$$= \frac{1}{2} \int_t^T \phi_0(s) \left(\int_t^s \phi_{j_1}(s_1) ds_1 \right)^2 ds; \ \ j_1 \geq 1. \tag{2.28}$$

Here $\phi_j(s)$ looks as follows:

$$\phi_j(x) = \sqrt{\frac{2j+1}{T-t}} P_j \left(\left(s - \frac{T+t}{2} \right) \frac{2}{T-t} \right); \ \ j \geq 0, \tag{2.29}$$

where $P_j(x)$ — is a Legendre polynomial.

Let's substitute (2.29) into (2.28) and calculate $C_{0j_1j_1}$; $j \geq 1$:

$$C_{0j_1j_1} = \frac{2j_1+1}{2(T-t)^{\frac{3}{2}}} \int_t^T \left(\int_{-1}^{z(s)} P_{j_1}(y) \frac{T-t}{2} dy \right)^2 ds =$$

$$= \frac{(2j_1+1)\sqrt{T-t}}{8} \int_t^T \left(\int_{-1}^{z(s)} \frac{1}{2j_1+1} \left(P'_{j_1+1}(y) - P'_{j_1-1}(y) \right) dy \right)^2 ds =$$

$$= \frac{\sqrt{T-t}}{8(2j_1+1)} \int_t^T \left(P_{j_1+1}(z(s)) - P_{j_1-1}(z(s)) \right)^2 ds, \tag{2.30}$$

where $z(s) = \left(s - \frac{T+t}{2} \right) \frac{2}{T-t}$, and we used following well-known properties of Legendre polynomials:

$$P_j(y) = \frac{1}{2j+1} \left(P'_{j+1}(y) - P'_{j-1}(y) \right); \ \ j \geq 1,$$

$$P_j(-1) = (-1)^j; \ \ j \geq 1.$$

Also, we denote $\frac{dP_j}{dy}(y) \stackrel{\text{def}}{=} P'_j(y)$.

From (2.30) using the property of orthogonality of Legendre polynomials we get the following relation

$$C_{0j_1j_1}=\frac{(T-t)^{\frac{3}{2}}}{16(2j_1+1)}\int\limits_{-1}^{1}\left(P_{j_1+1}^2(y)+P_{j_1-1}^2(y)\right)dy=$$

$$=\frac{(T-t)^{\frac{3}{2}}}{8(2j_1+1)}\left(\frac{1}{2j_1+3}+\frac{1}{2j_1-1}\right),$$

where we used the relation

$$\int\limits_{-1}^{1}P_j^2(y)dy=\frac{2}{2j+1}.$$

Then

$$\sum_{j_1=0}^{\infty}C_{0j_1j_1}=\frac{(T-t)^{\frac{3}{2}}}{6}+\frac{(T-t)^{\frac{3}{2}}}{8}\left(\sum_{j_1=1}^{\infty}\frac{1}{(2j_1+1)(2j_1+3)}+\sum_{j_1=1}^{\infty}\frac{1}{4j_1^2-1}\right)=$$

$$=\frac{(T-t)^{\frac{3}{2}}}{6}+\frac{(T-t)^{\frac{3}{2}}}{8}\left(\sum_{j_1=1}^{\infty}\frac{1}{4j_1^2-1}-\frac{1}{3}+\sum_{j_1=1}^{\infty}\frac{1}{4j_1^2-1}\right)=$$

$$=\frac{(T-t)^{\frac{3}{2}}}{6}+\frac{(T-t)^{\frac{3}{2}}}{8}\left(\frac{1}{2}-\frac{1}{3}+\frac{1}{2}\right)=\frac{(T-t)^{\frac{3}{2}}}{4}.$$

The relation (2.25) is proven.

Let's check correctness of (2.26). Represent $C_{1j_1j_1}$ in the form:

$$C_{1j_1j_1}=\frac{1}{2}\int\limits_t^T\phi_1(s)\left(\int\limits_t^s\phi_{j_1}(s_1)ds_1\right)^2ds=$$

$$=\frac{(T-t)^{\frac{3}{2}}(2j_1+1)\sqrt{3}}{16}\int\limits_{-1}^{1}P_1(y)\left(\int\limits_{-1}^{y}P_{j_1}(y_1)dy_1\right)^2dy;\ \ j_1\ge 1.$$

Since functions

$$\left(\int\limits_{-1}^{y}P_{j_1}(y_1)dy_1\right)^2;\ \ j_1\ge 1$$

are even, then, correspondently functions

$$P_1(y)\left(\int\limits_{-1}^{y}P_{j_1}(y_1)dy_1\right)^2dy;\ \ j_1\ge 1$$

are uneven.

It means, that $C_{1j_1j_1} = 0;\ j_1 \geq 1.$

Besides

$$C_{100} = \frac{\sqrt{3}(T-t)^{\frac{3}{2}}}{16} \int\limits_{-1}^{1} y(y+1)^2 dy = \frac{(T-t)^{\frac{3}{2}}}{4\sqrt{3}}.$$

Then

$$\sum_{j_1=0}^{\infty} C_{1j_1j_1} = C_{100} + \sum_{j_1=1}^{\infty} C_{1j_1j_1} = \frac{(T-t)^{\frac{3}{2}}}{4\sqrt{3}}.$$

The relation (2.26) is proven. Let's check correctness of formula (2.27). We have

$$\sum_{j_1=0}^{\infty} C_{j_3j_1j_1} = \sum_{j_1=0}^{\infty} \frac{1}{2} \int\limits_{t}^{T} \phi_{j_3}(s) \left(\int\limits_{t}^{s} \phi_{j_1}(s_1) ds_1 \right)^2 ds;\ j_3 \geq 2. \tag{2.31}$$

It is easy to see, that the integral $\int\limits_{t}^{s} \phi_{j_1}(s_1)ds_1$ is a Fourier coefficient for the function

$$K(s_1, s) = \begin{cases} 1, & s_1 < s \\ 0, & \text{otherwise} \end{cases};\ s_1, s \in [t, T].$$

The Parseval equality in this case looks as follows:

$$\sum_{j_1=0}^{\infty} \left(\int\limits_{t}^{s} \phi_{j_1}(s_1) ds_1 \right)^2 = \int\limits_{t}^{T} K^2(s_1, s) ds = \int\limits_{t}^{s} ds_1 = s - t. \tag{2.32}$$

Taking into account the nondecreasing of functional sequence

$$u_n(s) = \sum_{j_1=0}^{n} \left(\int\limits_{t}^{s} \phi_{j_1}(s_1) ds_1 \right)^2,$$

continuity of its members, as well as continuity of limit function $u(s) = s - t$ at the interval $[t, T]$ we have according to Dini test the uniform convergence of functional sequence $u_n(s)$ to the limit function $u(s) = s - t$ at the interval $[t, T]$.

Then from (2.31) and (2.32) using the uniform convergence of functional sequence $u_n(s)$ to the limit function $u(s)$ at the interval $[t, T]$ we have

$$\sum_{j_1=0}^{\infty} C_{j_3j_1j_1} = \frac{1}{2} \int\limits_{t}^{T} \phi_{j_3}(s) \sum_{j_1=0}^{\infty} \left(\int\limits_{t}^{s} \phi_{j_1}(s_1) ds_1 \right)^2 ds =$$

$$= \frac{1}{2}\int\limits_t^T \phi_{j_3}(s)(s-t)ds = 0;\ \ j_3 \ge 2. \tag{2.33}$$

Obtaining (2.33) we used the well-known feature of Legendre polynomials:

$$\int\limits_{-1}^{1} P_j(y)y^k dy = 0;\ \ j > k. \tag{2.34}$$

The relation (2.27) is proven.

Let's prove the equality (2.20). Using (2.26) we get

$$\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} = \sum_{j_1=0}^{p_1} C_{0j_1j_1}\zeta_0^{(i_3)} + \frac{(T-t)^{\frac{3}{2}}}{4\sqrt{3}}\zeta_1^{(i_3)} + \sum_{j_1=0}^{p_1}\sum_{j_3=2}^{p_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} =$$

$$= \sum_{j_1=0}^{p_1} C_{0j_1j_1}\zeta_0^{(i_3)} + \frac{(T-t)^{\frac{3}{2}}}{4\sqrt{3}}\zeta_1^{(i_3)} + \sum_{j_1=0}^{p_1}\sum_{j_3=2,j_3-\text{even}}^{2j_1+2} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}. \tag{2.35}$$

Since

$$C_{j_3j_1j_1} = \frac{(T-t)^{\frac{3}{2}}(2j_1+1)\sqrt{2j_3+1}}{16}\int\limits_{-1}^{1} P_{j_3}(y)\left(\int\limits_{-1}^{y} P_{j_1}(y_1)dy_1\right)^2 dy$$

and the degree of polynomial $\left(\int\limits_{-1}^{y} P_{j_1}(y_1)dy_1\right)^2$ equals to $2j_1+2$, then using (2.34) we get $C_{j_3j_1j_1} = 0$ for $j_3 > 2j_1+2$. It explains the circumstance, that we put $2j_1+2$ instead of p_3 in right part of the formula (2.35).

Moreover, the function $\left(\int\limits_{-1}^{y} P_{j_1}(y_1)dy_1\right)^2$ is even, it means, that the function $P_{j_3}(y)\left(\int\limits_{-1}^{y} P_{j_1}(y_1)dy_1\right)^2$ is uneven for uneven j_3. It means, that $C_{j_3j_1j_1} = 0$ for uneven j_3. That is why we summarize using even j_3 in the right part of the formula (2.35).

Then we have

$$\sum_{j_1=0}^{p_1}\sum_{j_3=2,j_3-\text{even}}^{2j_1+2} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} = \sum_{j_3=2,j_3-\text{even}}^{2p_1+2}\ \sum_{j_1=\frac{j_3-2}{2}}^{p_1} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} =$$

$$= \sum_{j_3=2,j_3-\text{even}}^{2p_1+2}\ \sum_{j_1=0}^{p_1} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}. \tag{2.36}$$

We replaced $\frac{j_3-2}{2}$ by zero in the right part of the formula (2.36), since $C_{j_3j_1j_1} = 0$ for $0 \le j_1 < \frac{j_3-2}{2}$.

Let's put (2.36) into (2.35):

$$\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3 j_1 j_1}\zeta_{j_3}^{(i_3)} = \sum_{j_1=0}^{p_1} C_{0 j_1 j_1}\zeta_0^{(i_3)} + \frac{(T-t)^{\frac{3}{2}}}{4\sqrt{3}}\zeta_1^{(i_3)} +$$

$$+\sum_{j_3=2, j_3-\text{even}}^{2p_1+2}\ \sum_{j_1=0}^{p_1} C_{j_3 j_1 j_1}\zeta_{j_3}^{(i_3)}. \tag{2.37}$$

It is easy to see, that the right part of the formula (2.37) doesn't depend on p_3.

If we prove, that

$$\lim_{p_1\to\infty} \mathrm{M}\left\{\left(\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3 j_1 j_1}\zeta_{j_3}^{(i_3)} - \frac{1}{4}(T-t)^{\frac{3}{2}}\left(\zeta_0^{(i_3)} + \frac{1}{\sqrt{3}}\zeta_1^{(i_3)}\right)\right)^2\right\} = 0, \tag{2.38}$$

then the relaion (2.20) will be proven.

Using (2.37) and (2.25) we may rewrite the left part of (2.38) in the following form:

$$\lim_{p_1\to\infty} \mathrm{M}\left\{\left(\left(\sum_{j_1=0}^{p_1} C_{0 j_1 j_1} - \frac{(T-t)^{\frac{3}{2}}}{4}\right)\zeta_0^{(i_3)} + \sum_{j_3=2, j_3-\text{even}}^{2p_1+2}\ \sum_{j_1=0}^{p_1} C_{j_3 j_1 j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\} =$$

$$= \lim_{p_1\to\infty}\left(\sum_{j_1=0}^{p_1} C_{0 j_1 j_1} - \frac{(T-t)^{\frac{3}{2}}}{4}\right)^2 + \lim_{p_1\to\infty}\sum_{j_3=2, j_3-\text{even}}^{2p_1+2}\left(\sum_{j_1=0}^{p_1} C_{j_3 j_1 j_1}\right)^2 =$$

$$= \lim_{p_1\to\infty}\sum_{j_3=2, j_3-\text{even}}^{2p_1+2}\left(\sum_{j_1=0}^{p_1} C_{j_3 j_1 j_1}\right)^2. \tag{2.39}$$

If we prove, that

$$\lim_{p_1\to\infty}\sum_{j_3=2, j_3-\text{even}}^{2p_1+2}\left(\sum_{j_1=0}^{p_1} C_{j_3 j_1 j_1}\right)^2 = 0, \tag{2.40}$$

then, the relation (2.20) will be proven.

We have

$$\sum_{j_3=2, j_3-\text{even}}^{2p_1+2}\left(\sum_{j_1=0}^{p_1} C_{j_3 j_1 j_1}\right)^2 =$$

$$= \frac{1}{4}\sum_{j_3=2, j_3-\text{even}}^{2p_1+2}\left(\int_t^T \phi_{j_3}(s)\sum_{j_1=0}^{p_1}\left(\int_t^s \phi_{j_1}(s_1)ds_1\right)^2 ds\right)^2 =$$

$$= \frac{1}{4} \sum_{j_3=2, j_3-\text{even}}^{2p_1+2} \left(\int_t^T \phi_{j_3}(s) \left((s-t) - \sum_{j_1=p_1+1}^{\infty} \left(\int_t^s \phi_{j_1}(s_1) ds_1 \right)^2 \right) ds \right)^2 =$$

$$= \frac{1}{4} \sum_{j_3=2, j_3-\text{even}}^{2p_1+2} \left(\int_t^T \phi_{j_3}(s) \sum_{j_1=p_1+1}^{\infty} \left(\int_t^s \phi_{j_1}(s_1) ds_1 \right)^2 ds \right)^2 \le$$

$$\le \frac{1}{4} \sum_{j_3=2, j_3-\text{even}}^{2p_1+2} \left(\int_t^T \mid \phi_{j_3}(s) \mid \sum_{j_1=p_1+1}^{\infty} \left(\int_t^s \phi_{j_1}(s_1) ds_1 \right)^2 ds \right)^2 . \qquad (2.41)$$

Obtaining (2.41) we used interrelations (2.32) and (2.33).

Then we have

$$\left(\int_t^s \phi_{j_1}(s_1) ds_1 \right)^2 = \frac{(T-t)(2j_1+1)}{4} \left(\int_{-1}^{z(s)} P_{j_1}(y) dy \right)^2 =$$

$$= \frac{T-t}{4(2j_1+1)} \left(\int_{-1}^{z(s)} \left(P'_{j_1+1}(y) - P'_{j_1-1}(y) \right) dy \right)^2 =$$

$$= \frac{T-t}{4(2j_1+1)} \left(P_{j_1+1}(z(s)) - P_{j_1-1}(z(s)) \right)^2$$

$$\le \frac{T-t}{2(2j_1+1)} \left(P^2_{j_1+1}(z(s)) + P^2_{j_1-1}(z(s)) \right), \qquad (2.42)$$

where $z(s) = \left(s - \frac{T+t}{2} \right) \frac{2}{T-t}$.

For the Legendre polynomials the following well-known estimation is correct:

$$\mid P_n(y) \mid < \frac{K}{\sqrt{n+1}(1-y^2)^{\frac{1}{4}}}; \; y \in (-1, 1); \; n \in N, \qquad (2.43)$$

where the constant K doesn't depend on y and n.

The estimation (2.43) may be rewritten for the function $\phi_n(s)$ in the following form:

$$\mid \phi_n(s) \mid < \sqrt{\frac{2n+1}{n+1}} \frac{K}{\sqrt{T-t}} \frac{1}{\left(1 - \left(\left(s - \frac{T+t}{2} \right) \frac{2}{T-t} \right)^2 \right)^{\frac{1}{4}}} <$$

$$< \frac{K_1}{\sqrt{T-t}} \frac{1}{\left(1 - \left(\left(s - \frac{T+t}{2} \right) \frac{2}{T-t} \right)^2 \right)^{\frac{1}{4}}}; \; K_1 = K\sqrt{2}; \; s \in (t, T). \qquad (2.44)$$

Let's estimate the right part (2.42) using the estimation (2.43):

$$\left(\int\limits_t^s \phi_{j_1}(s_1)ds_1\right)^2 < \frac{T-t}{2(2j_1+1)}\left(\frac{K^2}{j_1+2}+\frac{K^2}{j_1}\right)\frac{1}{\left(1-(z(s))^2\right)^{\frac{1}{2}}} <$$

$$< \frac{(T-t)K^2}{2j_1^2}\frac{1}{(1-(z(s))^2)^{\frac{1}{2}}};\ s\in(t,T), \qquad (2.45)$$

where $z(s)=\left(s-\frac{T+t}{2}\right)\frac{2}{T-t}$.

Substituting the estimation (2.45) into the relation (2.41) and using in (2.41) the estimation (2.44) for $\mid \phi_{j_3}(s)\mid$ we get:

$$\sum_{j_3=2,j_3-\text{even}}^{2p_1+2}\left(\sum_{j_1=0}^{p_1} C_{j_3j_1j_1}\right)^2 <$$

$$< \frac{(T-t)K^4K_1^2}{16}\sum_{j_3=2,j_3-\text{even}}^{2p_1+2}\left(\int\limits_t^T \frac{ds}{\left(1-\left(\left(s-\frac{T+t}{2}\right)\frac{2}{T-t}\right)^2\right)^{\frac{3}{4}}}\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1^2}\right)^2 =$$

$$= \frac{(T-t)^3K^4K_1^2(p_1+1)}{64}\left(\int\limits_{-1}^1\frac{dy}{(1-y^2)^{\frac{3}{4}}}\right)^2\left(\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1^2}\right)^2. \qquad (2.46)$$

Since

$$\int\limits_{-1}^1\frac{dy}{(1-y^2)^{\frac{3}{4}}}<\infty \qquad (2.47)$$

and

$$\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1^2}\le\int\limits_{p_1}^{\infty}\frac{dx}{x^2}=\frac{1}{p_1}, \qquad (2.48)$$

then from (2.46) we find:

$$\sum_{j_3=2,j_3-\text{even}}^{2p_1+2}\left(\sum_{j_1=0}^{p_1} C_{j_3j_1j_1}\right)^2 < \frac{C(T-t)^3(p_1+1)}{p_1^2}\to 0 \text{ with } p_1\to\infty, \qquad (2.49)$$

where the constant C doesn't depend on p_1 and $T-t$.

From (2.49) follows (2.40), and from (2.40) follows (2.20).

Let's examine proving of the equaity (2.21). From (2.24) we get

$$\sum_{j_3=0}^{\infty} C_{j_3j_30}=\frac{1}{4}(T-t)^{\frac{3}{2}}, \qquad (2.50)$$

$$\sum_{j_3=0}^{\infty} C_{j_3 j_3 j_1} = -\frac{1}{4\sqrt{3}}(T-t)^{\frac{3}{2}}, \qquad (2.51)$$

$$\sum_{j_3=0}^{\infty} C_{j_3 j_3 j_1} = 0,\ \ j_1 \ge 2. \qquad (2.52)$$

Let's check formulas (2.50) – (2.52) by direct calculation.
Let's examine (2.50). We have

$$\sum_{j_3=0}^{\infty} C_{j_3 j_3 0} = C_{000} + \sum_{j_3=1}^{\infty} C_{j_3 j_3 0};$$

$$C_{000} = \frac{(T-t)^{\frac{3}{2}}}{6};$$

$$C_{j_3 j_3 0} = \frac{(T-t)^{\frac{3}{2}}}{16(2j_3+1)} \int\limits_{-1}^{1} \left(P_{j_3+1}^2(y) + P_{j_3-1}^2(y)\right) dy =$$

$$= \frac{(T-t)^{\frac{3}{2}}}{8(2j_3+1)} \left(\frac{1}{2j_3+3} + \frac{1}{2j_3-1}\right);\ \ j_3 \ge 1.$$

Then

$$\sum_{j_3=0}^{\infty} C_{j_3 j_3 0} = \frac{(T-t)^{\frac{3}{2}}}{6} + \frac{(T-t)^{\frac{3}{2}}}{8}\left(\sum_{j_3=1}^{\infty} \frac{1}{(2j_3+1)(2j_3+3)} + \sum_{j_3=1}^{\infty} \frac{1}{4j_3^2-1}\right) =$$

$$= \frac{(T-t)^{\frac{3}{2}}}{6} + \frac{(T-t)^{\frac{3}{2}}}{8}\left(\sum_{j_3=1}^{\infty} \frac{1}{4j_3^2-1} - \frac{1}{3} + \sum_{j_3=1}^{\infty} \frac{1}{4j_3^2-1}\right) =$$

$$= \frac{(T-t)^{\frac{3}{2}}}{6} + \frac{(T-t)^{\frac{3}{2}}}{8}\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{2}\right) = \frac{(T-t)^{\frac{3}{2}}}{4}.$$

The relation (2.50) is proven. Let's check the equality (2.51). We have

$$C_{j_3 j_3 j_1} = \int\limits_t^T \phi_{j_3}(s) \int\limits_t^s \phi_{j_3}(s_1) \int\limits_t^{s_1} \phi_{j_1}(s_2) ds_2 ds_1 ds =$$

$$= \int\limits_t^T \phi_{j_1}(s_2) ds_2 \int\limits_{s_2}^T \phi_{j_3}(s_1) ds_1 \int\limits_{s_1}^T \phi_{j_3}(s) ds =$$

$$= \frac{1}{2} \int\limits_t^T \phi_{j_1}(s_2) \left(\int\limits_{s_2}^T \phi_{j_3}(s_1) ds_1\right)^2 ds_2 =$$

$$= \frac{(T-t)^{\frac{3}{2}}(2j_3+1)\sqrt{2j_1+1}}{16}\int\limits_{-1}^{1}P_{j_1}(y)\left(\int\limits_{y}^{1}P_{j_3}(y_1)dy_1\right)^2 dy;\ \ j_3 \ge 1. \quad (2.53)$$

Since functions

$$\left(\int\limits_{y}^{1}P_{j_3}(y_1)dy_1\right)^2;\ \ j_3 \ge 1$$

are even, then functions

$$P_1(y)\left(\int\limits_{y}^{1}P_{j_3}(y_1)dy_1\right)^2 dy;\ \ j_3 \ge 1$$

are uneven.

It means, that $C_{j_3j_31} = 0;\ j_3 \ge 1.$

Moreover

$$C_{001} = \frac{\sqrt{3}(T-t)^{\frac{3}{2}}}{16}\int\limits_{-1}^{1}y(1-y)^2dy = -\frac{(T-t)^{\frac{3}{2}}}{4\sqrt{3}}.$$

Then

$$\sum_{j_3=0}^{\infty}C_{j_3j_31} = C_{001} + \sum_{j_3=1}^{\infty}C_{j_3j_31} = -\frac{(T-t)^{\frac{3}{2}}}{4\sqrt{3}}.$$

The relation (2.51) is proven.

The equality (2.52) may be proven similarly to (2.27). We have

$$\sum_{j_3=0}^{\infty}C_{j_3j_3j_1} = \sum_{j_3=0}^{\infty}\frac{1}{2}\int\limits_{t}^{T}\phi_{j_1}(s_2)\left(\int\limits_{s_2}^{T}\phi_{j_3}(s_1)ds_1\right)^2 ds_2 =$$

$$= \frac{1}{2}\int\limits_{t}^{T}\phi_{j_1}(s_2)\sum_{j_3=0}^{\infty}\left(\int\limits_{s_2}^{T}\phi_{j_3}(s_1)ds_1\right)^2 ds_2 =$$

$$= \frac{1}{2}\int\limits_{t}^{T}\phi_{j_1}(s_2)(T-s_2)ds_2 = 0;\ \ j_1 \ge 2, \quad (2.54)$$

where we used the Parseval equality in the following form

$$\sum_{j_3=0}^{\infty}\left(\int\limits_{s_2}^{T}\phi_{j_3}(s_1)ds_1\right)^2 = \int\limits_{t}^{T}K^2(s_1,s_2)ds_1 = \int\limits_{s_2}^{T}ds_1 = T-s_2, \quad (2.55)$$

$$K(s_1,s_2) = \begin{cases}1, & s_2 < s_1\\ 0, & \text{otherwise}\end{cases};\ \ s_1,s_2 \in [t,T]$$

and the fact, that the series in the left part (2.55) converges uniformly according to Dini test. The relation (2.52) is proven.

Using the obtained results we have:

$$\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=\sum_{j_3=0}^{p_3} C_{j_3j_30}\zeta_0^{(i_1)}-\frac{(T-t)^{\frac{3}{2}}}{4\sqrt{3}}\zeta_1^{(i_1)}+\sum_{j_3=0}^{p_3}\sum_{j_1=2}^{p_1} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=$$

$$=\sum_{j_3=0}^{p_3} C_{j_3j_30}\zeta_0^{(i_1)}-\frac{(T-t)^{\frac{3}{2}}}{4\sqrt{3}}\zeta_1^{(i_1)}+\sum_{j_3=0}^{p_3}\sum_{j_1=2,j_1-\text{even}}^{2j_3+2} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}. \qquad (2.56)$$

Since

$$C_{j_3j_3j_1}=\frac{(T-t)^{\frac{3}{2}}(2j_3+1)\sqrt{2j_1+1}}{16}\int\limits_{-1}^{1} P_{j_1}(y)\left(\int\limits_{y}^{1} P_{j_3}(y_1)dy_1\right)^2 dy;\ \ j_3\ge 1,$$

and the degree of polynomial $\left(\int\limits_y^1 P_{j_3}(y_1)dy_1\right)^2$ equals to $2j_3+2$, then using (2.34) we get $C_{j_3j_3j_1}=0$ for $j_1>2j_3+2$. It explains the circumstance, that we put $2j_3+2$ instead of p_1 in the right part of the formula (2.56).

Moreover, the function $\left(\int\limits_y^1 P_{j_3}(y_1)dy_1\right)^2$ is even, it means, that the function $P_{j_1}(y)\left(\int\limits_y^1 P_{j_3}(y_1)dy_1\right)^2$ is uneven for uneven j_1. It means, that $C_{j_3j_3j_1}=0$ for uneven j_1. It explains summation of only even j_1 in the right part (2.56).

Then we have

$$\sum_{j_3=0}^{p_3}\sum_{j_1=2,j_1-\text{even}}^{2j_3+2} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\ \sum_{j_3=\frac{j_1-2}{2}}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=$$

$$=\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\ \sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}. \qquad (2.57)$$

We reptaced $\frac{j_1-2}{2}$ by zero in the right part of (2.57), since $C_{j_3j_3j_1}=0$ for $0\le j_3<\frac{j_1-2}{2}$.

Let's substitute (2.57) into (2.56):

$$\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=\sum_{j_3=0}^{p_3} C_{j_3j_30}\zeta_0^{(i_1)}-\frac{(T-t)^{\frac{3}{2}}}{4\sqrt{3}}\zeta_1^{(i_1)}+$$

$$+\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\ \sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}. \qquad (2.58)$$

It is easy to see, that the right part of the formula (2.58) doesn't depend on p_1.

If we prove, that

$$\lim_{p_3\to\infty} \mathrm{M}\left\{\left(\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)} - \frac{1}{4}(T-t)^{\frac{3}{2}}\left(\zeta_0^{(i_1)} - \frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right)\right)^2\right\} = 0, \tag{2.59}$$

then (2.21) will be proven.

Using (2.58) and (2.50), (2.51) we may rewrite the left part of the formula (2.59) in the following form:

$$\lim_{p_3\to\infty} \mathrm{M}\left\{\left(\left(\sum_{j_3=0}^{p_3} C_{j_3j_30} - \frac{(T-t)^{\frac{3}{2}}}{4}\right)\zeta_0^{(i_1)} + \sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\} =$$

$$= \lim_{p_3\to\infty}\left(\sum_{j_3=0}^{p_1} C_{j_3j_30} - \frac{(T-t)^{\frac{3}{2}}}{4}\right)^2 + \lim_{p_3\to\infty}\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\right)^2 =$$

$$= \lim_{p_3\to\infty}\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\right)^2.$$

If we prove, that

$$\lim_{p_3\to\infty}\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\right)^2 = 0, \tag{2.60}$$

then the relation (2.21) will be proven.

From (2.53) we get

$$\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\right)^2 =$$

$$= \frac{1}{4}\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\int\limits_t^T \phi_{j_1}(s_2)\sum_{j_3=0}^{p_3}\left(\int\limits_{s_2}^T \phi_{j_3}(s_1)ds_1\right)^2 ds_2\right)^2 =$$

$$= \frac{1}{4}\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\int\limits_t^T \phi_{j_1}(s_2)\left((T-s_2) - \sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_2}^T \phi_{j_3}(s_1)ds_1\right)^2\right)ds_2\right)^2 =$$

$$= \frac{1}{4}\sum_{j_1=2,j_1-\text{even}}^{2p_3+2}\left(\int\limits_t^T \phi_{j_1}(s_2)\sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_2}^T \phi_{j_3}(s_1)ds_1\right)^2 ds_2\right)^2 \le$$

$$\le \frac{1}{4}\sum_{j_1=2, j_1-\text{even}}^{2p_3+2}\left(\int\limits_t^T \mid \phi_{j_1}(s_2)\mid \sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_2}^T \phi_{j_3}(s_1)ds_1\right)^2 ds_2\right)^2. \qquad (2.61)$$

In order to get (2.61) we used the Parseval equality (2.55) and relation (2.54). Then we have

$$\left(\int\limits_{s_2}^T \phi_{j_3}(s_1)ds_1\right)^2 =$$

$$= \frac{(T-t)}{4(2j_3+1)}\left(P_{j_3+1}(z(s_2)) - P_{j_3-1}(z(s_2))\right)^2$$

$$\le \frac{T-t}{2(2j_3+1)}\left(P_{j_3+1}^2(z(s_2)) + P_{j_3-1}^2(z(s_2))\right)$$

$$< \frac{T-t}{2(2j_3+1)}\left(\frac{K^2}{j_3+2}+\frac{K^2}{j_3}\right)\frac{1}{\left(1-(z(s_2))^2\right)^{\frac{1}{2}}} <$$

$$< \frac{(T-t)K^2}{2j_3^2}\frac{1}{\left(1-(z(s_2))^2\right)^{\frac{1}{2}}};\ s\in(t,T), \qquad (2.62)$$

where $z(s_2) = \left(s_2 - \frac{T+t}{2}\right)\frac{2}{T-t}$.

In order to get (2.62) we used the estimation (2.43).

Substituting the estimation (2.62) into relation (2.61) and using in (2.61) the estimation (2.44) for $\mid \phi_{j_1}(s_2)\mid$ we get:

$$\sum_{j_1=2, j_1-\text{even}}^{2p_3+2}\left(\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\right)^2 <$$

$$< \frac{(T-t)K^4K_1^2}{16}\sum_{j_1=2, j_1-\text{even}}^{2p_3+2}\left(\int\limits_t^T \frac{ds}{\left(1-\left(\left(s_2-\frac{T+t}{2}\right)\frac{2}{T-t}\right)^2\right)^{\frac{3}{4}}}\sum_{j_3=p_3+1}^{\infty}\frac{1}{j_3^2}\right)^2 =$$

$$= \frac{(T-t)^3K^4K_1^2(p_3+1)}{64}\left(\int\limits_{-1}^1 \frac{dy}{(1-y^2)^{\frac{3}{4}}}\right)^2\left(\sum_{j_3=p_3+1}^{\infty}\frac{1}{j_3^2}\right)^2. \qquad (2.63)$$

Using (2.47) and (2.48) from (2.63) we find:

$$\sum_{j_1=2, j_1-\text{even}}^{2p_3+2}\left(\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\right)^2 < \frac{C(T-t)^3(p_3+1)}{p_3^2} \to 0 \text{ with } p_3\to\infty, \quad (2.64)$$

where the constant C doesn't depend on p_3 and $T-t$.

From (2.64) follows (2.60) and from (2.60) follows (2.21). Relation (2.21) is proven.

Let's prove the equality (2.22).

Since $\psi_1(\tau), \psi_2(\tau), \psi_3(\tau) \equiv 1$, then the following relation for the Fourier coefficients is correct (see section 1.2):

$$C_{j_1 j_1 j_3} + C_{j_1 j_3 j_1} + C_{j_3 j_1 j_1} = \frac{1}{2} C_{j_1}^2 C_{j_3},$$

where $C_j = 0$ for $j \geq 1$ and $C_0 = \sqrt{T-t}$. Then w.p.1

$$\sum_{j_1,j_3=0}^{\infty} C_{j_1 j_3 j_1} \zeta_{j_3}^{(i_2)} = \sum_{j_1,j_3=0}^{\infty} \left(\frac{1}{2} C_{j_1}^2 C_{j_3} - C_{j_1 j_1 j_3} - C_{j_3 j_1 j_1}\right) \zeta_{j_3}^{(i_2)}. \qquad (2.65)$$

Therefore, considering (2.20) and (2.21), w.p.1 we can write down the following:

$$\sum_{j_1,j_3=0}^{\infty} C_{j_1 j_3 j_1} \zeta_{j_3}^{(i_2)} = \frac{1}{2} C_0^3 \zeta_0^{(i_2)} - \sum_{j_1,j_3=0}^{\infty} C_{j_1 j_1 j_3} \zeta_{j_3}^{(i_2)} - \sum_{j_1,j_3=0}^{\infty} C_{j_3 j_1 j_1} \zeta_{j_3}^{(i_2)} =$$

$$= \frac{1}{2}(T-t)^{\frac{3}{2}} \zeta_0^{(i_2)} - \frac{1}{4}(T-t)^{\frac{3}{2}} \left(\zeta_0^{(i_2)} - \frac{1}{\sqrt{3}} \zeta_1^{(i_2)}\right) -$$

$$-\frac{1}{4}(T-t)^{\frac{3}{2}} \left(\zeta_0^{(i_2)} + \frac{1}{\sqrt{3}} \zeta_1^{(i_2)}\right) = 0. \qquad (2.66)$$

The relation (2.22) is proven. So, we have proven the following expansion for multiple stochastic Stratonovich integrals of 3rd multiplicity for the case of Legendre polynomials:

$$\int_t^{*T} \int_t^{*t_3} \int_t^{*t_2} d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} d\mathbf{f}_{t_3}^{(i_3)} = \sum_{j_1,j_2,j_3=0}^{\infty} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}, \qquad (2.67)$$

where the series converges in the mean-square sense,

$$C_{j_3 j_2 j_1} = \int_t^T \phi_{j_3}(s) \int_t^s \phi_{j_2}(s_1) \int_t^{s_1} \phi_{j_1}(s_2) ds_2 ds_1 ds$$

and $i_1, i_2, i_3 = 1, \ldots, m$.

It is easy to see, that the formula (2.67) may be proven for the case $i_1 = i_2 = i_3$ using the Ito formula (also see section 6.3):

$$\int_t^{*T} \int_t^{*t_3} \int_t^{*t_2} d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_1)} d\mathbf{f}_{t_3}^{(i_1)} = \frac{1}{6} \left(\int_t^T d\mathbf{f}_s^{(i_1)}\right)^3 = \frac{1}{6} \left(C_0 \zeta_0^{(i_1)}\right)^3 = C_{000} \zeta_0^{(i_1)} \zeta_0^{(i_1)} \zeta_0^{(i_1)},$$

where the equality is fulfilled with probability 1.

Let's analyze expansions of specific multiple stochastic Stratonovich and Ito integrals using obtained results and the system of Legendre polynomials.

Assume, that

$$I_{l_1\ldots l_k T,t}^{(i_1\ldots i_k)} = \int\limits_t^T (t-t_k)^{l_k}\ldots \int\limits_t^{t_2}(t-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)},$$

$$I_{l_1\ldots l_k T,t}^{*(i_1\ldots i_k)} = \int\limits_t^{*T} (t-t_k)^{l_k}\ldots \int\limits_t^{*t_2}(t-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)},$$

where $i_1,\ldots,i_k=1,\ldots,m;\ l_1,\ldots,l_k=0,\ 1,\ldots.$

The direct calculation according to theorem 1 provides:

$$I_{000_{T,t}}^{(i_1i_2i_3)} = -\frac{1}{T-t}\left(I_{0_{T,t}}^{(i_3)}I_{10_{T,t}}^{*(i_2i_1)}+I_{0_{T,t}}^{(i_1)}I_{10_{T,t}}^{*(i_2i_3)}\right)+$$
$$+\frac{1}{2}I_{0_{T,t}}^{(i_3)}\left(I_{00_{T,t}}^{*(i_1i_2)}-I_{00_{T,t}}^{*(i_2i_1)}\right)-$$
$$-(T-t)^{\frac{3}{2}}\left[\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_3)}\left(\zeta_0^{(i_2)}+\sqrt{3}\zeta_1^{(i_2)}-\frac{1}{\sqrt{5}}\zeta_2^{(i_2)}\right)+\right.$$
$$+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\}}\left(\zeta_0^{(i_3)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_3)}+\frac{1}{3\sqrt{5}}\zeta_2^{(i_3)}+G_{T,t}^{(i_3)}\right)+$$
$$+\frac{1}{4}\mathbf{1}_{\{i_2=i_3\}}\left(\zeta_0^{(i_1)}-\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}+\frac{1}{3\sqrt{5}}\zeta_2^{(i_1)}+G_{T,t}^{(i_1)}\right)+$$
$$\left.+\frac{1}{6}\mathbf{1}_{\{i_1=i_3\}}\left(\zeta_0^{(i_2)}-\frac{1}{\sqrt{5}}\zeta_2^{(i_2)}+Q_{T,t}^{(i_2)}\right)+\frac{1}{4}D_{T,t}^{(i_1i_2i_3)}\right],$$

$$I_{0_{T,t}}^{(i_1)}=\sqrt{T-t}\zeta_0^{(i_1)},$$

$$I_{00_{T,t}}^{*(i_1i_2)}=\frac{T-t}{2}\left[\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left\{\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right\}\right],$$

$$I_{10_{T,t}}^{*(i_1i_2)}=-\frac{T-t}{2}I_{00_{T,t}}^{*(i_1i_2)}-\frac{(T-t)^2}{4}\left[\frac{1}{\sqrt{3}}\zeta_0^{(i_2)}\zeta_1^{(i_1)}+\right.$$
$$\left.+\sum_{i=0}^{\infty}\left(\frac{(i+1)\zeta_{i+2}^{(i_2)}\zeta_i^{(i_1)}-(i+2)\zeta_i^{(i_2)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)}+\frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\right)\right],$$

$$D_{T,t}^{(i_1 i_2 i_3)} = \sum_{\substack{i=1,j=0,k=i \\ 2i\ge k+i-j\ge -2;\ k+i-j\ -\ \text{even}}}^{\infty} N_{ijk} K_{i+1,k+1,\frac{k+i-j}{2}+1} \zeta_i^{(i_1)} \zeta_j^{(i_2)} \zeta_k^{(i_3)} +$$

$$+ \sum_{\substack{i=1,j=0 \\ 2k\ge k+i-j\ge -2;\ k+i-j\ -\ \text{even}}}^{\infty} \sum_{k=1}^{i-1} N_{ijk} K_{k+1,i+1,\frac{k+i-j}{2}+1} \zeta_i^{(i_1)} \zeta_j^{(i_2)} \zeta_k^{(i_3)} -$$

$$- \sum_{\substack{i=1,j=0,k=i+2 \\ 2i+2\ge k+i-j\ge 0;\ k+i-j\ -\ \text{even}}}^{\infty} N_{ijk} K_{i+1,k-1,\frac{k+i-j}{2}} \zeta_i^{(i_1)} \zeta_j^{(i_2)} \zeta_k^{(i_3)} -$$

$$- \sum_{\substack{i=1,j=0 \\ 2k-2\ge k+i-j\ge 0;\ k+i-j\ -\ \text{even}}}^{\infty} \sum_{k=1}^{i+1} N_{ijk} K_{k-1,i+1,\frac{k+i-j}{2}} \zeta_i^{(i_1)} \zeta_j^{(i_2)} \zeta_k^{(i_3)} -$$

$$- \sum_{\substack{i=1,j=0,k=i-2,k\ge 1 \\ 2i-2\ge k+i-j\ge 0;\ k+i-j\ -\ \text{even}}}^{\infty} N_{ijk} K_{i-1,k+1,\frac{k+i-j}{2}} \zeta_i^{(i_1)} \zeta_j^{(i_2)} \zeta_k^{(i_3)} -$$

$$- \sum_{\substack{i=1,j=0 \\ 2k+2\ge k+i-j\ge 0;\ k+i-j\ -\ \text{even}}}^{\infty} \sum_{k=1}^{i-3} N_{ijk} K_{k+1,i-1,\frac{k+i-j}{2}} \zeta_i^{(i_1)} \zeta_j^{(i_2)} \zeta_k^{(i_3)} +$$

$$+ \sum_{\substack{i=1,j=0,k=i \\ 2i\ge k+i-j\ge 2;\ k+i-j\ -\ \text{even}}}^{\infty} N_{ijk} K_{i-1,k-1,\frac{k+i-j}{2}-1} \zeta_i^{(i_1)} \zeta_j^{(i_2)} \zeta_k^{(i_3)} +$$

$$+ \sum_{\substack{i=1,j=0 \\ 2k\ge k+i-j\ge 2;\ k+i-j\ -\ \text{even}}}^{\infty} \sum_{k=1}^{i-1} N_{ijk} K_{k-1,i-1,\frac{k+i-j}{2}-1} \zeta_i^{(i_1)} \zeta_j^{(i_2)} \zeta_k^{(i_3)},$$

$$G_{T,t}^{(i_3)} = -\sum_{j=1}^{\infty} \left\{ \sum_{k=j,\ k\ -\ \text{even}}^{2j} N_{jjk} K_{j+1,k+1,\frac{k}{2}+1} \zeta_k^{(i_3)} + \right.$$

$$+ \sum_{k=1,\ k\ -\ \text{even}}^{j-1} N_{jjk} K_{k+1,j+1,\frac{k}{2}+1} \zeta_k^{(i_3)} -$$

$$- \sum_{k=j-2,\ k\ge 1,\ k\ -\ \text{even}}^{2j-2} N_{jjk} K_{j-1,k+1,\frac{k}{2}} \zeta_k^{(i_3)} -$$

$$- \sum_{k=1,\ k\ -\ \text{even}}^{j-3} N_{jjk} K_{k+1,j-1,\frac{k}{2}} \zeta_k^{(i_3)} -$$

$$
-\sum_{k=j+2,\ k\ -\ \text{even}}^{2j+2} N_{jjk}K_{j+1,k-1,\frac{k}{2}}\zeta_k^{(i_3)}-
$$

$$
-\sum_{k=1,\ k\ -\ \text{even}}^{j+1} N_{jjk}K_{k-1,j+1,\frac{k}{2}}\zeta_k^{(i_3)}+
$$

$$
+\sum_{k=j,\ k\ -\ \text{even}}^{2j} N_{jjk}K_{j-1,k-1,\frac{k}{2}-1}\zeta_k^{(i_3)}+
$$

$$
+\sum_{k=1,\ k\ -\ \text{even}}^{j-1} N_{jjk}K_{k-1,j-1,\frac{k}{2}-1}\zeta_k^{(i_3)}\Biggr\},
$$

$$
Q_{T,t}^{(i_2)}=-\frac{3}{2}\sum_{i=1}^{\infty}\Biggl\{\sum_{j=0,\ j\ -\ \text{even}}^{2i+2} N_{iji}K_{i+1,i+1,i+1-\frac{j}{2}}\zeta_j^{(i_2)}-
$$

$$
-2\sum_{j=2,\ j\ -\ \text{even}}^{2i} N_{iji}K_{i-1,i-1,i-\frac{j}{2}}\zeta_j^{(i_2)}+
$$

$$
+\sum_{j=0,\ j\ -\ \text{even}}^{2i-2} N_{iji}K_{i-1,i-1,i-1-\frac{j}{2}}\zeta_j^{(i_2)}\Biggr\},
$$

where

$$
N_{ijk}=\sqrt{\frac{1}{(2k+1)(2j+1)(2i+1)}},
$$

$$
K_{m,n,k}=\frac{a_{m-k}a_k a_{n-k}}{a_{m+n-k}}\cdot\frac{2n+2m-4k+1}{2n+2m-2k+1};\ \ a_k=\frac{(2k-1)!!}{k!};\ \ m\le n.
$$

On the other hand, in accordance with (2.67) we may use more compact expression:

$$
I_{000_{T,t}}^{*(i_1i_2i_3)}=-\frac{1}{T-t}\left(I_{0_{T,t}}^{(i_3)}I_{10_{T,t}}^{*(i_2i_1)}+I_{0_{T,t}}^{(i_1)}I_{10_{T,t}}^{*(i_2i_3)}\right)+
$$

$$
+\frac{1}{2}I_{0_{T,t}}^{(i_3)}\left(I_{00_{T,t}}^{*(i_1i_2)}-I_{00_{T,t}}^{*(i_2i_1)}\right)-
$$

$$
-(T-t)^{\frac{3}{2}}\left[\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_3)}\left(\zeta_0^{(i_2)}+\sqrt{3}\zeta_1^{(i_2)}-\frac{1}{\sqrt{5}}\zeta_2^{(i_2)}\right)+\frac{1}{4}D_{T,t}^{(i_1i_2i_3)}\right]
$$

or

$$
I_{000_{T,t}}^{(i_1i_2i_3)}=-\frac{1}{T-t}\left(I_{0_{T,t}}^{(i_3)}I_{10_{T,t}}^{*(i_2i_1)}+I_{0_{T,t}}^{(i_1)}I_{10_{T,t}}^{*(i_2i_3)}\right)+
$$

$$
+\frac{1}{2}I_{0_{T,t}}^{(i_3)}\left(I_{00_{T,t}}^{*(i_1i_2)}-I_{00_{T,t}}^{*(i_2i_1)}\right)-
$$

$$-(T-t)^{\frac{3}{2}}\left[\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_3)}\left(\zeta_0^{(i_2)}+\sqrt{3}\zeta_1^{(i_2)}-\frac{1}{\sqrt{5}}\zeta_2^{(i_2)}\right)+\frac{1}{4}D_{T,t}^{(i_1i_2i_3)}\right]+$$

$$+\mathbf{1}_{\{i_1=i_2\}}\frac{1}{2}I_{1_{T,t}}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\}}\frac{1}{2}\left((T-t)I_{0_{T,t}}^{(i_1)}+I_{1_{T,t}}^{(i_1)}\right),$$

where

$$I_{1_{T,t}}^{(i)}=-\frac{1}{2}(T-t)^{\frac{3}{2}}\left(\zeta_0^{(i)}+\frac{1}{\sqrt{3}}\zeta_1^{(i)}\right),$$

$\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{f}_s^{(i)}$; $\{\phi_j(x)\}_{j=0}^{\infty}$ — is a full orthonormal systems of Legendre polynomials at the interval $[t,T]$.

Let's prove some generalizations of expansion (2.67) for the situation, when $\psi_i(\tau)\equiv(t-\tau)^{l_i}$; $l_i=0,\ 1,\ 2,\ldots$ are fixed natural numbers; $i=1,\ 2,\ 3$.

2.3.2 The case $\psi_1(\tau),\psi_2(\tau)\equiv(t-\tau)^l$, $\psi_3(\tau)\equiv(t-\tau)^{l_3}$; $i_1=i_2\neq i_3$

In this section we will prove the following expansion for multiple stochastic Stratonovich integral of 3rd multiplicity:

$$\int\limits_t^{*T}(t-s)^{l_3}\int\limits_t^{*s}(t-s_1)^l\int\limits_t^{*s_1}(t-s_2)^l d\mathbf{f}_{s_2}^{(i_1)}d\mathbf{f}_{s_1}^{(i_2)}d\mathbf{f}_s^{(i_3)}=$$

$$=\sum_{j_1,j_2,j_3=0}^{\infty}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\ \ (i_1=i_2\neq i_3;\ i_1,i_2,i_3=1,\ldots,m),\qquad(2.68)$$

where the series converges in the mean-square sense; $l,l_3=0,\ 1,\ 2,\ldots$ and

$$C_{j_3j_2j_1}=\int\limits_t^T\phi_{j_3}(s)(t-s)^{l_3}\int\limits_t^s(t-s_1)^l\phi_{j_2}(s_1)\int\limits_t^{s_1}(t-s_2)^l\phi_{j_1}(s_2)ds_2ds_1ds.\ (2.69)$$

If we prove the formula:

$$\sum_{j_1,j_3=0}^{\infty}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}=\frac{1}{2}\int\limits_t^T(t-s)^{l_3}\int\limits_t^s(t-s_1)^{2l}ds_1d\mathbf{f}_s^{(i_3)},\qquad(2.70)$$

where the series converges in the mean-square sense and the coefficients $C_{j_3j_1j_1}$ has the form (2.69), then using theorem 1 and standard relations between multiple stochastic Stratonovich and Ito integrals we get the expansion (2.68).

Using theorem 1 we may write down:

$$\frac{1}{2}\int\limits_t^T(t-s)^{l_3}\int\limits_t^s(t-s_1)^{2l}ds_1d\mathbf{f}_s^{(i_3)}=\frac{1}{2}\sum_{j_3=0}^{2l+l_3+1}\tilde{C}_{j_3}\zeta_{j_3}^{(i_3)}\ \text{w. p. 1},$$

where

$$\tilde{C}_{j_3}=\int\limits_t^T\phi_{j_3}(s)(t-s)^{l_3}\int\limits_t^s(t-s_1)^{2l}ds_1ds.$$

Then

$$\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}-\frac{1}{2}\sum_{j_3=0}^{2l+l_3+1}\tilde{C}_{j_3}\zeta_{j_3}^{(i_3)}=$$

$$=\sum_{j_3=0}^{2l+l_3+1}\left(\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}-\frac{1}{2}\tilde{C}_{j_3}\right)\zeta_{j_3}^{(i_3)}+\sum_{j_3=2l+l_3+2}^{p_3}\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}.$$

Therefore

$$\lim_{p_1,p_3\to\infty}\mathsf{M}\left\{\left(\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}-\frac{1}{2}\int\limits_t^T(t-s)^{l_3}\int\limits_t^s(t-s_1)^{2l}ds_1d\mathbf{f}_s^{(i_3)}\right)^2\right\}=$$

$$=\lim_{p_1\to\infty}\sum_{j_3=0}^{2l+l_3+1}\left(\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}-\frac{1}{2}\tilde{C}_{j_3}\right)^2+$$

$$+\lim_{p_1,p_3\to\infty}\mathsf{M}\left\{\left(\sum_{j_3=2l+l_3+2}^{p_3}\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\}. \tag{2.71}$$

Let's prove, that

$$\lim_{p_1\to\infty}\left(\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}-\frac{1}{2}\tilde{C}_{j_3}\right)^2=0. \tag{2.72}$$

We have

$$\left(\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}-\frac{1}{2}\tilde{C}_{j_3}\right)^2=$$

$$=\left(\frac{1}{2}\sum_{j_1=0}^{p_1}\int\limits_t^T\phi_{j_3}(s)(t-s)^{l_3}\left(\int\limits_t^s\phi_{j_1}(s_1)(t-s_1)^{l}ds_1\right)^2ds-\right.$$

$$\left.-\frac{1}{2}\int\limits_t^T\phi_{j_3}(s)(t-s)^{l_3}\int\limits_t^s(t-s_1)^{2l}ds_1ds\right)^2=$$

$$=\frac{1}{4}\left(\int\limits_t^T\phi_{j_3}(s)(t-s)^{l_3}\left(\sum_{j_1=0}^{p_1}\left(\int\limits_t^s\phi_{j_1}(s_1)(t-s_1)^{l}ds_1\right)^2-\int\limits_t^s(t-s_1)^{2l}ds_1\right)ds\right)^2$$

$$=\frac{1}{4}\left(\int\limits_t^T\phi_{j_3}(s)(t-s)^{l_3}\left(\int\limits_t^s(t-s_1)^{2l}ds_1-\right.\right.$$

$$-\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 - \int\limits_t^s (t-s_1)^{2l} ds_1\Biggr) ds\Biggr)^2 =$$

$$=\frac{1}{4}\left(\int\limits_t^T \phi_{j_3}(s)(t-s)^{l_3}\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 ds\right)^2. \qquad (2.73)$$

In order to get (2.73) we used the Parseval equality, which in this case may look as follows:

$$\sum_{j_1=0}^{\infty}\left(\int\limits_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 = \int\limits_t^T K^2(s,s_1)ds_1, \qquad (2.74)$$

where

$$K(s,s_1)=\begin{cases}(t-s_1)^l, & s_1<s\\ 0, & \text{otherwise}\end{cases};\ \ s,s_1\in[t,T].$$

Taking into account the nondecreasing of functional sequence

$$u_n(s)=\sum_{j_1=0}^{n}\left(\int\limits_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2,$$

continuity of its members and continuity of limit function

$$u(s)=\int\limits_t^s (t-s_1)^{2l}ds_1$$

at the interval $[t,T]$ in accordance with Dini test we have uniform convergence of functional sequences $u_n(s)$ to the limit function $u(s)$ at the interval $[t,T]$.

From (2.73) using the inequality of Cauchy-Bunyakovsky we get:

$$\left(\sum_{j_1=0}^{p_1} C_{j_3j_1j_1}-\frac{1}{2}\tilde C_{j_3}\right)^2 \le$$

$$\le \frac{1}{4}\int\limits_t^T \phi_{j_3}^2(s)(t-s)^{2l_3}ds\int\limits_t^T\left(\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2\right)^2 ds \le$$

$$\le \frac{1}{4}\varepsilon^2(T-t)^{2l_3}\int\limits_t^T \phi_{j_3}^2(s)ds(T-t)=\frac{1}{4}(T-t)^{2l_3+1}\varepsilon^2 \qquad (2.75)$$

when $p_1>N(\varepsilon)$, where $N(\varepsilon)$ is found for all $\varepsilon>0$.

From (2.75) it follows (2.72).

Further

$$\sum_{j_1=0}^{p_1}\sum_{j_3=2l+l_3+2}^{p_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} = \sum_{j_1=0}^{p_1}\sum_{j_3=2l+l_3+2}^{2(j_1+l+1)+l_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}. \qquad (2.76)$$

We put $2(j_1+l+1)+l_3$ instead of p_3, since $C_{j_3j_1j_1}=0$ for $j_3>2(j_1+l+1)+l_3$. This conclusion follows from the relation:

$$C_{j_3j_1j_1}=\frac{1}{2}\int\limits_t^T \phi_{j_3}(s)(t-s)^{l_3}\left(\int\limits_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 ds =$$

$$=\frac{1}{2}\int\limits_t^T \phi_{j_3}(s)Q_{2(j_1+l+1)+l_3}(s)ds,$$

where $Q_{2(j_1+l+1)+l_3}(s)$ is a polynomial of the degree $2(j_1+l+1)+l_3$.
It is easy to see, that

$$\sum_{j_1=0}^{p_1}\sum_{j_3=2l+l_3+2}^{2(j_1+l+1)+l_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)} = \sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\sum_{j_1=0}^{p_1} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}. \qquad (2.77)$$

Note, that we introduced some coefficients $C_{j_3j_1j_1}$ in the sum $\sum\limits_{j_1=0}^{p_1}$, which equals to zero. From (2.76) and (2.77) we get:

$$\mathrm{M}\left\{\left(\sum_{j_1=0}^{p_1}\sum_{j_3=2l+l_3+2}^{p_3} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\} = \mathrm{M}\left\{\left(\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\sum_{j_1=0}^{p_1} C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\} =$$

$$=\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\left(\sum_{j_1=0}^{p_1} C_{j_3j_1j_1}\right)^2 =$$

$$=\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\left(\frac{1}{2}\sum_{j_1=0}^{p_1}\int\limits_t^T \phi_{j_3}(s)(t-s)^{l_3}\left(\int\limits_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 ds\right)^2 =$$

$$=\frac{1}{4}\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\left(\int\limits_t^T \phi_{j_3}(s)(t-s)^{l_3}\sum_{j_1=0}^{p_1}\left(\int\limits_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 ds\right)^2 =$$

$$=\frac{1}{4}\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\left(\int\limits_t^T \phi_{j_3}(s)(t-s)^{l_3}\left(\int\limits_t^s (t-s_1)^{2l}ds_1-\right.\right.$$

$$
- \sum_{j_1=p_1+1}^{\infty} \left(\int_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1 \right)^2 \Biggr) ds \Biggr)^2 =
$$

$$
= \frac{1}{4} \sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3} \left(\int_t^T \phi_{j_3}(s)(t-s)^{l_3} \sum_{j_1=p_1+1}^{\infty} \left(\int_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1 \right)^2 ds \right)^2 . \tag{2.78}
$$

In order to get (2.78) we used the Parseval equality of type (2.74) and the following relation:

$$
\int_t^T \phi_{j_3}(s) Q_{2l+1+l_3}(s) ds = 0; \ \ j_3 > 2l+1+l_3,
$$

where $Q_{2l+1+l_3}(s)$ — is a polynomial of degree $2l+1+l_3$.

Further we have

$$
\left(\int_t^s \phi_{j_1}(s_1)(t-s_1)^l ds_1 \right)^2 =
$$

$$
= \frac{(T-t)^{2l+1}(2j_1+1)}{2^{2l+2}} \left(\int_{-1}^{z(s)} P_{j_1}(y)(1+y)^l dy \right)^2 =
$$

$$
= \frac{(T-t)^{2l+1}}{2^{2l+2}(2j_1+1)} \Biggl((1+z(s))^l \left(P_{j_1+1}(z(s)) - P_{j_1-1}(z(s)) \right) -
$$

$$
-l \int_{-1}^{z(s)} \left(P_{j_1+1}(y) - P_{j_1-1}(y) \right) (1+y)^{l-1} dy \Biggr)^2 \le
$$

$$
\le \frac{(T-t)^{2l+1} 2}{2^{2l+2}(2j_1+1)} \Biggl(\left(\frac{2(s-t)}{T-t} \right)^{2l} \left(P_{j_1+1}(z(s)) - P_{j_1-1}(z(s)) \right)^2 +
$$

$$
+ l^2 \left(\int_{-1}^{z(s)} \left(P_{j_1+1}(y) - P_{j_1-1}(y) \right) (1+y)^{l-1} dy \right)^2 \Biggr) \le
$$

$$
\le \frac{(T-t)^{2l+1}}{2^{2l+1}(2j_1+1)} \Biggl(2^{2l+1} \left(P_{j_1+1}^2(z(s)) + P_{j_1-1}^2(z(s)) \right) +
$$

$$
+ l^2 \int_{-1}^{z(s)} (1+y)^{2l-2} dy \int_{-1}^{z(s)} \left(P_{j_1+1}(y) - P_{j_1-1}(y) \right)^2 dy \Biggr) \le
$$

$$
\le \frac{(T-t)^{2l+1}}{2^{2l+1}(2j_1+1)} \Biggl(2^{2l+1} \left(P_{j_1+1}^2(z(s)) + P_{j_1-1}^2(z(s)) \right) +
$$

$$+\frac{2l^2}{2l-1}\left(\frac{2(s-t)}{T-t}\right)^{2l-1}\int\limits_{-1}^{z(s)}\left(P_{j_1+1}^2(y)+P_{j_1-1}^2(y)\right)dy\Biggr)\le$$

$$\le\frac{(T-t)^{2l+1}}{2(2j_1+1)}\Biggl(2\left(P_{j_1+1}^2\left(z(s)\right)+P_{j_1-1}^2\left(z(s)\right)\right)+$$

$$+\frac{l^2}{2l-1}\int\limits_{-1}^{z(s)}\left(P_{j_1+1}^2(y)+P_{j_1-1}^2(y)\right)dy\Biggr),\qquad(2.79)$$

where $z(s)=\left(s-\frac{T+t}{2}\right)\frac{2}{T-t}$.

Let's estimate the right part of (2.79) using (2.43):

$$\left(\int\limits_t^s\phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2<\frac{(T-t)^{2l+1}}{2(2j_1+1)}\left(\frac{K^2}{j_1+2}+\frac{K^2}{j_1}\right)\times$$

$$\times\left(\frac{2}{\left(1-(z(s))^2\right)^{\frac{1}{2}}}+\frac{l^2}{2l-1}\int\limits_{-1}^{z(s)}\frac{dy}{(1-y^2)^{\frac{1}{2}}}\right)<$$

$$<\frac{(T-t)^{2l+1}K^2}{2j_1^2}\left(\frac{2}{\left(1-(z(s))^2\right)^{\frac{1}{2}}}+\frac{l^2\pi}{2l-1}\right);\ s\in(t,T),\qquad(2.80)$$

where $z(s)=\left(s-\frac{T+t}{2}\right)\frac{2}{T-t}$.

From (2.78) and (2.80) we get:

$$\mathrm{M}\left\{\left(\sum_{j_1=0}^{p_1}\sum_{j_3=2l+l_3+2}^{p_3}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\right)^2\right\}\le$$

$$\le\frac{1}{4}\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\left(\int\limits_t^T\mid\phi_{j_3}(s)\mid(t-s)^{l_3}\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 ds\right)^2$$

$$\le\frac{1}{4}(T-t)^{2l_3}\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\left(\int\limits_t^T\mid\phi_{j_3}(s)\mid\sum_{j_1=p_1+1}^{\infty}\left(\int\limits_t^s\phi_{j_1}(s_1)(t-s_1)^l ds_1\right)^2 ds\right)^2$$

$$<\frac{(T-t)^{4l+2l_3+1}K^4K_1^2}{16}\sum_{j_3=2l+l_3+2}^{2(p_1+l+1)+l_3}\Biggl(\Biggl(\int\limits_t^T\frac{2ds}{\left(1-(z(s))^2\right)^{\frac{3}{4}}}+$$

$$+\frac{l^2\pi}{2l-1}\int\limits_t^T\frac{ds}{\left(1-(z(s))^2\right)^{\frac{1}{4}}}\Biggr)\sum_{j_1=p_1+1}^{\infty}\frac{1}{j_1^2}\Biggr)^2\le$$

$$\le \frac{(T-t)^{4l+2l_3+3}K^4K_1^2}{64}\cdot\frac{2p_1+1}{p_1^2}\left(\int\limits_{-1}^{1}\frac{2dy}{(1-y^2)^{\frac{3}{4}}}+\frac{l^2\pi}{2l-1}\int\limits_{-1}^{1}\frac{dy}{(1-y^2)^{\frac{1}{4}}}\right)^2\le$$

$$\le (T-t)^{4l+2l_3+3}C\frac{2p_1+1}{p_1^2}\to 0 \ \text{when}\ p_1\to\infty, \qquad (2.81)$$

where the constant C doesn't depend on p_1 and $T-t$, and

$$z(s)=\left(s-\frac{T+t}{2}\right)\frac{2}{T-t}.$$

From (2.71), (2.72) and (2.81) follows (2.70), and from (2.70) follows the expansion (2.68).

2.3.3 The case $\psi_3(\tau),\psi_2(\tau)\equiv(t-\tau)^l$, $\psi_1(\tau)\equiv(t-\tau)^{l_1}$; $i_3=i_2\ne i_1$

In this section we will prove the following expansion for multiple stochastic Stratonovich integral of 3rd multiplicity:

$$\int\limits_t^{*T}(t-s)^l\int\limits_t^{*s}(t-s_1)^l\int\limits_t^{*s_1}(t-s_2)^{l_1}d\mathbf{f}_{s_2}^{(i_1)}d\mathbf{f}_{s_1}^{(i_2)}d\mathbf{f}_{s}^{(i_3)}=$$

$$=\sum_{j_1,j_2,j_3=0}^{\infty}C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\ (i_3=i_2\ne i_1;\ i_1,i_2,i_3=1,\ldots,m), \qquad (2.82)$$

where the series converges in the mean-square sense; $l,l_1=0,\ 1,\ 2,\ldots$ and

$$C_{j_3j_2j_1}=\int\limits_t^T\phi_{j_3}(s)(t-s)^l\int\limits_t^s(t-s_1)^l\phi_{j_2}(s_1)\int\limits_t^{s_1}(t-s_2)^{l_1}\phi_{j_1}(s_2)ds_2ds_1ds. \qquad (2.83)$$

If we prove the formula:

$$\sum_{j_1,j_3=0}^{\infty}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=\frac{1}{2}\int\limits_t^T(t-s)^{2l}\int\limits_t^s(t-s_1)^{l_1}d\mathbf{f}_{s_1}^{(i_1)}ds, \qquad (2.84)$$

where the series converges in the mean-square sense and the coefficients $C_{j_3j_3j_1}$ has the form (2.83), then using theorem 1 and standard relations between multiple stochastic Ito and Stratonovich integrals we get the expansion (2.82).

Using theorem 1 we may write down:

$$\frac{1}{2}\int\limits_t^T(t-s)^{2l}\int\limits_t^s(t-s_1)^{l_1}d\mathbf{f}_{s_1}^{(i_1)}ds=$$

$$
= \frac{1}{2}\int\limits_t^T (t-s_1)^{l_1}\int\limits_{s_1}^T (t-s)^{2l} ds d\mathbf{f}_{s_1}^{(i_1)} =
$$

$$
= \frac{1}{2}\sum_{j_1=0}^{2l+l_1+1}\tilde{C}_{j_1}\zeta_{j_1}^{(i_1)} \ \text{w. p. 1},
$$

where

$$
\tilde{C}_{j_1} = \int\limits_t^T \phi_{j_1}(s_1)(t-s_1)^{l_1}\int\limits_{s_1}^T (t-s)^{2l} ds ds_1.
$$

Then

$$
\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)} - \frac{1}{2}\sum_{j_1=0}^{2l+l_1+1}\tilde{C}_{j_1}\zeta_{j_1}^{(i_1)} =
$$

$$
= \sum_{j_1=0}^{2l+l_1+1}\left(\sum_{j_3=0}^{p_3} C_{j_3j_3j_1} - \frac{1}{2}\tilde{C}_{j_1}\right)\zeta_{j_1}^{(i_1)} + \sum_{j_1=2l+l_1+2}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}.
$$

Therefore

$$
\lim_{p_1,p_3\to\infty} \mathrm{M}\left\{\left(\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)} - \frac{1}{2}\int\limits_t^T (t-s)^{2l}\int\limits_t^s (t-s_1)^{l_1} d\mathbf{f}_{s_1}^{(i_1)} ds\right)^2\right\} =
$$

$$
= \lim_{p_3\to\infty}\sum_{j_1=0}^{2l+l_1+1}\left(\sum_{j_3=0}^{p_3} C_{j_3j_3j_1} - \frac{1}{2}\tilde{C}_{j_1}\right)^2 +
$$

$$
+ \lim_{p_1,p_3\to\infty} \mathrm{M}\left\{\left(\sum_{j_1=2l+l_1+2}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\}. \tag{2.85}
$$

Let's prove, that

$$
\lim_{p_3\to\infty}\left(\sum_{j_3=0}^{p_3} C_{j_3j_3j_1} - \frac{1}{2}\tilde{C}_{j_1}\right)^2 = 0. \tag{2.86}
$$

We have

$$
\left(\sum_{j_3=0}^{p_3} C_{j_3j_3j_1} - \frac{1}{2}\tilde{C}_{j_1}\right)^2 =
$$

$$
= \left(\sum_{j_3=0}^{p_3}\int\limits_t^T \phi_{j_1}(s_2)(t-s_2)^{l_1} ds_2 \int\limits_{s_2}^T \phi_{j_3}(s_1)(t-s_1)^{l} ds_1 \int\limits_{s_1}^T \phi_{j_3}(s)(t-s)^{l} ds - \right.
$$

$$
\left. - \frac{1}{2}\int\limits_t^T \phi_{j_1}(s_1)(t-s_1)^{l_1}\int\limits_{s_1}^T (t-s)^{2l} ds ds_1\right)^2 =
$$

$$
=\left(\frac{1}{2}\sum_{j_3=0}^{p_3}\int\limits_t^T\phi_{j_1}(s_2)(t-s_2)^{l_1}\left(\int\limits_{s_2}^T\phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2 ds_2-\right.
$$

$$
\left.-\frac{1}{2}\int\limits_t^T\phi_{j_1}(s_1)(t-s_1)^{l_1}\int\limits_{s_1}^T(t-s)^{2l}dsds_1\right)^2=
$$

$$
=\frac{1}{4}\left(\int\limits_t^T\phi_{j_1}(s_1)(t-s_1)^{l_1}\left(\sum_{j_3=0}^{p_3}\left(\int\limits_{s_1}^T\phi_{j_3}(s)(t-s)^l ds\right)^2-\int\limits_{s_1}^T(t-s)^{2l}ds\right)ds_1\right)^2=
$$

$$
=\frac{1}{4}\left(\int\limits_t^T\phi_{j_1}(s_1)(t-s_1)^{l_1}\left(\int\limits_{s_1}^T(t-s)^{2l}ds-\right.\right.
$$

$$
\left.\left.-\sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_1}^T\phi_{j_3}(s)(t-s)^l ds\right)^2-\int\limits_{s_1}^T(t-s)^{2l}ds\right)ds_1\right)^2=
$$

$$
=\frac{1}{4}\left(\int\limits_t^T\phi_{j_1}(s_1)(t-s_1)^{l_1}\sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_1}^T\phi_{j_3}(s)(t-s)^l ds\right)^2 ds_1\right)^2. \qquad (2.87)
$$

In order to get (2.87) we used the Parseval equality, which in this case may look as follows:

$$
\sum_{j_3=0}^{\infty}\left(\int\limits_{s_1}^T\phi_{j_3}(s)(t-s)^l ds\right)^2=\int\limits_t^T K^2(s,s_1)ds, \qquad (2.88)
$$

where

$$
K(s,s_1)=\begin{cases}(t-s)^l, & s_1<s\\ 0, & \text{otherwise}\end{cases};\ \ s,s_1\in[t,T].
$$

Taking into account nondecreasing of functional sequence

$$
u_n(s_1)=\sum_{j_3=0}^{n}\left(\int\limits_{s_1}^T\phi_{j_3}(s)(t-s)^l ds\right)^2,
$$

continuity of its members and continuity of limit function

$$
u(s_1)=\int\limits_{s_1}^T(t-s)^{2l}ds
$$

at the interval $[t,T]$, according to Dini test we have uniform convergence of the functional sequence $u_n(s_1)$ to the limit function u(s) at the interval $[t,T]$.

From (2.87) using the inequality of Cauchy-Bunyakovsky we get:

$$\left(\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}-\frac{1}{2}\tilde{C}_{j_1}\right)^2\le$$

$$\le\frac{1}{4}\int\limits_t^T\phi_{j_1}^2(s_1)(t-s_1)^{2l_1}ds_1\int\limits_t^T\left(\sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_1}^T\phi_{j_3}(s)(t-s)^l ds\right)^2\right)^2 ds_1\le$$

$$\le\frac{1}{4}\varepsilon^2(T-t)^{2l_1}\int\limits_t^T\phi_{j_1}^2(s_1)ds_1(T-t)=\frac{1}{4}(T-t)^{2l_1+1}\varepsilon^2 \qquad (2.89)$$

when $p_3>N(\varepsilon)$, where $N(\varepsilon)$ is found for all $\varepsilon>0$.

From (2.89) follows (2.86).

We have

$$\sum_{j_3=0}^{p_3}\sum_{j_1=2l+l_1+2}^{p_1}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=\sum_{j_3=0}^{p_3}\sum_{j_1=2l+l_1+2}^{2(j_3+l+1)+l_1}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}. \qquad (2.90)$$

We put $2(j_3+l+1)+l_1$ instead of p_1, since $C_{j_3j_3j_1}=0$ when $j_1>2(j_3+l+1)+l_1$. It follows from the relation:

$$C_{j_3j_3j_1}=\frac{1}{2}\int\limits_t^T\phi_{j_1}(s_2)(t-s_2)^{l_1}\left(\int\limits_{s_2}^T\phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2 ds_2=$$

$$=\frac{1}{2}\int\limits_t^T\phi_{j_1}(s_2)Q_{2(j_3+l+1)+l_1}(s_2)ds_2,$$

where $Q_{2(j_3+l+1)+l_1}(s)$ — is a polynomial of degree $2(j_3+l+1)+l_1$.

It is easy to see, that

$$\sum_{j_3=0}^{p_3}\sum_{j_1=2l+l_1+2}^{2(j_3+l+1)+l_1}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}. \qquad (2.91)$$

Note, that we included some coefficients $C_{j_3j_3j_1}$ in the sum $\sum\limits_{j_3=0}^{p_3}$, which equals to zero.

From (2.90) and (2.91) we get:

$$\mathrm{M}\left\{\left(\sum_{j_3=0}^{p_3}\sum_{j_1=2l+l_1+2}^{p_1}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\}=\mathrm{M}\left\{\left(\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\}=$$

$$=\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\left(\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\right)^2=$$

$$=\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\left(\frac{1}{2}\sum_{j_3=0}^{p_3}\int\limits_t^T\phi_{j_1}(s_2)(t-s_2)^{l_1}\left(\int\limits_{s_2}^T\phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2 ds_2\right)^2=$$

$$=\frac{1}{4}\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\left(\int\limits_t^T\phi_{j_1}(s_2)(t-s_2)^{l_1}\sum_{j_3=0}^{p_3}\left(\int\limits_{s_2}^T\phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2 ds_2\right)^2=$$

$$=\frac{1}{4}\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\left(\int\limits_t^T\phi_{j_1}(s_2)(t-s_2)^{l_1}\left(\int\limits_{s_2}^T(t-s_1)^{2l}ds_1-\right.\right.$$

$$\left.\left.-\sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_2}^T\phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2\right)ds_2\right)^2=$$

$$=\frac{1}{4}\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\left(\int\limits_t^T\phi_{j_1}(s_2)(t-s_2)^{l_1}\sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_2}^T\phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2 ds_2\right)^2. \tag{2.92}$$

In order to get (2.92) we used the Parseval equality of type (2.88) and the following relation:

$$\int\limits_t^T\phi_{j_1}(s)Q_{2l+1+l_1}(s)ds=0;\ j_1>2l+1+l_1,$$

where $Q_{2l+1+l_1}(s)$ — is a polynomial of degree $2l+1+l_1$.

Further we have

$$\left(\int\limits_{s_2}^T\phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2=$$

$$=\frac{(T-t)^{2l+1}(2j_3+1)}{2^{2l+2}}\left(\int\limits_{z(s_2)}^1 P_{j_3}(y)(1+y)^l dy\right)^2=$$

$$=\frac{(T-t)^{2l+1}}{2^{2l+2}(2j_3+1)}\Biggl((1+z(s_2))^l\left(P_{j_3-1}(z(s_2))-P_{j_3+1}(z(s_2))\right)-$$

$$-l\int\limits_{z(s_2)}^1\left(P_{j_3+1}(y)-P_{j_3-1}(y)\right)(1+y)^{l-1}dy\Biggr)^2\le$$

$$\le \frac{(T-t)^{2l+1}2}{2^{2l+2}(2j_3+1)}\left(\left(\frac{2(s_2-t)}{T-t}\right)^{2l}\left(P_{j_3+1}\left(z(s_2)\right)-P_{j_3-1}\left(z(s_2)\right)\right)^2+\right.$$

$$\left.+l^2\left(\int\limits_{z(s_2)}^{1}\left(P_{j_3+1}(y)-P_{j_3-1}(y)\right)(1+y)^{l-1}dy\right)^2\right)\le$$

$$\le \frac{(T-t)^{2l+1}}{2^{2l+1}(2j_3+1)}\Biggl(2^{2l+1}\left(P_{j_3+1}^2\left(z(s_2)\right)+P_{j_3-1}^2\left(z(s_2)\right)\right)+$$

$$+l^2\int\limits_{z(s_2)}^{1}(1+y)^{2l-2}dy\int\limits_{z(s_2)}^{1}\left(P_{j_3+1}(y)-P_{j_3-1}(y)\right)^2dy\Biggr)\le$$

$$\le \frac{(T-t)^{2l+1}}{2^{2l+1}(2j_3+1)}\Biggl(2^{2l+1}\left(P_{j_3+1}^2\left(z(s_2)\right)+P_{j_3-1}^2\left(z(s_2)\right)\right)+$$

$$+\frac{2^{2l}l^2}{2l-1}\left(1-\left(\frac{(s_2-t)}{T-t}\right)^{2l-1}\right)\int\limits_{z(s_2)}^{1}\left(P_{j_3+1}^2(y)+P_{j_3-1}^2(y)\right)dy\Biggr)\le$$

$$\le \frac{(T-t)^{2l+1}}{2(2j_3+1)}\Biggl(2\left(P_{j_3+1}^2\left(z(s_2)\right)+P_{j_3-1}^2\left(z(s_2)\right)\right)+$$

$$+\frac{l^2}{2l-1}\int\limits_{z(s_2)}^{1}\left(P_{j_3+1}^2(y)+P_{j_3-1}^2(y)\right)dy\Biggr), \qquad (2.93)$$

where

$$z(s_2)=\left(s_2-\frac{T+t}{2}\right)\frac{2}{T-t}.$$

Let's estimate right part of (2.93) using of (2.43):

$$\left(\int\limits_{s_2}^{T}\phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2<\frac{(T-t)^{2l+1}}{2(2j_3+1)}\left(\frac{K^2}{j_3+2}+\frac{K^2}{j_3}\right)\times$$

$$\times\left(\frac{2}{\left(1-\left(z(s_2)\right)^2\right)^{\frac{1}{2}}}+\frac{l^2}{2l-1}\int\limits_{z(s_2)}^{1}\frac{dy}{\left(1-y^2\right)^{\frac{1}{2}}}\right)<$$

$$<\frac{(T-t)^{2l+1}K^2}{2j_3^2}\left(\frac{2}{\left(1-\left(z(s_2)\right)^2\right)^{\frac{1}{2}}}+\frac{l^2\pi}{2l-1}\right);\ s\in(t,T), \qquad (2.94)$$

where

$$z(s_2)=\left(s_2-\frac{T+t}{2}\right)\frac{2}{T-t}.$$

From (2.92) and (2.94) we get:

$$\mathrm{M}\left\{\left(\sum_{j_3=0}^{p_3}\sum_{j_1=2l+l_1+2}^{p_1} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\right)^2\right\}\le$$

$$\le\frac{1}{4}\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\left(\int\limits_t^T\mid\phi_{j_1}(s_2)\mid(t-s_2)^{l_1}\sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_2}^T\phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2 ds_2\right)^2$$

$$\le\frac{1}{4}(T-t)^{2l_1}\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\left(\int\limits_t^T\mid\phi_{j_1}(s_2)\mid\sum_{j_3=p_3+1}^{\infty}\left(\int\limits_{s_2}^T\phi_{j_3}(s_1)(t-s_1)^l ds_1\right)^2 ds_2\right)^2$$

$$<\frac{(T-t)^{4l+2l_1+1}K^4K_1^2}{16}\sum_{j_1=2l+l_1+2}^{2(p_3+l+1)+l_1}\left(\left(\int\limits_t^T\frac{2ds_2}{\left(1-(z(s_2))^2\right)^{\frac{3}{4}}}+\right.\right.$$

$$\left.\left.+\frac{l^2\pi}{2l-1}\int\limits_t^T\frac{ds_2}{\left(1-(z(s_2))^2\right)^{\frac{1}{4}}}\right)\sum_{j_3=p_3+1}^{\infty}\frac{1}{j_3^2}\right)^2\le$$

$$\le\frac{(T-t)^{4l+2l_1+3}K^4K_1^2}{64}\cdot\frac{2p_3+1}{p_3^2}\left(\int\limits_{-1}^1\frac{2dy}{(1-y^2)^{\frac{3}{4}}}+\frac{l^2\pi}{2l-1}\int\limits_{-1}^1\frac{dy}{(1-y^2)^{\frac{1}{4}}}\right)^2\le$$

$$\le(T-t)^{4l+2l_1+3}C\frac{2p_3+1}{p_3^2}\to 0\ \text{when}\ p_3\to\infty, \qquad (2.95)$$

where the constant C doesn't depend on p_3 and $T-t$, and

$$z(s_2)=\left(s_2-\frac{T+t}{2}\right)\frac{2}{T-t}.$$

From (2.85), (2.86) and (2.95) follows (2.84) and from (2.84) follows the expansion (2.82).

2.3.4 The case $\psi_1(\tau),\psi_2(\tau),\psi_3(\tau)\equiv(t-\tau)^l;\ i_1,i_2,i_3=1,\ldots,m$

In this section we will prove following expansion for multiple stochastic Stratonovich integral of 3rd multiplicity:

$$\int\limits_t^{*T}(t-s)^l\int\limits_t^{*s}(t-s_1)^l\int\limits_t^{*s_1}(t-s_2)^l d\mathbf{f}_{s_2}^{(i_1)}d\mathbf{f}_{s_1}^{(i_2)}d\mathbf{f}_{s}^{(i_3)}=$$

$$= \sum_{j_1,j_2,j_3=0}^{\infty} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\ \ (i_1, i_2, i_3 = 1, \ldots, m), \tag{2.96}$$

where the series converges in the mean-square sense; $l = 0,\ 1,\ 2, \ldots$ and

$$C_{j_3j_2j_1} = \int\limits_t^T \phi_{j_3}(s)(t-s)^l \int\limits_t^s (t-s_1)^l \phi_{j_2}(s_1) \int\limits_t^{s_1} (t-s_2)^l \phi_{j_1}(s_2) ds_2 ds_1 ds. \tag{2.97}$$

If we prove the formula:

$$\sum_{j_1,j_3=0}^{\infty} C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)} = 0, \tag{2.98}$$

where the series coverges in the mean-square sense and the coefficients $C_{j_3j_2j_1}$ have the form (2.97), then using theorem 1, relations (2.70), (2.84) when $l_1 = l_3 = l$ and standard relations between multiple stochastic Stratonovich and Ito integrals we have expansion (2.96).

Since $\psi_1(s),\ \psi_2(s),\ \psi_3(s) \equiv (t-s)^l$ the following relation for Fourier coefficients takes place

$$C_{j_1j_1j_3} + C_{j_1j_3j_1} + C_{j_3j_1j_1} = \frac{1}{2}C_{j_1}^2 C_{j_3},$$

where $C_{j_3j_2j_1}$ has the form (2.97) and

$$C_{j_1} = \int\limits_t^T \phi_{j_1}(s)(t-s)^l ds,$$

then w.p.1

$$\sum_{j_1,j_3=0}^{\infty} C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)} = \sum_{j_1,j_3=0}^{\infty}\left(\frac{1}{2}C_{j_1}^2 C_{j_3} - C_{j_1j_1j_3} - C_{j_3j_1j_1}\right)\zeta_{j_3}^{(i_2)}. \tag{2.99}$$

Taking into account (2.70) and (2.84) when $l_3 = l_1 = l$ and the Ito formula we have with probability 1:

$$\sum_{j_1,j_3=0}^{\infty} C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)} = \frac{1}{2}\sum_{j_1=0}^{l} C_{j_1}^2 \sum_{j_3=0}^{l} C_{j_3}\zeta_{j_3}^{(i_2)} - \sum_{j_1,j_3=0}^{\infty} C_{j_1j_1j_3}\zeta_{j_3}^{(i_2)} -$$

$$- \sum_{j_1,j_3=0}^{\infty} C_{j_3j_1j_1}\zeta_{j_3}^{(i_2)} =$$

$$= \frac{1}{2}\sum_{j_1=0}^{l} C_{j_1}^2 \int\limits_t^T (t-s)^l d\mathbf{f}_s^{(i_2)} - \frac{1}{2}\int\limits_t^T (t-s)^l \int\limits_t^s (t-s_1)^{2l} ds_1 d\mathbf{f}_s^{(i_2)} -$$

$$-\frac{1}{2}\int\limits_t^T (t-s)^{2l}\int\limits_t^s (t-s_1)^l d\mathbf{f}_{s_1}^{(i_2)}ds=$$

$$=\frac{1}{2}\sum_{j_1=0}^{l} C_{j_1}^2\int\limits_t^T (t-s)^l d\mathbf{f}_s^{(i_2)}+\frac{1}{2(2l+1)}\int\limits_t^T (t-s)^{3l+1}d\mathbf{f}_s^{(i_2)}-$$

$$-\frac{1}{2}\int\limits_t^T (t-s_1)^l\int\limits_{s_1}^T (t-s)^{2l}dsd\mathbf{f}_{s_1}^{(i_2)}=$$

$$=\frac{1}{2}\sum_{j_1=0}^{l} C_{j_1}^2\int\limits_t^T (t-s)^l d\mathbf{f}_s^{(i_2)}+\frac{1}{2(2l+1)}\int\limits_t^T (t-s)^{3l+1}d\mathbf{f}_s^{(i_2)}-$$

$$-\frac{1}{2(2l+1)}\left((T-t)^{2l+1}\int\limits_t^T (t-s)^l d\mathbf{f}_s^{(i_2)}+\int\limits_t^T (t-s)^{3l+1}d\mathbf{f}_s^{(i_2)}\right)=$$

$$=\frac{1}{2}\sum_{j_1=0}^{l} C_{j_1}^2\int\limits_t^T (t-s)^l d\mathbf{f}_s^{(i_2)}-\frac{(T-t)^{2l+1}}{2(2l+1)}\int\limits_t^T (t-s)^l d\mathbf{f}_s^{(i_2)}=$$

$$=\frac{1}{2}\left(\sum_{j_1=0}^{l} C_{j_1}^2-\int\limits_t^T (t-s)^{2l}ds\right)\int\limits_t^T (t-s)^l d\mathbf{f}_s^{(i_2)}=0.$$

Here, the Parseval equality looks as follows:

$$\sum_{j_1=0}^{\infty} C_{j_1}^2=\sum_{j_1=0}^{l} C_{j_1}^2=\int\limits_t^T (t-s)^{2l}ds=\frac{(T-t)^{2l+1}}{2l+1}$$

and

$$\int\limits_t^T (t-s)^l d\mathbf{f}_s^{(i_2)}=\sum_{j_3=0}^{l} C_{j_3}\zeta_{j_3}^{(i_2)}\ \text{w. p. 1.}$$

The expansion (2.96) is proven.

It is easy to see, that using Ito formula (see sect. 7.3) when $i_1=i_2=i_3$ we get:

$$\int\limits_t^{*T} (t-s)^l\int\limits_t^{*s} (t-s_1)^l\int\limits_t^{*s_1} (t-s_2)^l d\mathbf{f}_{s_2}^{(i_1)}d\mathbf{f}_{s_1}^{(i_1)}d\mathbf{f}_s^{(i_1)}=$$

$$=\frac{1}{6}\left(\int\limits_t^T (t-s)^l d\mathbf{f}_s^{(i_1)}\right)^3=\frac{1}{6}\left(\sum_{j_1=0}^{l} C_{j_1}\zeta_{j_1}^{(i_1)}\right)^3=$$

$$=\sum_{j_1,j_2,j_3=0}^{l} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}\zeta_{j_3}^{(i_1)}\ \text{w. p. 1.}\qquad(2.100)$$

The last step in the formula (2.100) was made on the basis of formula (1.36) derivation.

2.3.5 Theorem about expansion of multiple stochastic Stratonovich integrals of 3rd multiplicity, based on theorem 1. Case of Legendre polynomials

Let's combine in one statement the results obtained in the previous sections.

Theorem 4. *Assume, that* $\{\phi_j(x)\}_{j=0}^{\infty}$ — *is a full orthonormal system of Legendre polynomials at the interval* $[t, T]$. *Then, for multiple stochastic Stratonovich integral of* 3*rd multiplicity*

$$I_{l_1 l_2 l_3 T,t}^{*(i_1 i_2 i_3)} = \int\limits_t^{*T} (t-t_3)^{l_3} \int\limits_t^{*t_3} (t-t_2)^{l_2} \int\limits_t^{*t_2} (t-t_1)^{l_1} d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} d\mathbf{f}_{t_3}^{(i_3)}$$

$(i_1, i_2, i_3 = 1, \ldots, m)$ *the following converging in the mean-square sense expansion*

$$I_{l_1 l_2 l_3 T,t}^{*(i_1 i_2 i_3)} = \sum_{j_1, j_2, j_3 = 0}^{\infty} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \tag{2.101}$$

is reasonable for each of the following cases:
1. $i_1 \ne i_2,\ i_2 \ne i_3,\ i_1 \ne i_3$ *and* $l_1,\ l_2,\ l_3 = 0,\ 1,\ 2, \ldots$;
2. $i_1 = i_2 \ne i_3$ *and* $l_1 = l_2 \ne l_3$ *and* $l_1,\ l_2,\ l_3 = 0,\ 1,\ 2, \ldots$;
3. $i_1 \ne i_2 = i_3$ *and* $l_1 \ne l_2 = l_3$ *and* $l_1,\ l_2,\ l_3 = 0,\ 1,\ 2, \ldots$;
4. $i_1, i_2, i_3 = 1, \ldots, m$; $l_1 = l_2 = l_3 = l$ *and* $l = 0,\ 1,\ 2, \ldots$,

where

$$C_{j_3 j_2 j_1} = \int\limits_t^{T} (t-s)^{l_3} \phi_{j_3}(s) \int\limits_t^{s} (t-s_1)^{l_2} \phi_{j_2}(s_1) \int\limits_t^{s_1} (t-s_2)^{l_1} \phi_{j_1}(s_2) ds_2 ds_1 ds.$$

Let's note, that for expansion of multiple stochastic Stratonovich integrals of 2nd and 3rd multiplicity theorems 3 and 4 will be very useful.

2.4 Expansions of multiple stochastic Stratonovich integrals of 3rd multiplicity, based on theorem 1. Trigonometric case

In this section we will prove the following expansion for multiple stochastic Stratonovich integral of 3rd multiplicity:

$$\int\limits_t^{*T} \int\limits_t^{*t_3} \int\limits_t^{*t_2} d\mathbf{f}_{t_1}^{(i_1)} d\mathbf{f}_{t_2}^{(i_2)} d\mathbf{f}_{t_3}^{(i_3)} = \sum_{j_1, j_2, j_3 = 0}^{\infty} C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \tag{2.102}$$

where series converges in mean-square sense,

$$C_{j_3j_2j_1}=\int\limits_t^T\phi_{j_3}(s)\int\limits_t^s\phi_{j_2}(s_1)\int\limits_t^{s_1}\phi_{j_1}(s_2)ds_2ds_1ds$$

and $\{\phi_j(x)\}_{j=0}^{\infty}$ — is a full orthonormal system of trigonometric functions in the space $L_2([t,T])$.

If we prove the following formulas:

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}\stackrel{\rm def}{=}\sum_{j_1,j_3=0}^{\infty}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}=\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}dsd{\bf f}_{\tau}^{(i_3)},\qquad(2.103)$$

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}\stackrel{\rm def}{=}\sum_{j_1,j_3=0}^{\infty}C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}d{\bf f}_{s}^{(i_1)}d\tau,\qquad(2.104)$$

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}\stackrel{\rm def}{=}\sum_{j_1,j_3=0}^{\infty}C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)}=0.\qquad(2.105)$$

then from theorem 1, formulas (2.103) – (2.105) and standard relations between multiple stochastic Stratonovich and Ito integrals the expansion (2.102) will follow.

We have:

$$\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{p_1}C_{j_3j_1j_1}\zeta_{j_3}^{(i_3)}=\frac{(T-t)^{\frac{3}{2}}}{6}+\sum_{j_1=1}^{p_1}C_{0,2j_1,2j_1}\zeta_0^{(i_3)}+\sum_{j_1=1}^{p_1}C_{0,2j_1-1,2j_1-1}\zeta_0^{(i_3)}+$$

$$+\sum_{j_3=1}^{p_1}C_{2j_3,0,0}\zeta_{2j_3}^{(i_3)}+\sum_{j_3=1}^{p_3}\sum_{j_1=1}^{p_1}C_{2j_3,2j_1,2j_1}\zeta_{2j_3}^{(i_3)}+$$

$$+\sum_{j_3=1}^{p_3}\sum_{j_1=1}^{p_1}C_{2j_3,2j_1-1,2j_1-1}\zeta_{2j_3}^{(i_3)}+\sum_{j_3=1}^{p_3}C_{2j_3-1,0,0}\zeta_{2j_3-1}^{(i_3)}+$$

$$+\sum_{j_3=1}^{p_3}\sum_{j_1=1}^{p_1}C_{2j_3-1,2j_1,2j_1}\zeta_{2j_3-1}^{(i_3)}+\sum_{j_3=1}^{p_3}\sum_{j_1=1}^{p_1}C_{2j_3-1,2j_1-1,2j_1-1}\zeta_{2j_3-1}^{(i_3)},\qquad(2.106)$$

where the summation is stopped when $2j_1,\ 2j_1-1>p_1$ or $2j_3,\ 2j_3-1>p_3$ and

$$C_{0,2l,2l}=\frac{(T-t)^{\frac{3}{2}}}{8\pi^2l^2},\ C_{0,2l-1,2l-1}=\frac{3(T-t)^{\frac{3}{2}}}{8\pi^2l^2},\ C_{2l,0,0}=\frac{\sqrt{2}(T-t)^{\frac{3}{2}}}{4\pi^2l^2};\qquad(2.107)$$

$$C_{2r-1,2l,2l}=0, C_{2l-1,0,0}=-\frac{\sqrt{2}(T-t)^{\frac{3}{2}}}{4\pi l}, C_{2r-1,2l-1,2l-1}=0;\qquad(2.108)$$

$$C_{2r,2l,2l} = \begin{cases} -\frac{\sqrt{2}(T-t)^{\frac{3}{2}}}{16\pi^2 l^2}, \ r = 2l \\ 0, \ r \neq 2l \end{cases}, \ C_{2r,2l-1,2l-1} = \begin{cases} \frac{\sqrt{2}(T-t)^{\frac{3}{2}}}{16\pi^2 l^2}, \ r = 2l \\ -\frac{\sqrt{2}(T-t)^{\frac{3}{2}}}{4\pi^2 l^2}, \ r = l \\ 0, \ r \neq l, \ r \neq 2l \end{cases}. \tag{2.109}$$

After substituting (2.107) – (2.109) into (2.106) we get:

$$\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{p_1} C_{j_3 j_1 j_1}\zeta_{j_3}^{(i_3)} = (T-t)^{\frac{3}{2}}\left(\left(\frac{1}{6}+\frac{1}{2\pi^2}\sum_{j_1=1}^{p_1}\frac{1}{j_1^2}\right)\zeta_0^{(i_3)} - \right.$$

$$\left. -\frac{\sqrt{2}}{4\pi}\sum_{j_3=1}^{p_3}\frac{1}{j_3}\zeta_{2j_3-1}^{(i_3)}\right). \tag{2.110}$$

Using theorem 1 and the system of trigonometric functions we find

$$\frac{1}{2}\int\limits_t^T\int\limits_t^s ds_1 d\mathbf{f}_s^{(i_3)} = \frac{1}{2}\int\limits_t^T (s-t)d\mathbf{f}_s^{(i_3)} =$$

$$= \frac{1}{4}(T-t)^{\frac{3}{2}}\left(\zeta_0^{(i_3)} - \frac{\sqrt{2}}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_3)}\right). \tag{2.111}$$

From (2.110) and (2.111) it follows

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}} \ \mathsf{M}\left\{\left(\sum_{j_3=0}^{p_3}\sum_{j_1=0}^{p_1} C_{j_3 j_1 j_1}\zeta_{j_3}^{(i_3)} - \frac{1}{2}\int\limits_t^T\int\limits_t^s ds_1 d\mathbf{f}_s^{(i_3)}\right)^2\right\} =$$

$$= \underset{p_1,p_3\to\infty}{\text{l.i.m.}} \ (T-t)^3\left(\left(\frac{1}{6}+\frac{1}{2\pi^2}\sum_{j_1=1}^{p_1}\frac{1}{j_1^2}-\frac{1}{4}\right)^2 + \right.$$

$$\left. +\frac{1}{8\pi^2}\left(\frac{\pi^2}{6}-\sum_{j_3=1}^{p_3}\frac{1}{j_3^2}\right)\right) = 0.$$

So, the relation (2.103) is executed for the case of trigonometric system of functions.

Let's prove the relation (2.104). We have

$$\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3 j_3 j_1}\zeta_{j_1}^{(i_1)} = \frac{(T-t)^{\frac{3}{2}}}{6} + \sum_{j_3=1}^{p_3} C_{2j_3,2j_3,0}\zeta_0^{(i_1)} + \sum_{j_3=1}^{p_3} C_{2j_3-1,2j_3-1,0}\zeta_0^{(i_1)} +$$

$$+\sum_{j_1=1}^{p_1}\sum_{j_3=1}^{p_3} C_{2j_3,2j_3,2j_1-1}\zeta_{2j_1-1}^{(i_1)} + \sum_{j_1=1}^{p_1}\sum_{j_3=1}^{p_3} C_{2j_3-1,2j_3-1,2j_1-1}\zeta_{2j_1-1}^{(i_1)} +$$

$$+\sum_{j_1=1}^{p_1} C_{0,0,2j_1-1}\zeta_{2j_1-1}^{(i_1)}+\sum_{j_1=1}^{p_1}\sum_{j_3=1}^{p_3} C_{2j_3,2j_3,2j_1}\zeta_{2j_1}^{(i_1)}+$$

$$+\sum_{j_1=1}^{p_1}\sum_{j_3=1}^{p_3} C_{2j_3-1,2j_3-1,2j_1}\zeta_{2j_1}^{(i_1)}+\sum_{j_1=1}^{p_1} C_{0,0,2j_1}\zeta_{2j_1}^{(i_1)}, \qquad (2.112)$$

where the summation is stopped, when $2j_3,\ 2j_3-1>p_3$ or $2j_1,\ 2j_1-1>p_1$ and

$$C_{2l,2l,0}=\frac{(T-t)^{\frac{3}{2}}}{8\pi^2l^2},\ C_{2l-1,2l-1,0}=\frac{3(T-t)^{\frac{3}{2}}}{8\pi^2l^2},\ C_{0,0,2r}=\frac{\sqrt{2}(T-t)^{\frac{3}{2}}}{4\pi^2r^2}; \qquad (2.113)$$

$$C_{2l-1,2l-1,2r-1}=0, C_{0,0,2r-1}=\frac{\sqrt{2}(T-t)^{\frac{3}{2}}}{4\pi r}, C_{2l,2l,2r-1}=0; \qquad (2.114)$$

$$C_{2l,2l,2r}=\begin{cases}-\frac{\sqrt{2}(T-t)^{\frac{3}{2}}}{16\pi^2l^2},\ r=2l\\ 0,\ r\ne 2l\end{cases},\ C_{2l-1,2l-1,2r}=\begin{cases}-\frac{\sqrt{2}(T-t)^{\frac{3}{2}}}{16\pi^2l^2},\ r=2l\\ \frac{\sqrt{2}(T-t)^{\frac{3}{2}}}{4\pi^2l^2},\ r=l\\ 0,\ r\ne l,\ r\ne 2l\end{cases}. \qquad (2.115)$$

After substituting (2.113) – (2.115) into (2.112) we get:

$$\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}=(T-t)^{\frac{3}{2}}\left(\left(\frac{1}{6}+\frac{1}{2\pi^2}\sum_{j_3=1}^{p_3}\frac{1}{j_3^2}\right)\zeta_0^{(i_1)}+\right.$$

$$\left.+\frac{\sqrt{2}}{4\pi}\sum_{j_1=1}^{p_1}\frac{1}{j_1}\zeta_{2j_1-1}^{(i_1)}\right). \qquad (2.116)$$

Using the Ito formula, theorem 1 and the system of trigonometric functions we find

$$\frac{1}{2}\int_t^T\int_t^s d\mathbf{f}_{s_1}^{(i_1)}ds=\frac{1}{2}\left((T-t)\int_t^T d\mathbf{f}_s^{(i_1)}+\int_t^T(t-s)d\mathbf{f}_s^{(i_1)}\right)=$$

$$=\frac{1}{4}(T-t)^{\frac{3}{2}}\left(\zeta_0^{(i_1)}+\frac{\sqrt{2}}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_1)}\right). \qquad (2.117)$$

From (2.116) and (2.117) it follows

$$\underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ \mathsf{M}\left\{\left(\sum_{j_1=0}^{p_1}\sum_{j_3=0}^{p_3} C_{j_3j_3j_1}\zeta_{j_1}^{(i_1)}-\frac{1}{2}\int_t^T\int_t^s d\mathbf{f}_{s_1}^{(i_1)}ds\right)^2\right\}=$$

$$= \underset{p_1,p_3\to\infty}{\text{l.i.m.}}\ (T-t)^3 \left(\left(\frac{1}{6}+\frac{1}{2\pi^2}\sum_{j_3=1}^{p_3}\frac{1}{j_3^2}-\frac{1}{4}\right)^2 + \right.$$

$$\left. +\frac{1}{8\pi^2}\left(\frac{\pi^2}{6}-\sum_{j_1=1}^{p_1}\frac{1}{j_1^2}\right)\right) = 0.$$

So, the relation (2.104) is also correct for the case of trigonometric system of functions.

Let's prove the equality (2.105).

Since $\psi_1(\tau)$, $\psi_2(\tau)$, $\psi_3(\tau) \equiv 1$, then the following relation for the Fourier coefficients is correct:

$$C_{j_1j_1j_3} + C_{j_1j_3j_1} + C_{j_3j_1j_1} = \frac{1}{2}C_{j_1}^2 C_{j_3},$$

then w.p.1

$$\sum_{j_1,j_3=0}^{\infty} C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)} = \sum_{j_1,j_3=0}^{\infty}\left(\frac{1}{2}C_{j_1}^2 C_{j_3} - C_{j_1j_1j_3} - C_{j_3j_1j_1}\right)\zeta_{j_3}^{(i_2)}. \qquad (2.118)$$

Taking into account (2.103) and (2.104) w.p.1 let's write down the following:

$$\sum_{j_1,j_3=0}^{\infty} C_{j_1j_3j_1}\zeta_{j_3}^{(i_2)} = \frac{1}{2}C_0^3\zeta_0^{(i_2)} - \sum_{j_1,j_3=0}^{\infty} C_{j_1j_1j_3}\zeta_{j_3}^{(i_2)} - \sum_{j_1,j_3=0}^{\infty} C_{j_3j_1j_1}\zeta_{j_3}^{(i_2)} =$$

$$= \frac{1}{2}(T-t)^{\frac{3}{2}}\zeta_0^{(i_2)} - \frac{1}{4}(T-t)^{\frac{3}{2}}\left[\zeta_0^{(i_2)} + \frac{\sqrt{2}}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_2)}\right] -$$

$$-\frac{1}{4}(T-t)^{\frac{3}{2}}\left[\zeta_0^{(i_2)} - \frac{\sqrt{2}}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_2)}\right] = 0.$$

From (2.103) – (2.105) and theorem 1 we get the expansion (2.102).

The expansion (2.102) may be also obtained by direct calculation according to theorem 1:

$$\int_t^T\int_t^{t_3}\int_t^{t_2} d\mathbf{f}_{t_1}^{(i_3)} d\mathbf{f}_{t_2}^{(i_2)} d\mathbf{f}_{t_3}^{(i_1)} = \sum_{j_1,j_2,j_3=0}^{\infty} C_{j_3j_2j_1}\zeta_{j_1}^{(i_3)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_1)} -$$

$$-\mathbf{1}_{\{i_1=i_2\}}\left(-\frac{1}{4}(T-t)^{\frac{3}{2}}\left[\zeta_0^{(i_3)} - \frac{\sqrt{2}}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_3)}\right] + \right.$$

$$\left. +\frac{1}{2}(T-t)^{\frac{3}{2}}\zeta_0^{(i_3)} + \frac{1}{2\pi^2}(T-t)^{\frac{3}{2}}\left(\sum_{r=1}^{\infty}\frac{1}{r^2} - \frac{\pi^2}{6}\right)\zeta_0^{(i_3)}\right) -$$

$$
-\mathbf{1}_{\{i_2=i_3\}}\Biggl(\frac{1}{4}(T-t)^{\frac{3}{2}}\Biggl[\zeta_0^{(i_1)}-\frac{\sqrt{2}}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i_1)}\Biggr]+
$$

$$
+\frac{1}{2\pi^2}(T-t)^{\frac{3}{2}}\left(\sum_{r=1}^{\infty}\frac{1}{r^2}-\frac{\pi^2}{6}\right)\zeta_0^{(i_1)}\Biggr)+
$$

$$
+\mathbf{1}_{\{i_1=i_3\}}\frac{1}{\pi^2}(T-t)^{\frac{3}{2}}\left(\sum_{r=1}^{\infty}\frac{1}{r^2}-\frac{\pi^2}{6}\right)\zeta_0^{(i_2)}, \qquad (2.119)
$$

where

$$
\sum_{j_1,j_2,j_3=0}^{\infty}C_{j_3j_2j_1}\zeta_{j_1}^{(i_3)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_1)}=(T-t)^{\frac{3}{2}}\Biggl(\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}+
$$

$$
+\frac{1}{2\sqrt{2}}\sum_{r=1}^{\infty}\Biggl[\frac{1}{\pi r}\left\{\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_2)}\zeta_0^{(i_1)}-\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}\right\}+
$$

$$
+\frac{1}{\pi^2r^2}\left\{\zeta_{2r}^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-2\zeta_{2r}^{(i_2)}\zeta_0^{(i_3)}\zeta_0^{(i_1)}+\zeta_{2r}^{(i_3)}\zeta_0^{(i_2)}\zeta_0^{(i_1)}\right\}\Biggr]+
$$

$$
+\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\ne l}}^{\infty}\Biggl[\frac{1}{r^2-l^2}\Biggl\{\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_2)}\zeta_0^{(i_3)}-\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\zeta_{2l}^{(i_3)}+
$$

$$
+\frac{r}{l}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\zeta_0^{(i_3)}-\frac{l}{r}\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2l-1}^{(i_3)}\Biggr\}-\frac{1}{rl}\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}\zeta_{2l-1}^{(i_3)}\Biggr]
$$

$$
+\sum_{r=1}^{\infty}\Biggl[\frac{1}{4\pi r}\left\{\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_0^{(i_3)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_3)}-\zeta_{2r-1}^{(i_2)}\zeta_{2r}^{(i_3)}\zeta_0^{(i_1)}+\zeta_{2r-1}^{(i_3)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\right\}
$$

$$
+\frac{1}{8\pi^2r^2}\Biggl\{3\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_0^{(i_3)}+\zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_3)}-6\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_2)}+
$$

$$
+3\zeta_{2r-1}^{(i_2)}\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_1)}-2\zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_3)}\zeta_0^{(i_2)}+\zeta_{2r}^{(i_3)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\Biggr\}\Biggr]+
$$

$$
+\frac{1}{4\sqrt{2}\pi^2}\Biggl\{\sum_{r,m=1}^{\infty}\Biggl[\frac{2}{rm}\Bigl[-\zeta_{2r-1}^{(i_1)}\zeta_{2m-1}^{(i_2)}\zeta_{2m}^{(i_3)}+\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_{2m-1}^{(i_3)}+
$$

$$
+\zeta_{2r-1}^{(i_1)}\zeta_{2m}^{(i_2)}\zeta_{2m-1}^{(i_3)}-\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2m-1}^{(i_3)}\Bigr]+
$$

$$
+\frac{1}{m(r+m)}\Bigl[-\zeta_{2(m+r)}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_{2m}^{(i_3)}-\zeta_{2(m+r)-1}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2m}^{(i_3)}-
$$

$$
-\zeta_{2(m+r)-1}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_{2m-1}^{(i_3)}+\zeta_{2(m+r)}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2m-1}^{(i_3)}\Bigr]\Biggr]+
$$

$$
+\sum_{m=1,l=m+1}^{\infty}\Biggl[\frac{1}{m(l-m)}\Bigl[\zeta_{2(l-m)}^{(i_1)}\zeta_{2l}^{(i_2)}\zeta_{2m}^{(i_3)}+\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\zeta_{2m}^{(i_3)}-
$$

$$
-\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2l}^{(i_2)}\zeta_{2m-1}^{(i_3)}+\zeta_{2(l-m)}^{(i_1)}\zeta_{2l-1}^{(i_2)}\zeta_{2m-1}^{(i_3)}\Bigr]+
$$

$$
+\frac{1}{l(l-m)}\Bigl[-\zeta_{2(l-m)}^{(i_1)}\zeta_{2m}^{(i_2)}\zeta_{2l}^{(i_3)}+\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2m-1}^{(i_2)}\zeta_{2l}^{(i_3)}-
$$

$$
-\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2m}^{(i_2)}\zeta_{2l-1}^{(i_3)}-\zeta_{2(l-m)}^{(i_1)}\zeta_{2m-1}^{(i_2)}\zeta_{2l-1}^{(i_3)}\Bigr]\Bigr]\Bigr\}\Bigr).
$$

Since $\sum\limits_{r=1}^{\infty}\frac{1}{r^2}=\frac{\pi^2}{6}$ and according to theorem 1:

$$
\int\limits_t^T(t-\tau)d\mathbf{f}_\tau^{(i)}=-\frac{1}{2}(T-t)^{\frac{3}{2}}\Bigl[\zeta_0^{(i)}-\frac{\sqrt{2}}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\zeta_{2r-1}^{(i)}\Bigr],
$$

then, from (2.119) we get the required expansion:

$$
\sum_{j_1,j_2,j_3=0}^{\infty}C_{j_3j_2j_1}\zeta_{j_1}^{(i_3)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_1)}=\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}d\mathbf{f}_{t_1}^{(i_3)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_1)}+\mathbf{1}_{\{i_1=i_2\}}\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}d\mathbf{f}_s^{(i_3)}d\tau+
$$

$$
+\mathbf{1}_{\{i_2=i_3\}}\frac{1}{2}\int\limits_t^T\int\limits_t^{\tau}dsd\mathbf{f}_\tau^{(i_1)}=\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2}d\mathbf{f}_{t_1}^{(i_3)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_1)}.
$$

2.5 Expansion of multiple stochastic Stratonovich integrals of any fixed multiplicity k, based on generalized repeated Fourier series

2.5.1 The case of integrals of 2nd multiplicity

Let's analyze the approach to expansion of multiple stochastic integrals, which differs from the approaches examined before [34], [35], [46] (pp. 262 – 266), taking multiple stochastic Stratonovich integrals of 2nd multiplicity as an example.

Thus, let's analyze the multiple stochastic Stratonovich integral of the following type:

$$
J^*[\psi^{(2)}]_{T,t}=\int\limits_t^{*T}\psi_2(t_2)\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)};\ i_1,i_2=1,\ldots,m,
$$

where $\mathbf{w}_t^{(i)}=\mathbf{f}_t^{(i)}$ when $i=1,\ldots,m$; $\mathbf{w}_t^{(0)}=t$; $i_1,\ldots,i_k=0,\ 1,\ldots,m$; $\mathbf{f}_\tau^{(i)}$ $(i=1,\ldots,m)$ — are independent standard Wiener processes; $\psi_l(\tau)$ $(l=1,\ldots,k)$ — are continously differentiated functions at the interval $[t,T]$;

Let's analyze the function

$$K^*(t_1,t_2)=K(t_1,t_2)+\frac{1}{2}\mathbf{1}_{\{t_1=t_2\}}\psi_1(t_1)\psi_2(t_1),$$

where $t_1,t_2\in[t,T]$ and $K(t_1,t_2)$ has the form:

$$K(t_1,t_2)=\begin{cases}\psi_1(t_1)\psi_2(t_2), & t_1<t_2\\ 0, & \text{otherwise}\end{cases};\ \ t_1,t_2\in[t,T].$$

Due to lemmas proven in chapter 1 and formula

$$J^*[\psi^{(2)}]_{T,t}=\int\limits_t^T\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\neq 0\}}\int\limits_t^T\psi_2(t_2)\psi_1(t_2)dt_2$$

with probability 1, we have

$$J^*[\psi^{(2)}]_{T,t}=\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_1=0}^{N-1}\sum_{l_2=0}^{N-1}K^*(\tau_{l_1},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\ \text{w.p.1},\tag{2.120}$$

where the sense of formula (1.8) notations is kept.

Let's expand the function $K^*(t_1,t_2)$ using the variable t_1, when t_2 is fixed, into the Fourier series at the interval (t,T) :

$$K^*(t_1,t_2)=\sum_{j_1=0}^{\infty}C_{j_1}(t_2)\phi_{j_1}(t_1)\ \ (t_1\neq t,T),\tag{2.121}$$

where

$$C_{j_1}(t_2)=\int\limits_t^T K^*(t_1,t_2)\phi_{j_1}(t_1)dt_1=\int\limits_t^T K(t_1,t_2)\phi_{j_1}(t_1)dt_1=$$

$$=\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1,$$

$\{\phi_j(x)\}_{j=0}^{\infty}$ — is a full orthonormal system of continuous functions at the interval $[t,T]$.

The equality (2.121) is executed pointwise in each point of the interval (t,T) according to the variable t_1, when $t_2\in[t,T]$ is fixed due to sectionally smoothness of the function $K^*(t_1,t_2)$ according to the variable $t_1\in[t,T]$ (t_2 — is fixed).

Note also, that due to well-known features of the Fourier series, the series (2.121) converges when $t_1=t,T$ (not necessarily to the function $K^*(t_1,t_2)$).

Obtaining (2.121) we also used the fact that the right part of (2.121) converges when $t_1 = t_2$ (point of finite discontinuity of function $K(t_1, t_2)$) to the value

$$\frac{1}{2}\left(K(t_2-0,t_2)+K(t_2+0,t_2)\right)=\frac{1}{2}\psi_1(t_2)\psi_2(t_2)=K^*(t_2,t_2).$$

Function $C_{j_1}(t_2)$ is a continuously differentiated one at the interval $[t, T]$. Let's expand it into the Fourier series at the interval (t, T):

$$C_{j_1}(t_2)=\sum_{j_2=0}^{\infty}C_{j_2j_1}\phi_{j_2}(t_2)\quad (t_2\neq t,T), \tag{2.122}$$

where

$$C_{j_2j_1}=\int\limits_t^T C_{j_1}(t_2)\phi_{j_2}(t_2)dt_2=\int\limits_t^T\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2,$$

and the equality (2.122) is executed pointwise at any point of the interval (t, T); the right part of (2.122) converges when $t_2 = t, T$ (not necessarily to $C_{j_1}(t_2)$).

Let's substitute (2.122) into (2.121):

$$K^*(t_1,t_2)=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2);\quad (t_1,t_2)\in(t,T)^2, \tag{2.123}$$

moreover the series in the right part of (2.123) converges at the boundary of square $[t, T]^2$ (not necessarily to $K^*(t_1, t_2)$).

Hereafter, using the scheme of proving of theorem 1 and (2.120) w.p.1 we get:

$$J^*[\psi^{(2)}]_{T,t}=\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}+R_{T,t}^{p_1p_2}, \tag{2.124}$$

where

$$R_{T,t}^{p_1p_2}=\int\limits_t^T\int\limits_t^{t_2}G_{p_1p_2}(t_1,t_2)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}+\int\limits_t^T\int\limits_t^{t_1}G_{p_1p_2}(t_1,t_2)d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_1}^{(i_1)}+$$

$$+\mathbf{1}_{\{i_1=i_2\neq 0\}}\int\limits_t^T G_{p_1p_2}(t_1,t_1)dt_1;$$

$$G_{p_1p_2}(t_1,t_2)\stackrel{\rm def}{=}K^*(t_1,t_2)-\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}C_{j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2);$$

$p_1,\ p_2 < \infty$.

Using standard evaluations (7.3) and (7.4) for the moments of stochastic integrals, we obtain

$$\mathsf{M}\left\{\left(R_{T,t}^{p_1p_2}\right)^{2n}\right\} \le C_n\Biggl(\int\limits_t^T\int\limits_t^{t_2}\left(G_{p_1p_2}(t_1,t_2)\right)^{2n}dt_1dt_2+$$

$$+\int\limits_t^T\int\limits_t^{t_1}\left(G_{p_1p_2}(t_1,t_2)\right)^{2n}dt_2dt_1+\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\left(G_{p_1p_2}(t_1,t_1)\right)^{2n}dt_1\Biggr), \qquad (2.125)$$

where $C_n<\infty$ — is a constant which depends on n and $T-t$; $n=1,\ 2,\ldots$

Note, that due to assumptions proposed earlier, the function $G_{p_1p_2}(t_1,t_2)$ is continuous in the domains of integrating of integrals in the right part of (2.125) and it is bounded at the boundary of square $[t,T]^2$.

Let's estimate the integral in the right part of (2.125):

$$0\le\int\limits_t^T\int\limits_t^{t_2}\left(G_{p_1p_2}(t_1,t_2)\right)^{2n}dt_1dt_2=\Biggl(\int\limits_{D_\varepsilon}+\int\limits_{\Gamma_\varepsilon}\Biggr)\left(G_{p_1p_2}(t_1,t_2)\right)^{2n}dt_1dt_2\le$$

$$\le\sum_{i=0}^{N-1}\sum_{j=0}^{i}\max_{(t_1,t_2)\in[\tau_i,\tau_{i+1}]\mathsf{x}[\tau_j,\tau_{j+1}]}\left(G_{p_1p_2}(t_1,t_2)\right)^{2n}\Delta\tau_i\Delta\tau_j+MS_{\Gamma_\varepsilon}\le$$

$$\le\sum_{i=0}^{N-1}\sum_{j=0}^{i}\left(G_{p_1p_2}(\tau_i,\tau_j)\right)^{2n}\Delta\tau_i\Delta\tau_j+$$

$$+\sum_{i=0}^{N-1}\sum_{j=0}^{i}\left|\left(G_{p_1p_2}(t_i^{(p_1p_2)},t_j^{(p_1p_2)})\right)^{2n}-\left(G_{p_1p_2}(\tau_i,\tau_j)\right)^{2n}\right|\Delta\tau_i\Delta\tau_j+MS_{\Gamma_\varepsilon}\le$$

$$\le\sum_{i=0}^{N-1}\sum_{j=0}^{i}\left(G_{p_1p_2}(\tau_i,\tau_j)\right)^{2n}\Delta\tau_i\Delta\tau_j+\varepsilon_1\frac{1}{2}(T-t-3\varepsilon)^2\left(1+\frac{1}{N}\right)+MS_{\Gamma_\varepsilon}, \qquad (2.126)$$

where $D_\varepsilon=\{(t_1,t_2):\ t_2\in[t+2\varepsilon,T-\varepsilon],t_1\in[t+\varepsilon,t_2-\varepsilon]\}$; $\Gamma_\varepsilon=D\backslash D_\varepsilon$; $D=\{(t_1,t_2):t_2\in[t,T],t_1\in[t,t_2]\}$; ε — is any sufficiently small positive number; S_{Γ_ε} is area of Γ_ε; $M>0$ — is a positive constant limiting $\left(G_{p_1p_2}(t_1,t_2)\right)^{2n}$; $(t_i^{(p_1p_2)},t_j^{(p_1p_2)})$ is a point of maximum of this function, when $(t_1,t_2)\in[\tau_i,\tau_{i+1}]\mathsf{x}[\tau_j,\tau_{j+1}]$; $\tau_i=t+2\varepsilon+i\Delta$ $(i=0,1,\ldots,N)$; $\tau_N=T-\varepsilon$; $\Delta=(T-t-3\varepsilon)/N$; $\Delta<\varepsilon$; $\varepsilon_1>0$ — is any sufficiently small positive number.

Getting (2.126), we used well-known properties of integrals, the first and the second Weierstrass theorems for the function of two variables, as well as the continuity and as a result the uniform continuity of function

$\left(G_{p_1p_2}(t_1,t_2)\right)^{2n}$ in the domain D_ε ($\forall \varepsilon_1 > 0\ \exists \delta(\varepsilon_1) > 0$, which doesn't depend on t_1, t_2, p_1, p_2 and if $\sqrt{2}\Delta < \delta$, then the following inequality takes place:

$$\left|\left(G_{p_1p_2}(t_i^{(p_1p_2)}, t_j^{(p_1p_2)})\right)^{2n} - \left(G_{p_1p_2}(\tau_i,\tau_j)\right)^{2n}\right| < \varepsilon_1).$$

Considering (2.123) let's write down:

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\left(G_{p_1p_2}(t_1,t_2)\right)^{2n} = 0 \ \text{when}\ (t_1,t_2)\in D_\varepsilon$$

and execute the repeated passage to the limit $\lim\limits_{\varepsilon\to+0}\lim\limits_{p_1\to\infty}\lim\limits_{p_2\to\infty}$ in inequality (2.126). Then according to arbitrariness of ε_1 we have

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\int\limits_t^T\int\limits_t^{t_2}\left(G_{p_1p_2}(t_1,t_2)\right)^{2n}dt_1dt_2 = 0. \qquad (2.127)$$

Similarly to arguments given above we have:

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\int\limits_t^T\int\limits_t^{t_1}\left(G_{p_1p_2}(t_1,t_2)\right)^{2n}dt_2dt_1 = 0, \qquad (2.128)$$

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\int\limits_t^T\left(G_{p_1p_2}(t_1,t_1)\right)^{2n}dt_1 = 0. \qquad (2.129)$$

From (2.125), (2.127) – (2.129) we get

$$\lim_{p_1\to\infty}\lim_{p_2\to\infty}\mathsf{M}\left\{\left(R_{T,t}^{p_1p_2}\right)^{2n}\right\} = 0;\ n\in N.$$

The last equality and (2.124) provide a possibility to write down:

$$J^*[\psi^{(2)}]_{T,t} = \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)},$$

where convergence of the repeated series is regarded in the mean of degree $2n$; $n\in N$.

It is easy to note, that if we expand the function $K^*(t_1,t_2)$ into the Fourier series at the interval (t,T) at first according to the variable t_2 (t_1 is fixed), and then expand the Fourier coefficient of the obtained series

$$\psi_1(t_1)\int\limits_{t_1}^T\psi_2(t_2)\phi_{j_2}(t_2)dt_2$$

into the Fourier series at the interval (t,T) according to the variable t_1, then taking into account, that

$$\int\limits_t^T \psi_1(t_1)\phi_{j_1}(t_1)\int\limits_{t_1}^T \psi_2(t_2)\phi_{j_2}(t_2)dt_2dt_1 = C_{j_2j_1},$$

we will come to the following formula of expansion of multiple stochastic Stratononovich integral of second multiplicity:

$$J^*[\psi^{(2)}]_{T,t} = \sum_{j_2=0}^{\infty}\sum_{j_1=0}^{\infty} C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}.$$

2.5.2 The case of integrals of 3rd and 4th multiplicity

In the previous section we examined the following equality:

$$\psi_1(t_1)\left(\mathbf{1}_{\{t_1<t_2\}} + \frac{1}{2}\mathbf{1}_{\{t_1=t_2\}}\right) = \sum_{j_1=0}^{\infty}\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\phi_{j_1}(t_1), \qquad (2.130)$$

which is executed pointwise at the interval (t,T), besides the series in the right part of (2.130) converges when $t_1 = t,\ T$.

Using (2.130) we get:

$$\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2\phi_{j_2}(t_2)\phi_{j_1}(t_1) =$$

$$= \sum_{j_1=0}^{\infty}\psi_3(t_3)\psi_2(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\left(\mathbf{1}_{\{t_2<t_3\}} + \frac{1}{2}\mathbf{1}_{\{t_2=t_3\}}\right)\phi_{j_1}(t_1) =$$

$$= \psi_1(t_1)\psi_2(t_2)\psi_3(t_3)\left(\mathbf{1}_{\{t_1<t_2\}} + \frac{1}{2}\mathbf{1}_{\{t_1=t_2\}}\right)\left(\mathbf{1}_{\{t_2<t_3\}} + \frac{1}{2}\mathbf{1}_{\{t_2=t_3\}}\right). \quad (2.131)$$

On the other side, the left part (2.131) may be represented by expanding the function

$$\psi_3(t_3)\int\limits_t^{t_3}\psi_2(t_2)\phi_{j_2}(t_2)\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2$$

into the Fourier series at the interval (t,T) in the following form:

$$\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty} C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3), \qquad (2.132)$$

where

$$C_{j_3 j_2 j_1} = \int\limits_t^T \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2 dt_3.$$

So, we get the following equality:

$$\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty} C_{j_3 j_2 j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3) =$$

$$= \psi_1(t_1)\psi_2(t_2)\psi_3(t_3)\left(\mathbf{1}_{\{t_1<t_2\}} + \frac{1}{2}\mathbf{1}_{\{t_1=t_2\}}\right)\left(\mathbf{1}_{\{t_2<t_3\}} + \frac{1}{2}\mathbf{1}_{\{t_2=t_3\}}\right) =$$

$$= \prod_{l=1}^{3}\psi_l(t_l)\left(\mathbf{1}_{\{t_1<t_2\}}\mathbf{1}_{\{t_2<t_3\}} + \frac{1}{2}\mathbf{1}_{\{t_1=t_2\}}\mathbf{1}_{\{t_2<t_3\}} + \right.$$

$$\left. + \frac{1}{2}\mathbf{1}_{\{t_1<t_2\}}\mathbf{1}_{\{t_2=t_3\}} + \frac{1}{4}\mathbf{1}_{\{t_1=t_2\}}\mathbf{1}_{\{t_2=t_3\}}\right) \stackrel{\mathsf{def}}{=} K^*(t_1,t_2,t_3), \qquad (2.133)$$

which is executed pointwise in the open cube $(t,T)^3$, moreover series (2.132) converges at the boundary of the cube $[t,T]^3$.

Using (2.130) and (2.133) we get:

$$\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty} \psi_4(t_4)\int\limits_t^{t_4} \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2 dt_3 \times$$

$$\times \prod_{l=1}^{3}\phi_{j_l}(t_l) =$$

$$= \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty} \psi_4(t_4)\left(\mathbf{1}_{\{t_3<t_4\}} + \frac{1}{2}\mathbf{1}_{\{t_3=t_4\}}\right) \times$$

$$\times \psi_3(t_3)\int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2 \phi_{j_2}(t_2)\phi_{j_1}(t_1) =$$

$$= \psi_4(t_4)\left(\mathbf{1}_{\{t_3<t_4\}} + \frac{1}{2}\mathbf{1}_{\{t_3=t_4\}}\right) \times$$

$$\times \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty} \psi_3(t_3)\int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1) dt_1 dt_2 \phi_{j_2}(t_2)\phi_{j_1}(t_1) =$$

$$= \psi_4(t_4)\left(\mathbf{1}_{\{t_3<t_4\}} + \frac{1}{2}\mathbf{1}_{\{t_3=t_4\}}\right)\prod_{l=1}^{3}\psi_l(t_l)\prod_{l=1}^{2}\left(\mathbf{1}_{\{t_l<t_{l+1}\}} + \frac{1}{2}\mathbf{1}_{\{t_l=t_{l+1}\}}\right) =$$

$$= \prod_{l=1}^{4} \psi_l(t_l) \prod_{l=1}^{3} \left(\mathbf{1}_{\{t_l<t_{l+1}\}} + \frac{1}{2}\mathbf{1}_{\{t_l=t_{l+1}\}} \right). \qquad (2.134)$$

The left part of (2.134) may be lead by expanding the function

$$\psi_4(t_4) \int\limits_t^{t_4} \psi_3(t_3)\phi_{j_3}(t_3) \int\limits_t^{t_3} \psi_2(t_2)\phi_{j_2}(t_2) \int\limits_t^{t_2} \psi_1(t_1)\phi_{j_1}(t_1)dt_1dt_2dt_3$$

into the Fourier series at the interval (t, T) to the following form:

$$\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty}\sum_{j_4=0}^{\infty} C_{j_4j_3j_2j_1} \prod_{l=1}^{4} \phi_{j_l}(t_l), \qquad (2.135)$$

where $C_{j_4j_3j_2j_1}$ is defined using the formula (1.6).

As a result we get the following equality:

$$\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty}\sum_{j_4=0}^{\infty} C_{j_4j_3j_2j_1} \prod_{l=1}^{4} \phi_{j_l}(t_l) = \prod_{l=1}^{4} \psi_l(t_l) \prod_{l=1}^{3} \left(\mathbf{1}_{\{t_l<t_{l+1}\}} + \frac{1}{2}\mathbf{1}_{\{t_l=t_{l+1}\}} \right) \stackrel{\mathrm{def}}{=}$$

$$\stackrel{\mathrm{def}}{=} K^*(t_1,t_2,t_3,t_4), \qquad (2.136)$$

which is executed pointwise in the open hypercube $(t, T)^4$, moreover the series in the left part of (2.136) converges at the boundary of hypercube $[t, T]^4$.

Due to lemma 1, remark 2 and formula of connection between multiple stochastic Stratonovich and Ito integrals:

$$J^*[\psi^{(3)}] = \int\limits_t^{T} \psi_3(t_3) \int\limits_t^{t_3} \psi_2(t_2) \int\limits_t^{t_2} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}+$$

$$+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\neq 0\}} \int\limits_t^{T} \psi_3(t_3) \int\limits_t^{t_3} \psi_2(t_2)\psi_1(t_2)dt_2d\mathbf{w}_{t_3}^{(i_3)}+$$

$$+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\neq 0\}} \int\limits_t^{T} \psi_3(t_3)\psi_2(t_3) \int\limits_t^{t_3} \psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}dt_3$$

with probulity 1, we get:

$$J^*[\psi^{(3)}]_{T,t} = \underset{N\to\infty}{\text{l.i.m.}} \sum_{l_3=0}^{N-1}\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{N-1} K^*(\tau_{l_1},\tau_{l_2},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)} \qquad (2.137)$$

where the equality is fulfilled with probability 1.

Hereafter, using the scheme of theorem 1 proving and (2.137) w.p.1 we get:

$$J^*[\psi^{(3)}]_{T,t} = \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} + R_{T,t}^{p_1p_2p_3}, \qquad (2.138)$$

where

$$R_{T,t}^{p_1p_2p_3} = \underset{N\to\infty}{\text{l.i.m.}} \sum_{l_3=0}^{N-1}\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{N-1} G_{p_1p_2p_3}(\tau_{l_1},\tau_{l_2},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)},$$

$$G_{p_1p_2p_3}(t_1,t_2,t_3) \stackrel{\text{def}}{=} K^*(t_1,t_2,t_3) - \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\phi_{j_1}(t_1)\phi_{j_2}(t_2)\phi_{j_3}(t_3).$$

Using formula (2.163) for muliple sum we get:

$$R_{T,t}^{p_1p_2p_3} = \underset{N\to\infty}{\text{l.i.m.}} \sum_{l_3=0}^{N-1}\sum_{l_2=0}^{N-1}\sum_{l_1=0}^{N-1} G_{p_1p_2p_3}(\tau_{l_1},\tau_{l_2},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)} =$$

$$= \underset{N\to\infty}{\text{l.i.m.}} \sum_{l_3=0}^{N-1}\sum_{l_2=0}^{l_3-1}\sum_{l_1=0}^{l_2-1} \Biggl(G_{p_1p_2p_3}(\tau_{l_1},\tau_{l_2},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}+$$

$$+G_{p_1p_2p_3}(\tau_{l_1},\tau_{l_3},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_3)}+$$

$$+G_{p_1p_2p_3}(\tau_{l_2},\tau_{l_1},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}+$$

$$+G_{p_1p_2p_3}(\tau_{l_2},\tau_{l_3},\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_3)}+$$

$$+G_{p_1p_2p_3}(\tau_{l_3},\tau_{l_2},\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_3)}+$$

$$+G_{p_1p_2p_3}(\tau_{l_3},\tau_{l_1},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_3)}\Biggr)+$$

$$+\underset{N\to\infty}{\text{l.i.m.}} \sum_{l_3=0}^{N-1}\sum_{l_2=0}^{l_3-1} \Biggl(G_{p_1p_2p_3}(\tau_{l_2},\tau_{l_2},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}+$$

$$+G_{p_1p_2p_3}(\tau_{l_2},\tau_{l_3},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_2}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_3)}+$$

$$+G_{p_1p_2p_3}(\tau_{l_3},\tau_{l_2},\tau_{l_2})\Delta\mathbf{w}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_2}}^{(i_3)}\Biggr)+$$

$$+\underset{N\to\infty}{\text{l.i.m.}} \sum_{l_3=0}^{N-1}\sum_{l_1=0}^{l_3-1} \Biggl(G_{p_1p_2p_3}(\tau_{l_1},\tau_{l_3},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_1}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}+$$

$$+G_{p_1p_2p_3}(\tau_{l_3},\tau_{l_1},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}+$$

$$+G_{p_1p_2p_3}(\tau_{l_3},\tau_{l_3},\tau_{l_1})\Delta\mathbf{w}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_1}}^{(i_3)}\Bigr)+$$

$$+\underset{N\to\infty}{\text{l.i.m.}}\sum_{l_3=0}^{N-1}G_{p_1p_2p_3}(\tau_{l_3},\tau_{l_3},\tau_{l_3})\Delta\mathbf{w}_{\tau_{l_3}}^{(i_1)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_2)}\Delta\mathbf{w}_{\tau_{l_3}}^{(i_3)}=$$

$$=\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}G_{p_1p_2p_3}(t_1,t_2,t_3)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_3)}+$$

$$+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}G_{p_1p_2p_3}(t_1,t_3,t_2)d\mathbf{w}_{t_1}^{(i_1)}d\mathbf{w}_{t_2}^{(i_3)}d\mathbf{w}_{t_3}^{(i_2)}+$$

$$+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}G_{p_1p_2p_3}(t_2,t_1,t_3)d\mathbf{w}_{t_1}^{(i_2)}d\mathbf{w}_{t_2}^{(i_1)}d\mathbf{w}_{t_3}^{(i_3)}+$$

$$+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}G_{p_1p_2p_3}(t_2,t_3,t_1)d\mathbf{w}_{t_1}^{(i_3)}d\mathbf{w}_{t_2}^{(i_1)}d\mathbf{w}_{t_3}^{(i_2)}+$$

$$+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}G_{p_1p_2p_3}(t_3,t_2,t_1)d\mathbf{w}_{t_1}^{(i_3)}d\mathbf{w}_{t_2}^{(i_2)}d\mathbf{w}_{t_3}^{(i_1)}+$$

$$+\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}G_{p_1p_2p_3}(t_3,t_1,t_2)d\mathbf{w}_{t_1}^{(i_2)}d\mathbf{w}_{t_2}^{(i_3)}d\mathbf{w}_{t_3}^{(i_1)}+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\int\limits_t^{t_3}G_{p_1p_2p_3}(t_2,t_2,t_3)dt_2d\mathbf{w}_{t_3}^{(i_3)}+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_3}G_{p_1p_2p_3}(t_2,t_3,t_2)dt_2d\mathbf{w}_{t_3}^{(i_2)}+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_3}G_{p_1p_2p_3}(t_3,t_2,t_2)dt_2d\mathbf{w}_{t_3}^{(i_1)}+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_3}G_{p_1p_2p_3}(t_1,t_3,t_3)d\mathbf{w}_{t_1}^{(i_1)}dt_3+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\int\limits_t^T\int\limits_t^{t_3}G_{p_1p_2p_3}(t_3,t_1,t_3)d\mathbf{w}_{t_1}^{(i_2)}dt_3+$$

$$+\mathbf{1}_{\{i_1=i_2\ne 0\}}\int\limits_t^T\int\limits_t^{t_3}G_{p_1p_2p_3}(t_3,t_3,t_1)d\mathbf{w}_{t_1}^{(i_3)}dt_3.$$

Now, using standard estimations for moments of stochastic integrals we will come to the following inequality:

$$\mathsf{M}\left\{\left(R_{T,t}^{p_1p_2p_3}\right)^{2n}\right\} \le$$

$$\le C_n\Biggl(\int\limits_t^T\int\limits_t^{t_3}\int\limits_t^{t_2}\Bigl(\left(G_{p_1p_2p_3}(t_1,t_2,t_3)\right)^{2n} + \left(G_{p_1p_2p_3}(t_1,t_3,t_2)\right)^{2n} +$$

$$+\left(G_{p_1p_2p_3}(t_2,t_1,t_3)\right)^{2n} + \left(G_{p_1p_2p_3}(t_2,t_3,t_1)\right)^{2n} + \left(G_{p_1p_2p_3}(t_3,t_2,t_1)\right)^{2n} +$$

$$+\left(G_{p_1p_2p_3}(t_3,t_1,t_2)\right)^{2n}\Bigr)dt_1dt_2dt_3+$$

$$+\int\limits_t^T\int\limits_t^{t_3}\Bigl(\mathbf{1}_{\{i_1=i_2\ne 0\}}\Bigl(\left(G_{p_1p_2p_3}(t_2,t_2,t_3)\right)^{2n} + \left(G_{p_1p_2p_3}(t_3,t_3,t_2)\right)^{2n}\Bigr)+$$

$$+\mathbf{1}_{\{i_1=i_3\ne 0\}}\Bigl(\left(G_{p_1p_2p_3}(t_2,t_3,t_2)\right)^{2n} + \left(G_{p_1p_2p_3}(t_3,t_2,t_3)\right)^{2n}\Bigr)+$$

$$+\mathbf{1}_{\{i_2=i_3\ne 0\}}\Bigl(\left(G_{p_1p_2p_3}(t_3,t_2,t_2)\right)^{2n} + \left(G_{p_1p_2p_3}(t_2,t_3,t_3)\right)^{2n}\Bigr)dt_2dt_3\Bigr)\Biggr). \qquad (2.139)$$

It is important, that integands functions in the right part of (2.139) are continuous in the domains of integration of multiple integrals and in accordance with comment to the formula (2.133), are bounded at the boundaries of these domains, moreover, everywhere in $(t,T)^3$ the following formula takes place:

$$\lim_{p_1\to\infty}\ \lim_{p_2\to\infty}\ \lim_{p_3\to\infty} G_{p_1p_2p_3}(t_1,t_2,t_3) = 0. \qquad (2.140)$$

Further, similarly to (2.126) (two dimensional case) we realize the repeated passage to the limit $\lim\limits_{p_1\to\infty}\ \lim\limits_{p_2\to\infty}\ \lim\limits_{p_3\to\infty}$ under the integral signs in the right part and we get:

$$\lim_{p_1\to\infty}\ \lim_{p_2\to\infty}\ \lim_{p_3\to\infty} \mathsf{M}\left\{\left(R_{T,t}^{p_1p_2p_3}\right)^{2n}\right\} = 0.$$

The last relation in it's turn means, that

$$J^*[\psi^{(3)}]_{T,t} = \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}, \qquad (2.141)$$

where the repeated series converges in the mean of degree $2n$ (n — natural), that is

$$\lim_{p_1\to\infty}\ \lim_{p_2\to\infty}\ \lim_{p_3\to\infty} \mathsf{M}\left\{\left(J^*[\psi^{(3)}]_{T,t} - \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\sum_{j_3=0}^{p_3} C_{j_3j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)}\right)^{2n}\right\} = 0.$$

2.5.3 The case of integrals of multiplicity k

In this section we will formulate and prove the theorem about expansion of multiple stochastic Stratonovich integrals of any fixed multiplicity k of the form (1.2), based on the repeated Fourier series according to the Legendre polynomials or the system of trigonometric functions. This theorem provides a possibility to represent the multiple stochastic Stratonovich integral in the form of repeated series of products of standard Gaussian random variables.

Let's define the following function on the hypercube $[t,T]^k$

$$K(t_1,\ldots,t_k)=\prod_{l=1}^{k}\psi_l(t_l)\prod_{l=1}^{k-1}\mathbf{1}_{\{t_l<t_{l+1}\}},\ k\ge 2. \tag{2.142}$$

Let's formulate the following statement.

Theorem 5. *Assume, that the following conditions are met:*

1. $\psi_i(\tau);\ i=1,\ldots,k$ *— are continuously differentiated functions at the interval* $[t,T]$.

2. $\{\phi_j(x)\}_{j=0}^{\infty}$ *— is full orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$.

Then, the multiple stochastic Stratonovich integral $J^*[\psi^{(k)}]_{T,t}$ *of type* (1.2) *is expanded in the converging in the mean of degree* $2n;\ n\in N$, *repeated series*

$$J^*[\psi^{(k)}]_{T,t}=\sum_{j_1=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}, \tag{2.143}$$

where $\zeta_{j_l}^{(i_l)}=\int\limits_t^T\phi_{j_l}(s)d\mathbf{w}_s^{(i_l)}$ *— are independent standard Gaussian random variables for different* i_l *or* j_l *(if* $i_l\ne 0$*);*

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k. \tag{2.144}$$

Proving of the theorem will consist of several parts.

Let's define the function $K^*(t_1,\ldots,t_k)$ at the hypercube $[t,T]^k$ as follows:

$$K^*(t_1,\ldots,t_k)=\prod_{l=1}^{k}\psi_l(t_l)\prod_{l=1}^{k-1}\left(\mathbf{1}_{\{t_l<t_{l+1}\}}+\frac{1}{2}\mathbf{1}_{\{t_l=t_{l+1}\}}\right)=$$

$$=\prod_{l=1}^{k}\psi_l(t_l)\Biggl(\prod_{l=1}^{k-1}\mathbf{1}_{\{t_l<t_{l+1}\}}+\sum_{r=1}^{k-1}\frac{1}{2^r}\sum_{\substack{s_r,\ldots,s_1=1\\ s_r>\ldots>s_1}}^{k-1}\prod_{l=1}^{r}\mathbf{1}_{\{t_{s_l}=t_{s_l+1}\}}\prod_{\substack{l=1\\ l\ne s_1,\ldots,s_r}}^{k-1}\mathbf{1}_{\{t_l<t_{l+1}\}}\Biggr). \tag{2.145}$$

Particular cases of (2.145) for $k = 2,\ 3,\ 4$ were examined in detail earlier.

Theorem 6. *In conditions of theorem* 5 *the function* $K^*(t_1,\ldots,t_k)$ *is represented in any internal point of the hypercube* $[t,T]^k$ *by the repeated Fourier series*

$$K^*(t_1,\ldots,t_k)=\sum_{j_1=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l), \tag{2.146}$$

where $C_{j_k\ldots j_1}$ *has the form* (2.144)*. At that, the repeated series* (2.146) *converges at the boundary of hypercube* $[t,T]^k$*.*

We will perform proving using induction. This theorem is already proved for the cases $k = 2,\ 3$ and 4.

Let's introduce assumption of induction:

$$\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_{k-2}=0}^{\infty}\psi_{k-1}(t_{k-1})\int\limits_t^{t_{k-1}}\psi_{k-2}(t_{k-2})\phi_{j_{k-2}}(t_{k-2})\ldots$$

$$\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{k-2}\prod_{l=1}^{k-2}\phi_{j_l}(t_l)=$$

$$=\prod_{l=1}^{k-1}\psi_l(t_l)\prod_{l=1}^{k-2}\left(\mathbf{1}_{\{t_l<t_{l+1}\}}+\frac{1}{2}\mathbf{1}_{\{t_l=t_{l+1}\}}\right). \tag{2.147}$$

Then

$$\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_{k-1}=0}^{\infty}\psi_k(t_k)\int\limits_t^{t_k}\psi_{k-1}(t_{k-1})\phi_{j_{k-1}}(t_{k-1})\ldots$$

$$\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{k-1}\prod_{l=1}^{k-1}\phi_{j_l}(t_l)=$$

$$=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_{k-2}=0}^{\infty}\psi_k(t_k)\left(\mathbf{1}_{\{t_{k-1}<t_k\}}+\frac{1}{2}\mathbf{1}_{\{t_{k-1}=t_k\}}\right)\psi_{k-1}(t_{k-1})\times$$

$$\times\int\limits_t^{t_{k-1}}\psi_{k-2}(t_{k-2})\phi_{j_{k-2}}(t_{k-2})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{k-2}\prod_{l=1}^{k-2}\phi_{j_l}(t_l)=$$

$$=\psi_k(t_k)\left(\mathbf{1}_{\{t_{k-1}<t_k\}}+\frac{1}{2}\mathbf{1}_{\{t_{k-1}=t_k\}}\right)\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\ldots\sum_{j_{k-2}=0}^{\infty}\psi_{k-1}(t_{k-1})\times$$

$$\times\int\limits_t^{t_{k-1}}\psi_{k-2}(t_{k-2})\phi_{j_{k-2}}(t_{k-2})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{k-2}\prod_{l=1}^{k-2}\phi_{j_l}(t_l)=$$

$$= \psi_k(t_k)\left(\mathbf{1}_{\{t_{k-1}<t_k\}}+\frac{1}{2}\mathbf{1}_{\{t_{k-1}=t_k\}}\right)\prod_{l=1}^{k-1}\psi_l(t_l)\prod_{l=1}^{k-2}\left(\mathbf{1}_{\{t_l<t_{l+1}\}}+\frac{1}{2}\mathbf{1}_{\{t_l=t_{l+1}\}}\right)=$$

$$=\prod_{l=1}^{k}\psi_l(t_l)\prod_{l=1}^{k-1}\left(\mathbf{1}_{\{t_l<t_{l+1}\}}+\frac{1}{2}\mathbf{1}_{\{t_l=t_{l+1}\}}\right). \qquad (2.148)$$

On the other side, the left part of (2.148) may be represented by expanding the function

$$\psi_k(t_k)\int\limits_t^{t_k}\psi_{k-1}(t_{k-1})\phi_{j_{k-1}}(t_{k-1})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{k-1}$$

into the Fourier series at the interval (t,T) using the variable t_k to the following form:

$$\sum_{j_1=0}^{\infty}\ldots\sum_{j_k=0}^{\infty}C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l).$$

The theorem 6 is proven. □

Let's introduce the following notations:

$$J[\psi^{(k)}]_{T,t}^{s_l,\ldots,s_1}\stackrel{\rm def}{=}\prod_{p=1}^{l}\mathbf{1}_{\{i_{s_p}=i_{s_p+1}\ne 0\}}\times$$

$$\times\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_{s_l+3}}\psi_{s_l+2}(t_{s_l+2})\int\limits_t^{t_{s_l+2}}\psi_{s_l}(t_{s_l+1})\psi_{s_l+1}(t_{s_l+1})\times$$

$$\times\int\limits_t^{t_{s_l+1}}\psi_{s_l-1}(t_{s_l-1})\ldots\int\limits_t^{t_{s_1+3}}\psi_{s_1+2}(t_{s_1+2})\int\limits_t^{t_{s_1+2}}\psi_{s_1}(t_{s_1+1})\psi_{s_1+1}(t_{s_1+1})\times$$

$$\times\int\limits_t^{t_{s_1+1}}\psi_{s_1-1}(t_{s_1-1})\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{s_1-1}}^{(i_{s_1-1})}dt_{s_1+1}d\mathbf{w}_{t_{s_1+2}}^{(i_{s_1+2})}\ldots$$

$$\ldots d\mathbf{w}_{t_{s_l-1}}^{(i_{s_l-1})}dt_{s_l+1}d\mathbf{w}_{t_{s_l+2}}^{(i_{s_l+2})}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \qquad (2.149)$$

$$J^*[\psi^{(k)}]_{T,t}=\int\limits_t^{*T}\psi_k(t_k)\ldots\int\limits_t^{*t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \qquad (2.150)$$

$$J[\psi^{(k)}]_{T,t}=\int\limits_t^{T}\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_k}^{(i_k)}, \qquad (2.151)$$

where in (2.149)–(2.151): $\psi^{(k)}\stackrel{\rm def}{=}(\psi_k,\ldots,\psi_1)$, $\psi^{(1)}\stackrel{\rm def}{=}\psi_1$,

$$\mathcal{A}_{k,l}=\{(s_l,\ldots,s_1):\ \ s_l>s_{l-1}+1,\ldots,s_2>s_1+1;$$

$$s_l, \ldots, s_1 = 1, \ldots, k-1\}, \tag{2.152}$$

$(s_l, \ldots, s_1) \in \mathcal{A}_{k,l}$; $l = 1, \ldots, \left[\frac{k}{2}\right]$; $i_s = 0,\ 1, \ldots, m$; $s = 1, \ldots, k$; $[x]$ — is an integer part of number x; $\mathbf{1}_A$ — is an indicator of set A ($\mathbf{1}_A = 1$ if the condition A executed and $\mathbf{1}_A = 0$ otherwise).

Let's formulate the theorem about connection between multiple stochastic Ito and Stratonovich integrals $J[\psi^{(k)}]_{T,t},\ J^*[\psi^{(k)}]_{T,t}$ of fixed multiplicity k.

Theorem 7. *Assume, that $\psi_i(\tau)$; $i = 1, \ldots, k$ — are continuously differentiated functions at the interval $[t, T]$.*

Then, the following relation between multiple stochastic Ito and Stratonovich integrals is correct:

$$J^*[\psi^{(k)}]_{T,t} = J[\psi^{(k)}]_{T,t} + \sum_{r=1}^{\left[\frac{k}{2}\right]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in\mathcal{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1} \ \ w.p.1, \tag{2.153}$$

where $\sum\limits_{\emptyset}$ is supposed to be equal to zero.

Proof. Let's prove the equality (2.153) using induction. The case $k = 1$ is obvious.

If $k = 2$ from (2.153) we get

$$J^*[\psi^{(2)}]_{T,t} = J[\psi^{(2)}]_{T,t} + \frac{1}{2} J[\psi^{(2)}]_{T,t}^{1} \ \text{w.p.1.} \tag{2.154}$$

Let's demonstrate, that equality (2.154) is correct with probability 1. In order to do it let's examine the process $\eta_{t_2,t} = \psi_2(t_2) J[\psi_1]_{t_2,t}$; $t_2 \in [t, T]$ and find its stochastic differential using the Ito formula:

$$d\eta_{t_2,t} = J[\psi_1]_{t_2,t} d\psi_2(t_2) + \psi_1(t_2)\psi_2(t_2) d\mathbf{w}_{t_2}^{(i_1)}. \tag{2.155}$$

From the equality (2.155) it follows, that the diffusion coefficient of process $\eta_{t_2,t}$; $t_2 \in [t, T]$ equals to $\mathbf{1}_{\{i_1 \neq 0\}} \psi_1(t_2)\psi_2(t_2)$.

Further, using the standard relation between stochastic Stratonovich and Ito integrals (see sect. 7.2) with probubility 1 we will obtain the relation (2.154). Thus, predicating of this theorem is proven for $k = 1,\ 2$.

Assume, that predicating of this theorem is reasonable for certain $k > 2$,, and let's prove its rightness when the value k is greater by unity. In the assumption of induction with probability 1 we have

$$J^*[\psi^{(k+1)}]_{T,t} =$$

$$= \int\limits_t^{*T} \psi_{k+1}(\tau) \Biggl\{ J[\psi^k]_{\tau,t} + \sum_{r=1}^{\left[\frac{k}{2}\right]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in\mathcal{A}_{k,r}} J[\psi^{(k)}]_{\tau,t}^{s_r,\ldots,s_1} \Biggr\} d\mathbf{w}_{\tau}^{(i_{k+1})} =$$

$$= \int\limits_t^{*T} \psi_{k+1}(\tau) J[\psi^{(k)}]_{\tau,t} d\mathbf{w}_\tau^{(i_{k+1})} +$$

$$+ \sum_{r=1}^{\left[\frac{k}{2}\right]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in\mathcal{A}_{k,r}} \int\limits_t^{*T} \psi_{k+1}(\tau) J[\psi^{(k)}]_{\tau,t}^{s_r,\ldots,s_1} d\mathbf{w}_\tau^{(i_{k+1})}. \qquad (2.156)$$

Using the Ito formula and standard connection between stochastic Stratonovich and Ito integrals, we get with probability 1

$$\int\limits_t^{*T} \psi_{k+1}(\tau) J[\psi^{(k)}]_{\tau,t} d\mathbf{w}_\tau^{(i_{k+1})} = J[\psi^{(k+1)}]_{T,t} + \frac{1}{2} J[\psi^{(k+1)}]_{T,t}^{k}, \qquad (2.157)$$

$$\int\limits_t^{*T} \psi_{k+1}(\tau) J[\psi^{(k)}]_{\tau,t}^{s_r,\ldots,s_1} d\mathbf{w}_\tau^{(i_{k+1})} =$$

$$= \begin{cases} J[\psi^{(k+1)}]_{T,t}^{s_r,\ldots,s_1} & \text{if } s_r = k-1, \\ J[\psi^{(k+1)}]_{T,t}^{s_r,\ldots,s_1} + \frac{1}{2} J[\psi^{(k+1)}]_{T,t}^{k,s_r,\ldots,s_1} & \text{if } s_r < k-1. \end{cases} \qquad (2.158)$$

After insertion of (2.157) and (2.158) into (2.156) and regrouping of summands we pass to the relations which are reasonable with probability 1

$$J^*[\psi^{(k+1)}]_{T,t} = J[\psi^{(k+1)}]_{T,t} + \sum_{r=1}^{\left[\frac{k}{2}\right]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in\mathcal{A}_{k+1,r}} J[\psi^{(k+1)}]_{T,t}^{s_r,\ldots,s_1} \qquad (2.159)$$

when k is even and

$$J^*[\psi^{(k'+1)}]_{T,t} = J[\psi^{(k'+1)}]_{T,t} + \sum_{r=1}^{\left[\frac{k'}{2}\right]+1} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in\mathcal{A}_{k'+1,r}} J[\psi^{(k'+1)}]_{T,t}^{s_r,\ldots,s_1} \qquad (2.160)$$

when $k' = k+1$ is uneven.

From (2.159) and (2.160) with probubility 1 we have

$$J^*[\psi^{(k+1)}]_{T,t} = J[\psi^{(k+1)}]_{T,t} + \sum_{r=1}^{\left[\frac{k+1}{2}\right]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in\mathcal{A}_{k+1,r}} J[\psi^{(k+1)}]_{T,t}^{s_r,\ldots,s_1}. \qquad (2.161)$$

The relation (2.161) accomplishes proving of the theorem. □

Let's analyze the stochastic integral of type (1.12) and find its representation, convenient for following verbal proof. In order to do it we introduce several notations. Assume, that

$$S_N^{(k)}(a) = \sum_{j_k=0}^{N-1} \ldots \sum_{j_1=0}^{j_2-1} \sum_{(j_1,\ldots,j_k)} a_{(j_1,\ldots,j_k)},$$

$$\mathrm{C}_{s_r} \ldots \mathrm{C}_{s_1} S_N^{(k)}(a) = \sum_{j_k=0}^{N-1} \ldots \sum_{j_{s_r+1}=0}^{j_{s_r+2}-1} \sum_{j_{s_r-1}=0}^{j_{s_r+1}-1} \ldots \sum_{j_{s_1+1}=0}^{j_{s_1+2}-1} \sum_{j_{s_1-1}=0}^{j_{s_1+1}-1} \ldots \sum_{j_1=0}^{j_2-1} \times$$

$$\times \sum_{\prod\limits_{l=1}^{r} \mathbf{I}_{j_{s_l}, j_{s_l+1}}(j_1,\ldots,j_k)} a_{\prod\limits_{l=1}^{r} \mathbf{I}_{j_{s_l}, j_{s_l+1}}(j_1,\ldots,j_k)},$$

where

$$\prod_{l=1}^{r} \mathbf{I}_{j_{s_l}, j_{s_l+1}}(j_1,\ldots,j_k) \stackrel{\text{def}}{=} \mathbf{I}_{j_{s_r}, j_{s_r+1}} \ldots \mathbf{I}_{j_{s_1}, j_{s_1+1}}(j_1,\ldots,j_k),$$

$$\mathrm{C}_{s_0} \ldots \mathrm{C}_{s_1} S_N^{(k)}(a) = S_N^{(k)}(a); \ \prod_{l=1}^{0} \mathbf{I}_{j_{s_l}, j_{s_l+1}}(j_1,\ldots,j_k) = (j_1,\ldots,j_k);$$

$$\mathbf{I}_{j_l, j_{l+1}}(j_{q_1},\ldots,j_{q_2},j_l,j_{q_3},\ldots,j_{q_{k-2}},j_l,j_{q_{k-1}},\ldots,j_{q_k}) \stackrel{\text{def}}{=}$$

$$\stackrel{\text{def}}{=} (j_{q_1},\ldots,j_{q_2},j_{l+1},j_{q_3},\ldots,j_{q_{k-2}},j_{l+1},j_{q_{k-1}},\ldots,j_{g_k});$$

here $l = 1, 2, \ldots;$ $l \ne q_1,\ldots,q_2,q_3,\ldots,q_{k-2},q_{k-1},\ldots,q_k = 1, 2, \ldots;$ $s_1,\ldots,s_r = 1,\ldots,k-1;$ $s_r > \ldots > s_1;$ $a_{(j_{q_1},\ldots,j_{q_k})}$ — scalars; $q_1,\ldots,q_k = 1,\ldots,k;$ expression $\sum\limits_{(j_{q_1},\ldots,j_{q_k})}$ means the sum according to all possible derangements $(j_{q_1},\ldots,j_{q_k})$.

Using induction it is possible to prove the following equality:

$$\sum_{j_k=0}^{N-1} \ldots \sum_{j_1=0}^{N-1} a_{(j_1,\ldots,j_k)} = \sum_{r=0}^{k-1} \sum_{\substack{s_r,\ldots,s_1=1 \\ s_r>\ldots>s_1}}^{k-1} \mathrm{C}_{s_r} \ldots \mathrm{C}_{s_1} S_N^{(k)}(a), \tag{2.162}$$

where $k = 1, 2, \ldots;$ the sum according to empty set supposed as equal to 1.

Hereafter, we will identify the following records:

$$a_{(j_1,\ldots,j_k)} = a_{(j_1\ldots j_k)} = a_{j_1\ldots j_k}.$$

In particular, from (2.162) when $k = 2, 3, 4$ we get the following formulas

$$\sum_{j_2=0}^{N-1} \sum_{j_1=0}^{N-1} a_{(j_1,j_2)} = S_N^{(2)}(a) + \mathrm{C}_1 S_N^{(2)}(a) =$$

$$=\sum_{j_2=0}^{N-1}\sum_{j_1=0}^{j_2-1}\sum_{(j_1,j_2)}a_{(j_1j_2)}+\sum_{j_2=0}^{N-1}a_{(j_2j_2)}=$$

$$=\sum_{j_2=0}^{N-1}\sum_{j_1=0}^{j_2-1}(a_{j_1j_2}+a_{j_2j_1})+\sum_{j_2=0}^{N-1}a_{j_2j_2};$$

$$\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{N-1}\sum_{j_1=0}^{N-1}a_{(j_1,j_2,j_3)}=S_N^{(3)}(a)+\mathrm{C}_1S_N^{(3)}(a)+$$

$$+\mathrm{C}_2S_N^{(3)}(a)+\mathrm{C}_2\mathrm{C}_1S_N^{(3)}(a)=$$

$$=\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1}\sum_{(j_1,j_2,j_3)}a_{(j_1j_2j_3)}+\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{(j_2,j_2,j_3)}a_{(j_2j_2j_3)}+$$

$$+\sum_{j_3=0}^{N-1}\sum_{j_1=0}^{j_3-1}\sum_{(j_1,j_3,j_3)}a_{(j_1j_3j_3)}+\sum_{j_3=0}^{N-1}a_{(j_3j_3j_3)}=$$

$$=\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1}(a_{j_1j_2j_3}+a_{j_1j_3j_2}+a_{j_2j_1j_3}+a_{j_2j_3j_1}+a_{j_3j_2j_1}+a_{j_3j_1j_2})+$$

$$+\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{j_3-1}(a_{j_2j_2j_3}+a_{j_2j_3j_2}+a_{j_3j_2j_2})+$$

$$+\sum_{j_3=0}^{N-1}\sum_{j_1=0}^{j_3-1}(a_{j_1j_3j_3}+a_{j_3j_1j_3}+a_{j_3j_3j_1})+$$

$$+\sum_{j_3=0}^{N-1}a_{j_3j_3j_3}; \tag{2.163}$$

$$\sum_{j_4=0}^{N-1}\sum_{j_3=0}^{N-1}\sum_{j_2=0}^{N-1}\sum_{j_1=0}^{N-1}a_{(j_1,j_2,j_3,j_4)}=S_N^{(4)}(a)+\mathrm{C}_1S_N^{(4)}(a)+\mathrm{C}_2S_N^{(4)}(a)+$$

$$+\mathrm{C}_3S_N^{(4)}(a)+\mathrm{C}_2\mathrm{C}_1S_N^{(4)}(a)+\mathrm{C}_3\mathrm{C}_1S_N^{(4)}(a)+\mathrm{C}_3\mathrm{C}_2S_N^{(4)}(a)+\mathrm{C}_3\mathrm{C}_2\mathrm{C}_1S_N^{(4)}(a)=$$

$$=\sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1}\sum_{(j_1,j_2,j_3,j_4)}a_{(j_1j_2j_3j_4)}+\sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}\sum_{j_2=0}^{j_3-1}\sum_{(j_2,j_2,j_3,j_4)}a_{(j_2j_2j_3j_4)}+$$

$$+\sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}\sum_{j_1=0}^{j_3-1}\sum_{(j_1,j_3,j_3,j_4)}a_{(j_1j_3j_3j_4)}+\sum_{j_4=0}^{N-1}\sum_{j_2=0}^{j_4-1}\sum_{j_1=0}^{j_2-1}\sum_{(j_1,j_2,j_4,j_4)}a_{(j_1j_2j_4j_4)}+$$

$$+\sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}\sum_{(j_3,j_3,j_3,j_4)} a_{(j_3j_3j_3j_4)}+\sum_{j_4=0}^{N-1}\sum_{j_2=0}^{j_4-1}\sum_{(j_2,j_2,j_4,j_4)} a_{(j_2j_2j_4j_4)}+$$

$$+\sum_{j_4=0}^{N-1}\sum_{j_1=0}^{j_4-1}\sum_{(j_1,j_4,j_4,j_4)} a_{(j_1j_4j_4j_4)}+\sum_{j_4=0}^{N-1} a_{j_4j_4j_4j_4}=$$

$$=\sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1}\bigl(a_{j_1j_2j_3j_4}+a_{j_1j_2j_4j_3}+a_{j_1j_3j_2j_4}+a_{j_1j_3j_4j_2}+$$

$$+a_{j_1j_4j_3j_2}+a_{j_1j_4j_2j_3}+a_{j_2j_1j_3j_4}+a_{j_2j_1j_4j_3}+a_{j_2j_4j_1j_3}+a_{j_2j_4j_3j_1}+a_{j_2j_3j_1j_4}+$$

$$+a_{j_2j_3j_4j_1}+a_{j_3j_1j_2j_4}+a_{j_3j_1j_4j_2}+a_{j_3j_2j_1j_4}+a_{j_3j_2j_4j_1}+a_{j_3j_4j_1j_2}+a_{j_3j_4j_2j_1}+$$

$$+a_{j_4j_1j_2j_3}+a_{j_4j_1j_3j_2}+a_{j_4j_2j_1j_3}+a_{j_4j_2j_3j_1}+a_{j_4j_3j_1j_2}+a_{j_4j_3j_2j_1}\bigr)+$$

$$+\sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}\sum_{j_2=0}^{j_3-1}\bigl(a_{j_2j_2j_3j_4}+a_{j_2j_2j_4j_3}+a_{j_2j_3j_2j_4}+a_{j_2j_4j_2j_3}+a_{j_2j_3j_4j_2}+a_{j_2j_4j_3j_2}+$$

$$+a_{j_3j_2j_2j_4}+a_{j_4j_2j_2j_3}+a_{j_3j_2j_4j_2}+a_{j_4j_2j_3j_2}+a_{j_4j_3j_2j_2}+a_{j_3j_4j_2j_2}\bigr)+$$

$$+\sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}\sum_{j_1=0}^{j_3-1}\bigl(a_{j_3j_3j_1j_4}+a_{j_3j_3j_4j_1}+a_{j_3j_1j_3j_4}+a_{j_3j_4j_3j_1}+a_{j_3j_4j_1j_3}+a_{j_3j_1j_4j_3}+$$

$$+a_{j_1j_3j_3j_4}+a_{j_4j_3j_3j_1}+a_{j_4j_3j_1j_3}+a_{j_1j_3j_4j_3}+a_{j_1j_4j_3j_3}+a_{j_4j_1j_3j_3}\bigr)+$$

$$+\sum_{j_4=0}^{N-1}\sum_{j_2=0}^{j_4-1}\sum_{j_1=0}^{j_2-1}\bigl(a_{j_4j_4j_1j_2}+a_{j_4j_4j_2j_1}+a_{j_4j_1j_4j_2}+a_{j_4j_2j_4j_1}+a_{j_4j_2j_1j_4}+a_{j_4j_1j_2j_4}+$$

$$+a_{j_1j_4j_4j_2}+a_{j_2j_4j_4j_1}+a_{j_2j_4j_1j_4}+a_{j_1j_4j_2j_4}+a_{j_1j_2j_4j_4}+a_{j_2j_1j_4j_4}\bigr)+$$

$$+\sum_{j_4=0}^{N-1}\sum_{j_3=0}^{j_4-1}\bigl(a_{j_3j_3j_3j_4}+a_{j_3j_3j_4j_3}+a_{j_3j_4j_3j_3}+a_{j_4j_3j_3j_3}\bigr)+$$

$$+\sum_{j_4=0}^{N-1}\sum_{j_2=0}^{j_4-1}\bigl(a_{j_2j_2j_4j_4}+a_{j_2j_4j_2j_4}+a_{j_2j_4j_4j_2}+a_{j_4j_2j_2j_4}+a_{j_4j_2j_4j_2}+a_{j_4j_4j_2j_2}\bigr)+$$

$$+\sum_{j_4=0}^{N-1}\sum_{j_1=0}^{j_4-1}\bigl(a_{j_1j_4j_4j_4}+a_{j_4j_1j_4j_4}+a_{j_4j_4j_1j_4}+a_{j_4j_4j_4j_1}\bigr)+$$

$$+\sum_{j_4=0}^{N-1} a_{j_4j_4j_4j_4}.$$

Possibly, the formula (2.162) for any k was founded by the author for the first time.

The relation (2.162) will be used frequently in the future.

Assume, that $a_{(j_1,\ldots,j_k)} = \Phi\left(\tau_{j_1},\ldots,\tau_{j_k}\right)\prod\limits_{l=1}^{k}\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}$, where $\Phi\left(t_1,\ldots,t_k\right)$ — is a nonrandom function of k variables. Then from (1.12) and (2.162) we have

$$J[\Phi]_{T,t}^{(k)} = \sum_{r=0}^{k-1}\ \sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}} \times$$

$$\times \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_k=0}^{N-1}\ldots\sum_{j_{s_r+1}=0}^{j_{s_r+2}-1}\sum_{j_{s_r-1}=0}^{j_{s_r+1}-1}\ldots\sum_{j_{s_1+1}=0}^{j_{s_1+2}-1}\sum_{j_{s_1-1}=0}^{j_{s_1+1}-1}\ldots\sum_{j_1=0}^{j_2-1}\ \sum_{\prod\limits_{l=1}^{r}\mathbf{I}_{j_{s_l},j_{s_l+1}}(j_1,\ldots,j_k)} \times$$

$$\times\Biggl[\Phi\left(\tau_{j_1},\ldots,\tau_{j_{s_1-1}},\tau_{j_{s_1+1}},\tau_{j_{s_1+1}},\ldots,\tau_{j_{s_r-1}},\tau_{j_{s_r+1}},\tau_{j_{s_r+1}},\ldots,\tau_{j_k}\right)\times$$

$$\times\Delta\mathbf{w}_{\tau_{j_1}}^{(i_1)}\ldots\Delta\mathbf{w}_{\tau_{j_{s_1-1}}}^{(i_{s_1-1})}\Delta\mathbf{w}_{\tau_{j_{s_1+1}}}^{(i_{s_1})}\Delta\mathbf{w}_{\tau_{j_{s_1+1}}}^{(i_{s_1+1})}\ldots$$

$$\ldots\Delta\mathbf{w}_{\tau_{j_{s_r-1}}}^{(i_{s_r-1})}\Delta\mathbf{w}_{\tau_{j_{s_r+1}}}^{(i_{s_r})}\Delta\mathbf{w}_{\tau_{j_{s_r+1}}}^{(i_{s_r+1})}\ldots\Delta\mathbf{w}_{\tau_{j_k}}^{(i_k)}\Biggr] =$$

$$= \sum_{r=0}^{k-1}\ \sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}} I[\Phi]_{T,t}^{(k)s_1,\ldots,s_r}\ \text{ w.p.1}, \qquad (2.164)$$

where

$$I[\Phi]_{T,t}^{(k)s_1,\ldots,s_r} = \int\limits_t^T\ldots\int\limits_t^{t_{s_r+3}}\int\limits_t^{t_{s_r+2}}\int\limits_t^{t_{s_r}}\ldots\int\limits_t^{t_{s_1+3}}\int\limits_t^{t_{s_1+2}}\int\limits_t^{t_{s_1}}\ldots\int\limits_t^{t_2}\ \sum_{\prod\limits_{l=1}^{r}\mathbf{I}_{t_{s_l},t_{s_l+1}}(t_1,\ldots,t_k)} \times$$

$$\times\Biggl[\Phi\left(t_1,\ldots,t_{s_1-1},t_{s_1+1},t_{s_1+1},\ldots,t_{s_r-1},t_{s_r+1},t_{s_r+1},\ldots,t_k\right)\times$$

$$\times d\mathbf{w}_{t_1}^{(i_1)}\ldots d\mathbf{w}_{t_{s_1-1}}^{(i_{s_1-1})}d\mathbf{w}_{t_{s_1+1}}^{(i_{s_1})}d\mathbf{w}_{t_{s_1+1}}^{(i_{s_1+1})}d\mathbf{w}_{t_{s_1+2}}^{(i_{s_1+2})}\ldots$$

$$\ldots d\mathbf{w}_{t_{s_r-1}}^{(i_{s_r-1})}d\mathbf{w}_{t_{s_r+1}}^{(i_{s_r})}d\mathbf{w}_{t_{s_r+1}}^{(i_{s_r+1})}d\mathbf{w}_{t_{s_r+2}}^{(i_{s_r+2})}\ldots d\mathbf{w}_{t_k}^{(i_k)}\Biggr], \qquad (2.165)$$

and $\sum\limits_{\emptyset} \stackrel{\text{def}}{=} 1$, $k\ge 2$; the set $\mathrm{A}_{k,r}$ is defined in theorem 11 (see (2.152); we suppose, that right part of (2.165) exists as Ito stochastic integral.

Remark 3. *The summands in the right part of* (2.165) *should be understood as follows: for each derangement from the set* $\prod\limits_{l=1}^{r}\mathbf{I}_{t_{s_l},t_{s_l+1}}(t_1,\ldots,t_k)$ *it is necessary to perform replacement in the right part of* (2.165) *of all pairs* (*their number is* r) *of differentials with similar lower indexes of type* $d\mathbf{w}_{t_p}^{(i)}d\mathbf{w}_{t_p}^{(j)}$ *by values* $\mathbf{1}_{\{i=j\ne 0\}}dt_p$.

Using standard evaluations for the moments of stochastic integrals we get:

$$\mathsf{M}\left\{\left|J[\Phi]_{T,t}^{(k)}\right|^{2n}\right\} \le C_{nk}\sum_{r=0}^{k-1}\ \sum_{(s_r,\ldots,s_1)\in \mathrm{A}_{k,r}} \mathsf{M}\left\{\left|I[\Phi]_{T,t}^{(k)s_1,\ldots,s_r}\right|^{2n}\right\}, \qquad (2.166)$$

$$\mathsf{M}\left\{\left|I[\Phi]_{T,t}^{(k)s_1,\ldots,s_r}\right|^{2n}\right\} \le$$

$$\le C_{nk}^{s_1\ldots s_r}\int\limits_t^T \ldots \int\limits_t^{t_{s_r+3}}\int\limits_t^{t_{s_r+2}}\int\limits_t^{t_{s_r}} \ldots \int\limits_t^{t_{s_1+3}}\int\limits_t^{t_{s_1+2}}\int\limits_t^{t_{s_1}} \ldots \int\limits_t^{t_2} \sum_{\prod\limits_{l=1}^{r}\mathbf{I}_{t_{s_l},t_{s_l+1}}(t_1,\ldots,t_k)} \times$$

$$\times \Phi^{2n}\left(t_1,\ldots,t_{s_1-1},t_{s_1+1},t_{s_1+1},\ldots,t_{s_r-1},t_{s_r+1},t_{s_r+1},\ldots,t_k\right) \times$$

$$\times dt_1\ldots dt_{s_1-1}dt_{s_1+1}dt_{s_1+2}\ldots dt_{s_r-1}dt_{s_r+1}dt_{s_r+2}\ldots dt_k, \qquad (2.167)$$

where derangements in the course of summation in (2.167) are performed only in $\Phi^{2n}(\ldots)$; $C_{nk},\ C_{nk}^{s_1\ldots s_r}<\infty$.

Lemma 4. *In conditions of theorem* 5 *valid the following relation*

$$J[K^*]_{T,t}^{(k)} = J^*[\psi^{(k)}]_{T,t}\ \textit{w. p. } 1. \qquad (2.168)$$

Proof. Substituting (2.145) in (1.12), using lemma 1 and remark 2, it is easy to see, that w. p. 1

$$J[K^*]_{T,t}^{(k)} = J[\psi^{(k)}]_{T,t} + \sum_{r=1}^{\left[\frac{k}{2}\right]} \frac{1}{2^r} \sum_{(s_r,\ldots,s_1)\in\mathrm{A}_{k,r}} J[\psi^{(k)}]_{T,t}^{s_r,\ldots,s_1}, \qquad (2.169)$$

where the meaning of theorem 7 notations is kept.

The affirmation of lemma results from (2.169) in accordance with theorem 7. □

Using lemmas from the proof of theorem 1 we get:

$$J^*[\psi^{(k)}]_{T,t} = \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)} + J[R_{p_1\ldots p_k}]_{T,t}^{(k)} \text{ w. p. } 1,$$

where stochastic integral $J[R_{p_1\ldots p_k}]_{T,t}^{(k)}$ defined in accordance with (1.12) and

$$R_{p_1\ldots p_k}(t_1,\ldots,t_k) = K^*(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l), \quad (2.170)$$

$\zeta_{j_l}^{(i_l)} = \int\limits_t^T \phi_{j_l}(s) d\mathbf{w}_s^{(i_l)},\ p_1, \ldots, p_k < \infty.$

At that, the following equation is executed pointwise in $(t, T)^k$ in accordance with theorem 6:

$$\lim_{p_1 \to \infty} \ldots \lim_{p_k \to \infty} R_{p_1 \ldots p_k}(t_1, \ldots, t_k) = 0. \tag{2.171}$$

Lemma 5. *In conditions of theorem* 5

$$\lim_{p_1 \to \infty} \ldots \lim_{p_k \to \infty} \mathsf{M}\left\{\left|J[R_{p_1 \ldots p_k}]_{T,t}^{(k)}\right|^{2n}\right\} = 0,\ n \in N.$$

Proof. According to (2.145) and (2.170) we have the following in all internal points of the hypercube $[t, T]^k$:

$$R_{p_1 \ldots p_k}(t_1, \ldots, t_k) =$$

$$= \prod_{l=1}^{k} \psi_l(t_l) \Biggl(\prod_{l=1}^{k-1} \mathbf{1}_{\{t_l < t_{l+1}\}} + \sum_{r=1}^{k-1} \frac{1}{2^r} \sum_{\substack{s_r, \ldots, s_1 = 1 \\ s_r > \ldots > s_1}}^{k-1} \prod_{l=1}^{r} \mathbf{1}_{\{t_{s_l} = t_{s_l + 1}\}} \prod_{\substack{l=1 \\ l \ne s_1, \ldots, s_r}}^{k-1} \mathbf{1}_{\{t_l < t_{l+1}\}} \Biggr) -$$

$$- \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l). \tag{2.172}$$

Due to (2.172) the function $R_{p_1 \ldots p_k}(t_1, \ldots, t_k)$ is continuous in the domains of integration of stochastic integrals in the right part of (2.164) and it is bounded at the boundaries of these domains (let's remind, that the repeated series

$$\sum_{j_1=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} C_{j_k \ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l)$$

converges at the boundary of hypercube $[t, T]^k$).

Then, taking $R_{p_1 \ldots p_k}(t_1, \ldots, t_k)$ instead of $\Phi(t_1, \ldots, t_k)$ in (2.166), (2.167) and performing the repeated passage to the limit $\lim\limits_{p_1 \to \infty} \ldots \lim\limits_{p_k \to \infty}$ under the integral signs in these estimations (like it was performed for the two-dimensional case), considering (2.171), we get the required result. Lemma 5 and theorem 5 are proven. □

Note, that in accordance with theorem 9 we may approximate the multiple stochastic Stratonovich integral $J^*[\psi^{(k)}]_{T,t}$ using the expression

$$J^*[\psi^{(k)}]_{T,t}^{p_1 \ldots p_k} = \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1} \prod_{l=1}^{k} \zeta_{j_l}^{(i_l)};\ p_1, \ldots, p_k < \infty. \tag{2.173}$$

Chapter 3

Expansions of multiple stochastic integrals of other types, based on generalized multiple Fourier series

In this chapter we demonstrate, that approach to expansion of multiple stochastic Ito integrals considered in chapter 1 (theorem 1) is essentially general and allows some transformation for other types of multiple stochastic integrals. Here we consider the versions of the theorem 1 for multiple stochastic integrals according to martingale Poisson measures and for multiple stochastic integrals according to martingales. Considered theorems are sufficiently natural according to general properties of martingales.

3.1 Expansion of multiple stochastic integrals according to martingale Poisson's measures

Let's introduce the following stochastic integral in the analysis:

$$P[\chi^{(k)}]_{T,t} = \int\limits_t^T \int\limits_X \chi_k(t_k, \mathbf{y}_k) \ldots \int\limits_t^{t_2} \int\limits_X \chi_1(t_1, \mathbf{y}_1) \tilde{\nu}^{(i_1)}(dt_1, d\mathbf{y}_1) \ldots \tilde{\nu}^{(i_k)}(dt_k, d\mathbf{y}_k), \tag{3.1}$$

where $\Re^n \stackrel{\rm def}{=} X$; $i_1, \ldots, i_k = 0,\ 1, \ldots, m$; $\nu^{(i)}(dt, d\mathbf{y})$ — are independent Poisson measures, which defines on $[0, T] \times X$ (see sect. 7.6); $\tilde{\nu}^{(i)}(dt, d\mathbf{y}) = \nu^{(i)}(dt, d\mathbf{y}) - \Pi(d\mathbf{y})dt$ — are martingale Poisson measures; $i = 1, \ldots, m$; $\tilde{\nu}^{(0)}(dt, d\mathbf{y}) \stackrel{\rm def}{=} \Pi(d\mathbf{y})dt$; $\chi_l(\tau, \mathbf{y}) = \psi_l(\tau)\varphi_l(\mathbf{y})$; $\psi_l(\tau) : [t, T] \to \Re^1$; $\varphi_l(\mathbf{y}) : X \to \Re^1$; $\chi_l(s, \mathbf{y}) \in H_2(\Pi, [t, T])$, $l = 1, \ldots, k$; — is a class of nonanticipative random functions $\varphi : [0, T] \times \mathbf{Y} \times \Omega \to \Re^1$, for which

$$\int\limits_0^T \int\limits_{\mathbf{Y}} \mathsf{M}\{|\varphi(t, \mathbf{y})|^2\} \Pi(d\mathbf{y}) dt < \infty$$

(see sect 7.6).

Theorem 8. *Assume, that the following conditions are met:*

1. $\psi_i(\tau);\ i=1,\ 2,\ldots,k$ — *are continuouse functions at the interval* $[t,T]$.

2. $\{\phi_j(x)\}_{j=0}^{\infty}$ — *is a full orthonormal system of functions in the space* $L_2([t,T])$ *each function of which for finite* j *satisfies the condition* $(\star)$ *(see p.39).*

3. $\int\limits_X |\varphi_l(\mathbf{y})|^s\,\Pi(dy)<\infty;\ l=1,\ldots,k;\ s=1,2,\ldots,2^{k+1}$.

Then, the multiple stochastic integral according to martingale Poisson measures $P[\chi^{(k)}]_{T,t}$ *is expanded into the multiple series converging in the mean-square sense*

$$P[\chi^{(k)}]_{T,t}=\sum_{j_1,\ldots,j_k=0}^{\infty} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\pi_{j_l}^{(l,i_l)}-$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k}\prod_{r=1}^{k}\phi_{j_r}(\tau_{l_r})\int\limits_X \varphi_r(\mathbf{y})\tilde{\nu}^{(i_r)}([\tau_{l_r},\tau_{l_r+1}),d\mathbf{y})\Biggr), \tag{3.2}$$

$\mathrm{G}_k=\mathrm{H}_k\backslash \mathrm{L}_k;\ \mathrm{H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\}$,

$$\mathrm{L}_k=\Biggl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;$$

$$l_g\ne l_r (g\ne r);\ g,r=1,\ldots,k\Biggr\};$$

$\pi_j^{(l,i_l)}=\int\limits_t^T\int\limits_X \phi_j(\tau)\varphi_l(\mathbf{y})\tilde{\nu}^{(i_l)}(d\tau,d\mathbf{y})$ — *are independent for different* $i_l\ne 0$ *and uncorrelated random variables for different* j;

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k} K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k;$$

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k),\ t_1<\ldots<t_k\\ 0,\ \textit{otherwise}\end{cases};\ t_1,\ldots,t_k\in[t,T].$$

Proof. This theorem may be proven as well as theorem 1. Small differences will take place only in proving of analogues of lemmas 1 – 3 for the considered case.

Lemma 6. *Assume, that* $\psi_l(\tau)$ — *are continuous functions at the interval* $[t,T]$, *and the functions* $\varphi_l(\mathbf{y})$ *are such, that* $\int\limits_X |\varphi_l(\mathbf{y})|^p\,\Pi(d\mathbf{y})<\infty$; $p=1,\ 2;\ l=1,\ldots,k$. *Then, we have with probability* 1*:*

$$P[\bar{\chi}^{(k)}]_{T,t}=\underset{N\to\infty}{\text{l.i.m.}}\sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{j_2-1}\prod_{l=1}^{k}\int\limits_X \chi_l(\tau_{j_l},\mathbf{y})\bar{\nu}^{(i_l)}([\tau_{j_l},\tau_{j_l+1}),d\mathbf{y}), \tag{3.3}$$

where $\{\tau_{j_l}\}_{j_l=0}^{N-1}$ *— is partition of the interval* $[t,T]$, *which satisfies the condition* (1.7), $\bar\nu^{(i)}([\tau,s),d\mathbf{y})=\tilde\nu^{(i)}([\tau,s),d\mathbf{y})$ *or* $\nu^{(i)}([\tau,s),d\mathbf{y})$; *the integral* $P[\bar\chi^{(k)}]_{T,t}$ *differs from the integral* $P[\chi^{(k)}]_{T,t}$ *by the fact, that in* $P[\bar\chi^{(k)}]_{T,t}$ *instead of* $\tilde\nu^{(i_l)}(dt_l,d\mathbf{y}_l)$ *stay* $\bar\nu^{(i_l)}(dt_l,d\mathbf{y}_l)$; $l=1,\ldots,k$.

Proof. Using estimations of stochastic integrals according to Poisson measures (see sect. 7.7), and the conditions of lemma 6, it is easy to note, that the integral sum of the integral $P[\bar\chi^{(k)}]_{T,t}$ under conditions of lemma 6 may be represented in the form of prelimit expression from the right part of (3.3) and of value, which converges to zero in the mean-square sense if $N\to\infty$. □

Let's introduce the following stochastic integrals in the analysis:

$$\underset{N\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{N-1}\Phi(\tau_{j_1},\ldots,\tau_{j_k})\prod_{l=1}^{k}\int\limits_X\varphi_l(\mathbf{y})\tilde\nu^{(i_l)}([\tau_{j_l},\tau_{j_l+1}),d\mathbf{y})\stackrel{\text{def}}{=}P[\Phi]_{T,t}^{(k)},$$

$$\int\limits_t^T\ldots\int\limits_t^{t_2}\Phi(t_1,\ldots,t_k)\int\limits_X\varphi_1(\mathbf{y})\tilde\nu^{(i_1)}(dt_1,d\mathbf{y})\ldots\int\limits_X\varphi_k(\mathbf{y})\tilde\nu^{(i_k)}(dt_k,d\mathbf{y})\stackrel{\text{def}}{=}$$

$$\stackrel{\text{def}}{=}\hat P[\Phi]_{T,t}^{(k)},$$

where the sense of notations included in (3.3) is kept; $\Phi(t_1,\ldots,t_k):[t,T]^k\to\Re^1$ — is bounded nonrandom function.

Note, that if the functions $\varphi_l(\mathbf{y})$; $l=1,\ldots,k$ satisfy the conditions of lemma 6, and the function $\Phi(t_1,\ldots,t_k)$ is continuous, then for the integral $\hat P[\Phi]_{T,t}^{(k)}$ the equality of type (3.3) is reasonable with probability 1.

Lemma 7. *Assume, that for* $l=1,\ldots,k$ *the following conditions are executed:* $g_l(\tau,\mathbf{y})=h_l(\tau)\varphi_l(\mathbf{y})$; *the functions* $h_l(\tau):[t,T]\to\Re^1$ *satisfy the condition* $(\star)$ *(see p.39) and the functions* $\varphi_l(\mathbf{y}):X\to\Re^1$ *satisfy the condition* $\int\limits_X|\varphi_l(\mathbf{y})|^p\,\Pi(d\mathbf{y})<\infty$; $p=1,2,3,\ldots,2^{k+1}$. *Then*

$$\prod_{l=1}^{k}\int\limits_t^T\int\limits_X g_l(s,\mathbf{y})\bar\nu^{(i_l)}(ds,d\mathbf{y})=P[\Phi]_{T,t}^{(k)}\ \text{w.p.1},\ \ \Phi(t_1,\ldots,t_k)=\prod_{l=1}^{k}h_l(t_l).$$

Proof. Let's introduce the following notations:

$$J[\bar g_l]_N\stackrel{\text{def}}{=}\sum_{j=0}^{N-1}\int\limits_X g_l(\tau_j,\mathbf{y})\bar\nu^{(i_l)}([\tau_j,\tau_{j+1}),d\mathbf{y}),\ \ J[\bar g_l]_{T,t}\stackrel{\text{def}}{=}\int\limits_t^T\int\limits_X g_l(s,\mathbf{y})\bar\nu^{(i_l)}(ds,d\mathbf{y}).$$

It is easy to see, that

$$\prod_{l=1}^{k}J[\bar g_l]_N-\prod_{l=1}^{k}J[\bar g_l]_{T,t}=\sum_{l=1}^{k}\left(\prod_{q=1}^{l-1}J[\bar g_q]_{T,t}\right)\left(J[\bar g_l]_N-J[\bar g_l]_{T,t}\right)\left(\prod_{q=l+1}^{k}J[\bar g_q]_N\right).$$

Using Minkowsky inequality and inequality of Cauchy-Buniakovsky together with estimations of moments of integrals according to Poisson measures [2] (see sect. 7.7) and conditions of lemma 7, we get

$$\left(\mathsf{M}\left\{\left|\prod_{l=1}^{k} J[\bar{g}_l]_N - \prod_{l=1}^{k} J[\bar{g}_l]_{T,t}\right|^2\right\}\right)^{\frac{1}{2}} \le C_k \sum_{l=1}^{k} \left(\mathsf{M}\left\{\left|J[\bar{g}_l]_N - J[\bar{g}_l]_{T,t}\right|^4\right\}\right)^{\frac{1}{4}}, \quad (3.4)$$

where $C_k < \infty$.

Since it is clear, that

$$J[\bar{g}_l]_N - J[\bar{g}_l]_{T,t} = \sum_{q=0}^{N-1} J[\Delta\bar{g}_l]_{\tau_{q+1},\tau_q},$$

$$J[\Delta\bar{g}_l]_{\tau_{q+1},\tau_q} = \int_{\tau_q}^{\tau_{q+1}} \int_X \left(g_l(\tau_q,\mathbf{y}) - g_l(s,\mathbf{y})\right) \tilde{\nu}^{(i_l)}(ds,d\mathbf{y}),$$

then due to independence of $J[\Delta\bar{g}_l]_{\tau_{q+1},\tau_q}$ for different q we have [27]:

$$\mathsf{M}\left\{\left|\sum_{j=0}^{N-1} J[\Delta\bar{g}_l]_{\tau_{j+1},\tau_j}\right|^4\right\} = \sum_{j=0}^{N-1} \mathsf{M}\left\{\left|J[\Delta\bar{g}_l]_{\tau_{j+1},\tau_j}\right|^4\right\} +$$

$$+6\sum_{j=0}^{N-1} \mathsf{M}\left\{\left|J[\Delta\bar{g}_l]_{\tau_{j+1},\tau_j}\right|^2\right\} \sum_{q=0}^{j-1} \mathsf{M}\left\{\left|J[\Delta\bar{g}_l]_{\tau_{q+1},\tau_q}\right|^2\right\}. \quad (3.5)$$

Then, using estimations of moments of stochastic integrals according to Poisson measures [2] (see sect. 7.7) and the conditions of lemma 7, we get, that the right part of (3.5) converges to zero when $N \to \infty$. Considering this fact and (3.4) we come to the affirmation of lemma. $\square$

Proving of theorem 8 according to the scheme used for proving of theorem 1 using lemmas 6, 7 and estimations of moments of stochastic integrals according to Poisson measures (see sect. 7.7), we get:

$$\mathsf{M}\left\{\left(R_{T,t}^{p_1,\ldots,p_k}\right)^2\right\} \le C_k \prod_{l=1}^{k} \int_X \varphi_l^2(\mathbf{y})\Pi(\mathbf{y})\times$$

$$\times \int_{[t,T]^k} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l)\right)^2 dt_1\ldots dt_k \le$$

$$\le \bar{C}_k \int_{[t,T]^k} \left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k\ldots j_1} \prod_{l=1}^{k} \phi_{j_l}(t_l)\right)^2 dt_1\ldots dt_k \to 0$$

if $p_1,\ldots,p_k\to\infty$, where constant $\bar{C}_k$ depends only on k (multiplicity of multiple stochastic integral according to martingale Poisson measures) and

$$R_{T,t}^{p_1,\ldots,p_k}=P[\chi^{(k)}]_{T,t}-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k}C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\pi_{j_l}^{(l,i_l)}-$$

$$-\underset{N\to\infty}{\text{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathrm{G}_k}\prod_{r=1}^{k}\phi_{j_r}(\tau_{l_r})\int_{X}\varphi_r(\mathbf{y})\tilde{\nu}^{(i_r)}([\tau_{l_r},\tau_{l_r+1}),d\mathbf{y})\Biggr).$$

Theorem 8 is proven. $\square$

Let's give an example of theorem 13 usage. When $i_1\ne i_2,\ i_1,i_2=1,\ldots,m$ according to theorem 13 using the system of Legendre polynomials we get

$$\int_t^T\int_X\varphi_2(\mathbf{y}_1)\int_t^{t_1}\int_X\varphi_1(\mathbf{y}_2)\tilde{\nu}^{(i_2)}(dt_2,d\mathbf{y}_2)\tilde{\nu}^{(i_1)}(dt_1,d\mathbf{y}_1)=$$

$$=\frac{T-t}{2}\Biggl(\pi_0^{(1,i_1)}\pi_0^{(2,i_2)}+\sum_{i=1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left(\pi_{i-1}^{(2,i_2)}\pi_i^{(1,i_1)}-\pi_{i-1}^{(1,i_1)}\pi_i^{(2,i_2)}\right)\Biggr),$$

$$\int_t^T\int_X\varphi_1(\mathbf{y}_1)\tilde{\nu}^{(i_1)}(dt_1,d\mathbf{y}_1)=\sqrt{T-t}\,\pi_0^{(1,i_1)},$$

where $\pi_j^{(l,i_l)}=\int_t^T\int_X\phi_j(\tau)\varphi_l(\mathbf{y})\tilde{\nu}^{(i_l)}(d\tau,d\mathbf{y});\ l=1,\ 2;\ \{\phi_j(\tau)\}_{j=0}^{\infty}$ — is a full orthonormal system of Legendre polynomials in the space $L_2([t,T])$.

3.2 Expansion of multiple stochastic integrals according to martingales

Assume, that the fixed probability space $(\Omega,\mathcal{F},\mathsf{P})$ is preset and assume, that $\{\mathcal{F}_t,t\in[0,T]\}$ — is a non-decreasing collection of σ-subalgebras $\mathcal{F}$. Through $\mathrm{M}_2(\rho,[0,T])$ we will denote a class of $\mathcal{F}_t$-measurable for each $t\in[0,T]$ martingales M_t, and satisfying the conditions $\mathsf{M}\left\{(M_s-M_t)^2\right\}=\int_t^s\rho(\tau)d\tau,\ \mathsf{M}\left\{|M_s-M_t|^p\right\}\le C_p|s-t|$, where $0\le t<s\le T$; $\rho(\tau)$ — is a non-negative, continuously differentiated nonrandom function at the interval $[0,T]$; $C_p<\infty$ — is a constant; $p=3,4,\ldots$.

It is obvious, that the martingale from class $\mathrm{M}_2(\rho,[0,T])$ is D-martingale [2].

Assume, that $\{\tau_j\}_{j=0}^{N}$ — is a partition of interval $[0,T]$, for which

$$0=\tau_0<\tau_1<\ldots<\tau_N=T,\ \max_{0\le j\le N-1}|\tau_{j+1}-\tau_j|\to 0 \text{ when } N\to\infty. \quad (3.6)$$

In accordance with features of the function $\rho(\tau)$ we will write the condition of membership of $\mathcal{F}_t$-measurable for each $t \in [0,T]$ stochastic process $\xi_t; t \in [0,T]$ to the class $H_2(\rho,[0,T])$ (see sect. 7.5) in the form $\int\limits_0^T \mathsf{M}\{|\xi_t|^2\}\rho(t)dt < \infty$.

Let's assume the step random function $\xi_t^{(N)}$ at the partition $\{\tau_j\}_{j=0}^N$ as follows: $\xi_t^{(N)} = \xi_{\tau_{j-1}}$ with probability 1 when $t \in [\tau_{j-1},\tau_j)$; $j = 1,\ldots,N$. In the section 7.5 (see also [2]) we defined the stochastic integral from the process $\xi_t \in H_2(\rho,[0,T])$ according to martingale. In accordance with it, the stochastic integral according to the martingale $M_t \in \mathrm{M}_2(\rho,[0,T])$ is defined by the following equality

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{j=0}^{N-1} \xi_{\tau_j}^{(N)}\left(M_{\tau_{j+1}} - M_{\tau_j}\right) \stackrel{\text{def}}{=} \int\limits_0^T \xi_t dM_t, \tag{3.7}$$

where $\{\tau_j\}_{j=0}^N$ — is a partition of the interval $[0,T]$, satisfying the condition (3.6); $\xi_t^{(N)}$ — is any sequence of step functions from $H_2(\rho,[0,T])$, for which

$$\int\limits_0^T \mathsf{M}\{|\xi_t^{(N)} - \xi_t|^2\}\rho(t)dt \to 0 \text{ when } N\to\infty.$$

Using $Q_4(\rho,[0,T])$ let's denote the subclass $\mathrm{M}_2(\rho,[0,T])$ of martingales $M_t, t\in[0,T]$, for which in case of some $\alpha > 0$ the following estimation is true:

$$\mathsf{M}\left\{\left|\int\limits_\theta^\tau g(s)dM_s\right|^4\right\} \le K_4\int\limits_\theta^\tau |g(s)|^\alpha ds,$$

where $0 \le \theta < \tau \le T$; $g(s)$ — is a bounded non-random function at the interval $[0,T]$; $K_4 < \infty$ — is a constant.

Using $G_n(\rho,[0,T])$ let's denote the subclass $\mathrm{M}_2(\rho,[0,T])$ of martingales $M_t, t\in[0,T]$, for which

$$\mathsf{M}\left\{\left|\int\limits_\theta^\tau g(s)dM_s\right|^n\right\} < \infty,$$

where $0 \le \theta < \tau \le T$; $n \in N$; $g(s)$ — is the same function, as in the definition of $Q_4(\rho,[0,T])$.

Let's remind (see sect. 7.1), that if $(\xi_t)^n \in H_2(\rho,[0,T])$ when $\rho(t)\equiv 1$, then the estimation [2] is correct:

$$\mathsf{M}\left\{\left|\int\limits_\theta^\tau \xi_t dt\right|^{2n}\right\} \le (\tau-\theta)^{2n-1}\int\limits_\theta^\tau \mathsf{M}\{|\xi_t|^{2n}\}dt,\ 0\le\theta<\tau\le T. \tag{3.8}$$

Assume, that

$$J[\psi^{(k)}]_{T,t}^{M} \stackrel{\rm def}{=} \int\limits_t^T \psi_k(t_k)\ldots \int\limits_t^{t_2} \psi_1(t_1)dM_{t_1}^{(1,i_1)}\ldots dM_{t_k}^{(k,i_k)};\, i_1,\ldots,i_k=0,1,\ldots,m,$$

where $M^{(r,i)}$ $(r=1,\ldots,k)$ — are independent for different $i=1,\ 2,\ldots,m.$

Let's prove the following theorem.

Theorem 9. *Assume, that the following conditions are met:*

1. $M_{\tau}^{(l,i_l)}\in Q_4(\rho,[t,T]),\ G_n(\rho,[t,T]);\ n=2,4,\ldots,2^k;\ k\in N;\ i_l=1,\ldots,m;\ l=1,\ldots,k.$

2. $\{\phi_j(x)\}_{j=0}^{\infty}$ *— is a full orthonormal system of functions in the space* $L_2([t,T])$*, each function of which for finite* j *satisfies the condition* $(\star)$ *(see p.39).*

3. $\psi_i(\tau);\ i=1,\ 2,\ldots,k$ *— are continuous functions at the interval* $[t,T]$.

Then the multiple stochastic integral $J[\psi^{(k)}]_{T,t}^{M}$ *according to martingales is expanded into the converging in the mean-square sense multiple series*

$$J[\psi^{(k)}]_{T,t}^{M}=\sum_{j_1,\ldots,j_k=0}^{\infty} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\xi_{j_l}^{(l,i_l)}-$$

$$-\underset{N\to\infty}{\hbox{l.i.m.}}\sum_{(l_1,\ldots,l_k)\in\mathcal{G}_k}\phi_{j_1}(\tau_{l_1})\Delta M_{\tau_{l_1}}^{(1,i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta M_{\tau_{l_k}}^{(k,i_k)}\Biggr),$$

where $\mathcal{G}_k=\mathcal{H}_k\backslash\mathcal{L}_k;\ \mathcal{H}_k=\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1\},$

$$\mathcal{L}_k=\Biggl\{(l_1,\ldots,l_k):\ l_1,\ldots,l_k=0,\ 1,\ldots,N-1;$$
$$l_g\ne l_r(g\ne r);\ g,r=1,\ldots,k\Biggr\};$$

$\xi_{j_l}^{(l,i_l)}=\int\limits_t^T\phi_{j_l}(s)dM_s^{(l,i_l)}$ *— are independent for different* $i_l=1,\ldots,m;\ l=1,\ldots,k$ *and uncorrelated for various* j_l *(if* $\rho(\tau)$ *— is a constant,* $i_l\ne 0$*) random variables;*

$$C_{j_k\ldots j_1}=\int\limits_{[t,T]^k}K(t_1,\ldots,t_k)\prod_{l=1}^{k}\phi_{j_l}(t_l)dt_1\ldots dt_k;$$

$$K(t_1,\ldots,t_k)=\begin{cases}\psi_1(t_1)\ldots\psi_k(t_k),\ t_1<\ldots<t_k\\ 0,\ \hbox{otherwise}\end{cases};\ t_1,\ldots,t_k\in[t,T].$$

Proof. In order to prove theorem 14 let's analyze several lemmas.

Lemma 8. *Assume, that* $M_\tau^{(l,i_l)} \in \mathrm{M}_2(\rho,[t,T])$; $i_l = 1,\ldots,m$; $l = 1,\ldots,k$, *and* $\psi_i(\tau)$; $i = 1, 2,\ldots,k$ *— are continuous functions at the interval* $[t,T]$.

Then

$$J[\psi^{(k)}]_{T,t}^M = \underset{N\to\infty}{\text{l.i.m.}} \sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{j_2-1}\prod_{l=1}^{k}\psi_l(\tau_{j_l})\Delta M_{\tau_{j_l}}^{(l,i_l)} \ w.p.1, \tag{3.9}$$

where $\{\tau_j\}_{j=0}^N$ *— is a partition of the interval* $[0,T]$, *satisfying the condition of type* (3.6).

Proof. Since (see sect.7.5)

$$\mathsf{M}\left\{\left(\int_\theta^\tau \xi_s dM_s^{(l,i_l)}\right)^2\right\} = \int_\theta^\tau \mathsf{M}\{|\xi_s|^2\}\rho(s)ds,$$

$$\mathsf{M}\left\{\left(\int_\theta^\tau \xi_s ds\right)^2\right\} \le (\tau-\theta)\int_\theta^\tau \mathsf{M}\{|\xi_s|^2\}ds,$$

where $\xi_t \in H_2(\rho,[0,T])$; $t \le \theta < \tau \le T$; $i_l = 1,\ldots,m$; $l = 1,\ldots,k$, then the integral sum of integral $J[\psi^{(k)}]_{T,t}^M$ in conditions of lemma 8 may be represented in the form of prelimit expression from the right part (3.9) and the value, which converges to zero in the mean-square sense when $N\to\infty$. $\square$

Assume, that

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{j_1,\ldots,j_k=0}^{N-1}\Phi(\tau_{j_1},\ldots,\tau_{j_k})\prod_{l=1}^{k}\Delta M_{\tau_{j_l}}^{(l,i_l)} \stackrel{\text{def}}{=} I[\Phi]_{T,t}^{(k)}, \tag{3.10}$$

where $\{\tau_j\}_{j=0}^N$ is a partition of the interval $[0,T]$, satisfying the condition of type (3.6).

Lemma 9. *Assume, that* $M_s^{(l,i_l)} \in Q_4(\rho,[t,T])$, $G_r(\rho,[t,T])$; $r = 2,4,\ldots,2^k$; $i_l = 1,\ldots,m$; $l = 1,\ldots,k$, *and* $g_1(s),\ldots,g_k(s)$ *— are functions satisfying the condition* $(\star)$ *(see p.39).*

Then

$$\prod_{l=1}^{k}\int_t^T g_l(s)dM_s^{(l,i_l)} = I[\Phi]_{T,t}^{(k)} \ w.p.1,\ \Phi(t_1,\ldots,t_k) = \prod_{l=1}^{k} g_l(t_l).$$

Proof. Let's denote

$$I[g_l]_N \stackrel{\text{def}}{=} \sum_{j=0}^{N-1} g_l(\tau_j)\Delta M_{\tau_j}^{(l,i_l)},\ I[g_l]_{T,t} \stackrel{\text{def}}{=} \int_t^T g_l(s)dM_s^{(l,i_l)}.$$

Note, that

$$\prod_{l=1}^{k} I[g_l]_N - \prod_{l=1}^{k} I[g_l]_{T,t} = \sum_{l=1}^{k}\left(\prod_{q=1}^{l-1} I[g_q]_{T,t}\right)\left(I[g_l]_N - I[g_l]_{T,t}\right)\left(\prod_{q=l+1}^{k} I[g_q]_N\right).$$

Using the Minkowsky inequality and inequality of Cauchy-Bunyakovsky, as well as the conditions of lemma 9, we get

$$\left(\mathsf{M}\left\{\left|\prod_{l=1}^{k} I[g_l]_N - \prod_{l=1}^{k} I[g_l]_{T,t}\right|^2\right\}\right)^{\frac{1}{2}} \le C_k \sum_{l=1}^{k}\left(\mathsf{M}\left\{\left|I[g_l]_N - I[g_l]_{T,t}\right|^4\right\}\right)^{\frac{1}{4}}, \quad (3.11)$$

where $C_k < \infty$ — is a constant.

Since

$$I[g_l]_N - I[g_l]_{T,t} = \sum_{q=0}^{N-1} I[\Delta g_l]_{\tau_{q+1},\tau_q},\ I[\Delta g_l]_{\tau_{q+1},\tau_q} = \int\limits_{\tau_q}^{\tau_{q+1}} \left(g_l(\tau_q) - g_l(s)\right) dM_s^{(l,i_l)},$$

then due to independence of $I[\Delta g_l]_{\tau_{q+1},\tau_q}$ for different q we have [27]:

$$\mathsf{M}\left\{\left|\sum_{j=0}^{N-1} I[\Delta g_l]_{\tau_{j+1},\tau_j}\right|^4\right\} = \sum_{j=0}^{N-1} \mathsf{M}\left\{\left|I[\Delta g_l]_{\tau_{j+1},\tau_j}\right|^4\right\} +$$

$$+6\sum_{j=0}^{N-1} \mathsf{M}\left\{\left|I[\Delta g_l]_{\tau_{j+1},\tau_j}\right|^2\right\}\sum_{q=0}^{j-1} \mathsf{M}\left\{\left|I[\Delta g_l]_{\tau_{q+1},\tau_q}\right|^2\right\}. \quad (3.12)$$

Then, using the conditions of lemma 9, we get, that the right part of (3.12) converges to zero when $N \to \infty$. Considering this fact and (3.11) we come to the affirmation of lemma. □

Then, using the proven lemmas and repeating the proof of theorem 1 with correspondent changes we get:

$$\mathsf{M}\left\{\left(R_{T,t}^{p_1,\ldots,p_k}\right)^2\right\} \le C_k \int\limits_{[t,T]^k}\Biggl(K(t_1,\ldots,t_k) -$$

$$-\sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\Biggr)^2 \rho(t_1)dt_1\ldots\rho(t_k)dt_k \le$$

$$\le \bar{C}_k \int\limits_{[t,T]^k}\left(K(t_1,\ldots,t_k) - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\phi_{j_l}(t_l)\right)^2 dt_1\ldots dt_k \to 0$$

when $p_1,\ldots,p_k \to \infty$, where the constant $\bar{C}_k$ depends only on k (multiplicity of multiple stochastic integral according to martingales) and

$$R_{T,t}^{p_1,\ldots,p_k} = J[\psi^{(k)}]_{T,t}^{M} - \sum_{j_1=0}^{p_1}\ldots\sum_{j_k=0}^{p_k} C_{j_k\ldots j_1}\Biggl(\prod_{l=1}^{k}\xi_{j_l}^{(l,i_l)} -$$

$$-\underset{N\to\infty}{\text{l.i.m.}} \sum_{(l_1,\ldots,l_k)\in \mathrm{G}_k} \phi_{j_1}(\tau_{l_1})\Delta M_{\tau_{l_1}}^{(1,i_1)}\ldots\phi_{j_k}(\tau_{l_k})\Delta M_{\tau_{l_k}}^{(k,i_k)}\Biggr).$$

Theorem 9 is proven. □

3.3 Remark about full orthonormal systems of functions with weight in the space $L_2([t,T])$

Let's note, that in theorem 14 we may use the full orthonormal systems of functions not only with weight 1 but with some other weight in the space $L_2([t,T])$.

Let's analyze the following boundary-value problem

$$(p(x)\Phi'(x))' + q(x)\Phi(x) = -\lambda r(x)\Phi(x),$$

$$\alpha\Phi(a) + \beta\Phi'(a) = 0,\ \gamma\Phi(a) + \delta\Phi'(a) = 0,$$

where functions $p(x), q(x), r(x)$ satisfy the well-known conditions and α, β, γ, δ, λ — are real numbers.

It has been known (V.A. Steklov), that eigenfunctions $\Phi_0(x)$, $\Phi_1(x),\ldots$ of this boundary-value problem create a full orthonormal system of functions with weight $r(x)$ in the space $L_2([a,b])$, as well as the Fourier series of function $\sqrt{r(x)}f(x) \in L_2([a,b])$ according to the system of functions $\sqrt{r(x)}\Phi_0(x)$, $\sqrt{r(x)}\Phi_1(x),\ldots$ converges in the mean to this function at this interval, moreover the Fourier coefficients are defined using the formula

$$C_j = \int\limits_a^b r(x)f(x)\Phi_j(x)dx. \tag{3.13}$$

Note, that if we expand the function $f(x) \in L_2([a,b])$ into the Fourier series in accordance with the system of functions $\Phi_0(x)$, $\Phi_1(x),\ldots$, then the expansion coefficients will also be defined using the formula (3.13) and the convergence of Fourier series will take place in the mean with weight $r(x)$ to the function $f(x)$ at the interval $[a,b]$.

It is known, that analyzing the task about fluctuations of circular membrane (common case) the boundary-value problem appears for the equation of Euler-Bessel with the parameter λ and integer index $n:$

$$r^2R''(r) + rR'(r) + \left(\lambda^2r^2 - n^2\right)R(r) = 0. \tag{3.14}$$

The eigenfunctions of this task considering specific boundary conditions are the following functions

$$J_n\Bigl(\mu_j\frac{r}{L}\Bigr), \tag{3.15}$$

where $r \in [0, L]$, μ_j; $j = 0,\ 1,\ 2, \ldots$ — are ordered in ascending order positive roots of the Bessel function $J_n(\mu)$; $n = 0,\ 1,\ 2, \ldots$.

In the task about radial fluctuations of the circular membrane the boundary-value task appears for the equation (3.14) when $n = 0$, the eigenfunctions of which are functions (3.15) when $n = 0$.

Let's analyze the system of functions

$$\Psi_j(\tau) = \frac{\sqrt{2}}{T J_{n+1}(\mu_j)} J_n\Bigl(\frac{\mu_j}{T}\tau\Bigr);\ j = 0,\ 1,\ 2, \ldots, \tag{3.16}$$

where

$$J_n(x) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{x}{2}\right)^{n+2m} (\Gamma(m+1)\Gamma(m+n+1))^{-1}$$

— is the Bessel function of first genus and

$$\Gamma(z) = \int\limits_0^{\infty} e^{-x} x^{z-1} dx$$

— is a gamma-function; μ_j — are numbered in ascending order positive roots of the function $J_n(x)$, n — is a natural number or zero.

Due to the well-known features of the Bessel functions, the system $\{\Psi_j(\tau)\}_{j=0}^{\infty}$ is a full orthonormal system of continuous functions with weight τ in the space $L_2([0, T])$.

Let's use the system of functions (3.16) in the theorem 14.

Let's analyze the multiple stochastic integral

$$\int\limits_0^T \int\limits_0^s dM_\tau^{(1)} dM_s^{(2)},$$

where

$$M_s^{(i)} = \int\limits_0^s \sqrt{\tau} d\mathbf{f}_\tau^{(i)};$$

$\mathbf{f}_\tau^{(i)}$ $(i = 1, 2)$ — are independent standart Wiener processes, $0 \le s \le T$; $M_s^{(i)}$ — is a martingale (see sect. 7.5), where $\rho(\tau) = \tau$. In addition, $M_s^{(i)}$ has a Gaussian distribution. It is obvious, that the conditions of theorem 14 when $k = 2$ are executed.

Repeating the proof of the theorem 14 when $k = 2$ for the system of functions (3.16), we get

$$\int\limits_0^T \int\limits_0^s dM_\tau^{(1)} dM_s^{(2)} = \sum_{j_1, j_2 = 0}^{\infty} C_{j_2 j_1} \zeta_{j_1}^{(1)} \zeta_{j_2}^{(2)},$$

where the multiple series converges in the mean-square sense and

$$\zeta_j^{(i)}=\int\limits_0^T \Psi_j(\tau)dM_\tau^{(i)}$$

— are standard Gaussian random variables; $j=0,1,2,\ldots$; $i=1,2$; $\mathsf{M}\{\zeta_{j_1}^{(1)}\zeta_{j_2}^{(2)}\}=0$;

$$C_{j_2j_1}=\int\limits_0^T s\Psi_{j_2}(s)\int\limits_0^s \tau\Psi_{j_1}(\tau)d\tau ds.$$

It is obvious, that we may get this result using another method: we can use theorem 1 for multiple stochastic Ito integral

$$\int\limits_0^T \sqrt{s}\int\limits_0^s \sqrt{\tau}d\mathbf{f}_\tau^{(1)}d\mathbf{f}_s^{(2)},$$

and as a system of functions $\{\phi_j(s)\}_{j=0}^{\infty}$ in the theorem 1 we may take

$$\phi_j(s)=\frac{\sqrt{2s}}{TJ_{n+1}(\mu_j)}J_n\Bigl(\frac{\mu_j}{T}s\Bigr);\ \ j=0,1,2,\ldots.$$

As a result, we would obtain

$$\int\limits_0^T \sqrt{s}\int\limits_0^s \sqrt{\tau}d\mathbf{f}_\tau^{(1)}d\mathbf{f}_s^{(2)}=\sum_{j_1,j_2=0}^{\infty}C_{j_2j_1}\zeta_{j_1}^{(1)}\zeta_{j_2}^{(2)},$$

where the multiple series converges in the mean-square sense and

$$\zeta_j^{(i)}=\int\limits_0^T \phi_j(\tau)d\mathbf{f}_\tau^{(i)}$$

— are standard Gaussian random variables; $j=0,1,2,\ldots$; $i=1,2$; $\mathsf{M}\{\zeta_{j_1}^{(1)}\zeta_{j_2}^{(2)}\}=0$;

$$C_{j_2j_1}=\int\limits_0^T \sqrt{s}\phi_{j_2}(s)\int\limits_0^s \sqrt{\tau}\phi_{j_1}(\tau)d\tau ds.$$

Easy calculation demonstrates, that

$$\tilde{\phi}_j(s)=\frac{\sqrt{2(s-t)}}{(T-t)J_{n+1}(\mu_j)}J_n\left(\frac{\mu_j}{T-t}(s-t)\right);\ j=0,1,2,\ldots$$

— is a full orthonormal system of functions in the space $L_2([t,T])$.

Then, using theorem 1 we get

$$\int\limits_t^T \sqrt{s-t}\int\limits_t^s \sqrt{\tau-t}d\mathbf{f}_\tau^{(1)}d\mathbf{f}_s^{(2)}=\sum_{j_1,j_2=0}^{\infty}\tilde{C}_{j_2j_1}\tilde{\zeta}_{j_1}^{(1)}\tilde{\zeta}_{j_2}^{(2)},$$

where the multiple series converges in the mean-square sense and

$$\tilde{\zeta}_j^{(i)}=\int\limits_t^T\tilde{\phi}_j(\tau)d\mathbf{f}_\tau^{(i)}$$

— are standard Gaussian random variables; $j=0,1,2,\ldots$; $i=1,2$; $\mathsf{M}\{\tilde{\zeta}_{j_1}^{(1)}\tilde{\zeta}_{j_2}^{(2)}\}=0$;

$$\tilde{C}_{j_2j_1}=\int\limits_t^T\sqrt{s-t}\tilde{\phi}_{j_2}(s)\int\limits_t^s\sqrt{\tau-t}\tilde{\phi}_{j_1}(\tau)d\tau ds.$$

Chapter 4

The exact calculation of mean-square errors of approximation of multiple stochastic Ito integrals

This chapter is based on results of chapter 1 (theorems 1 and 2) and adapt this results to practical needs (numerical integration of Ito stochastic differential equations). Using Parseval equality and relations for multiple sums we derive the exact (not estimate) expressions for mean-square errors of approximations of multiple Ito stochastic integrals. Exact formulas for such mean-square errors for stochastic integrals of multiplicity 1 – 4 are derived.

4.1 The case of any k and pairwise different $i_1,\ldots,i_k = 1,\ldots,m$

At first, let's build mean-square approximations of multiple stochastic Ito integrals $J[\psi^{(k)}]_{T,t}$ of type (1.1) for pairwise different $i_1,\ldots,i_k = 1,\ldots,m$ (in this case they coinside with the correspondent stochastic Stratonovich integrals) in the form of truncated multiple series, into which they expand in accordance with the approach, based on multiple Fourier series, converging in the mean (theorem 1).

Assume, that $J[\psi^{(k)}]_{T,t}^{q_1,\ldots,q_k}$ — is approximation of multiple stochastic Ito integral $J[\psi^{(k)}]_{T,t}$ for pairwise different $i_1,\ldots,i_k = 1,\ldots,m$, which looks as follows

$$J[\psi^{(k)}]_{T,t}^{q_1,\ldots,q_k} = \sum_{j_1=0}^{q_1}\ldots\sum_{j_k=0}^{q_k} C_{j_k\ldots j_1}\prod_{l=1}^{k}\zeta_{j_l}^{(i_l)}, \qquad (4.1)$$

where numbers $q_i < \infty$ satisfy the following condition on the mean-square accuracy of approximation:

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{q_1,\ldots,q_k}-J[\psi^{(k)}]_{T,t}\right)^2\right\}\le\varepsilon, \tag{4.2}$$

ε — is a fixed small positive number.

Theorem 1 provides a possibility to calculate accurately the mean-square error of approximation of multiple stochastic Ito integral of any fixed multiplicity k.

Let's examine the case of pairwise different $i_1,\ldots,i_k=1,\ldots,m$.

Lemma 10. *Assume, that* $i_1,\ldots,i_k=1,\ldots,m$ *and pairwise different. Then, the mean square error of approximation* (4.1) *of the multiple stochastic Ito integral* $J[\psi^{(k)}]_{T,t}$ *is detected using the formula*

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{q_1,\ldots,q_k}-J[\psi^{(k)}]_{T,t}\right)^2\right\}=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-$$

$$-\sum_{j_1=0}^{q_1}\ldots\sum_{j_k=0}^{q_k}C_{j_k\ldots j_1}^2; \tag{4.3}$$

convergence in (4.3) *takes place in the sense of limit when* $q_1,\ldots,q_k\to\infty$.

Proof. We have

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}\right)^2\right\}=\int\limits_t^T\psi_k^2(t_k)\ldots\int\limits_t^{t_2}\psi_1^2(t_1)dt_1\ldots dt_k=$$

$$=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k. \tag{4.4}$$

The Parseval equality in our case looks as follows

$$\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k=\lim_{q_1,\ldots,q_k\to\infty}\sum_{j_1=0}^{q_1}\ldots\sum_{j_k=0}^{q_k}C_{j_k\ldots j_1}^2.$$

Then

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{q_1,\ldots,q_k}\right)^2\right\}=\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}\right)^2\right\}-$$

$$-2\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}J[\psi^{(k)}]_{T,t}^{q_1,\ldots,q_k}\right\}+\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{q_1,\ldots,q_k}\right)^2\right\}. \tag{4.5}$$

Since according to theorem 1 we have

$$J[\psi^{(k)}]_{T,t}=\left\{\sum_{j_1,\ldots,j_k=0}^{\infty}-\sum_{j_1=0}^{q_1}\ldots\sum_{j_k=0}^{q_k}\right\}C_{j_k\ldots j_1}\prod_{l=1}^k\zeta_{j_l}^{(i_l)}+J[\psi^{(k)}]_{T,t}^{q_1,\ldots,q_k},$$

then

$$\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}J[\psi^{(k)}]_{T,t}^{q_1,\ldots,q_k}\right\}=\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{q_1,\ldots,q_k}\right)^2\right\}. \qquad (4.6)$$

Substituting (4.4) and (4.6) into (4.5) we get

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}-J[\psi^{(k)}]_{T,t}^{q_1,\ldots,q_k}\right)^2\right\}=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k-$$

$$-\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{q_1,\ldots,q_k}\right)^2\right\}. \qquad (4.7)$$

Considering (4.7) and the relation

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{q_1,\ldots,q_k}\right)^2\right\}=\sum_{j_1=0}^{q_1}\ldots\sum_{j_k=0}^{q_k}C_{j_k\ldots j_1}^2,$$

which is reasonable for pairwise different $i_1,\ldots,i_k=1,\ldots,m$, we come to affirmation of lemma 10. The lemma is proven. □

4.2 The case of non-pairwise different $i_1,\ldots,i_k=1,\ldots,m$

Proving lemma 10, we practically established the following formula for any $i_1,\ldots,i_k=1,\ldots,m$:

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{q_1,\ldots,q_k}-J[\psi^{(k)}]_{T,t}\right)^2\right\}=\int\limits_{[t,T]^k}K^2(t_1,\ldots,t_k)dt_1\ldots dt_k+$$

$$+2\left(\mathsf{M}\left\{J[\psi^{(k)}]_{T,t}^{q_1,\ldots,q_k}\right\}\right)^2-\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t}^{q_1,\ldots,q_k}\right)^2\right\}. \qquad (4.8)$$

Hereafter, calculating the mean-square error of approximation of multiple stochastic Ito integrals we have to calculate mathematical expectations from multiple sums of random variables with complex multiindex coefficients. Accordingly, formula (2.162) will be useful.

We will demonstrate systematically, by the example of multiple stochastic Ito integrals of 1st, 2nd, 3rd and 4th multiplicities, that there are no any technical problems for getting the analogue of (4.3) for any $i_1,\ldots,i_k=1,\ldots,m$ (for simplicity hereafter we assume, that $q_i=p$).

4.2.1 The case $k=1$

In this case according to lemma 10 we get

$$\mathsf{M}\left\{\left(J[\psi^{(1)}]_{T,t}^{q}-J[\psi^{(1)}]_{T,t}\right)^2\right\}=\int\limits_{[t,T]^1}K^2(t_1)dt_1-\sum_{j_1=0}^{q}C_{j_1}^2.$$

4.2.2 The case $k=2$ and any $i_1,i_2=1,\ldots,m$

When $i_1\ne i_2$ we get the required formula from lemma 10. Let $i_1=i_2=i=1,\ldots,m.$

We have

$$\mathsf{M}\left\{\left(J[\psi^{(2)}]_{T,t}\right)^2\right\}=\int\limits_t^T\psi_2^2(t_2)\int\limits_t^{t_2}\psi_1^2(t_1)dt_1dt_2=\int\limits_{[t,T]^2}K^2(t_1,t_2)dt_1dt_2.$$

Since

$$\mathsf{M}\left\{J[\psi^{(2)}]_{T,t}^p\right\}=0,$$

where

$$J[\psi^{(2)}]_{T,t}^p=\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}-\sum_{j_1=0}^{p}C_{j_1j_1},$$

then, according to (4.8) we get

$$\mathsf{M}\left\{\left(J[\psi^{(2)}]_{T,t}^p-J[\psi^{(2)}]_{T,t}\right)^2\right\}=\int\limits_{[t,T]^2}K^2(t_1,t_2)dt_1dt_2-\mathsf{M}\left\{\left(J[\psi^{(2)}]_{T,t}^p\right)^2\right\}. \tag{4.9}$$

Then, using (2.162) we get

$$\mathsf{M}\left\{\left(J[\psi^{(2)}]_{T,t}^p\right)^2\right\}=\mathsf{M}\Biggl\{\Biggl(\sum_{j_1,j_2=0}^{p}C_{j_2j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}-\sum_{j_1=0}^{p}C_{j_1j_1}\Biggr)^2\Biggr\}=$$

$$=\mathsf{M}\Biggl\{\sum_{j_1,j_1',j_2,j_2'=0}^{q}C_{j_2j_1}C_{j_2'j_1'}\zeta_{j_1}^{(i)}\zeta_{j_1'}^{(i)}\zeta_{j_2}^{(i)}\zeta_{j_2'}^{(i)}\Biggr\}-$$

$$-2\Biggl(\sum_{j_1=0}^{p}C_{j_1j_1}\Biggr)^2+\Biggl(\sum_{j_1=0}^{p}C_{j_1j_1}\Biggr)^2=3\sum_{j_1=0}^{p}C_{j_1j_1}^2+$$

$$+\sum_{j_2=0}^{p}\sum_{j_1=0}^{j_2-1}\left(C_{j_2j_1}^2+C_{j_1j_2}^2+2C_{j_1j_1}C_{j_2j_2}+2C_{j_2j_1}C_{j_1j_2}\right)-\Biggl(\sum_{j_1=0}^{p}C_{j_1j_1}\Biggr)^2. \tag{4.10}$$

Substituting (4.10) into (4.9) and considering, that

$$\sum_{j_1=0}^{p}C_{j_1j_1}^2+\sum_{j_2=0}^{p}\sum_{j_1=0}^{j_2-1}\left(C_{j_2j_1}^2+C_{j_1j_2}^2\right)=\sum_{j_1,j_2=0}^{p}C_{j_2j_1}^2,$$

$$\sum_{j_1=0}^{p}C_{j_1j_1}^2+2\sum_{j_2=0}^{p}\sum_{j_1=0}^{j_2-1}C_{j_1j_1}C_{j_2j_2}=\sum_{j_1,j_2=0}^{p}C_{j_1j_1}C_{j_2j_2}=\Biggl(\sum_{j_1=0}^{p}C_{j_1j_1}\Biggr)^2,$$

$$\sum_{j_1=0}^{p} C_{j_1j_1}^2 + 2\sum_{j_2=0}^{p}\sum_{j_1=0}^{j_2-1} C_{j_1j_2}C_{j_2j_1} = \sum_{j_1,j_2=0}^{p} C_{j_1j_2}C_{j_2j_1},$$

finally we get

$$\mathsf{M}\left\{\left(J[\psi^{(2)}]_{T,t}^p - J[\psi^{(2)}]_{T,t}\right)^2\right\} = \int\limits_{[t,T]^2} K^2(t_1,t_2)dt_1dt_2 - \sum_{j_1,j_2=0}^{p} C_{j_2j_1}^2 -$$

$$- \sum_{j_1,j_2=0}^{p} C_{j_1j_2}C_{j_2j_1}. \qquad (4.11)$$

4.2.3 The case $k = 3$ and any $i_1, i_2, i_3 = 1, \ldots, m$

The case of pairwise different i_1, i_2, i_3 is analyzed in lemma 10, that is why we have to analyze 4 cases (it is assumed, that $i_1, i_2, i_3 = 1, \ldots, m$): 1. $i_1 = i_2 \neq i_3$; 2. $i_1 \neq i_2 = i_3$; 3. $i_1 = i_3 \neq i_2$; 4. $i_1 = i_2 = i_3$.

Let's start from the first case:

$$\mathsf{M}\left\{\left(J[\psi^{(3)}]_{T,t}\right)^2\right\} = \int\limits_{[t,T]^3} K^2(t_1,t_2,t_3)dt_1dt_2dt_3;\ \ \mathsf{M}\left\{J[\psi^{(3)}]_{T,t}^p\right\} = 0,$$

where

$$J[\psi^{(3)}]_{T,t}^p = \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\right).$$

So, in our case we have

$$\mathsf{M}\left\{\left(J[\psi^{(3)}]_{T,t} - J[\psi^{(3)}]_{T,t}^p\right)^2\right\} = \int\limits_{[t,T]^3} K^2(t_1,t_2,t_3)dt_1dt_2dt_3 - \mathsf{M}\left\{\left(J[\psi^{(3)}]_{T,t}^p\right)^2\right\}. \qquad (4.12)$$

Further using of (2.162) we get

$$\mathsf{M}\left\{\left(J[\psi^{(3)}]_{T,t}^p\right)^2\right\} = \mathsf{M}\Biggl\{\sum_{j_3,j_3',j_2,j_2',j_1,j_1'=0}^{p} C_{j_3j_2j_1}C_{j_3'j_2'j_1'}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)} - \mathbf{1}_{\{j_1=j_2\}}\right)\times$$

$$\times\left(\zeta_{j_1'}^{(i_1)}\zeta_{j_2'}^{(i_1)} - \mathbf{1}_{\{j_1'=j_2'\}}\right)\zeta_{j_3}^{(i_3)}\zeta_{j_3'}^{(i_3)}\Biggr\} =$$

$$= \mathsf{M}\Biggl\{\sum_{j_3=0}^{p}\sum_{j_2,j_2',j_1,j_1'=0}^{p} C_{j_3j_2j_1}C_{j_3j_2'j_1'}\zeta_{j_1}^{(i_1)}\zeta_{j_1'}^{(i_1)}\zeta_{j_2}^{(i_1)}\zeta_{j_2'}^{(i_1)}\Biggr\} -$$

$$
-2\sum_{j_3=0}^{p}\Biggl(\sum_{j_1=0}^{p}C_{j_3j_1j_1}\Biggr)^2+\sum_{j_3=0}^{p}\Biggl(\sum_{j_1=0}^{p}C_{j_3j_1j_1}\Biggr)^2=
$$

$$
=3\sum_{j_3=0}^{p}\sum_{j_1=0}^{p}(C_{j_3j_1j_1})^2+\sum_{j_3=0}^{p}\sum_{j_2=0}^{p}\sum_{j_1=0}^{j_2-1}\Bigl(C^2_{j_3j_2j_1}+C^2_{j_3j_1j_2}+
$$

$$
+2C_{j_3j_1j_1}C_{j_3j_2j_2}+2C_{j_3j_2j_1}C_{j_3j_1j_2}\Bigr)-\sum_{j_3=0}^{p}\Biggl(\sum_{j_1=0}^{p}C_{j_3j_1j_1}\Biggr)^2. \qquad (4.13)
$$

Substituting (4.13) into (4.12) and considering, that

$$
\sum_{j_3,j_1=0}^{p}C^2_{j_3j_1j_1}+\sum_{j_3,j_2=0}^{p}\sum_{j_1=0}^{j_2-1}\Bigl(C^2_{j_3j_2j_1}+C^2_{j_3j_1j_2}\Bigr)=\sum_{j_3,j_2,j_1=0}^{p}C^2_{j_3j_2j_1},
$$

$$
\sum_{j_3,j_1=0}^{p}C^2_{j_3j_1j_1}+2\sum_{j_3,j_2=0}^{p}\sum_{j_1=0}^{j_2-1}C_{j_3j_1j_1}C_{j_3j_2j_2}=\sum_{j_3,j_2,j_1=0}^{p}C_{j_3j_1j_1}C_{j_3j_2j_2}=
$$

$$
=\sum_{j_3=0}^{p}\Biggl(\sum_{j_1=0}^{p}C_{j_3j_1j_1}\Biggr)^2,
$$

$$
\sum_{j_3,j_1=0}^{p}C^2_{j_3j_1j_1}+2\sum_{j_3,j_2=0}^{p}\sum_{j_1=0}^{j_2-1}C_{j_3j_1j_2}C_{j_3j_2j_1}=\sum_{j_3,j_2,j_1=0}^{p}C_{j_3j_1j_2}C_{j_3j_2j_1},
$$

finally we get

$$
\mathsf{M}\left\{\Bigl(J[\psi^{(3)}]^p_{T,t}-J[\psi^{(3)}]_{T,t}\Bigr)^2\right\}=\int\limits_{[t,T]^3}K^2(t_1,t_2,t_3)dt_1dt_2dt_3-\sum_{j_3,j_2,j_1=0}^{p}C^2_{j_3j_2j_1}-
$$

$$
-\sum_{j_3,j_2,j_1=0}^{p}C_{j_3j_1j_2}C_{j_3j_2j_1}\ \ (i_1=i_2\ne i_3).
$$

In the 2nd and 3rd case similarly to the previous reasoning we correspondently get

$$
\mathsf{M}\left\{\Bigl(J[\psi^{(3)}]^p_{T,t}-J[\psi^{(3)}]_{T,t}\Bigr)^2\right\}=\int\limits_{[t,T]^3}K^2(t_1,t_2,t_3)dt_1dt_2dt_3-\sum_{j_3,j_2,j_1=0}^{p}C^2_{j_3j_2j_1}-
$$

$$
-\sum_{j_3,j_2,j_1=0}^{p}C_{j_2j_3j_1}C_{j_3j_2j_1}\ \ (i_1\ne i_2=i_3).
$$

$$
\mathsf{M}\left\{\Bigl(J[\psi^{(3)}]^p_{T,t}-J[\psi^{(3)}]_{T,t}\Bigr)^2\right\}=\int\limits_{[t,T]^3}K^2(t_1,t_2,t_3)dt_1dt_2dt_3-\sum_{j_3,j_2,j_1=0}^{p}C^2_{j_3j_2j_1}-
$$

$$-\sum_{j_3,j_2,j_1=0}^{p} C_{j_3j_2j_1}C_{j_1j_2j_3}\ \ (i_1=i_3\ne i_2).$$

In the 4th particular case when $\psi_1(s),\psi_2(s),\psi_3(s)\equiv 1$ with probubility 1 we have (see sect.6.2):

$$I_{000_{T,t}}^{(i_1i_1i_1)}=\frac{1}{6}(T-t)^{\frac{3}{2}}\left(\left(\zeta_0^{(i_1)}\right)^3-3\zeta_0^{(i_1)}\right).$$

In more general case, when $\psi_1(s),\psi_2(s),\psi_3(s)\equiv (t-s)^l$; l — is a fixed natural number or zero, with probability 1 we may right down (see section 6.2):

$$I_{lll_{T,t}}^{(i_1i_1i_1)}=\frac{1}{6}\left(\left(I_{l_{T,t}}^{(i_1)}\right)^3-3I_{l_{T,t}}^{(i_1)}\Delta_{l_{T,t}}\right),$$

$$I_{lll_{T,t}}^{*(i_1i_1i_1)}=\frac{1}{6}\left(I_{l_{T,t}}^{(i_1)}\right)^3,\ \ I_{l_{T,t}}^{(i_1)}=\sum_{j=0}^{l}C_j\zeta_j^{(i_1)},$$

where $\Delta_{l_{T,t}}=\int\limits_t^T(t-s)^{2l}ds,\ C_j=\int\limits_t^T(t-s)^l\phi_j(s)ds;\ \{\phi_j(s)\}_{j=0}^{\infty}$ — is a full orthonormal system of Legendre polynomials at the interval $[t,T]$.

If the functions $\psi_1(s),\ldots,\psi_3(s)$ are different in the 4th particular case, then calculation of the value $\mathsf{M}\left\{\left(J[\psi^{(3)}]_{T,t}^p-J[\psi^{(3)}]_{T,t}\right)^2\right\}$ becomes more difficult then in all cases analyzed previously.

Nevertheless we will calculate the specified mean square error of approximation.

According to theorem 1 when $k=3$ and $i_1=i_2=i_3=1,\ldots,m$ we have:

$$J[\psi^{(3)}]_{T,t}=\sum_{j_1,j_2,j_3=0}^{\infty}C_{j_3j_2j_1}\Bigl(\zeta_{j_1}\zeta_{j_2}\zeta_{j_3}-\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}-\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}-\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}\Bigr)$$

and correspondently

$$J[\psi^{(3)}]_{T,t}^p=\sum_{j_1,j_2,j_3=0}^{p}C_{j_3j_2j_1}\Bigl(\zeta_{j_1}\zeta_{j_2}\zeta_{j_3}-\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}-\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}-\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}\Bigr),$$

where for the sake of simplicity we assumed, that $\zeta_j^{(i)}=\zeta_j$.

According to (4.8) it is enough to calculate only $\mathsf{M}\left\{\left(J[\psi^{(3)}]_{T,t}^p\right)^2\right\}$.

We have

$$\mathsf{M}\left\{\left(J[\psi^{(3)}]_{T,t}^p\right)^2\right\}=\mathsf{M}\Biggl\{\sum_{j_1,j_1',j_2,j_2',j_3,j_3'=0}^{p}C_{j_3j_2j_1}C_{j_3'j_2'j_1'}\zeta_{j_1}\zeta_{j_1'}\zeta_{j_2}\zeta_{j_2'}\zeta_{j_3}\zeta_{j_3'}\Biggr\}-$$

$$
-2\sum_{j_2'=0}^{p}\mathsf{M}\Biggl\{\sum_{j_1,j_2,j_3,j_3'=0}^{p} C_{j_3j_2j_1}C_{j_3'j_2'j_2'}\zeta_{j_1}\zeta_{j_2}\zeta_{j_3}\zeta_{j_3'}\Biggr\}-
$$

$$
-2\sum_{j_3'=0}^{p}\mathsf{M}\Biggl\{\sum_{j_1,j_2,j_3,j_1'=0}^{p} C_{j_3j_2j_1}C_{j_3'j_3'j_1'}\zeta_{j_1}\zeta_{j_2}\zeta_{j_3}\zeta_{j_1'}\Biggr\}-
$$

$$
-2\sum_{j_3'=0}^{p}\mathsf{M}\Biggl\{\sum_{j_1,j_2,j_3,j_2'=0}^{p} C_{j_3j_2j_1}C_{j_3'j_2'j_3'}\zeta_{j_1}\zeta_{j_2}\zeta_{j_3}\zeta_{j_2'}\Biggr\}+
$$

$$
+\sum_{j_1,j_2,j_3=0}^{p}\bigl(C_{j_3j_2j_2}C_{j_3j_1j_1}+C_{j_3j_3j_1}C_{j_2j_2j_1}+C_{j_3j_2j_3}C_{j_1j_2j_1}+
$$

$$
+2C_{j_3j_2j_2}C_{j_1j_1j_3}+2C_{j_3j_2j_2}C_{j_1j_3j_1}+2C_{j_3j_3j_1}C_{j_2j_1j_2}\bigr)\,.
$$

According to (2.162) when $k=4$ and $k=6$ we have:

$$
\mathsf{M}\Biggl\{\sum_{j_1,j_1',j_2,j_2',j_3,j_3'=0}^{p} C_{j_3j_2j_1}C_{j_3'j_2'j_1'}\zeta_{j_1}\zeta_{j_1'}\zeta_{j_2}\zeta_{j_2'}\zeta_{j_3}\zeta_{j_3'}\Biggr\}=
$$

$$
=15\sum_{j_3=0}^{p}C_{j_3j_3j_3}^2+\sum_{j_3=0}^{p}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1}\sum_{(j_1,j_1,j_2,j_2,j_3,j_3)} C_{(j_3j_2j_1}C_{j_3j_2j_1)}+
$$

$$
+3\sum_{j_3=0}^{p}\sum_{j_1=0}^{j_3-1}\sum_{(j_1,j_1,j_3,j_3,j_3,j_3)} C_{(j_3j_3j_1}C_{j_3j_3j_1)}+
$$

$$
+3\sum_{j_3=0}^{p}\sum_{j_1=0}^{j_3-1}\sum_{(j_1,j_1,j_1,j_1,j_3,j_3)} C_{(j_3j_1j_1}C_{j_3j_1j_1)};
$$

$$
\mathsf{M}\Biggl\{\sum_{j_1,j_2,j_3,j_3'=0}^{p} C_{j_3j_2j_1}C_{j_3'j_2'j_2'}\zeta_{j_1}\zeta_{j_2}\zeta_{j_3}\zeta_{j_3'}\Biggr\}=
$$

$$
=3\sum_{j_3=0}^{p}C_{j_3j_3j_3}C_{j_3j_2'j_2'}+\sum_{j_3=0}^{p}\sum_{j_1=0}^{j_3-1}\sum_{(j_1,j_1,j_3,j_3)} C_{(j_3j_1j_1}C_{j_3j_2'j_2')};
$$

$$
\mathsf{M}\Biggl\{\sum_{j_1,j_2,j_3,j_1'=0}^{p} C_{j_3j_2j_1}C_{j_3'j_3'j_1'}\zeta_{j_1}\zeta_{j_2}\zeta_{j_3}\zeta_{j_1'}\Biggr\}=
$$

$$
=3\sum_{j_3=0}^{p}C_{j_3j_3j_3}C_{j_3'j_3'j_3}+\sum_{j_3=0}^{p}\sum_{j_1=0}^{j_3-1}\sum_{(j_1,j_1,j_3,j_3)} C_{(j_3j_3j_1}C_{j_3'j_3'j_1)};
$$

$$
\mathsf{M}\Biggl\{\sum_{j_1,j_2,j_3,j_2'=0}^{p} C_{j_3j_2j_1}C_{j_3'j_2'j_3'}\zeta_{j_1}\zeta_{j_2}\zeta_{j_3}\zeta_{j_2'}\Biggr\}=
$$

$$
= 3\sum_{j_3=0}^{p} C_{j_3j_3j_3}C_{j_3'j_3j_3'} + \sum_{j_3=0}^{p}\sum_{j_1=0}^{j_3-1}\sum_{(j_1,j_1,j_3,j_3)} C_{(j_3j_3j_1}C_{j_3'j_1j_3')},
$$

where, as usual $\sum\limits_{(j_1,\ldots,j_k)}$ — is a sum according to all possible derangements $(j_1,\ldots,\ j_k)$, new symbol of type $C_{(j_1\ldots j_l}C_{j_{l+1}\ldots j_k)}$ means, that performing derangements the index of value $C_{(j_1\ldots j_l}C_{j_{l+1}\ldots j_k)}$ is $(j_1\ldots j_lj_{l+1}\ldots j_k) = (j_1\ldots,j_k)$, as, if we assumed $a_{(j_1\ldots j_k)} = C_{(j_1\ldots j_l}C_{j_{l+1}\ldots j_k)}$ in (2.162).

In principle, the mean-square error which is a subject of interest for us may be considered as found according to formula

$$
\mathsf{M}\left\{\left(J[\psi^{(3)}]^p_{T,t} - J[\psi^{(3)}]_{T,t}\right)^2\right\} = \int\limits_{[t,T]^3} K^2(t_1,t_2,t_3)dt_1dt_2dt_3 - \mathsf{M}\left\{\left(J[\psi^{(3)}]^p_{T,t}\right)^2\right\},
$$

since the value $\mathsf{M}\left\{\left(J[\psi^{(3)}]^p_{T,t}\right)^2\right\}$ is calculated. For the avoidance of doubts let's right down the line of sums including in the obtained formulas in the expanded form

$$
\sum_{(j_1,j_1,j_3,j_3)} C_{(j_3j_1j_1}C_{j_3j_2'j_2')} = C_{j_3j_1j_1}C_{j_3j_2'j_2'} + C_{j_3j_1j_3}C_{j_1j_2'j_2'} +
$$

$$
+C_{j_3j_3j_1}C_{j_1j_2'j_2'} + C_{j_1j_3j_3}C_{j_1j_2'j_2'} + C_{j_1j_3j_1}C_{j_3j_2'j_2'} + C_{j_1j_1j_3}C_{j_3j_2'j_2'};
$$

$$
\sum_{(j_1,j_1,j_3,j_3)} C_{(j_3j_1j_1}C_{j_3'j_3'j_1)} = C_{j_3j_3j_1}C_{j_3'j_3'j_1} + C_{j_3j_1j_3}C_{j_3'j_3'j_1} +
$$

$$
+C_{j_3j_1j_1}C_{j_3'j_3'j_3} + C_{j_1j_3j_3}C_{j_3'j_3'j_1} + C_{j_1j_3j_1}C_{j_3'j_3'j_3} + C_{j_1j_1j_3}C_{j_3'j_3'j_3};
$$

$$
\sum_{(j_1,j_1,j_3,j_3)} C_{(j_3j_3j_1}C_{j_3'j_1j_3')} = C_{j_3j_3j_1}C_{j_3'j_1j_3'} + C_{j_3j_1j_3}C_{j_3'j_1j_3'} +
$$

$$
+C_{j_3j_1j_1}C_{j_3'j_3j_3'} + C_{j_1j_3j_3}C_{j_3'j_1j_3'} + C_{j_1j_3j_1}C_{j_3'j_1j_3'} + C_{j_1j_1j_3}C_{j_3'j_3j_3'};
$$

$$
\sum_{(j_1,j_1,j_3,j_3,j_3,j_3)} C_{(j_3j_3j_1}C_{j_3j_3j_1)} = 2C_{j_1j_1j_3}C_{j_3j_3j_3} + 2C_{j_1j_3j_1}C_{j_3j_3j_3} +
$$

$$
+C^2_{j_1j_3j_3} + 2C_{j_1j_3j_3}C_{j_3j_1j_3} + 2C_{j_3j_1j_1}C_{j_3j_3j_3} + C^2_{j_3j_1j_3} +
$$

$$
+C^2_{j_3j_3j_1} + 2C_{j_1j_3j_3}C_{j_3j_3j_1} + 2C_{j_3j_1j_3}C_{j_3j_3j_1};
$$

$$
\sum_{(j_1,j_1,j_1,j_1,j_3,j_3)} C_{(j_3j_1j_1}C_{j_3j_1j_1)} = 2C_{j_3j_3j_1}C_{j_1j_1j_1} + 2C_{j_3j_1j_3}C_{j_1j_1j_1} +
$$

$$
+C^2_{j_3j_1j_1} + 2C_{j_3j_1j_1}C_{j_1j_3j_1} + 2C_{j_3j_1j_1}C_{j_1j_1j_3} + 2C_{j_1j_3j_3}C_{j_1j_1j_1} +
$$

$$+C_{j_1j_3j_1}^2+2C_{j_1j_3j_1}C_{j_1j_1j_3}+C_{j_1j_1j_3}^2;$$

$$\sum_{(j_1,j_1,j_2,j_2,j_3,j_3)}C_{(j_3j_2j_1}C_{j_3j_2j_1)}=2C_{j_1j_1j_2}C_{j_2j_3j_3}+2C_{j_1j_1j_2}C_{j_3j_2j_3}+$$

$$+2C_{j_1j_1j_2}C_{j_3j_3j_2}+2C_{j_1j_1j_3}C_{j_2j_2j_3}+2C_{j_1j_1j_3}C_{j_2j_3j_2}+2C_{j_1j_1j_3}C_{j_3j_2j_2}+$$

$$+2C_{j_1j_2j_1}C_{j_2j_3j_3}+2C_{j_1j_2j_1}C_{j_3j_2j_3}+2C_{j_1j_2j_1}C_{j_3j_3j_2}+2C_{j_1j_3j_1}C_{j_3j_2j_2}+$$

$$+2C_{j_2j_3j_2}C_{j_1j_3j_1}+2C_{j_1j_3j_1}C_{j_2j_2j_3}+2C_{j_1j_2j_2}C_{j_1j_3j_3}+2C_{j_1j_2j_3}C_{j_1j_3j_2}+$$

$$+2C_{j_1j_2j_2}C_{j_3j_1j_3}+2C_{j_1j_2j_3}C_{j_2j_1j_3}+2C_{j_1j_3j_2}C_{j_2j_1j_3}+2C_{j_1j_3j_3}C_{j_2j_1j_2}+$$

$$+2C_{j_1j_3j_2}C_{j_3j_1j_2}+2C_{j_1j_2j_3}C_{j_3j_1j_2}+2C_{j_1j_2j_2}C_{j_3j_3j_1}+2C_{j_1j_2j_3}C_{j_2j_3j_1}+$$

$$+2C_{j_1j_3j_2}C_{j_2j_3j_1}+2C_{j_1j_3j_3}C_{j_2j_2j_1}+2C_{j_1j_3j_2}C_{j_3j_2j_1}+2C_{j_1j_2j_3}C_{j_3j_2j_1}+$$

$$+2C_{j_2j_1j_1}C_{j_2j_3j_3}+2C_{j_2j_1j_1}C_{j_3j_2j_3}+2C_{j_2j_1j_1}C_{j_3j_3j_2}+2C_{j_3j_1j_1}C_{j_3j_2j_2}+$$

$$+2C_{j_3j_1j_1}C_{j_2j_3j_2}+2C_{j_3j_1j_1}C_{j_2j_2j_3}+2C_{j_2j_1j_2}C_{j_3j_1j_3}+2C_{j_2j_1j_3}C_{j_3j_1j_2}+$$

$$+2C_{j_2j_1j_2}C_{j_3j_3j_1}+2C_{j_2j_1j_3}C_{j_2j_3j_1}+2C_{j_2j_1j_3}C_{j_3j_2j_1}+2C_{j_3j_1j_2}C_{j_3j_2j_1}+$$

$$+2C_{j_3j_1j_3}C_{j_2j_2j_1}+2C_{j_3j_1j_2}C_{j_2j_3j_1}+2C_{j_3j_3j_1}C_{j_2j_2j_1}+2C_{j_2j_3j_1}C_{j_3j_2j_1}+$$

$$+C_{j_1j_3j_2}^2+C_{j_1j_2j_3}^2+C_{j_2j_3j_1}^2+C_{j_2j_1j_3}^2+C_{j_3j_2j_1}^2+C_{j_3j_1j_2}^2.$$

We will make one remark concerning calculation of the mean-square error of approximation for the multiple stochastic Stratonovich integral of 3rd multiplicity of the following form:

$$\int\limits_t^{*T}\int\limits_t^{*t_3}\int\limits_t^{*t_2}d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}d\mathbf{f}_{t_3}^{(i_3)}\stackrel{\rm def}{=}I_{000_{T,t}}^{*(i_1i_2i_3)};\ i_1,i_2,i_3=1,\ldots,m.$$

Since

$$I_{000_{T,t}}^{*(i_1i_2i_3)}=I_{000_{T,t}}^{(i_1i_2i_3)}+\frac{1}{2}\mathbf{1}_{\{i_1=i_2\}}\int\limits_t^T\int\limits_t^{\tau}dsd\mathbf{f}_{\tau}^{(i_3)}+\frac{1}{2}\mathbf{1}_{\{i_2=i_3\}}\int\limits_t^T\int\limits_t^{\tau}d\mathbf{f}_s^{(i_1)}d\tau=$$

$$=I_{000_{T,t}}^{(i_1i_2i_3)}+\frac{1}{4}\mathbf{1}_{\{i_1=i_2\}}(T-t)^{\frac{3}{2}}\left(\zeta_0^{(i_3)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_3)}\right)+$$

$$+\frac{1}{4}\mathbf{1}_{\{i_2=i_3\}}(T-t)^{\frac{3}{2}}\left(\zeta_0^{(i_1)}-\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right),\qquad (4.14)$$

where $\zeta_j^{(i)} = \int\limits_t^T \phi_j(s)d\mathbf{f}_s^{(i)}$; $\{\phi_j(x)\}_{j=0}^{\infty}$ — is a full orthonormal system of Legendre polynomials at the interval $[t,T]$, then

$$I_{000_{T,t}}^{*(i_1i_2i_3)p} = I_{000_{T,t}}^{(i_1i_2i_3)p} + \frac{1}{4}\mathbf{1}_{\{i_1=i_2\}}(T-t)^{\frac{3}{2}}\left(\zeta_0^{(i_3)} + \frac{1}{\sqrt{3}}\zeta_1^{(i_3)}\right) +$$

$$+\frac{1}{4}\mathbf{1}_{\{i_2=i_3\}}(T-t)^{\frac{3}{2}}\left(\zeta_0^{(i_1)} - \frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right), \tag{4.15}$$

where $I_{000_{T,t}}^{(i_1i_2i_3)p}$ — is the approximation of multiple stochastic Ito integral $I_{000_{T,t}}^{(i_1i_2i_3)}$, which has the following form:

$$I_{000_{T,t}}^{(i_1i_2i_3)p} = \sum_{j_1,j_2,j_3=0}^{p} C_{j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)} -$$

$$-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\Biggr).$$

From (4.14) and (4.15) we finally get:

$$\mathsf{M}\left\{\left(I_{000_{T,t}}^{*(i_3i_2i_1)} - I_{000_{T,t}}^{*(i_3i_2i_1)p}\right)^2\right\} = \mathsf{M}\left\{\left(I_{000_{T,t}}^{(i_3i_2i_1)} - I_{000_{T,t}}^{(i_3i_2i_1)p}\right)^2\right\}. \tag{4.16}$$

It is obvious, that formula (4.16) will also be correct for trigonometric system of functions.

4.2.4 The case $k=4$ and any $i_1,i_2,i_3,i_4=1,\ldots,m$

The case of pairwise different $i_1,\ldots,i_4$ is examined in lemma 10, so we just need to analyze the following particular cases: 1. $i_1=i_2\ne i_3,i_4$; $i_3\ne i_4$; 2. $i_1=i_3\ne i_2,i_4$; $i_2\ne i_4$; 3. $i_1=i_4\ne i_2,i_3$; $i_2\ne i_3$; 4. $i_2=i_3\ne i_1,i_4$; $i_1\ne i_4$; 5. $i_2=i_4\ne i_1,i_3$; $i_1\ne i_3$; 6. $i_3=i_4\ne i_1,i_2$; $i_1\ne i_2$; 7. $i_1=i_2=i_3\ne i_4$; 8. $i_2=i_3=i_4\ne i_1$; 9. $i_1=i_2=i_4\ne i_3$; 10. $i_1=i_3=i_4\ne i_2$; 11. $i_1=i_2=i_3=i_4$; 12. $i_1=i_2\ne i_3=i_4$; 13. $i_1=i_3\ne i_2=i_4$; 14. $i_1=i_4\ne i_2=i_3$.

Let's start from the 1st particular case. According to theorem 1 we have:

$$J[\psi^{(4)}]_{T,t} = \sum_{j_1,j_2,j_3,j_4=0}^{\infty} C_{j_4j_3j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right),$$

$$J[\psi^{(4)}]_{T,t}^{p} = \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\left(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_4)}\right).$$

Since $\mathsf{M}\left\{J[\psi^{(4)}]_{T,t}^p\right\}=0$, then according to (4.8) for calculating the mean-square error $\mathsf{M}\left\{\left(J[\psi^{(4)}]_{T,t}^p-J[\psi^{(4)}]_{T,t}\right)^2\right\}$ we need to calculate $\mathsf{M}\left\{\left(J[\psi^{(4)}]_{T,t}^p\right)^2\right\}$.

Using (2.162) when $k=4$ we have:

$$\mathsf{M}\left\{\left(J[\psi^{(4)}]_{T,t}^p\right)^2\right\}=$$

$$=\sum_{j_3,j_4=0}^{p}\mathsf{M}\Biggl\{\sum_{j_1,j_1',j_2,j_2'=0}^{q}C_{j_4j_3j_2j_1}C_{j_4j_3j_2'j_1'}\zeta_{(j_1)T,t}^{(i_1)}\zeta_{(j_1')T,t}^{(i_1)}\zeta_{(j_2)T,t}^{(i_1)}\zeta_{(j_2')T,t}^{(i_1)}\Biggr\}-$$

$$-\sum_{j_2,j_3,j_4,j_1'=0}^{q}C_{j_4j_3j_2j_2}C_{j_4j_3j_1'j_1'}-\sum_{j_2,j_3,j_4,j_2'=0}^{q}C_{j_4j_3j_2j_2}C_{j_4j_3j_2'j_2'}+$$

$$+\sum_{j_2,j_3,j_4,j_2'=0}^{q}C_{j_4j_3j_2j_2}C_{j_4j_3j_2'j_2'}=$$

$$=\sum_{j_3,j_4=0}^{p}\Biggl(\mathsf{M}\Biggl\{\sum_{j_1,j_1',j_2,j_2'=0}^{q}C_{j_4j_3j_2j_1}C_{j_4j_3j_2'j_1'}\zeta_{(j_1)T,t}^{(i_1)}\zeta_{(j_1')T,t}^{(i_1)}\zeta_{(j_2)T,t}^{(i_1)}\zeta_{(j_2')T,t}^{(i_1)}\Biggr\}-$$

$$-\Biggl(\sum_{j_2=0}^{p}C_{j_4j_3j_2j_2}\Biggr)^2\Biggr)=$$

$$=\sum_{j_3,j_4=0}^{p}\Biggl(3\sum_{j_1=0}^{p}C_{j_4j_3j_1j_1}^2+\sum_{j_2=0}^{p}\sum_{j_1=0}^{j_2-1}\Bigl(C_{j_4j_3j_2j_1}^2+C_{j_4j_3j_1j_2}^2+$$

$$+2C_{j_4j_3j_1j_1}C_{j_4j_3j_2j_2}+2C_{j_4j_3j_2j_1}C_{j_4j_3j_1j_2}\Bigr)-\Biggl(\sum_{j_2=0}^{p}C_{j_4j_3j_2j_2}\Biggr)^2\Biggr)=$$

$$=\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}^2+\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_1j_2}C_{j_4j_3j_2j_1}.$$

Therefore

$$\mathsf{M}\left\{\left(J[\psi^{(4)}]_{T,t}^p-J[\psi^{(4)}]_{T,t}\right)^2\right\}=\int\limits_{[t,T]^4}K^2(t_1,\ldots,t_4)dt_1\ldots dt_4-$$

$$-\sum_{j_4,j_3,j_2,j_1=0}^{p}C_{j_4j_3j_2j_1}^2-\sum_{j_4,j_3,j_2,j_1=0}^{p}C_{j_4j_3j_1j_2}C_{j_4j_3j_2j_1}\ \ (i_1=i_2\neq i_3,i_4;\ \ i_3\neq i_4).$$

For particular cases 2–6 we get in the complete analogy:

$$\mathsf{M}\left\{\left(J[\psi^{(4)}]_{T,t}^p-J[\psi^{(4)}]_{T,t}\right)^2\right\}=\int\limits_{[t,T]^4}K^2(t_1,\ldots,t_4)dt_1\ldots dt_4-$$

$$
-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}^2-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}C_{j_4j_1j_2j_3}\ \ (i_1=i_3\ne i_2,i_4;\ \ i_2\ne i_4);
$$

$$
\mathsf{M}\left\{\left(J[\psi^{(4)}]_{T,t}^p-J[\psi^{(4)}]_{T,t}\right)^2\right\}=\int\limits_{[t,T]^4} K^2(t_1,\ldots,t_4)dt_1\ldots dt_4-
$$

$$
-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}^2-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}C_{j_1j_3j_2j_4}\ \ (i_1=i_4\ne i_2,i_3;\ \ i_2\ne i_3);
$$

$$
\mathsf{M}\left\{\left(J[\psi^{(4)}]_{T,t}^p-J[\psi^{(4)}]_{T,t}\right)^2\right\}=\int\limits_{[t,T]^4} K^2(t_1,\ldots,t_4)dt_1\ldots dt_4-
$$

$$
-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}^2-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}C_{j_4j_2j_3j_1}\ \ (i_2=i_3\ne i_1,i_4;\ \ i_1\ne i_4);
$$

$$
\mathsf{M}\left\{\left(J[\psi^{(4)}]_{T,t}^p-J[\psi^{(4)}]_{T,t}\right)^2\right\}=\int\limits_{[t,T]^4} K^2(t_1,\ldots,t_4)dt_1\ldots dt_4-
$$

$$
-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}^2-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}C_{j_2j_3j_4j_1}\ \ (i_2=i_4\ne i_1,i_3;\ \ i_1\ne i_3);
$$

$$
\mathsf{M}\left\{\left(J[\psi^{(4)}]_{T,t}^p-J[\psi^{(4)}]_{T,t}\right)^2\right\}=\int\limits_{[t,T]^4} K^2(t_1,\ldots,t_4)dt_1\ldots dt_4-
$$

$$
-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}^2-\sum_{j_4,j_3,j_2,j_1=0}^{p} C_{j_4j_3j_2j_1}C_{j_3j_4j_2j_1}\ \ (i_3=i_4\ne i_1,i_2;\ \ i_1\ne i_2).
$$

For the 7th particular case $(i_1=i_2=i_3\ne i_4)$ we have:

$$
J[\psi^{(4)}]_{T,t}^p=\sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\Biggl(\zeta_{j_1}\zeta_{j_2}\zeta_{j_3}\tilde\zeta_{j_4}-\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}\tilde\zeta_{j_4}-
$$

$$
-\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}\tilde\zeta_{j_4}-\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}\tilde\zeta_{j_4}\Biggr),
$$

где $\zeta_j^{(i_1)}=\zeta_j^{(i_2)}=\zeta_j^{(i_3)}=\zeta_j$, $\zeta_j^{(i_4)}=\tilde\zeta_j$;

$$
\mathsf{M}\left\{\left(J[\psi^{(4)}]_{T,t}^p\right)^2\right\}=\sum_{j_4=0}^{p}\mathsf{M}\Biggl\{\sum_{j_1,j_1',j_2,j_2',j_3,j_3'=0}^{p} C_{j_4j_3j_2j_1}C_{j_4j_3'j_2'j_1'}\zeta_{j_1}\zeta_{j_1'}\zeta_{j_2}\zeta_{j_2'}\zeta_{j_3}\zeta_{j_3'}\Biggr\}-
$$

$$-2\sum_{j_2',j_4=0}^{p}\mathsf{M}\left\{\sum_{j_1,j_2,j_3,j_3'=0}^{p}C_{j_4j_3j_2j_1}C_{j_4j_3'j_2'j_2'}\zeta_{j_1}\zeta_{j_2}\zeta_{j_3}\zeta_{j_3'}\right\}-$$

$$-2\sum_{j_3',j_4=0}^{p}\mathsf{M}\left\{\sum_{j_1,j_2,j_3,j_2'=0}^{p}C_{j_4j_3j_2j_1}C_{j_4j_3'j_2'j_3'}\zeta_{j_1}\zeta_{j_2}\zeta_{j_3}\zeta_{j_2'}\right\}-$$

$$-2\sum_{j_3',j_4=0}^{p}\mathsf{M}\left\{\sum_{j_1,j_2,j_3,j_1'=0}^{p}C_{j_4j_3j_2j_1}C_{j_4j_3'j_3'j_1'}\zeta_{j_1}\zeta_{j_2}\zeta_{j_3}\zeta_{j_1'}\right\}+$$

$$+\sum_{j_1,j_2,j_3,j_4=0}^{p}\left(C_{j_4j_3j_2j_2}C_{j_4j_3j_1j_1}+C_{j_4j_3j_2j_3}C_{j_4j_1j_2j_1}+C_{j_4j_3j_3j_1}C_{j_4j_2j_2j_1}+\right.$$
$$\left.+2C_{j_4j_3j_2j_2}C_{j_4j_1j_3j_1}+2C_{j_4j_3j_2j_2}C_{j_4j_1j_1j_3}+2C_{j_4j_3j_2j_3}C_{j_4j_1j_1j_2}\right);$$

Then, according to (2.162) when $k=6$ and $k=4$ we have:

$$\mathsf{M}\left\{\sum_{j_1,j_1',j_2,j_2',j_3,j_3'=0}^{p}C_{j_4j_3j_2j_1}C_{j_4j_3'j_2'j_1'}\zeta_{j_1}\zeta_{j_1'}\zeta_{j_2}\zeta_{j_2'}\zeta_{j_3}\zeta_{j_3'}\right\}=$$

$$=15\sum_{j_3=0}^{p}C_{j_4j_3j_3j_3}^2+\sum_{j_3=0}^{p}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1}\sum_{(j_1,j_1,j_2,j_2,j_3,j_3)}C_{(j_4j_3j_2j_1}C_{j_4j_3j_2j_1)}+$$

$$+3\sum_{j_3=0}^{p}\sum_{j_1=0}^{j_3-1}\sum_{(j_1,j_1,j_3,j_3,j_3,j_3)}C_{(j_4j_3j_3j_1}C_{j_4j_3j_3j_1)}+$$

$$+3\sum_{j_3=0}^{p}\sum_{j_1=0}^{j_3-1}\sum_{(j_1,j_1,j_1,j_1,j_3,j_3)}C_{(j_4j_3j_1j_1}C_{j_4j_3j_1j_1)};$$

$$\mathsf{M}\left\{\sum_{j_1,j_2,j_3,j_3'=0}^{p}C_{j_4j_3j_2j_1}C_{j_4j_3'j_2'j_2'}\zeta_{j_1}\zeta_{j_2}\zeta_{j_3}\zeta_{j_3'}\right\}=$$

$$=3\sum_{j_3=0}^{p}C_{j_4j_3j_3j_3}C_{j_4j_3j_2'j_2'}+\sum_{j_3=0}^{p}\sum_{j_1=0}^{j_3-1}\sum_{(j_1,j_1,j_3,j_3)}C_{(j_4j_3j_1j_1}C_{j_4j_3j_2'j_2')};$$

$$\mathsf{M}\left\{\sum_{j_1,j_2,j_3,j_2'=0}^{p}C_{j_4j_3j_2j_1}C_{j_4j_3'j_2'j_3'}\zeta_{j_1}\zeta_{j_2}\zeta_{j_3}\zeta_{j_2'}\right\}=$$

$$=3\sum_{j_3=0}^{p}C_{j_4j_3j_3j_3}C_{j_4j_3'j_3j_3'}+\sum_{j_3=0}^{p}\sum_{j_1=0}^{j_3-1}\sum_{(j_1,j_1,j_3,j_3)}C_{(j_4j_3j_3j_1}C_{j_4j_3'j_1j_3')};$$

$$\mathsf{M}\left\{\sum_{j_1,j_2,j_3,j_1'=0}^{p}C_{j_4j_3j_2j_1}C_{j_4j_3'j_3'j_1'}\zeta_{j_1}\zeta_{j_2}\zeta_{j_3}\zeta_{j_1'}\right\}=$$

$$= 3\sum_{j_3=0}^{p} C_{j_4j_3j_3j_3}C_{j_4j_3'j_3'j_3} + \sum_{j_3=0}^{p}\sum_{j_1=0}^{j_3-1}\sum_{(j_1,j_1,j_3,j_3)} C_{(j_4j_3j_3j_1}C_{j_4j_3'j_3'j_1)}.$$

Combining obtained equalities we calculate the mean square error of approximation, which is of interest to us, using the formula:

$$\mathsf{M}\left\{\left(J[\psi^{(4)}]_{T,t}^{p} - J[\psi^{(4)}]_{T,t}\right)^2\right\} = \int\limits_{[t,T]^4} K^2(t_1,\ldots,t_4)dt_1\ldots dt_4 -$$

$$-\mathsf{M}\left\{\left(J[\psi^{(4)}]_{T,t}^{p}\right)^2\right\}. \tag{4.17}$$

For the 8th particular case $(i_2 = i_3 = i_4 \neq i_1)$ we have:

$$J[\psi^{(4)}]_{T,t}^{p} = \sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\Biggl(\zeta_{j_2}\zeta_{j_3}\zeta_{j_4}\tilde{\zeta}_{j_1} - \mathbf{1}_{\{j_2=j_3\}}\zeta_{j_4}\tilde{\zeta}_{j_1} -$$

$$-\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_3}\tilde{\zeta}_{j_1} - \mathbf{1}_{\{j_3=j_4\}}\zeta_{j_2}\tilde{\zeta}_{j_1}\Biggr),$$

where $\zeta_j^{(i_2)} = \zeta_j^{(i_3)} = \zeta_j^{(i_4)} = \zeta_j$, $\zeta_j^{(i_1)} = \tilde{\zeta}_j$;

$$\mathsf{M}\left\{\left(J[\psi^{(4)}]_{T,t}^{p}\right)^2\right\} = \sum_{j_1=0}^{p}\mathsf{M}\Biggl\{\sum_{j_2,j_2',j_3,j_3',j_4,j_4'=0}^{p} C_{j_4j_3j_2j_1}C_{j_4'j_3'j_2'j_1}\zeta_{j_2}\zeta_{j_2'}\zeta_{j_3}\zeta_{j_3'}\zeta_{j_4}\zeta_{j_4'}\Biggr\} -$$

$$-2\sum_{j_3',j_1=0}^{p}\mathsf{M}\Biggl\{\sum_{j_2,j_3,j_4,j_4'=0}^{p} C_{j_4j_3j_2j_1}C_{j_4'j_3'j_3'j_1}\zeta_{j_2}\zeta_{j_3}\zeta_{j_4}\zeta_{j_4'}\Biggr\} -$$

$$-2\sum_{j_1,j_4'=0}^{p}\mathsf{M}\Biggl\{\sum_{j_2,j_3,j_4,j_3'=0}^{p} C_{j_4j_3j_2j_1}C_{j_4'j_3'j_4'j_1}\zeta_{j_2}\zeta_{j_3}\zeta_{j_4}\zeta_{j_3'}\Biggr\} -$$

$$-2\sum_{j_4',j_1=0}^{p}\mathsf{M}\Biggl\{\sum_{j_2,j_3,j_4,j_2'=0}^{p} C_{j_4j_3j_2j_1}C_{j_4'j_4'j_2'j_1}\zeta_{j_2}\zeta_{j_3}\zeta_{j_4}\zeta_{j_2'}\Biggr\} +$$

$$+\sum_{j_1,j_2,j_3,j_4=0}^{p}\bigl(C_{j_4j_2j_2j_1}C_{j_4j_3j_3j_1} + C_{j_4j_3j_4j_1}C_{j_2j_3j_2j_1} + C_{j_4j_4j_2j_1}C_{j_3j_3j_2j_1} +$$

$$+2C_{j_4j_3j_4j_1}C_{j_3j_2j_2j_1} + 2C_{j_4j_4j_2j_1}C_{j_2j_3j_3j_1} + 2C_{j_4j_4j_2j_1}C_{j_3j_2j_3j_1}\bigr).$$

Then, according to (2.162) when $k = 6$ and $k = 4$ we have:

$$\mathsf{M}\Biggl\{\sum_{j_2,j_2',j_3,j_3',j_4,j_4'=0}^{p} C_{j_4j_3j_2j_1}C_{j_4'j_3'j_2'j_1}\zeta_{j_2}\zeta_{j_2'}\zeta_{j_3}\zeta_{j_3'}\zeta_{j_4}\zeta_{j_4'}\Biggr\} =$$

$$=15\sum_{j_4=0}^{p}C_{j_4j_4j_4j_1}^2+\sum_{j_4=0}^{p}\sum_{j_3=0}^{j_4-1}\sum_{j_2=0}^{j_3-1}\sum_{(j_2,j_2,j_3,j_3,j_4,j_4)}C_{(j_4j_3j_2j_1}C_{j_4j_3j_2j_1)}+$$

$$+3\sum_{j_4=0}^{p}\sum_{j_2=0}^{j_4-1}\sum_{(j_2,j_2,j_4,j_4,j_4,j_4)}C_{(j_4j_4j_2j_1}C_{j_4j_4j_2j_1)}+$$

$$+3\sum_{j_4=0}^{p}\sum_{j_2=0}^{j_4-1}\sum_{(j_2,j_2,j_2,j_2,j_4,j_4)}C_{(j_4j_2j_2j_1}C_{j_4j_2j_2j_1)};$$

$$\mathsf{M}\left\{\sum_{j_2,j_3,j_4,j_4'=0}^{p}C_{j_4j_3j_2j_1}C_{j_4'j_3'j_3'j_1}\zeta_{j_2}\zeta_{j_3}\zeta_{j_4}\zeta_{j_4'}\right\}=$$

$$=3\sum_{j_4=0}^{p}C_{j_4j_4j_4j_1}C_{j_4j_3'j_3'j_1}+\sum_{j_4=0}^{p}\sum_{j_2=0}^{j_4-1}\sum_{(j_2,j_2,j_4,j_4)}C_{(j_4j_2j_2j_1}C_{j_4j_3'j_3'j_1)};$$

$$\mathsf{M}\left\{\sum_{j_2,j_3,j_3',j_4=0}^{p}C_{j_4j_3j_2j_1}C_{j_4'j_3'j_4'j_1}\zeta_{j_2}\zeta_{j_3}\zeta_{j_4}\zeta_{j_3'}\right\}=$$

$$=3\sum_{j_4=0}^{p}C_{j_4j_4j_4j_1}C_{j_4'j_4j_4'j_1}+\sum_{j_4=0}^{p}\sum_{j_2=0}^{j_4-1}\sum_{(j_2,j_2,j_4,j_4)}C_{(j_4j_2j_2j_1}C_{j_4'j_4j_4'j_1)};$$

$$\mathsf{M}\left\{\sum_{j_2,j_2',j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}C_{j_4'j_4'j_2'j_1}\zeta_{j_2}\zeta_{j_3}\zeta_{j_4}\zeta_{j_2'}\right\}=$$

$$=3\sum_{j_4=0}^{p}C_{j_4j_4j_4j_1}C_{j_4'j_4'j_4j_1}+\sum_{j_4=0}^{p}\sum_{j_2=0}^{j_4-1}\sum_{(j_2,j_2,j_4,j_4)}C_{(j_4j_4j_2j_1}C_{j_4'j_4'j_2j_1)}.$$

Combining the obtained equalities and using (4.17) we will get the mean square error of approximation which is of interest to us.

Let's analyze the 9th particular case ($i_1=i_2=i_4\ne i_3$):

$$J[\psi^{(4)}]_{T,t}^p=\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\Biggl(\zeta_{j_1}\zeta_{j_2}\zeta_{j_4}\tilde\zeta_{j_3}-\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_4}\tilde\zeta_{j_3}-$$

$$-\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}\tilde\zeta_{j_3}-\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}\tilde\zeta_{j_3}\Biggr),$$

where $\zeta_j^{(i_1)}=\zeta_j^{(i_2)}=\zeta_j^{(i_4)}=\zeta_j$, $\zeta_j^{(i_3)}=\tilde\zeta_j$;

$$\mathsf{M}\left\{\left(J[\psi^{(4)}]_{T,t}^p\right)^2\right\}=\sum_{j_3=0}^{p}\mathsf{M}\left\{\sum_{j_1,j_1',j_2,j_2',j_4,j_4'=0}^{p}C_{j_4j_3j_2j_1}C_{j_4'j_3j_2'j_1'}\zeta_{j_1}\zeta_{j_2}\zeta_{j_4}\zeta_{j_1'}\zeta_{j_2'}\zeta_{j_4'}\right\}-$$

$$-2\sum_{j_3,j_2'=0}^{p}\mathsf{M}\left\{\sum_{j_4,j_2,j_1,j_4'=0}^{p}C_{j_4j_3j_2j_1}C_{j_4'j_3j_2'j_2'}\zeta_{j_1}\zeta_{j_2}\zeta_{j_4}\zeta_{j_4'}\right\}-$$

$$-2\sum_{j_3,j_4'=0}^{p}\mathsf{M}\left\{\sum_{j_1,j_2,j_4,j_2'=0}^{p}C_{j_4j_3j_2j_1}C_{j_4'j_3j_2'j_4'}\zeta_{j_1}\zeta_{j_2}\zeta_{j_4}\zeta_{j_2'}\right\}-$$

$$-2\sum_{j_4',j_3=0}^{p}\mathsf{M}\left\{\sum_{j_1,j_2,j_4,j_1'=0}^{p}C_{j_4j_3j_2j_1}C_{j_4'j_3j_4'j_1'}\zeta_{j_1}\zeta_{j_2}\zeta_{j_4}\zeta_{j_1'}\right\}+$$

$$+\sum_{j_1,j_2,j_3,j_4=0}^{p}\left(C_{j_4j_3j_2j_2}C_{j_4j_3j_1j_1}+C_{j_4j_3j_2j_4}C_{j_1j_3j_2j_1}+C_{j_4j_3j_4j_1}C_{j_2j_3j_2j_1}+\right.$$

$$\left.+2C_{j_4j_3j_2j_4}C_{j_2j_3j_1j_1}+2C_{j_4j_3j_4j_1}C_{j_1j_3j_2j_2}+2C_{j_4j_3j_4j_1}C_{j_2j_3j_1j_2}\right);$$

$$\mathsf{M}\left\{\sum_{j_1,j_1',j_2,j_2',j_4,j_4'=0}^{p}C_{j_4j_3j_2j_1}C_{j_4'j_3j_2'j_1'}\zeta_{j_1}\zeta_{j_2}\zeta_{j_4}\zeta_{j_1'}\zeta_{j_2'}\zeta_{j_4'}\right\}=$$

$$=15\sum_{j_4=0}^{p}C^2_{j_4j_3j_4j_4}+\sum_{j_4=0}^{p}\sum_{j_2=0}^{j_4-1}\sum_{j_1=0}^{j_2-1}\sum_{(j_1,j_1,j_2,j_2,j_4,j_4)}C_{(j_4j_3j_2j_1}C_{j_4j_3j_2j_1)}+$$

$$+3\sum_{j_4=0}^{p}\sum_{j_1=0}^{j_4-1}\sum_{(j_1,j_1,j_4,j_4,j_4,j_4)}C_{(j_4j_3j_4j_1}C_{j_4j_3j_4j_1)}+$$

$$+3\sum_{j_4=0}^{p}\sum_{j_1=0}^{j_4-1}\sum_{(j_1,j_1,j_1,j_1,j_4,j_4)}C_{(j_4j_3j_1j_1}C_{j_4j_3j_1j_1)};$$

$$\mathsf{M}\left\{\sum_{j_1,j_2,j_4,j_4'=0}^{p}C_{j_4j_3j_2j_1}C_{j_4'j_3j_2'j_2'}\zeta_{j_1}\zeta_{j_2}\zeta_{j_4}\zeta_{j_4'}\right\}=$$

$$=3\sum_{j_4=0}^{p}C_{j_4j_3j_4j_4}C_{j_4j_3j_2'j_2'}+\sum_{j_4=0}^{p}\sum_{j_1=0}^{j_4-1}\sum_{(j_1,j_1,j_4,j_4)}C_{(j_4j_3j_1j_1}C_{j_4j_3j_2'j_2')};$$

$$\mathsf{M}\left\{\sum_{j_1,j_2,j_4,j_2'=0}^{p}C_{j_4j_3j_2j_1}C_{j_4'j_3j_2'j_4'}\zeta_{j_1}\zeta_{j_2}\zeta_{j_4}\zeta_{j_2'}\right\}=$$

$$=3\sum_{j_4=0}^{p}C_{j_4j_3j_4j_4}C_{j_4'j_3j_4j_4'}+\sum_{j_4=0}^{p}\sum_{j_1=0}^{j_4-1}\sum_{(j_1,j_1,j_4,j_4)}C_{(j_4j_3j_1j_1}C_{j_4'j_3j_4j_4')};$$

$$\mathsf{M}\left\{\sum_{j_1,j_2,j_4,j_1'=0}^{p}C_{j_4j_3j_2j_1}C_{j_4'j_3j_4'j_1'}\zeta_{j_1}\zeta_{j_2}\zeta_{j_4}\zeta_{j_1'}\right\}=$$

$$
=3\sum_{j_4=0}^{p}C_{j_4j_3j_4j_4}C_{j_4'j_3j_4'j_4}+\sum_{j_4=0}^{p}\sum_{j_1=0}^{j_4-1}\sum_{(j_1,j_1,j_4,j_4)}C_{(j_4j_3j_1j_1}C_{j_4'j_3j_4'j_4)}.
$$

Similarly in the 10th particular case ($i_1=i_3=i_4\ne i_2$) we have:

$$
J[\psi^{(4)}]_{T,t}^{p}=\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\Biggl(\zeta_{j_1}\zeta_{j_3}\zeta_{j_4}\tilde{\zeta}_{j_2}-\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_4}\tilde{\zeta}_{j_2}-
$$

$$
-\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_3}\tilde{\zeta}_{j_2}-\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}\tilde{\zeta}_{j_2}\Biggr),
$$

where $\zeta_j^{(i_1)}=\zeta_j^{(i_3)}=\zeta_j^{(i_4)}=\zeta_j$, $\zeta_j^{(i_2)}=\tilde{\zeta}_j$;

$$
\mathsf{M}\left\{\left(J[\psi^{(4)}]_{T,t}^{p}\right)^2\right\}=\sum_{j_2=0}^{p}\mathsf{M}\Biggl\{\sum_{j_1,j_1',j_3,j_3',j_4,j_4'=0}^{p}C_{j_4j_3j_2j_1}C_{j_4'j_3'j_2j_1'}\zeta_{j_1}\zeta_{j_1'}\zeta_{j_3}\zeta_{j_3'}\zeta_{j_4}\zeta_{j_4'}\Biggr\}-
$$

$$
-2\sum_{j_3',j_2=0}^{p}\mathsf{M}\Biggl\{\sum_{j_1,j_3,j_4,j_4'=0}^{p}C_{j_4j_3j_2j_1}C_{j_4'j_3'j_2j_3'}\zeta_{j_1}\zeta_{j_3}\zeta_{j_4}\zeta_{j_4'}\Biggr\}-
$$

$$
-2\sum_{j_2,j_4'=0}^{p}\mathsf{M}\Biggl\{\sum_{j_1,j_3,j_4,j_3'=0}^{p}C_{j_4j_3j_2j_1}C_{j_4'j_3'j_2j_4'}\zeta_{j_1}\zeta_{j_3}\zeta_{j_4}\zeta_{j_3'}\Biggr\}-
$$

$$
-2\sum_{j_4',j_2=0}^{p}\mathsf{M}\Biggl\{\sum_{j_1,j_3,j_4,j_1'=0}^{p}C_{j_4j_3j_2j_1}C_{j_4'j_4'j_2j_1'}\zeta_{j_1}\zeta_{j_3}\zeta_{j_4}\zeta_{j_1'}\Biggr\}+
$$

$$
+\sum_{j_1,j_2,j_3,j_4=0}^{p}(C_{j_4j_3j_2j_3}C_{j_4j_1j_2j_1}+C_{j_4j_3j_2j_4}C_{j_1j_3j_2j_1}+C_{j_4j_4j_2j_1}C_{j_3j_3j_2j_1}+
$$

$$
+2C_{j_4j_3j_2j_4}C_{j_3j_1j_2j_1}+2C_{j_4j_4j_2j_1}C_{j_1j_3j_2j_3}+2C_{j_4j_4j_2j_1}C_{j_3j_1j_2j_3});
$$

$$
\mathsf{M}\Biggl\{\sum_{j_1,j_1',j_3,j_3',j_4,j_4'=0}^{p}C_{j_4j_3j_2j_1}C_{j_4'j_3'j_2j_1'}\zeta_{j_1}\zeta_{j_1'}\zeta_{j_3}\zeta_{j_3'}\zeta_{j_4}\zeta_{j_4'}\Biggr\}=
$$

$$
=15\sum_{j_4=0}^{p}C_{j_4j_4j_2j_4}^2+\sum_{j_4=0}^{p}\sum_{j_3=0}^{j_4-1}\sum_{j_1=0}^{j_3-1}\sum_{(j_1,j_1,j_3,j_3,j_4,j_4)}C_{(j_4j_3j_2j_1}C_{j_4j_3j_2j_1)}+
$$

$$
+3\sum_{j_4=0}^{p}\sum_{j_1=0}^{j_4-1}\sum_{(j_1,j_1,j_4,j_4,j_4,j_4)}C_{(j_4j_4j_2j_1}C_{j_4j_4j_2j_1)}+
$$

$$
+3\sum_{j_4=0}^{p}\sum_{j_1=0}^{j_4-1}\sum_{(j_1,j_1,j_1,j_1,j_4,j_4)}C_{(j_4j_1j_2j_1}C_{j_4j_1j_2j_1)};
$$

$$\mathsf{M}\left\{\sum_{j_1,j_3,j_4,j_4'=0}^{p} C_{j_4j_3j_2j_1}C_{j_4'j_3'j_2j_3'}\zeta_{j_1}\zeta_{j_3}\zeta_{j_4}\zeta_{j_4'}\right\}=$$

$$=3\sum_{j_4=0}^{p} C_{j_4j_4j_2j_4}C_{j_4j_3'j_2j_3'}+\sum_{j_4=0}^{p}\sum_{j_1=0}^{j_4-1}\sum_{(j_1,j_1,j_4,j_4)} C_{(j_4j_1j_2j_1}C_{j_4j_3'j_2j_3')};$$

$$\mathsf{M}\left\{\sum_{j_1,j_3,j_4,j_3'=0}^{p} C_{j_4j_3j_2j_1}C_{j_4'j_3'j_2j_4'}\zeta_{j_1}\zeta_{j_3}\zeta_{j_4}\zeta_{j_3'}\right\}=$$

$$=3\sum_{j_4=0}^{p} C_{j_4j_4j_2j_4}C_{j_4'j_4j_2j_4'}+\sum_{j_4=0}^{p}\sum_{j_1=0}^{j_4-1}\sum_{(j_1,j_1,j_4,j_4)} C_{(j_4j_4j_2j_1}C_{j_4'j_1j_2j_4')};$$

$$\mathsf{M}\left\{\sum_{j_1,j_3,j_4,j_1'=0}^{p} C_{j_4j_3j_2j_1}C_{j_4'j_4'j_2j_1'}\zeta_{j_1}\zeta_{j_3}\zeta_{j_4}\zeta_{j_1'}\right\}=$$

$$=3\sum_{j_4=0}^{p} C_{j_4j_4j_2j_4}C_{j_4'j_4'j_2j_4}+\sum_{j_4=0}^{p}\sum_{j_1=0}^{j_4-1}\sum_{(j_1,j_1,j_4,j_4)} C_{(j_4j_4j_2j_1}C_{j_4'j_4'j_2j_1)}.$$

Let's go to the 11th–14th particular cases. Using theorem 1, we will take the expression for approximation in this cases. For the 11th particular case $(i_1=i_2=i_3=i_4)$:

$$J[\psi^{(4)}]_{T,t}^{p}=\sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\Biggl(\prod_{l=1}^{4}\zeta_{j_l}-\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}\zeta_{j_4}-\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}\zeta_{j_4}-$$

$$-\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}\zeta_{j_3}-\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}\zeta_{j_4}-\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}\zeta_{j_3}-\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}\zeta_{j_2}+$$

$$+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{j_3=j_4\}}+\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{j_2=j_4\}}+\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr), \qquad (4.18)$$

where $\zeta_j^{(i_1)}=\zeta_j^{(i_2)}=\zeta_j^{(i_3)}=\zeta_j^{(i_4)}=\zeta_j$.

For the 12th particular case $(i_1=i_2\neq i_3=i_4)$:

$$J[\psi^{(4)}]_{T,t}^{p}=\sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_3)}-\mathbf{1}_{\{j_1=j_2\}}\zeta_{j_3}^{(i_3)}\zeta_{j_4}^{(i_3)}-$$

$$-\mathbf{1}_{\{j_3=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_1)}+\mathbf{1}_{\{j_1=j_2\}}\mathbf{1}_{\{j_3=j_4\}}\Biggr).$$

For the 13th particular case $(i_1=i_3\neq i_2=i_4)$:

$$J[\psi^{(4)}]_{T,t}^{p}=\sum_{j_1,j_2,j_3,j_4=0}^{p} C_{j_4j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_2)}-\mathbf{1}_{\{j_1=j_3\}}\zeta_{j_2}^{(i_2)}\zeta_{j_4}^{(i_2)}-$$

$$
-\mathbf{1}_{\{j_2=j_4\}}\zeta_{j_1}^{(i_1)}\zeta_{j_3}^{(i_1)}+\mathbf{1}_{\{j_1=j_3\}}\mathbf{1}_{\{j_2=j_4\}}\Biggr).
$$

For the 14th particular case ($i_1=i_4\ne i_2=i_3$):

$$
J[\psi^{(4)}]_{T,t}^p=\sum_{j_1,j_2,j_3,j_4=0}^{p}C_{j_4j_3j_2j_1}\Biggl(\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_2)}-\mathbf{1}_{\{j_1=j_4\}}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_2)}-
$$

$$
-\mathbf{1}_{\{j_2=j_3\}}\zeta_{j_1}^{(i_1)}\zeta_{j_4}^{(i_1)}+\mathbf{1}_{\{j_1=j_4\}}\mathbf{1}_{\{j_2=j_3\}}\Biggr).
$$

Due to some bulkiness we will not examine these particular cases and will make several remarks (it will not be principally difficult to analyze particular cases 11–14 in comparison with the previous).

It is easy to see, that in the 11th–14th particular cases the following expression is executed:

$$
\mathsf{M}\left\{J[\psi^{(4)}]_{T,t}^p\right\}=0.
$$

Within the frame of 11th particular case when $\psi_1(s),\ldots,\psi_4(s)\equiv 1$, as we mentioned before, with probability 1 the following formula is correct (see sect.6.2):

$$
J[\psi^{(4)}]_{T,t}=I_{0000_{T,t}}^{(i_1i_1i_1i_1)}=\frac{(T-t)^2}{24}\left(\left(\zeta_0^{(i_1)}\right)^4-6\left(\zeta_0^{(i_1)}\right)^2+3\right).
$$

In more general case, when $\psi_1(s),\ldots,\psi_4(s)\equiv(t-s)^l$; l — is a fixed natural number or zero, with probubility 1 we may write down (see. sect.6.2):

$$
J[\psi^{(4)}]_{T,t}=I_{llll_{T,t}}^{(i_1i_1i_1i_1)}=\frac{1}{24}\left(\left(I_{l_{T,t}}^{(i_1)}\right)^4-6\left(I_{l_{T,t}}^{(i_1)}\right)^2\Delta_{l_{T,t}}+3\left(\Delta_{l_{T,t}}\right)^2\right),
$$

$$
J^{*}[\psi^{(4)}]_{T,t}=I_{llll_{T,t}}^{*(i_1i_1i_1i_1)}=\frac{1}{24}\left(I_{l_{T,t}}^{(i_1)}\right)^4,
$$

$$
I_{l_{T,t}}^{(i)}=\sum_{j=0}^{l}C_j\zeta_j^{(i_1)},\ \ \Delta_{l_{T,t}}=\int\limits_t^T(t-s)^{2l}ds,
$$

where in the next-to-last formula we propose, that the expansion of stochastic integral is performed using Legendre polynomials.

It is obvious, that the main difficulty which will be met in the 11th–14th particular cases when calculating $\mathsf{M}\left\{\left(J[\psi^{(4)}]_{T,t}^p-J[\psi^{(4)}]_{T,t}\right)^2\right\}$ or, according to formula (4.17), when calculating $\mathsf{M}\left\{\left(J[\psi^{(4)}]_{T,t}^p\right)^2\right\}$, it will be connected with calculation of the following mathematical expectation (11th particular case):

$$
\mathsf{M}\Biggl\{\sum_{j_1,j_1',j_2,j_2',j_3,j_3',j_4,j_4'=0}^{p}C_{j_4j_3j_2j_1}C_{j_4'j_3'j_2'j_1'}\zeta_{j_1}\zeta_{j_1'}\zeta_{j_2}\zeta_{j_2'}\zeta_{j_3}\zeta_{j_3'}\zeta_{j_4}\zeta_{j_4'}\Biggr\}.
$$

According to (2.162) when $k=8$ we have:

$$\mathsf{M}\left\{\sum_{j_1,j_1',j_2,j_2',j_3,j_3',j_4,j_4'=0}^{p} C_{j_4j_3j_2j_1}C_{j_4'j_3'j_2'j_1'}\zeta_{j_1}\zeta_{j_1'}\zeta_{j_2}\zeta_{j_2'}\zeta_{j_3}\zeta_{j_3'}\zeta_{j_4}\zeta_{j_4'}\right\}=$$

$$=105\sum_{j_4=0}^{p}C^2_{j_4j_4j_4j_4}+15\sum_{j_4=0}^{p}\sum_{j_1=0}^{j_4-1}\sum_{(j_1,j_1,j_4,j_4,j_4,j_4,j_4,j_4)}C_{(j_4j_4j_4j_1}C_{j_4j_4j_4j_1)}+$$

$$+15\sum_{j_4=0}^{p}\sum_{j_1=0}^{j_4-1}\sum_{(j_1,j_1,j_1,j_1,j_1,j_1,j_4,j_4)}C_{(j_4j_1j_1j_1}C_{j_4j_1j_1j_1)}+$$

$$+9\sum_{j_4=0}^{p}\sum_{j_1=0}^{j_4-1}\sum_{(j_1,j_1,j_1,j_1,j_4,j_4,j_4,j_4)}C_{(j_4j_4j_1j_1}C_{j_4j_4j_1j_1)}+$$

$$+3\sum_{j_4=0}^{p}\sum_{j_2=0}^{j_4-1}\sum_{j_1=0}^{j_2-1}\sum_{(j_1,j_1,j_2,j_2,j_4,j_4,j_4,j_4)}C_{(j_4j_4j_2j_1}C_{j_4j_4j_2j_1)}+$$

$$+3\sum_{j_4=0}^{p}\sum_{j_2=0}^{j_4-1}\sum_{j_1=0}^{j_2-1}\sum_{(j_1,j_1,j_2,j_2,j_2,j_2,j_4,j_4)}C_{(j_4j_2j_2j_1}C_{j_4j_2j_2j_1)}+$$

$$+3\sum_{j_4=0}^{p}\sum_{j_3=0}^{j_4-1}\sum_{j_1=0}^{j_3-1}\sum_{(j_1,j_1,j_1,j_1,j_3,j_3,j_4,j_4)}C_{(j_4j_3j_1j_1}C_{j_4j_3j_1j_1)}+$$

$$+\sum_{j_4=0}^{p}\sum_{j_3=0}^{j_4-1}\sum_{j_2=0}^{j_3-1}\sum_{j_1=0}^{j_2-1}\sum_{(j_1,j_1,j_2,j_2,j_3,j_3,j_4,j_4)}C_{(j_4j_3j_2j_1}C_{j_4j_3j_2j_1)}.$$

According the scheme proposed above we may, increasing metodically the multiplicity k of the multiple stochastic Ito integral and separating various particular cases which correspond to various combinations of indexes $i_1,\ldots,i_k=1,\ldots,m$, calculate accurately the mean-square errors of approximations of the multiple stochastic integrals, obtained in accordance with theorem 1.

4.3 Some peculiarities of calculation of mean square error of approximation for the systems of polynomial and trigonometric functions

Using the example we will demonstrate, that for the case of trigonometric system of functions the approximation on the basis of formula (4.1) may be developed in such manner, that the error $\mathsf{M}\left\{\left(J[\psi^{(2)}]^q_{T,t}-J[\psi^{(2)}]_{T,t}\right)^2\right\}$ $(i_1\ne i_2)$ will turn out to be significantly less, than the right part of (4.3).

Assume, that the following trigonometric system of functions is taken as the system of functions $\{\phi_j(s)\}_{j=0}^{\infty}$

$$\phi_j(s) = \frac{1}{\sqrt{T-t}} \begin{cases} 1 & \text{when } j = 0 \\ \sqrt{2}\sin\frac{2\pi r(s-t)}{T-t} & \text{when } j = 2r-1 \\ \sqrt{2}\cos\frac{2\pi r(s-t)}{T-t} & \text{when } j = 2r \end{cases}, \tag{4.19}$$

where $r = 1,\ 2, \ldots$

Using the theorem 1 for the system of functions (4.19) to the multiple stochastic Ito integral of type

$$I_{00_{T,t}}^{(i_2 i_1)} = \int\limits_t^T \int\limits_t^s d\mathbf{f}_\tau^{(i_2)} d\mathbf{f}_s^{(i_1)},\ i_1, i_2 = 1, \ldots, m;\ i_1 \neq i_2,$$

we get

$$I_{00_{T,t}}^{(i_2 i_1)} = \frac{1}{2}(T-t)\Biggl[\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \frac{1}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\Bigl\{\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)} - \zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)} +$$

$$+\sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)} - \zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\Bigr\}\Biggr], \tag{4.20}$$

where $\zeta_j^{(i)} \stackrel{\text{def}}{=} \int\limits_t^T \phi_j(s) d\mathbf{f}_s^{(i)}$; $\mathbf{f}_s^{(i)}$ $(i = 1, \ldots, m)$ — are independent standard Wiener processes. At that, the series (4.20) converges in the mean-square sense.

According to (4.1) it is necessary to write down

$$I_{00_{T,t}}^{(i_2 i_1)q} = \frac{1}{2}(T-t)\Biggl[\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \frac{1}{\pi}\sum_{r=1}^{q}\frac{1}{r}\Bigl\{\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)} - \zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)} +$$

$$+\sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)} - \zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\Bigr\}\Biggr]. \tag{4.21}$$

From (4.20) and (4.21) when $i_1 \neq i_2$ we have:

$$\mathsf{M}\left\{\left(I_{00_{T,t}}^{(i_2 i_1)} - I_{00_{T,t}}^{(i_2 i_1)q}\right)^2\right\} = \frac{3(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6} - \sum_{r=1}^{q}\frac{1}{r^2}\right). \tag{4.22}$$

It is easy to see, that the right part of (4.22) may be decreased threefold, if instead of the approximation of type (4.21) we take the following approximation [23]:

$$I_{00_{T,t}}^{(i_2 i_1)q} = \frac{1}{2}(T-t)\Biggl[\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \frac{1}{\pi}\sum_{r=1}^{q}\frac{1}{r}\Bigl\{\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)} - \zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)} +$$

$$+\sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\Biggr\}+\frac{\sqrt{2}}{\pi}\sqrt{\alpha_q}\left(\xi_q^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\xi_q^{(i_2)}\right)\Biggr], \qquad (4.23)$$

where

$$\xi_q^{(i)}=\frac{1}{\sqrt{\alpha_q}}\sum_{r=q+1}^{\infty}\frac{1}{r}\,\zeta_{2r-1}^{(i)}\sim \mathrm{N}(0,1);\ \ \alpha_q=\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}.$$

At that, the Gaussian random variables $\zeta_0^{(i)}$, $\zeta_{2r}^{(i)}$, $\zeta_{2r-1}^{(i)}$, $\xi_q^{(i)}$; $r=1,\ldots,q$; $i=1,\ldots,m$ are independent in total.

From (4.20) and (4.23) when $i_1\ne i_2$ we get

$$\mathsf{M}\left\{\left(I_{00_{T,t}}^{(i_2i_1)}-I_{00_{T,t}}^{(i_2i_1)q}\right)^2\right\}=\frac{(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}\right), \qquad (4.24)$$

i.e the right part of equation (4.24) is tree-times less then the right part of equation (4.22).

The given method of advancing approximations of multiple stochastic integrals [23] is generalized for the case of integrals of third multiplicity [24]. Apparently, analyzing stochastic integrals of higher multiplicity, than the third one, we cannot propose the universal method for introducing additional random variables as it was made in (4.23). As a result, in each case we have to act individually.

You may omit it selecting the full orthonormal system of Legendre polynomials at the space $L_2([t,T])$ as a system of functions $\{\phi_j(s)\}_{j=0}^{\infty}$.

Let's remind, that in chapter 2 using the system of Legendre polynomials for $i_1\ne i_2$ we got the expantion:

$$I_{00_{T,t}}^{(i_1i_2)}=\frac{T-t}{2}\Biggl[\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left\{\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right\}\Biggr],$$

which doesn't require perfection, as in the case of trigonometric system of functions.

It is easy to see, that

$$\mathsf{M}\left\{\left(I_{00_{T,t}}^{(i_1i_2)}-I_{00_{T,t}}^{(i_1i_2)q}\right)^2\right\}=\frac{(T-t)^2}{2}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right). \qquad (4.25)$$

From (4.24) and (4.25) we get:

$$\mathsf{M}\left\{\left(I_{00_{T,t}}^{(i_2i_1)}-I_{00_{T,t}}^{(i_2i_1)q}\right)^2\right\}=\frac{(T-t)^2}{2\pi^2}\sum_{r=q+1}^{\infty}\frac{1}{r^2}\le$$

$$\le\frac{(T-t)^2}{2\pi^2}\int\limits_{q}^{\infty}\frac{dx}{x^2}=\frac{(T-t)^2}{2\pi^2 q}\le C_1\frac{(T-t)^2}{q} \qquad (4.26)$$

and

$$\mathsf{M}\left\{\left(I_{00_{T,t}}^{(i_1 i_2)} - I_{00_{T,t}}^{(i_1 i_2)q}\right)^2\right\} = \frac{(T-t)^2}{2}\sum_{i=q+1}^{\infty}\frac{1}{4i^2-1} \le$$

$$\le \frac{(T-t)^2}{2}\int\limits_q^{\infty}\frac{1}{4x^2-1}dx = -\frac{(T-t)^2}{8}\ln\left|1-\frac{2}{2q+1}\right| \le C_2\frac{(T-t)^2}{q} \qquad (4.27)$$

correspondently, where $C_1,\ C_2$ — are constant.

Since the value $T-t$ plays the role of integration step in the numerical procedures for stochastic differential Ito equations, then this value is sufficiently small.

Keeping in mind this circumstance, it is easy to note, that there is such constant C_3, that

$$\mathsf{M}\left\{\left(I_{l_1\ldots l_{k}T,t}^{(i_1\ldots i_k)} - I_{l_1\ldots l_{k}T,t}^{(i_1\ldots i_k)q}\right)^2\right\} \le C_3\mathsf{M}\left\{\left(I_{00_{T,t}}^{(i_1 i_2)} - I_{00_{T,t}}^{(i_1 i_2)q}\right)^2\right\}, \qquad (4.28)$$

where $I_{l_1\ldots l_{k}T,t}^{(i_1\ldots i_k)q}$ — is the approximation of multiple stochastic integral $I_{l_1\ldots l_{k}T,t}^{(i_1\ldots i_k)}$ from the class (5.1), which has the form (4.1) for $q_1=\ldots=q_k=q$ and $i_1,\ldots,i_k=1,\ldots,m;\ k\ge 2$.

From (4.26), (4.27) and (4.28) we finally get:

$$\mathsf{M}\left\{\left(I_{l_1\ldots l_{k}T,t}^{(i_1\ldots i_k)} - I_{l_1\ldots l_{k}T,t}^{(i_1\ldots i_k)q}\right)^2\right\} \le C\frac{(T-t)^2}{q}, \qquad (4.29)$$

where C — is a constant.

Note, that the estimation (4.29) is general enough, and at the same time it is rather rough. Significant part of this chapter has been devoted to obtaining the exact expressions for the left part of (4.29) when $k=1,\ldots,4$. These exact expressions provide a possibility to minimize the length of sequence of standard Gaussian random variables, required for combined approximation of multiple stochastic integrals.

Chapter 5

Approximation of specific multiple stochastic Stratonovich and Ito integrals

In this chapter we give huge practical material about expansions and approximations of specific multiple Ito and Stratonovich stochastic integrals using theorem 1 and systems of Legendre polynomials and system of trigonometric functions. Considered multiple Ito and Stratonovich integrals are included into stochastic Taylor expansions (Taylor-Ito and Taylor-Stratonovich expansions). Therefore, results of this chapter may be very useful for numerical solution of stochastic differential Ito equations. Expansions of multiple stochastic Ito and Stratonovich integrals of multiplicity 1 – 5 using of Legendre polynomials and expansions of multiple stochastic Ito and Stratonovich integrals of multiplicity 1 – 3 using of trigonometric functions are derived.

5.1 Approximation of specific multiple stochastic integrals of multiplicities 1–5 using Legendre polynomials

In this chapter we provide considerable practical material (based on theorems 1 – 7) about expansions of multiple stochastic Ito and Stratonovich integrals of the following form:

$$I_{l_1\ldots l_k T,t}^{(i_1\ldots i_k)}=\int\limits_t^T(t-t_k)^{l_k}\ldots\int\limits_t^{t_2}(t-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)}, \qquad (5.1)$$

$$I_{l_1\ldots l_k T,t}^{*(i_1\ldots i_k)}=\int\limits_t^{*T}(t-t_k)^{l_k}\ldots\int\limits_t^{*t_2}(t-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)},$$

where $i_1,\ldots,i_k=1,\ldots,m;\ l_1,\ldots,l_k=0,\ 1,\ldots.$

The full orthonormal system of Legendre polynomials in the space $L_2([t,T])$ looks as follows

$$\{\phi_j(x)\}_{j=0}^{\infty},\ \phi_j(x)=\sqrt{\frac{2j+1}{T-t}}P_j\left(\left(x-\frac{T+t}{2}\right)\frac{2}{T-t}\right), \tag{5.2}$$

where $P_j(x)$ — is a Legendre polynomial. It is well-known [28], that the polynomials $P_j(x)$ may be represented for example in the form

$$P_j(x)=\frac{1}{2^j j!}\frac{d^j}{dx^j}\left(x^2-1\right)^j.$$

This representation, as we know, is called an equality of Rodrigues Note some well-known features of polynomials $P_j(x)$:

$$P_j(1)=1;\ P_{j+1}(-1)=-P_j(-1);\ j=0,\ 1,\ 2,\ \ldots,$$

$$\frac{dP_{j+1}(x)}{dx}-\frac{dP_{j-1}(x)}{dx}=(2j+1)P_j(x);\ j=1,\ 2,\ldots,$$

$$\int\limits_{-1}^{1}x^kP_j(x)dx=0;\ k=0,\ 1,\ 2,\ldots,j-1,$$

$$\int\limits_{-1}^{1}P_k(x)P_j(x)dx=\begin{cases}0 & \text{if } k\neq j\\ \frac{2}{2j+1} & \text{if } k=j\end{cases},$$

$$xP_j(x)=\frac{(j+1)P_{j+1}(x)+jP_{j-1}(x)}{2j+1};\ j=1,\ 2,\ldots,$$

$$P_n(x)P_m(x)=\sum_{k=0}^{m}K_{m,n,k}P_{n+m-2k}(x),$$

where

$$K_{m,n,k}=\frac{a_{m-k}a_ka_{n-k}}{a_{m+n-k}}\cdot\frac{2n+2m-4k+1}{2n+2m-2k+1};\ a_k=\frac{(2k-1)!!}{k!};\ m\le n.$$

Considering these features and using the system of functions (5.2) we get the following expansions of multiple stochastic Ito and Stratonovich integrals:

$$I_{0_{T,t}}^{(i_1)}=\sqrt{T-t}\zeta_0^{(i_1)}, \tag{5.3}$$

$$I_{1_{T,t}}^{(i_1)}=-\frac{(T-t)^{3/2}}{2}\left(\zeta_0^{(i_1)}+\frac{1}{\sqrt{3}}\zeta_1^{(i_1)}\right), \tag{5.4}$$

$$I_{2_{T,t}}^{(i_1)} = \frac{(T-t)^{5/2}}{3}\left(\zeta_0^{(i_1)} + \frac{\sqrt{3}}{2}\zeta_1^{(i_1)} + \frac{1}{2\sqrt{5}}\zeta_2^{(i_1)}\right), \tag{5.5}$$

$$I_{00_{T,t}}^{*(i_1 i_2)} = \frac{T-t}{2}\left[\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \sum_{i=1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left\{\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)} - \zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right\}\right], \tag{5.6}$$

$$I_{01_{T,t}}^{*(i_1 i_2)} = -\frac{T-t}{2}I_{00_{T,t}}^{*(i_1 i_2)} - \frac{(T-t)^2}{4}\Biggl[\frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_2)}+$$

$$+\sum_{i=0}^{\infty}\Biggl(\frac{(i+2)\zeta_i^{(i_1)}\zeta_{i+2}^{(i_2)} - (i+1)\zeta_{i+2}^{(i_1)}\zeta_i^{(i_2)}}{\sqrt{(2i+1)(2i+5)}(2i+3)} - \frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\Biggr)\Biggr], \tag{5.7}$$

$$I_{10_{T,t}}^{*(i_1 i_2)} = -\frac{T-t}{2}I_{00_{T,t}}^{*(i_1 i_2)} - \frac{(T-t)^2}{4}\Biggl[\frac{1}{\sqrt{3}}\zeta_0^{(i_2)}\zeta_1^{(i_1)}+$$

$$+\sum_{i=0}^{\infty}\Biggl(\frac{(i+1)\zeta_{i+2}^{(i_2)}\zeta_i^{(i_1)} - (i+2)\zeta_i^{(i_2)}\zeta_{i+2}^{(i_1)}}{\sqrt{(2i+1)(2i+5)}(2i+3)} + \frac{\zeta_i^{(i_1)}\zeta_i^{(i_2)}}{(2i-1)(2i+3)}\Biggr)\Biggr], \tag{5.8}$$

$$I_{10_{T,t}}^{(i_1 i_2)} = I_{10_{T,t}}^{*(i_1 i_2)} + \frac{1}{4}\mathbf{1}_{\{i_1=i_2\}}(T-t)^2;\ I_{01_{T,t}}^{(i_1 i_2)} = I_{01_{T,t}}^{*(i_1 i_2)} + \frac{1}{4}\mathbf{1}_{\{i_1=i_2\}}(T-t)^2, \tag{5.9}$$

$$I_{000_{T,t}}^{*(i_1 i_2 i_3)} = -\frac{1}{T-t}\left(I_{0_{T,t}}^{(i_3)}I_{10_{T,t}}^{*(i_2 i_1)} + I_{0_{T,t}}^{(i_1)}I_{10_{T,t}}^{*(i_2 i_3)}\right) +$$

$$+\frac{1}{2}I_{0_{T,t}}^{(i_3)}\left(I_{00_{T,t}}^{*(i_1 i_2)} - I_{00_{T,t}}^{*(i_2 i_1)}\right) -$$

$$-(T-t)^{\frac{3}{2}}\left[\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_3)}\left(\zeta_0^{(i_2)} + \sqrt{3}\zeta_1^{(i_2)} - \frac{1}{\sqrt{5}}\zeta_2^{(i_2)}\right) +\right.$$

$$\left.+\frac{1}{4}D_{T,t}^{(i_1 i_2 i_3)}\right] \tag{5.10}$$

or in general form:

$$I_{000_{T,t}}^{*(i_1 i_2 i_3)} = \sum_{j_1,j_2,j_3=0}^{\infty} C_{j_3 j_2 j_1}\zeta_{j_1}^{(i_1)}\zeta_{j_2}^{(i_2)}\zeta_{j_3}^{(i_3)};$$

$$C_{j_3 j_2 j_1} = \int\limits_t^T \phi_{j_3}(s)\int\limits_t^s \phi_{j_2}(s_1)\int\limits_t^{s_1}\phi_{j_1}(s_2)ds_2 ds_1 ds,$$

$$I_{000_{T,t}}^{(i_1i_2i_3)}=I_{000_{T,t}}^{*(i_1i_2i_3)}+\mathbf{1}_{\{i_1=i_2\}}\frac{1}{2}I_{1_{T,t}}^{(i_3)}-\mathbf{1}_{\{i_2=i_3\}}\frac{1}{2}\left((T-t)I_{0_{T,t}}^{(i_1)}+I_{1_{T,t}}^{(i_1)}\right),$$

$$I_{02_{T,t}}^{*(i_1i_2)}=-\frac{(T-t)^2}{4}I_{00_{T,t}}^{*(i_1i_2)}-(T-t)I_{01_{T,t}}^{*(i_1i_2)}+\frac{(T-t)^3}{8}\Biggl[\frac{2}{3\sqrt{5}}\zeta_2^{(i_2)}\zeta_0^{(i_1)}+$$

$$+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=0}^{\infty}\Biggl(\frac{(i+2)(i+3)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-(i+1)(i+2)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+$$

$$+\frac{(i^2+i-3)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-(i^2+3i-1)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\Biggr)\Biggr], \qquad (5.11)$$

$$I_{20_{T,t}}^{*(i_1i_2)}=-\frac{(T-t)^2}{4}I_{00_{T,t}}^{*(i_1i_2)}-(T-t)I_{10_{T,t}}^{*(i_1i_2)}+\frac{(T-t)^3}{8}\Biggl[\frac{2}{3\sqrt{5}}\zeta_0^{(i_2)}\zeta_2^{(i_1)}+$$

$$+\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=0}^{\infty}\Biggl(\frac{(i+1)(i+2)\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-(i+2)(i+3)\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+$$

$$+\frac{(i^2+3i-1)\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-(i^2+i-3)\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\Biggr)\Biggr], \qquad (5.12)$$

$$I_{11_{T,t}}^{*(i_1i_2)}=-\frac{(T-t)^2}{4}I_{00_{T,t}}^{*(i_1i_2)}-\frac{(T-t)}{2}\left(I_{10_{T,t}}^{*(i_1i_2)}+I_{01_{T,t}}^{*(i_1i_2)}\right)+\frac{(T-t)^3}{8}\Biggl[\frac{1}{3}\zeta_1^{(i_1)}\zeta_1^{(i_2)}+$$

$$+\sum_{i=0}^{\infty}\Biggl(\frac{(i+1)(i+3)\left(\zeta_{i+3}^{(i_2)}\zeta_i^{(i_1)}-\zeta_i^{(i_2)}\zeta_{i+3}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+7)}(2i+3)(2i+5)}+$$

$$+\frac{(i+1)^2\left(\zeta_{i+1}^{(i_2)}\zeta_i^{(i_1)}-\zeta_i^{(i_2)}\zeta_{i+1}^{(i_1)}\right)}{\sqrt{(2i+1)(2i+3)}(2i-1)(2i+5)}\Biggr)\Biggr], \qquad (5.13)$$

$$I_{02_{T,t}}^{(i_1i_2)}=I_{02_{T,t}}^{*(i_1i_2)}-\frac{1}{6}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3;\ I_{20_{T,t}}^{(i_1i_2)}=I_{20_{T,t}}^{*(i_1i_2)}-\frac{1}{6}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3, \qquad (5.14)$$

$$I_{11_{T,t}}^{(i_1i_2)}=I_{11_{T,t}}^{*(i_1i_2)}-\frac{1}{6}\mathbf{1}_{\{i_1=i_2\}}(T-t)^3, \qquad (5.15)$$

$$I_{3_{T,t}}^{(i_1)}=-\frac{(T-t)^{7/2}}{4}\left(\zeta_0^{(i_1)}+\frac{3\sqrt{3}}{5}\zeta_1^{(i_1)}+\frac{1}{\sqrt{5}}\zeta_2^{(i_1)}+\frac{1}{5\sqrt{7}}\zeta_3^{(i_1)}\right), \qquad (5.16)$$

$$D_{T,t}^{(i_1 i_2 i_3)}=\sum_{\substack{i=1,j=0,k=i\\ 2i\ge k+i-j\ge -2;\ k+i-j\ -\ \text{even}}}^{\infty} N_{ijk}K_{i+1,k+1,\frac{k+i-j}{2}+1}\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}+$$

$$+\sum_{\substack{i=1,j=0\\ 2k\ge k+i-j\ge -2;\ k+i-j\ -\ \text{even}}}^{\infty}\sum_{k=1}^{i-1} N_{ijk}K_{k+1,i+1,\frac{k+i-j}{2}+1}\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}-$$

$$-\sum_{\substack{i=1,j=0,k=i+2\\ 2i+2\ge k+i-j\ge 0;\ k+i-j\ -\ \text{even}}}^{\infty} N_{ijk}K_{i+1,k-1,\frac{k+i-j}{2}}\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}-$$

$$-\sum_{\substack{i=1,j=0\\ 2k-2\ge k+i-j\ge 0;\ k+i-j\ -\ \text{even}}}^{\infty}\sum_{k=1}^{i+1} N_{ijk}K_{k-1,i+1,\frac{k+i-j}{2}}\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}-$$

$$-\sum_{\substack{i=1,j=0,k=i-2,k\ge 1\\ 2i-2\ge k+i-j\ge 0;\ k+i-j\ -\ \text{even}}}^{\infty} N_{ijk}K_{i-1,k+1,\frac{k+i-j}{2}}\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}-$$

$$-\sum_{\substack{i=1,j=0\\ 2k+2\ge k+i-j\ge 0;\ k+i-j\ -\ \text{even}}}^{\infty}\sum_{k=1}^{i-3} N_{ijk}K_{k+1,i-1,\frac{k+i-j}{2}}\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}+$$

$$+\sum_{\substack{i=1,j=0,k=i\\ 2i\ge k+i-j\ge 2;\ k+i-j\ -\ \text{even}}}^{\infty} N_{ijk}K_{i-1,k-1,\frac{k+i-j}{2}-1}\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}+$$

$$+\sum_{\substack{i=1,j=0\\ 2k\ge k+i-j\ge 2;\ k+i-j\ -\ \text{even}}}^{\infty}\sum_{k=1}^{i-1} N_{ijk}K_{k-1,i-1,\frac{k+i-j}{2}-1}\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}, \qquad (5.17)$$

where

$$N_{ijk}=\sqrt{\frac{1}{(2k+1)(2j+1)(2i+1)}},$$

$$K_{m,n,k}=\frac{a_{m-k}a_k a_{n-k}}{a_{m+n-k}}\cdot\frac{2n+2m-4k+1}{2n+2m-2k+1};\ a_k=\frac{(2k-1)!!}{k!};\ m\le n.$$

Let's analyze approximation $I_{00_{T,t}}^{*(i_1 i_2)q}$ of multiple stochastic integral $I_{00_{T,t}}^{*(i_1 i_2)}$, obtained from (5.6) replacing ∞ on q.

It is easy to prove, that

$$\mathsf{M}\left\{\left(I_{00_{T,t}}^{*(i_1 i_2)}-I_{00_{T,t}}^{*(i_1 i_2)q}\right)^2\right\}=\frac{(T-t)^2}{2}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right). \qquad (5.18)$$

Then, using lemma 10 we get:

$$\mathsf{M}\left\{\left(I_{10_{T,t}}^{*(i_1 i_2)}-I_{10_{T,t}}^{*(i_1 i_2)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{01_{T,t}}^{*(i_1 i_2)}-I_{01_{T,t}}^{*(i_1 i_2)q}\right)^2\right\}=$$

$$=\frac{(T-t)^4}{16}\left(\frac{5}{9}-2\sum_{i=2}^{q}\frac{1}{4i^2-1}-\sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2}-\right.$$

$$\left.-\sum_{i=0}^{q}\frac{(i+2)^2+(i+1)^2}{(2i+1)(2i+5)(2i+3)^2}\right). \qquad (5.19)$$

We proposed $i_1 \neq i_2$ in formulas (5.18), (5.19). Let's examine (5.7), (5.8) for $i_1 = i_2$:

$$I_{01_{T,t}}^{*(i_1 i_1)}=-\frac{(T-t)^2}{4}\left[\left(\zeta_0^{(i_1)}\right)^2+\frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_1)}+\right.$$

$$\left.+\sum_{i=0}^{\infty}\left\{\frac{1}{\sqrt{(2i+1)(2i+5)}(2i+3)}\zeta_i^{(i_1)}\zeta_{i+2}^{(i_1)}-\frac{1}{(2i-1)(2i+3)}\left(\zeta_i^{(i_1)}\right)^2\right\}\right], \qquad (5.20)$$

$$I_{10_{T,t}}^{*(i_1 i_1)}=-\frac{(T-t)^2}{4}\left[\left(\zeta_0^{(i_1)}\right)^2+\frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_1)}+\right.$$

$$\left.+\sum_{i=0}^{\infty}\left\{-\frac{1}{\sqrt{(2i+1)(2i+5)}(2i+3)}\zeta_i^{(i_1)}\zeta_{i+2}^{(i_1)}+\frac{1}{(2i-1)(2i+3)}\left(\zeta_i^{(i_1)}\right)^2\right\}\right], \qquad (5.21)$$

from which, considering (5.3) and (5.4), we get

$$I_{10_{T,t}}^{*(i_1 i_1)}+I_{01_{T,t}}^{*(i_1 i_1)}=-\frac{(T-t)^2}{2}\left(\left(\zeta_0^{(i_1)}\right)^2+\frac{1}{\sqrt{3}}\zeta_0^{(i_1)}\zeta_1^{(i_1)}\right)=I_{0_{T,t}}^{(i_1)}I_{1_{T,t}}^{(i_1)}\ \text{w p. 1.} \qquad (5.22)$$

Obtaining (5.22) we supposed, that equations (5.7), (5.8) are executed with probability 1. Complete proof of this fact will be given in this chapter.

Note, that it is easy to get equality (5.22) using Ito formula and formulas of connection between multiple stochastic Ito and Stratonovich integrals.

Direct calculation using (5.20), (5.21) gives:

$$\mathsf{M}\left\{\left(I_{10_{T,t}}^{*(i_1 i_1)}-I_{10_{T,t}}^{*(i_1 i_1)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{01_{T,t}}^{*(i_1 i_1)}-I_{01_{T,t}}^{*(i_1 i_1)q}\right)^2\right\}=$$

$$
=\frac{(T-t)^4}{16}\Biggl[\sum_{i=q+1}^{\infty}\frac{1}{(2i+1)(2i+5)(2i+3)^2}+\sum_{i=q+1}^{\infty}\frac{2}{(2i-1)^2(2i+3)^2}+
$$

$$
+\left(\sum_{i=q+1}^{\infty}\frac{1}{(2i-1)(2i+3)}\right)^2\Biggr],
$$

where $I_{01_{T,t}}^{*(i_1i_1)q}$, $I_{10_{T,t}}^{*(i_1i_1)q}$ detected from (5.20), (5.21) replacing ∞ by q.

On the other side, formula (4.11) provides a possibility to get more comfortable expressions from the practical point of view, but for multiple stochastic Ito integrals:

$$
\mathsf{M}\left\{\left(I_{10_{T,t}}^{(i_1i_1)}-I_{10_{T,t}}^{(i_1i_1)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{01_{T,t}}^{(i_1i_1)}-I_{01_{T,t}}^{(i_1i_1)q}\right)^2\right\}=
$$

$$
=\frac{(T-t)^4}{16}\left(\frac{1}{9}-\sum_{i=0}^{q}\frac{1}{(2i+1)(2i+5)(2i+3)^2}-2\sum_{i=1}^{q}\frac{1}{(2i-1)^2(2i+3)^2}\right). \tag{5.23}
$$

In tables 5.1 – 5.3 we have calculations according to formulas (5.18), (5.19), (5.23) for various values of q. In the given tables ε means right parts of these formulas. It follows from (6.36), that

$$
I_{11_{T,t}}^{*(i_1i_1)}=\frac{\left(I_{1_{T,t}}^{(i_1)}\right)^2}{2}\ \text{w. p. 1.} \tag{5.24}
$$

In addition, using the Ito formula we have:

$$
I_{20_{T,t}}^{(i_1i_1)}+I_{02_{T,t}}^{(i_1i_1)}=I_{0_{T,t}}^{(i_1)}I_{2_{T,t}}^{(i_1)}-\frac{(T-t)^3}{3}\ \text{w. p. 1,}
$$

from which, considering formula (5.14) we get:

$$
I_{20_{T,t}}^{*(i_1i_1)}+I_{02_{T,t}}^{*(i_1i_1)}=I_{0_{T,t}}^{(i_1)}I_{2_{T,t}}^{(i_1)}\ \text{w. p. 1.} \tag{5.25}
$$

Let's check whether formulas (5.24), (5.25) follow from (5.11) – (5.13), if we suppose $i_1=i_2$ in the last ones.

From (5.11) – (5.13) when $i_1=i_2$ we get:

$$
I_{20_{T,t}}^{*(i_1i_1)}+I_{02_{T,t}}^{*(i_1i_1)}=-\frac{(T-t)^2}{2}I_{00_{T,t}}^{*(i_1i_1)}-(T-t)\left(I_{10_{T,t}}^{*(i_1i_1)}+I_{01_{T,t}}^{*(i_1i_1)}\right)+
$$

$$
+\frac{(T-t)^3}{4}\left(\frac{1}{3}\left(\zeta_0^{(i_1)}\right)^2+\frac{2}{3\sqrt{5}}\zeta_2^{(i_1)}\zeta_0^{(i_1)}\right), \tag{5.26}
$$

$$I_{11_{T,t}}^{*(i_1i_1)} = -\frac{(T-t)^2}{4} I_{00_{T,t}}^{*(i_1i_1)} - \frac{T-t}{2}\left(I_{10_{T,t}}^{*(i_1i_1)} + I_{01_{T,t}}^{*(i_1i_1)}\right) + \frac{(T-t)^3}{24}\left(\zeta_1^{(i_1)}\right)^2. \tag{5.27}$$

It is easy to see, that from (5.26) and (5.27), considering (5.22) and (5.3) – (5.6), we actually obtain equalities (5.24) and (5.25), and it indirectly confirm rightness of formulas (5.11) – (5.13).

On the basis of presented expansions of multiple stochastic integrals we can see, that increasing of multiplicities of these integrals or degree indexes of their weight functions leads to noticeable complication of formulas intended for mentioned expansions.

However, increasing of mentioned parameters lead to increasing of orders of smallness according to $T-t$ in the mean-square sense for multiple stochastic integrals, that lead to sharp decrease of member quantities in the expansions of multiple stochastic integrals, which are required for achieving acceptable accuracies of approximation. In the context of it let's examine the approach to approximation of multiple stochastic integrals, which provides a possibility to obtain mean-square approximations of the required accuracy without using common expansions of type (5.10).

Let's analyze the following approximation:

$$I_{000_{T,t}}^{(i_1i_2i_3)q_1} = \sum_{i,j,k=0}^{q_1} C_{kji}\Biggl(\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)} -$$

$$-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{i=j\}}\zeta_k^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j=k\}}\zeta_i^{(i_1)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{i=k\}}\zeta_j^{(i_2)}\Biggr), \tag{5.28}$$

where $i_1, i_2, i_3 = 1, \ldots, m$ and

$$C_{kji} = \int\limits_t^T \phi_k(z)\int\limits_t^z \phi_j(y)\int\limits_t^y \phi_i(x)dxdydz =$$

$$= \frac{\sqrt{(2i+1)(2j+1)(2k+1)}}{8}(T-t)^{3/2}\bar{C}_{kji};$$

$$\bar{C}_{kji} = \int\limits_{-1}^{1} P_k(z)\int\limits_{-1}^{z} P_j(y)\int\limits_{-1}^{y} P_i(x)dxdydz;$$

$P_i(x);\ i = 0,\ 1,\ 2, \ldots$ — are Legendre polynomials.

In particular, from (5.28) when $i_1 \ne i_2$, $i_2 \ne i_3$, $i_1 \ne i_3$ we get:

$$I_{000_{T,t}}^{(i_1i_2i_3)q_1} = \sum_{i,j,k=0}^{q_1} C_{kji}\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}. \tag{5.29}$$

Note, that due to the results obtained in chapter 2, the right part of the formula (5.29) determines approximation $I_{000_{T,t}}^{*(i_1 i_2 i_3)q_1}$ of multiple stochastic Stratonovich integral $I_{000_{T,t}}^{*(i_1 i_2 i_3)}$, but for any possible $i_1, i_2, i_3 = 1, \ldots, m$:

$$I_{000_{T,t}}^{*(i_1 i_2 i_3)q_1} = \sum_{i,j,k=0}^{q_1} C_{kji} \zeta_i^{(i_1)} \zeta_j^{(i_2)} \zeta_k^{(i_3)} \ (i_1, i_2, i_3 = 1, \ldots, m).$$

We will remind, that formulas for mean-square errors of approximations of type (5.28) obtained at the beginning of chapter 4 for various combinations i_1, i_2, i_3 are look as follows:

$$\mathsf{M}\left\{\left(I_{000_{T,t}}^{(i_1 i_2 i_3)} - I_{000_{T,t}}^{(i_1 i_2 i_3)q_1}\right)^2\right\} = \frac{(T-t)^3}{6} - \sum_{i,j,k=0}^{q_1} C_{kji}^2 \ (i_1 \ne i_2, i_1 \ne i_3, i_2 \ne i_3), \tag{5.30}$$

$$\mathsf{M}\left\{\left(I_{000_{T,t}}^{(i_1 i_2 i_3)} - I_{000_{T,t}}^{(i_1 i_2 i_3)q_1}\right)^2\right\} = \frac{(T-t)^3}{6} - \sum_{i,j,k=0}^{q_1} C_{kji}^2 - \sum_{i,j,k=0}^{q_1} C_{jki} C_{kji} \ (i_1 \ne i_2 = i_3), \tag{5.31}$$

$$\mathsf{M}\left\{\left(I_{000_{T,t}}^{(i_1 i_2 i_3)} - I_{000_{T,t}}^{(i_1 i_2 i_3)q_1}\right)^2\right\} = \frac{(T-t)^3}{6} - \sum_{i,j,k=0}^{q_1} C_{kji}^2 - \sum_{i,j,k=0}^{q_1} C_{kji} C_{ijk} \ (i_1 = i_3 \ne i_2), \tag{5.32}$$

$$\mathsf{M}\left\{\left(I_{000_{T,t}}^{(i_1 i_2 i_3)} - I_{000_{T,t}}^{(i_1 i_2 i_3)q_1}\right)^2\right\} = \frac{(T-t)^3}{6} - \sum_{i,j,k=0}^{q_1} C_{kji}^2 - \sum_{i,j,k=0}^{q_1} C_{kij} C_{kji} \ (i_1 = i_2 \ne i_3). \tag{5.33}$$

For the case $i_1 = i_2 = i_3 = i$ it is comfortable to use the following formulas:

$$I_{000_{T,t}}^{*(iii)} = \frac{1}{6}(T-t)^{\frac{3}{2}} \left(\zeta_0^{(i)}\right)^3, I_{000_{T,t}}^{(iii)} = \frac{1}{6}(T-t)^{\frac{3}{2}} \left(\left(\zeta_0^{(i)}\right)^3 - 3\zeta_0^{(i)}\right) \text{ w. p. 1.} \tag{5.34}$$

In more general case, when $\psi_1(s), \psi_2(s), \psi_3(s) \equiv (t-s)^l$; l — is a fixed natural number or zero, with probubility 1 we may write down

$$I_{lll_{T,t}}^{(iii)} = \frac{1}{6}\left(\left(I_{l_{T,t}}^{(i)}\right)^3 - 3 I_{l_{T,t}}^{(i)} \Delta_{l_{T,t}}\right),$$

$$I_{lll_{T,t}}^{*(iii)} = \frac{1}{6}\left(I_{l_{T,t}}^{(i)}\right)^3, \ I_{l_{T,t}}^{(i)} = \sum_{j=0}^{l} C_j\zeta_j^{(i)}, \ \Delta_{l_{T,t}} = \int_t^T (t-s)^{2l}ds.$$

where $C_j = \int_t^T (t-s)^l\phi_j(s)ds$; $\{\phi_j(s)\}_{j=0}^{\infty}$ — is a full orthonormal system of Legendre polynomials at the interval $[t, T]$.

Now, it is clear, that for approximation of stochastic integral $I_{000_{T,t}}^{(i_1i_2i_3)}$ we may use formulas (5.28) – (5.34) instead of complex expansion (5.10). We may act similarly with more complicated multiple stochastic integrals. For example, for the approximation of stochastic integral $I_{0000_{T,t}}^{(i_1i_2i_3i_4)}$ according to theorem 1, we may write down:

$$I_{0000_{T,t}}^{(i_1i_2i_3i_4)q_2} = \sum_{i,j,k,l=0}^{q_2} C_{lkji}\Biggl(\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}\zeta_l^{(i_4)} -$$

$$-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{i=j\}}\zeta_k^{(i_3)}\zeta_l^{(i_4)} - \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{i=k\}}\zeta_j^{(i_2)}\zeta_l^{(i_4)} -$$

$$-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{i=l\}}\zeta_j^{(i_2)}\zeta_k^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j=k\}}\zeta_i^{(i_1)}\zeta_l^{(i_4)} -$$

$$-\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j=l\}}\zeta_i^{(i_1)}\zeta_k^{(i_3)} - \mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{k=l\}}\zeta_i^{(i_1)}\zeta_j^{(i_2)} +$$

$$+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{i=j\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{k=l\}} + \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{i=k\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j=l\}} +$$

$$+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{i=l\}}\mathbf{1}_{\{j_2=j_3\}}\mathbf{1}_{\{j=k\}}\Biggr),$$

where

$$C_{lkji} = \int_t^T \phi_l(u)\int_t^u \phi_k(z)\int_t^z \phi_j(y)\int_t^y \phi_i(x)dxdydzdu =$$

$$= \frac{\sqrt{(2i+1)(2j+1)(2k+1)(2l+1)}}{16}(T-t)^2\bar{C}_{lkji};$$

$$\bar{C}_{lkji} = \int_{-1}^{1} P_l(u)\int_{-1}^{u} P_k(z)\int_{-1}^{z} P_j(y)\int_{-1}^{y} P_i(x)dxdydzdu.$$

In chapter 4 we obtained accurate formulas for the mean-square error of approximation $\mathsf{M}\left\{\left(J[\psi^{(4)}]_{T,t}^{q_2} - J[\psi^{(4)}]_{T,t}\right)^2\right\}$ for various combinations of indexes i_1, i_2, i_3, i_4. Considering in these formulas, that $\psi_1(\tau), \ldots, \psi_4(\tau) \equiv 1$ we will get $\mathsf{M}\left\{\left(I_{0000_{T,t}}^{(i_1i_2i_3i_4)q_2} - I_{0000_{T,t}}^{(i_1i_2i_3i_4)}\right)^2\right\}$.

Table 5.1: Check of formula (5.18)

$2\varepsilon/(T-t)^2$	0.1667	0.0238	0.0025	$2.4988 \cdot 10^{-4}$	$2.4999 \cdot 10^{-5}$
q	1	10	100	1000	10000

Table 5.2: Check of formula (5.19)

$16\varepsilon/(T-t)^4$	0.3797	0.0581	0.0062	$6.2450 \cdot 10^{-4}$	$6.2495 \cdot 10^{-5}$
q	1	10	100	1000	10000

The case when $i_1 = \ldots = i_4 = i$ in chapter 4 was analyzed partly, however in this case with probability 1

$$I_{0000_{T,t}}^{(iiii)} = \frac{(T-t)^2}{24}\left(\left(\zeta_0^{(i)}\right)^4 - 6\left(\zeta_0^{(i)}\right)^2 + 3\right), I_{0000_{T,t}}^{*(iiii)} = \frac{(T-t)^2}{24}\left(\zeta_0^{(i)}\right)^4.$$

In more general case, when $\psi_1(\tau), \ldots, \psi_4(\tau) \equiv (t-\tau)^l$; l — is a fixed natural number or zero, with probability 1 we may right down the following:

$$I_{llll_{T,t}}^{(iiii)} = \frac{1}{24}\left(\left(I_{l_{T,t}}^{(i)}\right)^4 - 6\left(I_{l_{T,t}}^{(i)}\right)^2 \Delta_{l_{T,t}} + 3\left(\Delta_{l_{T,t}}\right)^2\right),$$

$$I_{llll_{T,t}}^{*(iiii)} = \frac{1}{24}\left(I_{l_{T,t}}^{(i)}\right)^4,\ I_{l_{T,t}}^{(i)} = \sum_{j=0}^{l} C_j \zeta_j^{(i)},\ \Delta_{l_{T,t}} = \int\limits_t^T (t-s)^{2l} ds,$$

where in the next-to-last formula we propose, that the expansion of stochastic integral is performed using Legendre polynomials.

Assume, that $q_1 = 6$. In tables 5.4–5.10 are given the exact values of coefficients $\bar{C}_{kji}$ when $i, j, k = 0, 1, \ldots, 6$.

Calculating the value of expression (5.30) when $q_1 = 6$, $i_1 \neq i_2$, $i_1 \neq i_3$, $i_3 \neq i_2$ we get the following approximate equality:

$$\mathsf{M}\left\{\left(I_{000_{T,t}}^{(i_1 i_2 i_3)} - I_{000_{T,t}}^{(i_1 i_2 i_3) q_1}\right)^2\right\} \approx 0.01956(T-t)^3.$$

Let's choose, for example, $q_2 = 2$. In tables 5.11–5.19 we have the exact values of coefficients $\bar{C}_{lkji}$; $i, j, k, l = 0, 1, 2$. In case of pairwise different

Table 5.3: Check of formula (5.23)

$16\varepsilon/(T-t)^4$	0.0070	$4.3551 \cdot 10^{-5}$	$6.0076 \cdot 10^{-8}$	$6.2251 \cdot 10^{-11}$	$6.3178 \cdot 10^{-14}$
q	1	10	100	1000	10000

Table 5.4: Coefficients $\bar{C}_{ojk}$

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$j=0$	$\frac{4}{3}$	$\frac{-2}{3}$	$\frac{2}{15}$	0	0	0	0
$j=1$	0	$\frac{2}{15}$	$\frac{-2}{15}$	$\frac{4}{105}$	0	0	0
$j=2$	$\frac{-4}{15}$	$\frac{2}{15}$	$\frac{2}{105}$	$\frac{-2}{35}$	$\frac{2}{105}$	0	0
$j=3$	0	$\frac{-2}{35}$	$\frac{2}{35}$	$\frac{2}{315}$	$\frac{-2}{63}$	$\frac{8}{693}$	0
$j=4$	0	0	$\frac{-8}{315}$	$\frac{2}{63}$	$\frac{2}{693}$	$\frac{-2}{99}$	$\frac{10}{1287}$
$j=5$	0	0	0	$\frac{-10}{693}$	$\frac{2}{99}$	$\frac{2}{1287}$	$\frac{-2}{143}$
$j=6$	0	0	0	0	$\frac{-4}{429}$	$\frac{2}{143}$	$\frac{2}{2145}$

Table 5.5: Coefficients $\bar{C}_{1jk}$

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$j=0$	$\frac{2}{3}$	$\frac{-4}{15}$	0	$\frac{2}{105}$	0	0	0
$j=1$	$\frac{2}{15}$	0	$\frac{-4}{105}$	0	$\frac{2}{315}$	0	0
$j=2$	$\frac{-2}{15}$	$\frac{8}{105}$	0	$\frac{-2}{105}$	0	$\frac{4}{1155}$	0
$j=3$	$\frac{-2}{35}$	0	$\frac{8}{315}$	0	$\frac{-38}{3465}$	0	$\frac{20}{9009}$
$j=4$	0	$\frac{-4}{315}$	0	$\frac{46}{3465}$	0	$\frac{-64}{9009}$	0
$j=5$	0	0	$\frac{-4}{693}$	0	$\frac{74}{9009}$	0	$\frac{-32}{6435}$
$j=6$	0	0	0	$\frac{-10}{3003}$	0	$\frac{4}{715}$	0

i_1, i_2, i_3, i_4 we have the following equality:

$$\mathsf{M}\left\{\left(I_{0000_{T,t}}^{(i_1 i_2 i_3 i_4)} - I_{0000_{T,t}}^{(i_1 i_2 i_3 i_4)q_2}\right)^2\right\} = \frac{(T-t)^4}{24} - \sum_{i,j,k,l=0}^{2} C_{lkji}^2. \tag{5.35}$$

Note, that it is easy to check correctness of the following equalities (see (5.7), (5.8), (5.11) – (5.13)):

$$\sum_{j=0}^{\infty} C_{jj}^{10} = \sum_{j=0}^{\infty} C_{jj}^{01} = -\frac{(T-t)^2}{4}, \tag{5.36}$$

$$\sum_{j=0}^{\infty} C_{jj}^{20} = \sum_{j=0}^{\infty} C_{jj}^{11} = \sum_{j=0}^{\infty} C_{jj}^{02} = \frac{(T-t)^3}{6}, \tag{5.37}$$

where

$$C_{jj}^{10} = \int_t^T \phi_j(x) \int_t^x \phi_j(y)(t-y) dy dx,$$

$$C_{jj}^{01} = \int_t^T \phi_j(x)(t-x) \int_t^x \phi_j(y) dy dx,$$

Table 5.6: Coefficients $\bar{C}_{2jk}$

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$j=0$	$\frac{2}{15}$	0	$\frac{-4}{105}$	0	$\frac{2}{315}$	0	0
$j=1$	$\frac{2}{15}$	$\frac{-4}{105}$	0	$\frac{-2}{315}$	0	$\frac{8}{3465}$	0
$j=2$	$\frac{2}{105}$	0	0	0	$\frac{-2}{495}$	0	$\frac{4}{3003}$
$j=3$	$\frac{-2}{35}$	$\frac{8}{315}$	0	$\frac{-2}{3465}$	0	$\frac{-116}{45045}$	0
$j=4$	$\frac{-8}{315}$	0	$\frac{4}{495}$	0	$\frac{-2}{6435}$	0	$\frac{-16}{9009}$
$j=5$	0	$\frac{-4}{693}$	0	$\frac{38}{9009}$	0	$\frac{-8}{45045}$	0
$j=6$	0	0	$\frac{-8}{3003}$	0	$\frac{-118}{45045}$	0	$\frac{-4}{36465}$

Table 5.7: Coefficients $\bar{C}_{3jk}$

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$j=0$	0	$\frac{2}{105}$	0	$\frac{-4}{315}$	0	$\frac{2}{693}$	0
$j=1$	$\frac{4}{105}$	0	$\frac{-2}{315}$	0	$\frac{-8}{3465}$	0	$\frac{10}{9009}$
$j=2$	$\frac{2}{35}$	$\frac{-2}{105}$	0	$\frac{4}{3465}$	0	$\frac{-74}{45045}$	0
$j=3$	$\frac{2}{315}$	0	$\frac{-2}{3465}$	0	$\frac{16}{45045}$	0	$\frac{-10}{9009}$
$j=4$	$\frac{-2}{63}$	$\frac{46}{3465}$	0	$\frac{-32}{45045}$	0	$\frac{2}{9009}$	0
$j=5$	$\frac{-10}{693}$	0	$\frac{38}{9009}$	0	$\frac{-4}{9009}$	0	$\frac{122}{765765}$
$j=6$	0	$\frac{-10}{3003}$	0	$\frac{20}{9009}$	0	$\frac{-226}{765765}$	0

Table 5.8: Coefficients $\bar{C}_{4jk}$

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$j=0$	0	0	$\frac{2}{315}$	0	$\frac{-4}{693}$	0	$\frac{2}{1287}$
$j=1$	0	$\frac{2}{315}$	0	$\frac{-8}{3465}$	0	$\frac{-10}{9009}$	0
$j=2$	$\frac{2}{105}$	0	$\frac{-2}{495}$	0	$\frac{4}{6435}$	0	$\frac{-38}{45045}$
$j=3$	$\frac{2}{63}$	$\frac{-38}{3465}$	0	$\frac{16}{45045}$	0	$\frac{2}{9009}$	0
$j=4$	$\frac{2}{693}$	0	$\frac{-2}{6435}$	0	0	0	$\frac{2}{13923}$
$j=5$	$\frac{-2}{99}$	$\frac{74}{9009}$	0	$\frac{-4}{9009}$	0	$\frac{-2}{153153}$	0
$j=6$	$\frac{-4}{429}$	0	$\frac{118}{45045}$	0	$\frac{-4}{13923}$	0	$\frac{-2}{188955}$

Table 5.9: Coefficients $\bar{C}_{5jk}$

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$j=0$	0	0	0	$\frac{2}{693}$	0	$\frac{-4}{1287}$	0
$j=1$	0	0	$\frac{8}{3465}$	0	$\frac{-10}{9009}$	0	$\frac{-4}{6435}$
$j=2$	0	$\frac{4}{1155}$	0	$\frac{-74}{45045}$	0	$\frac{16}{45045}$	0
$j=3$	$\frac{8}{693}$	0	$\frac{-116}{45045}$	0	$\frac{2}{9009}$	0	$\frac{8}{58905}$
$j=4$	$\frac{2}{99}$	$\frac{-64}{9009}$	0	$\frac{2}{9009}$	0	$\frac{4}{153153}$	0
$j=5$	$\frac{2}{1287}$	0	$\frac{-8}{45045}$	0	$\frac{-2}{153153}$	0	$\frac{4}{415701}$
$j=6$	$\frac{-2}{143}$	$\frac{4}{715}$	0	$\frac{-226}{765765}$	0	$\frac{-8}{415701}$	0

Table 5.10: Coefficients $\bar{C}_{6jk}$

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$j=0$	0	0	0	0	$\frac{2}{1287}$	0	$\frac{-4}{2145}$
$j=1$	0	0	0	$\frac{10}{9009}$	0	$\frac{-4}{6435}$	0
$j=2$	0	0	$\frac{4}{3003}$	0	$\frac{-38}{45045}$	0	$\frac{8}{36465}$
$j=3$	0	$\frac{20}{9009}$	0	$\frac{-10}{9009}$	0	$\frac{8}{58905}$	0
$j=4$	$\frac{10}{1287}$	0	$\frac{-16}{9009}$	0	$\frac{2}{13923}$	0	$\frac{4}{188955}$
$j=5$	$\frac{2}{143}$	$\frac{-32}{6435}$	0	$\frac{122}{765765}$	0	$\frac{4}{415701}$	0
$j=6$	$\frac{2}{2145}$	0	$\frac{-4}{36465}$	0	$\frac{-2}{188955}$	0	0

Table 5.11: Coefficients $\bar{C}_{00kl}$

	$l=0$	$l=1$	$l=2$
$k=0$	$\frac{2}{3}$	$\frac{-2}{5}$	$\frac{2}{15}$
$k=1$	$\frac{-2}{15}$	$\frac{2}{15}$	$\frac{-2}{21}$
$k=2$	$\frac{-2}{15}$	$\frac{2}{35}$	$\frac{2}{105}$

Table 5.12: Coefficients $\bar{C}_{10kl}$

	$l=0$	$l=1$	$l=2$
$k=0$	$\frac{2}{5}$	$\frac{-2}{9}$	$\frac{2}{35}$
$k=1$	$\frac{-2}{45}$	$\frac{2}{35}$	$\frac{-2}{45}$
$k=2$	$\frac{-2}{21}$	$\frac{2}{45}$	$\frac{2}{315}$

Table 5.13: Coefficients $\bar{C}_{02kl}$

	$l=0$	$l=1$	$l=2$
$k=0$	$\frac{-2}{15}$	$\frac{2}{21}$	$\frac{-4}{105}$
$k=1$	$\frac{2}{35}$	$\frac{-4}{105}$	$\frac{2}{105}$
$k=2$	$\frac{4}{105}$	$\frac{-2}{105}$	0

Table 5.14: Coefficients $\bar{C}_{01kl}$

	$l=0$	$l=1$	$l=2$
$k=0$	$\frac{2}{15}$	$\frac{-2}{45}$	$\frac{-2}{105}$
$k=1$	$\frac{2}{45}$	$\frac{-2}{105}$	$\frac{2}{315}$
$k=2$	$\frac{-2}{35}$	$\frac{2}{63}$	$\frac{-2}{315}$

Table 5.15: Coefficients $\bar{C}_{11kl}$

	$l=0$	$l=1$	$l=2$
$k=0$	$\frac{2}{15}$	$\frac{-2}{35}$	0
$k=1$	$\frac{2}{105}$	0	$\frac{-2}{315}$
$k=2$	$\frac{-4}{105}$	$\frac{2}{105}$	0

Table 5.16: Coefficients $\bar{C}_{20kl}$

	$l=0$	$l=1$	$l=2$
$k=0$	$\frac{2}{15}$	$\frac{-2}{35}$	0
$k=1$	$\frac{2}{105}$	0	$\frac{-2}{315}$
$k=2$	$\frac{-4}{105}$	$\frac{2}{105}$	0

Table 5.17: Coefficients $\bar{C}_{21kl}$

	$l=0$	$l=1$	$l=2$
$k=0$	$\frac{2}{21}$	$\frac{-2}{45}$	$\frac{2}{315}$
$k=1$	$\frac{2}{315}$	$\frac{2}{315}$	$\frac{-2}{225}$
$k=2$	$\frac{-2}{105}$	$\frac{2}{225}$	$\frac{2}{1155}$

Table 5.18: Coefficients $\bar{C}_{12kl}$

	$l=0$	$l=1$	$l=2$
$k=0$	$\frac{-2}{35}$	$\frac{2}{45}$	$\frac{-2}{105}$
$k=1$	$\frac{2}{63}$	$\frac{-2}{105}$	$\frac{2}{225}$
$k=2$	$\frac{2}{105}$	$\frac{-2}{225}$	$\frac{-2}{3465}$

$$C_{jj}^{11}=\int\limits_t^T\phi_j(x)(t-x)\int\limits_t^x\phi_j(y)(t-y)dydx,$$

$$C_{jj}^{20}=\int\limits_t^T\phi_j(x)\int\limits_t^x\phi_j(y)(t-y)^2dydx,$$

$$C_{jj}^{02}=\int\limits_t^T\phi_j(x)(t-x)^2\int\limits_t^x\phi_j(y)dydx,$$

$\{\phi_j(x)\}_{j=0}^{\infty}$ — is a full orthonormal system of Legendre polynomials in the space $L_2([t,T])$.

Note, that equalities (5.36) and (5.37) together with the theorem 1 when $k=2$ and formulas (5.9), (5.14), (5.15) confirm formula (??) for multiple stochastic Stratonovich integrals $I_{10_{T,t}}^{*(i_1i_2)}$, $I_{01_{T,t}}^{*(i_1i_2)}$, $I_{20_{T,t}}^{*(i_1i_2)}$, $I_{11_{T,t}}^{*(i_1i_2)}$, $I_{02_{T,t}}^{*(i_1i_2)}$; $i_1,i_2=1,\ldots,m$.

Let's analyze approximations for the following four multiple stochastic Ito integrals:

$$I_{001_{T,t}}^{(i_1i_2i_3)q_3}=\sum_{i,j,k=0}^{q_3}C_{kji}^{001}\Biggl(\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}-$$

$$-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{i=j\}}\zeta_k^{(i_3)}-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j=k\}}\zeta_i^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{i=k\}}\zeta_j^{(i_2)}\Biggr),$$

$$I_{010_{T,t}}^{(i_1i_2i_3)q_4}=\sum_{i,j,k=0}^{q_4}C_{kji}^{010}\Biggl(\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}-$$

$$-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{i=j\}}\zeta_k^{(i_3)}-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j=k\}}\zeta_i^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{i=k\}}\zeta_j^{(i_2)}\Biggr),$$

$$I_{100_{T,t}}^{(i_1i_2i_3)q_5}=\sum_{i,j,k=0}^{q_5}C_{kji}^{100}\Biggl(\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}-$$

$$-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{i=j\}}\zeta_k^{(i_3)}-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j=k\}}\zeta_i^{(i_1)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{i=k\}}\zeta_j^{(i_2)}\Biggr),$$

$$I_{00000_{T,t}}^{(i_1i_2i_3i_4i_5)q_6}=\sum_{i,j,k,l,r=0}^{q_6}C_{rlkji}\Biggl(\zeta_r^{(i_5)}\zeta_l^{(i_4)}\zeta_k^{(i_3)}\zeta_j^{(i_2)}\zeta_i^{(i_1)}-$$

$$-\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{i=j\}}\zeta_k^{(i_3)}\zeta_l^{(i_4)}\zeta_r^{(i_5)}-\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{i=k\}}\zeta_j^{(i_2)}\zeta_l^{(i_4)}\zeta_r^{(i_5)}-$$

$$
-\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{i=l\}}\zeta_j^{(i_2)}\zeta_k^{(i_3)}\zeta_r^{(i_5)} - \mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{i=r\}}\zeta_j^{(i_2)}\zeta_k^{(i_3)}\zeta_l^{(i_4)}-
$$

$$
-\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j=k\}}\zeta_i^{(i_1)}\zeta_l^{(i_4)}\zeta_r^{(i_5)} - \mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j=l\}}\zeta_i^{(i_1)}\zeta_k^{(i_3)}\zeta_r^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j=r\}}\zeta_i^{(i_1)}\zeta_k^{(i_3)}\zeta_l^{(i_4)} - \mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{k=l\}}\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_r^{(i_5)}-
$$

$$
-\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{k=r\}}\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_l^{(i_4)} - \mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{l=r\}}\zeta_i^{(i_1)}\zeta_j^{(i_2)}\zeta_k^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{i=j\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{k=l\}}\zeta_r^{(i_5)} + \mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{i=j\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{k=r\}}\zeta_l^{(i_4)}+
$$

$$
+\mathbf{1}_{\{i_1=i_2\}}\mathbf{1}_{\{i=j\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{l=r\}}\zeta_k^{(i_3)} + \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{i=k\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j=l\}}\zeta_r^{(i_5)}+
$$

$$
+\mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{i=k\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j=r\}}\zeta_l^{(i_4)} + \mathbf{1}_{\{i_1=i_3\}}\mathbf{1}_{\{i=k\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{l=r\}}\zeta_j^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{i=l\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j=k\}}\zeta_r^{(i_5)} + \mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{i=l\}}\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j=r\}}\zeta_k^{(i_3)}+
$$

$$
+\mathbf{1}_{\{i_1=i_4\}}\mathbf{1}_{\{i=l\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{k=r\}}\zeta_j^{(i_2)} + \mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{i=r\}}\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j=k\}}\zeta_l^{(i_4)}+
$$

$$
+\mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{i=r\}}\mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j=l\}}\zeta_k^{(i_3)} + \mathbf{1}_{\{i_1=i_5\}}\mathbf{1}_{\{i=r\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{k=l\}}\zeta_j^{(i_2)}+
$$

$$
+\mathbf{1}_{\{i_2=i_3\}}\mathbf{1}_{\{j=k\}}\mathbf{1}_{\{i_4=i_5\}}\mathbf{1}_{\{l=r\}}\zeta_i^{(i_1)} + \mathbf{1}_{\{i_2=i_4\}}\mathbf{1}_{\{j=l\}}\mathbf{1}_{\{i_3=i_5\}}\mathbf{1}_{\{k=r\}}\zeta_i^{(i_1)}+
$$

$$
+\mathbf{1}_{\{i_2=i_5\}}\mathbf{1}_{\{j=r\}}\mathbf{1}_{\{i_3=i_4\}}\mathbf{1}_{\{k=l\}}\zeta_i^{(i_1)}\Biggr),
$$

where

$$
C_{kji}^{001} = \int\limits_t^T (t-z)\phi_k(z)\int\limits_t^z \phi_j(y)\int\limits_t^y \phi_i(x)dxdydz =
$$

$$
= \frac{\sqrt{(2i+1)(2j+1)(2k+1)}}{16}(T-t)^{\frac{5}{2}}\bar{C}_{kji}^{001};
$$

$$
C_{kji}^{010} = \int\limits_t^T \phi_k(z)\int\limits_t^z (t-y)\phi_j(y)\int\limits_t^y \phi_i(x)dxdydz =
$$

$$
= \frac{\sqrt{(2i+1)(2j+1)(2k+1)}}{16}(T-t)^{\frac{5}{2}}\bar{C}_{kji}^{010};
$$

$$
C_{kji}^{100} = \int\limits_t^T \phi_k(z)\int\limits_t^z \phi_j(y)\int\limits_t^y (t-x)\phi_i(x)dxdydz =
$$

$$
= \frac{\sqrt{(2i+1)(2j+1)(2k+1)}}{16}(T-t)^{\frac{5}{2}}\bar{C}_{kji}^{100};
$$

$$
C_{rlkji} = \int\limits_t^T \phi_r(v)\int\limits_t^v \phi_l(u)\int\limits_t^u \phi_k(z)\int\limits_t^z \phi_j(y)\int\limits_t^y \phi_i(x)dxdydzdudv =
$$

Table 5.19: Coefficients $\bar{C}_{22kl}$

	$l=0$	$l=1$	$l=2$
$k=0$	$\frac{-2}{105}$	$\frac{-2}{315}$	0
$k=1$	$\frac{2}{315}$	0	$\frac{-2}{1155}$
$k=2$	0	$\frac{2}{3465}$	0

Table 5.20: Coefficients $\bar{C}_{0jk}^{001}$

	$k=0$	$k=1$	$k=2$
$j=0$	-2	$\frac{14}{15}$	$\frac{-2}{15}$
$j=1$	$\frac{-2}{15}$	$\frac{-2}{15}$	$\frac{6}{35}$
$j=2$	$\frac{2}{5}$	$\frac{-22}{105}$	$\frac{-2}{105}$

$$
=\frac{\sqrt{(2i+1)(2j+1)(2k+1)(2l+1)(2r+1)}}{32}(T-t)^{\frac{5}{2}}\bar{C}_{rlkji},
$$

where

$$
\bar{C}_{kji}^{100}=-\int\limits_{-1}^{1}P_k(z)\int\limits_{-1}^{z}P_j(y)\int\limits_{-1}^{y}P_i(x)(x+1)dxdydz;
$$

$$
\bar{C}_{kji}^{010}=-\int\limits_{-1}^{1}P_k(z)\int\limits_{-1}^{z}P_j(y)(y+1)\int\limits_{-1}^{y}P_i(x)dxdydz;
$$

$$
\bar{C}_{kji}^{001}=-\int\limits_{-1}^{1}P_k(z)(z+1)\int\limits_{-1}^{z}P_j(y)\int\limits_{-1}^{y}P_i(x)dxdydz;
$$

$$
\bar{C}_{rlkji}=\int\limits_{-1}^{1}P_r(v)\int\limits_{-1}^{v}P_l(u)\int\limits_{-1}^{u}P_k(z)\int\limits_{-1}^{z}P_j(y)\int\limits_{-1}^{y}P_i(x)dxdydzdudv.
$$

Assume, that $q_3=q_4=q_5=2$, $q_6=1$. In tables 5.20–5.36 we have the exact values of coefficients $\bar{C}_{kji}^{001}$, $\bar{C}_{kji}^{010}$, $\bar{C}_{kji}^{001}$; $i,j,k=0,1,2$; $\bar{C}_{rlkji}$; $i,j,k,l,r=0,1$.

Table 5.21: Coefficients $\bar{C}_{1jk}^{001}$

	$k=0$	$k=1$	$k=2$
$j=0$	$\frac{-6}{5}$	$\frac{22}{45}$	$\frac{-2}{105}$
$j=1$	$\frac{-2}{9}$	$\frac{-2}{105}$	$\frac{26}{315}$
$j=2$	$\frac{22}{105}$	$\frac{-38}{315}$	$\frac{-2}{315}$

Table 5.22: Coefficients $\bar{C}_{2jk}^{001}$

	$k=0$	$k=1$	$k=2$
$j=0$	$\frac{-2}{5}$	$\frac{2}{21}$	$\frac{4}{105}$
$j=1$	$\frac{-22}{105}$	$\frac{4}{105}$	$\frac{2}{105}$
$j=2$	0	$\frac{-2}{105}$	0

Table 5.23: Coefficients $\bar{C}_{0jk}^{100}$

	$k=0$	$k=1$	$k=2$
$j=0$	$\frac{-2}{3}$	$\frac{2}{15}$	$\frac{2}{15}$
$j=1$	$\frac{-2}{15}$	$\frac{-2}{45}$	$\frac{2}{35}$
$j=2$	$\frac{2}{15}$	$\frac{-2}{35}$	$\frac{-4}{105}$

Table 5.24: Coefficients $\bar{C}_{1jk}^{100}$

	$k=0$	$k=1$	$k=2$
$j=0$	$\frac{-2}{5}$	$\frac{2}{45}$	$\frac{2}{21}$
$j=1$	$\frac{-2}{15}$	$\frac{-2}{105}$	$\frac{4}{105}$
$j=2$	$\frac{2}{35}$	$\frac{-2}{63}$	$\frac{-2}{105}$

Table 5.25: Coefficients $\bar{C}_{2jk}^{100}$

	$k=0$	$k=1$	$k=2$
$j=0$	$\frac{-2}{15}$	$\frac{-2}{105}$	$\frac{4}{105}$
$j=1$	$\frac{-2}{21}$	$\frac{-2}{315}$	$\frac{2}{105}$
$j=2$	$\frac{-2}{105}$	$\frac{-2}{315}$	0

Table 5.26: Coefficients $\bar{C}_{0jk}^{010}$

	$k=0$	$k=1$	$k=2$
$j=0$	$\frac{-4}{3}$	$\frac{8}{15}$	0
$j=1$	$\frac{-4}{15}$	0	$\frac{8}{105}$
$j=2$	$\frac{4}{15}$	$\frac{-16}{105}$	0

Table 5.27: Coefficients $\bar{C}_{1jk}^{010}$

	$k=0$	$k=1$	$k=2$
$j=0$	$\frac{-4}{5}$	$\frac{4}{15}$	$\frac{4}{105}$
$j=1$	$\frac{-4}{15}$	$\frac{4}{105}$	$\frac{4}{105}$
$j=2$	$\frac{4}{35}$	$\frac{-8}{105}$	0

Table 5.28: Coefficients $\bar{C}_{2jk}^{010}$

	$k=0$	$k=1$	$k=2$
$j=0$	$\frac{-4}{15}$	$\frac{4}{105}$	$\frac{4}{105}$
$j=1$	$\frac{-4}{21}$	$\frac{4}{105}$	$\frac{4}{315}$
$j=2$	$\frac{-4}{105}$	0	0

Table 5.29: Coefficients $\bar{C}_{000lr}$

	$r=0$	$r=1$
$l=0$	$\frac{4}{15}$	$\frac{-8}{45}$
$l=1$	$\frac{-4}{45}$	$\frac{8}{105}$

Table 5.30: Coefficients $\bar{C}_{010lr}$

	$r=0$	$r=1$
$l=0$	$\frac{4}{45}$	$\frac{-16}{315}$
$l=1$	$\frac{-4}{315}$	$\frac{4}{315}$

Table 5.31: Coefficients $\bar{C}_{110lr}$

	$r=0$	$r=1$
$l=0$	$\frac{8}{105}$	$\frac{-2}{45}$
$l=1$	$\frac{-4}{315}$	$\frac{4}{315}$

Table 5.32: Coefficients $\bar{C}_{011lr}$

	$r=0$	$r=1$
$l=0$	$\frac{8}{315}$	$\frac{-4}{315}$
$l=1$	0	$\frac{2}{945}$

Table 5.33: Coefficients $\bar{C}_{001lr}$

	$r=0$	$r=1$
$l=0$	0	$\frac{4}{315}$
$l=1$	$\frac{8}{315}$	$\frac{-2}{105}$

Table 5.34: Coefficients $\bar{C}_{100lr}$

	$r=0$	$r=1$
$l=0$	$\frac{8}{45}$	$\frac{-4}{35}$
$l=1$	$\frac{-16}{315}$	$\frac{2}{45}$

Table 5.35: Coefficients $\bar{C}_{101lr}$

	$r=0$	$r=1$
$l=0$	$\frac{4}{315}$	0
$l=1$	$\frac{4}{315}$	$\frac{-8}{945}$

Table 5.36: Coefficients $\bar{C}_{111lr}$

	$r=0$	$r=1$
$l=0$	$\frac{2}{105}$	$\frac{-8}{945}$
$l=1$	$\frac{2}{945}$	0

In case of pairwise different $i_1, \ldots, i_5$ from tables 5.20–5.36 we have

$$\mathsf{M}\left\{\left(I_{100_{T,t}}^{(i_1 i_2 i_3)} - I_{100_{T,t}}^{(i_1 i_2 i_3)q_3}\right)^2\right\} = \frac{(T-t)^5}{60} - \sum_{i,j,k=0}^{q_3}\left(C_{kji}^{100}\right)^2 \approx$$

$$\approx 0.00815429(T-t)^5,$$

$$\mathsf{M}\left\{\left(I_{010_{T,t}}^{(i_1 i_2 i_3)} - I_{010_{T,t}}^{(i_1 i_2 i_3)q_4}\right)^2\right\} = \frac{(T-t)^5}{20} - \sum_{i,j,k=0}^{q_4}\left(C_{kji}^{010}\right)^2 \approx$$

$$\approx 0.0173903(T-t)^5,$$

$$\mathsf{M}\left\{\left(I_{001_{T,t}}^{(i_1 i_2 i_3)} - I_{001_{T,t}}^{(i_1 i_2 i_3)q_5}\right)^2\right\} = \frac{(T-t)^5}{10} - \sum_{i,j,k=0}^{q_5}\left(C_{kji}^{001}\right)^2 \approx$$

$$\approx 0.0252801(T-t)^5,$$

$$\mathsf{M}\left\{\left(I_{00000_{T,t}}^{(i_1 i_2 i_3 i_4 i_5)} - I_{00000_{T,t}}^{(i_1 i_2 i_3 i_4 i_5)q_6}\right)^2\right\} = \frac{(T-t)^5}{120} - \sum_{i,j,k,l,r=0}^{q_6} C_{rlkji}^2 \approx$$

$$\approx 0.00759105(T-t)^5.$$

5.2 About Fourier-Legendre coefficients

As we can see from the results of this chapter, the most labor-intensive work while building approximations of multiple stochastic integrals is connected with calculation of coefficients

$$C_{j_k \ldots j_1} = \int\limits_{[t,T]^k} K(t_1, \ldots, t_k) \prod_{l=1}^{k} \phi_{j_l}(t_l) dt_1 \ldots dt_k; \tag{5.38}$$

$$K(t_1, \ldots, t_k) = \begin{cases} \prod\limits_{l=1}^{k} \psi_l(t_l) \prod\limits_{l=1}^{k-1} \mathbf{1}_{\{t_l < t_{l+1}\}} & \text{if } k \geq 2 \\ \psi_1(t_1) & \text{if } k = 1 \end{cases}.$$

Here $(t_1, \ldots, t_k) \in [t, T]^k$; $\{\phi_j(x)\}_{j=0}^{\infty}$ — is full orthonormal system of functions in the space $L_2([t, T])$.

The aim of this section is to identify some features of calculation of Fourier coefficients $C_{j_k\ldots j_1}$ (k — is fixed) for expansions of multiple stochastic integrals from the stochastic Taylor-Ito and Taylor-Stratonovich expansions (see sect.7.8) when using the system of Legendre polynomials.

For classical Taylor-Ito and Taylor-Stratonovich expansions [22], [24]) (see sect.7.8) in (5.38) it is necessary to assume, that $\psi_1(s),\ldots,\psi_k(s)\equiv 1$, and for unified Taylor-Ito and Taylor-Stratonovich expansions [43], [46], [48]) (see sect.7.9) — $\psi_q(s)\equiv (t-s)^{l_q}$; $q=1,\ldots,k$; $l_q=0,\ 1,\ 2,\ldots$.

So, we will calculate the integrals

$$\bar{C}_{j_k\ldots j_1}=\int\limits_t^T\phi_{j_k}(t_k)\int\limits_t^{t_k}\phi_{j_{k-1}}(t_{k-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_{k-1}dt_k;$$

$$\hat{C}_{j_k\ldots j_1}^{l_1\ldots l_k}=\int\limits_t^T(t-t_k)^{l_k}\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}(t-t_1)^{l_1}\phi_{j_1}(t_1)dt_1\ldots dt_k,$$

where $\{\phi_j(x)\}_{j=0}^{\infty}$ — is a full orthonormal system of Legendre polynomials in the space $L_2([t,T])$.

We have

$$\bar{C}_{j_k\ldots j_1}=\frac{(T-t)^{\frac{k}{2}}}{2^k}\prod_{l=1}^k\sqrt{2j_l+1}\cdot A_{j_k\ldots j_1},$$

where

$$A_{j_k\ldots j_1}=\int\limits_{-1}^1 P_{j_k}(t_k)\int\limits_{-1}^{t_k}P_{j_{k-1}}(t_{k-1})\ldots\int\limits_{-1}^{t_2}P_{j_1}(t_1)dt_1\ldots dt_{k-1}dt_k; \qquad (5.39)$$

$\{P_n(x)\}_{n=0}^{\infty}$ — is a full orthonormal system of Legendre polynomials in the space $L_2([-1,1])$:

$$P_n(x)=\frac{1}{2^n}\sum_{q=0}^{[\frac{n}{2}]}\frac{(-1)^q(2n-2q)!}{q!(n-q)!(n-2q)!}x^{n-2q}. \qquad (5.40)$$

Substituting (5.40) into (5.39) we get

$$A_{j_k\ldots j_1}=\frac{1}{2^{j_1+\ldots+j_k}}\sum_{q_1=0}^{[\frac{j_1}{2}]}\ldots\sum_{q_k=0}^{[\frac{j_k}{2}]}\prod_{l=1}^k\frac{(-1)^{q_l}(2j_l-2q_l)!}{q_l!(j_l-q_l)!(j_l-2q_l)!}\times$$

$$\times\int\limits_{-1}^1(t_k)^{j_k-2q_k}\ldots\int\limits_{-1}^{t_2}(t_1)^{j_1-2q_1}dt_1\ldots dt_k.$$

So, calculation of $\bar{C}_{j_k\ldots j_1}$ reduces to calculation of integral

$$I_{p_k\ldots p_1}=\int\limits_{-1}^{1}(t_k)^{p_k}\ldots\int\limits_{-1}^{t_2}(t_1)^{p_1}dt_1\ldots dt_k;\ \ p_1,\ldots,p_k=0,\ 1,\ 2,\ldots. \qquad (5.41)$$

Now, examine $\hat{C}_{j_k\ldots j_1}^{l_1\ldots l_k}$. We have

$$\hat{C}_{j_k\ldots j_1}^{l_1\ldots l_k}=\frac{(T-t)^{\frac{k}{2}}}{2^k}\prod_{l=1}^{k}\sqrt{2j_l+1}\cdot\left(\frac{(-1)(T-t)}{2}\right)^{l_1+\ldots+l_k}I_{j_k\ldots j_1}^{l_1\ldots l_k},$$

where

$$I_{j_k\ldots j_1}^{l_1\ldots l_k}=\int\limits_{-1}^{1}(1+t_k)^{l_k}P_{j_k}(t_k)\ldots\int\limits_{-1}^{t_2}(1+t_1)^{l_1}P_{j_1}(t_1)dt_1\ldots dt_k=$$

$$=\sum_{s_k=0}^{l_k}\ldots\sum_{s_1=0}^{l_1}C_{l_k}^{s_k}\ldots C_{l_1}^{s_1}\int\limits_{-1}^{1}(t_k)^{s_k}P_{j_k}(t_k)\ldots\int\limits_{-1}^{t_2}(t_1)^{s_1}P_{j_1}(t_1)dt_1\ldots dt_k,$$

where C_n^k — is a binomial coefficient.

Further

$$\int\limits_{-1}^{1}(t_k)^{s_k}P_{j_k}(t_k)\ldots\int\limits_{-1}^{t_2}(t_1)^{s_1}P_{j_1}(t_1)dt_1\ldots dt_k=$$

$$=\frac{1}{2^{j_1+\ldots+j_k}}\sum_{q_1=0}^{\left[\frac{j_1}{2}\right]}\ldots\sum_{q_k=0}^{\left[\frac{j_k}{2}\right]}\prod_{l=1}^{k}\frac{(-1)^{q_l}(2j_l-2q_l)!}{q_l!(j_l-q_l)!(j_l-2q_l)!}\times$$

$$\times\int\limits_{-1}^{1}(t_k)^{j_k-2q_k+s_k}\ldots\int\limits_{-1}^{t_2}(t_1)^{j_1-2q_1+s_1}dt_1\ldots dt_k.$$

Consequently, calculation $\hat{C}_{j_k\ldots j_1}^{l_1\ldots l_k}$ again reduces to calculation of integral (5.41). Calculation of integral (5.41) is not a problem:

$$I_{p_1}=\frac{1}{p_1+1}\left(1-(-1)^{p_1+1}\right);$$

$$I_{p_2p_1}=\frac{1}{p_1+1}\left(\frac{1}{p_1+p_2+2}\left(1-(-1)^{p_1+p_2+2}\right)-\frac{(-1)^{p_1+1}}{p_2+1}\left(1-(-1)^{p_2+1}\right)\right);$$

$$I_{p_3p_2p_1}=\frac{1}{p_1+1}\left(\frac{1}{p_1+p_2+2}\left(\frac{1}{p_1+p_2+p_3+3}\left(1-(-1)^{p_1+p_2+p_3+3}\right)-\right.\right.$$

$$
-\frac{(-1)^{p_1+p_2+2}}{p_3+1}\left(1-(-1)^{p_3+1}\right)\Biggr)-
$$

$$
-\frac{(-1)^{p_1+1}}{p_2+1}\Biggl(\frac{1}{p_3+p_2+2}\left(1-(-1)^{p_3+p_2+2}\right)-
$$

$$
-\frac{(-1)^{p_2+1}}{p_3+1}\left(1-(-1)^{p_3+1}\right)\Biggr)\Biggr)
$$

$$
\ldots
$$

Actually, the integral of type $I_{p_k\ldots p_1}$ may be calculated for various values k using computer packs of symbol transformations of type DERIVE or MAPLE.

It will not be easy if we use trigonometric functions instead of Legendre polynomials. It is connected with the fact, that integrals

$$
\bar{C}_{j_k\ldots j_1}=\int\limits_t^T\phi_{j_k}(t_k)\int\limits_t^{t_k}\phi_{j_{k-1}}(t_{k-1})\ldots\int\limits_t^{t_2}\phi_{j_1}(t_1)dt_1\ldots dt_{k-1}dt_k;
$$

$$
\hat{C}_{j_k\ldots j_1}^{l_1\ldots l_k}=\int\limits_t^T(t-t_k)^{l_k}\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}(t-t_1)^{l_1}\phi_{j_1}(t_1)dt_1\ldots dt_k,
$$

where $\{\phi_j(x)\}_{j=0}^{\infty}$ — is a full orthonormal system of trigonometric functions in the space $L_2([t,T])$:

$$
\phi_j(s)=\frac{1}{\sqrt{T-t}}\begin{cases}1 & \text{for } j=0\\ \sqrt{2}\sin\frac{2\pi r(s-t)}{T-t} & \text{when } j=2r-1\\ \sqrt{2}\cos\frac{2\pi r(s-t)}{T-t} & \text{when } j=2r\end{cases};
$$

$r=1,\ 2,\ldots$ "ramify intensively" for various combinations of indexes $j_1,\ldots,j_k$ i.e. in cases of various combinations of indexes $j_1,\ldots,j_k$ the mentioned integrals are calculated using significantly different formulas, moreover the number of these formulas grows abruptly with the growth of multiplicity of stochastic integral. It is obvious, that even when $k=4$, calculations become very complicated.

Let's explain the mentioned idea using an example.

Using trigonometric functions, for example, there could be a necessity to integrate the product of the following form:

$$
\sin\frac{2\pi r(s-t)}{T-t}\sin\frac{2\pi q(s-t)}{T-t}\ \ (r,q\ge 0),
$$

which equals to

$$
\frac{1}{2}\left(-\cos\frac{2\pi(r+q)(s-t)}{T-t}+\cos\frac{2\pi(r-q)q(s-t)}{T-t}\right)
$$

It is clear, that integrating the last expression the following cases may occur:
1. $r+q\neq 0$ and $r-q\neq 0$;
2. $r+q\neq 0$ and $r-q=0$;
3. $r+q=0$ and $r-q=0$.

In each of three cases, the primitive function will be calculated using "its own formula".

If we use in previous reasonings the system of Legendre polynomials, then there are no different cases and integration will be more simple.

Since the product of two polynomials is a polynomial and integrating polynomials we actually use only the formula of primitive function from the power function with non-negative degree index.

5.3 Approximation of specific multiple stochastic integrals of multiplicities 1–3 using the trigonometric system of functions

Let's examine approximations of some multiple stochastic integrals of the following types

$$I_{l_1\ldots l_k T,t}^{(i_1\ldots i_k)}=\int\limits_t^T (t-t_k)^{l_k}\ldots\int\limits_t^{t_2}(t-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)},$$

$$I_{l_1\ldots l_k T,t}^{*(i_1\ldots i_k)}=\int\limits_t^{*T} (t-t_k)^{l_k}\ldots\int\limits_t^{*t_2}(t-t_1)^{l_1}d\mathbf{f}_{t_1}^{(i_1)}\ldots d\mathbf{f}_{t_k}^{(i_k)},$$

obtained using theorems 1 – 4 and using the trigonometric system of functions:

$$I_{0_{T,t}}^{(i_1)}=\sqrt{T-t}\zeta_0^{(i_1)}, \tag{5.42}$$

$$I_{1_{T,t}}^{(i_1)q}=-\frac{(T-t)^{\frac{3}{2}}}{2}\Biggl[\zeta_0^{(i_1)}-\frac{\sqrt{2}}{\pi}\Biggl(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)}+\sqrt{\alpha_q}\xi_q^{(i_1)}\Biggr)\Biggr], \tag{5.43}$$

$$I_{00_{T,t}}^{*(i_2i_1)q}=\frac{1}{2}(T-t)\Biggl[\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\frac{1}{\pi}\sum_{r=1}^{q}\frac{1}{r}\Bigl\{\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}+$$

$$+\sqrt{2}\left(\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\right)\Bigr\}+\frac{\sqrt{2}}{\pi}\sqrt{\alpha_q}\left(\xi_q^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\xi_q^{(i_2)}\right)\Biggr], \tag{5.44}$$

$$
I_{000_{T,t}}^{*(i_3i_2i_1)q} = (T-t)^{\frac{3}{2}}\Biggl(\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}+\frac{\sqrt{\alpha_q}}{2\sqrt{2}\pi}\left(\xi_q^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-\xi_q^{(i_3)}\zeta_0^{(i_2)}\zeta_0^{(i_1)}\right)
$$

$$
+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\left(\mu_q^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-2\mu_q^{(i_2)}\zeta_0^{(i_1)}\zeta_0^{(i_3)}+\mu_q^{(i_3)}\zeta_0^{(i_1)}\zeta_0^{(i_2)}\right)+
$$

$$
+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\Biggl[\frac{1}{\pi r}\left\{\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_2)}\zeta_0^{(i_1)}-\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}\right\}+
$$

$$
+\frac{1}{\pi^2r^2}\left\{\zeta_{2r}^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-2\zeta_{2r}^{(i_2)}\zeta_0^{(i_3)}\zeta_0^{(i_1)}+\zeta_{2r}^{(i_3)}\zeta_0^{(i_2)}\zeta_0^{(i_1)}\right\}\Biggr]+
$$

$$
+\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\ne l}}^{q}\Biggl[\frac{1}{r^2-l^2}\Biggl\{\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_2)}\zeta_0^{(i_3)}-\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\zeta_{2l}^{(i_3)}+
$$

$$
+\frac{r}{l}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\zeta_0^{(i_3)}-\frac{l}{r}\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2l-1}^{(i_3)}\Biggr\}-\frac{1}{rl}\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}\zeta_{2l-1}^{(i_3)}\Biggr]+
$$

$$
+\sum_{r=1}^{q}\Biggl[\frac{1}{4\pi r}\left\{\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_0^{(i_3)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_3)}-\zeta_{2r-1}^{(i_2)}\zeta_{2r}^{(i_3)}\zeta_0^{(i_1)}+\zeta_{2r-1}^{(i_3)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\right\}
$$

$$
+\frac{1}{8\pi^2r^2}\Biggl\{3\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_0^{(i_3)}+\zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_3)}-6\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_2)}+
$$

$$
+3\zeta_{2r-1}^{(i_2)}\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_1)}-2\zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_3)}\zeta_0^{(i_2)}+\zeta_{2r}^{(i_3)}\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\Biggr\}\Biggr]+
$$

$$
+\frac{1}{4\sqrt{2}\pi^2}\Biggl\{\sum_{r,m=1}^{q}\Biggl[\frac{2}{rm}\Biggl[-\zeta_{2r-1}^{(i_1)}\zeta_{2m-1}^{(i_2)}\zeta_{2m}^{(i_3)}+\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_{2m-1}^{(i_3)}+
$$

$$
+\zeta_{2r-1}^{(i_1)}\zeta_{2m}^{(i_2)}\zeta_{2m-1}^{(i_3)}-\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2m-1}^{(i_3)}\Biggr]+
$$

$$
+\frac{1}{m(r+m)}\Biggl[-\zeta_{2(m+r)}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_{2m}^{(i_3)}-\zeta_{2(m+r)-1}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2m}^{(i_3)}-
$$

$$
-\zeta_{2(m+r)-1}^{(i_1)}\zeta_{2r}^{(i_2)}\zeta_{2m-1}^{(i_3)}+\zeta_{2(m+r)}^{(i_1)}\zeta_{2r-1}^{(i_2)}\zeta_{2m-1}^{(i_3)}\Biggr]\Biggr]+
$$

$$
+\sum_{m=1}^{q}\sum_{l=m+1}^{q}\Biggl[\frac{1}{m(l-m)}\Biggl[\zeta_{2(l-m)}^{(i_1)}\zeta_{2l}^{(i_2)}\zeta_{2m}^{(i_3)}+\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\zeta_{2m}^{(i_3)}-
$$

$$
-\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2l}^{(i_2)}\zeta_{2m-1}^{(i_3)}+\zeta_{2(l-m)}^{(i_1)}\zeta_{2l-1}^{(i_2)}\zeta_{2m-1}^{(i_3)}\Biggr]+
$$

$$
+\frac{1}{l(l-m)}\Biggl[-\zeta_{2(l-m)}^{(i_1)}\zeta_{2m}^{(i_2)}\zeta_{2l}^{(i_3)}+\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2m-1}^{(i_2)}\zeta_{2l}^{(i_3)}-
$$

$$
-\zeta_{2(l-m)-1}^{(i_1)}\zeta_{2m}^{(i_2)}\zeta_{2l-1}^{(i_3)}-\zeta_{2(l-m)}^{(i_1)}\zeta_{2m-1}^{(i_2)}\zeta_{2l-1}^{(i_3)}\Biggr]\Biggr]\Biggr\}\Biggr), \qquad (5.45)
$$

$$I_{10_{T,t}}^{*(i_2 i_1)q} = (T-t)^2\Biggl(-\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_2)} - \frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\xi_q^{(i_1)}\zeta_0^{(i_2)} +$$

$$+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\left(2\mu_q^{(i_2)}\zeta_0^{(i_1)} - \mu_q^{(i_1)}\zeta_0^{(i_2)}\right) +$$

$$+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\Biggl[-\frac{1}{\pi r}\Biggl\{\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)} - \frac{1}{\sqrt{2}}\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)} + \frac{1}{\sqrt{2}}\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}\Biggr\} +$$

$$+\frac{1}{\pi^2 r^2}\Biggl(-\zeta_{2r}^{(i_1)}\zeta_0^{(i_2)} + 2\zeta_0^{(i_1)}\zeta_{2r}^{(i_2)} - \frac{3}{2\sqrt{2}}\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_2)} - \frac{1}{2\sqrt{2}}\zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_2)}\Biggr)\Biggr] +$$

$$+\frac{1}{2\pi^2}\sum_{\substack{k,l=1\\k\ne l}}^{q}\frac{1}{l^2-k^2}\Biggl[\zeta_{2k}^{(i_1)}\zeta_{2l}^{(i_2)} - \frac{k}{l}\zeta_{2k-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\Biggr]\Biggr), \qquad (5.46)$$

$$I_{01_{T,t}}^{*(i_2 i_1)q} = (T-t)^2\Biggl(-\frac{1}{3}\zeta_0^{(i_1)}\zeta_0^{(i_2)} + \frac{1}{\sqrt{2}\pi}\sqrt{\alpha_q}\left(\frac{1}{2}\xi_q^{(i_2)}\zeta_0^{(i_1)} - \xi_q^{(i_1)}\zeta_0^{(i_2)}\right) +$$

$$+\frac{1}{\sqrt{2}\pi^2}\sqrt{\beta_q}\left(\frac{1}{2}\mu_q^{(i_2)}\zeta_0^{(i_1)} - \mu_q^{(i_1)}\zeta_0^{(i_2)}\right) +$$

$$+\frac{1}{\sqrt{2}}\sum_{r=1}^{q}\left[\frac{1}{\pi r}\left\{\frac{1}{2}\zeta_0^{(i_1)}\zeta_{2r-1}^{(i_2)} - \zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)} + \frac{1}{2\sqrt{2}}\left\{\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)} - \zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}\right\}\right\} +\right.$$

$$\left.+\frac{1}{\pi^2 r^2}\left(\frac{1}{2}\zeta_0^{(i_1)}\zeta_{2r}^{(i_2)} - \zeta_{2r}^{(i_1)}\zeta_0^{(i_2)} + \frac{3}{4\sqrt{2}}\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_2)} + \frac{1}{4\sqrt{2}}\zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_2)}\right)\right] +$$

$$+\frac{1}{2\pi^2}\sum_{\substack{k,l=1\\k\ne l}}^{q}\frac{1}{l^2-k^2}\Biggl[\frac{l}{k}\zeta_{2k-1}^{(i_1)}\zeta_{2l-1}^{(i_2)} + \zeta_{2k}^{(i_1)}\zeta_{2l}^{(i_2)}\Biggr]\Biggr), \qquad (5.47)$$

$$I_{2_{T,t}}^{(i_1)} = (T-t)^{\frac{5}{2}}\Biggl[\frac{1}{3}\zeta_0^{(i_1)} + \frac{1}{\sqrt{2}\pi^2}\Biggl(\sum_{r=1}^{q}\frac{1}{r^2}\zeta_{2r}^{(i_1)} + \sqrt{\beta_q}\mu_q^{(i_1)}\Biggr) +$$

$$+\frac{1}{\sqrt{2}\pi}\Biggl(\sum_{r=1}^{q}\frac{1}{r}\zeta_{2r-1}^{(i_1)} + \sqrt{\alpha_q}\xi_q^{(i_1)}\Biggr)\Biggr], \qquad (5.48)$$

where

$$\xi_q^{(i)} = \frac{1}{\sqrt{\alpha_q}}\sum_{r=q+1}^{\infty}\frac{1}{r}\,\zeta_{2r-1}^{(i)};\ \ \alpha_q = \frac{\pi^2}{6} - \sum_{r=1}^{q}\frac{1}{r^2};\ \ \mu_q^{(i)} = \frac{1}{\sqrt{\beta_q}}\sum_{r=q+1}^{\infty}\frac{1}{r^2}\,\zeta_{2r}^{(i)};$$

$$\beta_q = \frac{\pi^4}{90} - \sum_{r=1}^{q}\frac{1}{r^4};\ \ \zeta_j^{(i)} = \int_t^T \phi_j(s)d\mathbf{f}_s^{(i)};$$

$\phi_j(s)$ has the form (4.19); $\zeta_0^{(i)}$, $\zeta_{2r}^{(i)}$, $\zeta_{2r-1}^{(i)}$, $\xi_q^{(i)}$, $\mu_q^{(i)}$; $r=1,\ldots,q$; $i=1,\ldots,m$ — are independent standard Gaussian random variables; $i_1,\ i_2,\ i_3 = 1,\ldots,m$.

Let's analyze the mean-square errors of approximations (5.44)–(5.47). From relations (5.44)–(5.47) when $i_1\ne i_2,\ i_2\ne i_3,\ i_1\ne i_3$ we get

$$\mathsf{M}\left\{\left(I_{00_{T,t}}^{*(i_2i_1)}-I_{00_{T,t}}^{*(i_2i_1)q}\right)^2\right\}=\frac{(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}\right), \tag{5.49}$$

$$\mathsf{M}\left\{\left(I_{000_{T,t}}^{(i_3i_2i_1)}-I_{000_{T,t}}^{(i_3i_2i_1)q}\right)^2\right\}=$$

$$=(T-t)^3\left\{\frac{1}{4\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}\right)+\frac{55}{32\pi^4}\left(\frac{\pi^4}{90}-\sum_{r=1}^{q}\frac{1}{r^4}\right)+\right.$$

$$\left.+\frac{1}{4\pi^4}\left(\sum_{\substack{r,l=1\\ r\ne l}}^{\infty}-\sum_{\substack{r,l=1\\ r\ne l}}^{q}\right)\frac{5l^4+4r^4-3l^2r^2}{r^2l^2(r^2-l^2)^2}\right\}, \tag{5.50}$$

$$\mathsf{M}\left\{\left(I_{01_{T,t}}^{*(i_1)}-I_{01_{T,t}}^{*(i_1)q}\right)^2\right\}=(T-t)^4\left\{\frac{1}{8\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}\right)+\right.$$

$$\left.+\frac{5}{32\pi^4}\left(\frac{\pi^4}{90}-\sum_{r=1}^{q}\frac{1}{r^4}\right)+\frac{1}{4\pi^4}\left(\sum_{\substack{k,l=1\\ k\ne l}}^{\infty}-\sum_{\substack{k,l=1\\ k\ne l}}^{q}\right)\frac{l^2+k^2}{k^2(l^2-k^2)^2}\right\}, \tag{5.51}$$

$$\mathsf{M}\left\{\left(I_{10_{T,t}}^{*(i_2i_1)}-I_{10_{T,t}}^{*(i_2i_1)q}\right)^2\right\}=(T-t)^4\left\{\frac{1}{8\pi^2}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}\right)+\right.$$

$$\left.+\frac{5}{32\pi^4}\left(\frac{\pi^4}{90}-\sum_{r=1}^{q}\frac{1}{r^4}\right)+\frac{1}{4\pi^4}\left(\sum_{\substack{k,l=1\\ k\ne l}}^{\infty}-\sum_{\substack{k,l=1\\ k\ne l}}^{q}\right)\frac{l^2+k^2}{l^2(l^2-k^2)^2}\right\}. \tag{5.52}$$

It is easy to demonstrate, that relations (5.50), (5.51) and (5.52) may be represented using lemma 10 in the following form:

$$\mathsf{M}\left\{\left(I_{000_{T,t}}^{(i_3i_2i_1)}-I_{000_{T,t}}^{(i_3i_2i_1)q}\right)^2\right\}=(T-t)^3\left\{\frac{4}{45}-\frac{1}{4\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}-\right.$$

$$\left.-\frac{55}{32\pi^4}\sum_{r=1}^{q}\frac{1}{r^4}-\frac{1}{4\pi^4}\sum_{\substack{r,l=1\\ r\ne l}}^{q}\frac{5l^4+4r^4-3r^2l^2}{r^2l^2\left(r^2-l^2\right)^2}\right\}, \tag{5.53}$$

$$
\mathsf{M}\left\{\left(I_{10_{T,t}}^{*(i_2i_1)}-I_{10_{T,t}}^{*(i_2i_1)q}\right)^2\right\}=\frac{(T-t)^4}{4}\left\{\frac{1}{9}-\frac{1}{2\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}-\right.
$$

$$
\left.-\frac{5}{8\pi^4}\sum_{r=1}^{q}\frac{1}{r^4}-\frac{1}{\pi^4}\sum_{\substack{k,l=1\\k\ne l}}^{q}\frac{k^2+l^2}{l^2\left(l^2-k^2\right)^2}\right\}, \tag{5.54}
$$

$$
\mathsf{M}\left\{\left(I_{01_{T,t}}^{*(i_2i_1)}-I_{01_{T,t}}^{*(i_2i_1)q}\right)^2\right\}=\frac{(T-t)^4}{4}\left\{\frac{1}{9}-\frac{1}{2\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}-\right.
$$

$$
\left.-\frac{5}{8\pi^4}\sum_{r=1}^{q}\frac{1}{r^4}-\frac{1}{\pi^4}\sum_{\substack{k,l=1\\k\ne l}}^{q}\frac{l^2+k^2}{k^2\left(l^2-k^2\right)^2}\right\}. \tag{5.55}
$$

Comparing (5.53)–(5.55) and (5.50)–(5.52), note, that

$$
\sum_{\substack{k,l=1\\k\ne l}}^{\infty}\frac{l^2+k^2}{k^2\left(l^2-k^2\right)^2}=\sum_{\substack{k,l=1\\k\ne l}}^{\infty}\frac{l^2+k^2}{l^2\left(l^2-k^2\right)^2}=\frac{\pi^4}{48}, \tag{5.56}
$$

$$
\sum_{\substack{r,l=1\\r\ne l}}^{\infty}\frac{5l^4+4r^4-3r^2l^2}{r^2l^2\left(r^2-l^2\right)^2}=\frac{9\pi^4}{80}. \tag{5.57}
$$

We will mention approximations of stochastic integrals $I_{10_{T,t}}^{*(i_1i_1)}$, $I_{01_{T,t}}^{*(i_1i_1)}$ and the conditions of selecting the number q using the trigonometric system of functions:

$$
I_{10_{T,t}}^{*(i_1i_1)q}=(T-t)^2\Biggl(-\frac{1}{6}\left(\zeta_0^{(i_1)}\right)^2-\frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\xi_q^{(i_1)}\zeta_0^{(i_1)}+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\mu_q^{(i_1)}\zeta_0^{(i_1)}+
$$

$$
+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\left[\frac{1}{\pi^2r^2}\left(\zeta_{2r}^{(i_1)}\zeta_0^{(i_1)}-\frac{3}{2\sqrt{2}}\left(\zeta_{2r-1}^{(i_1)}\right)^2-\frac{1}{2\sqrt{2}}\left(\zeta_{2r}^{(i_1)}\right)^2\right)-\frac{1}{\pi r}\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_1)}\right]+
$$

$$
+\frac{1}{2\pi^2}\sum_{\substack{k,l=1\\k\ne l}}^{q}\frac{1}{l^2-k^2}\left[\zeta_{2k}^{(i_1)}\zeta_{2l}^{(i_1)}-\frac{k}{l}\zeta_{2k-1}^{(i_1)}\zeta_{2l-1}^{(i_1)}\right]\Biggr),
$$

$$
I_{01_{T,t}}^{*(i_1i_1)q}=(T-t)^2\Biggl(-\frac{1}{3}\left(\zeta_0^{(i_1)}\right)^2-\frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\xi_q^{(i_1)}\zeta_0^{(i_1)}-\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\mu_q^{(i_1)}\zeta_0^{(i_1)}+
$$

$$
+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\left[\frac{1}{\pi^2r^2}\left(\frac{3}{2\sqrt{2}}\left(\zeta_{2r-1}^{(i_1)}\right)^2+\frac{1}{2\sqrt{2}}\left(\zeta_{2r}^{(i_1)}\right)^2-\zeta_{2r}^{(i_1)}\zeta_0^{(i_1)}\right)-\frac{1}{\pi r}\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_1)}\right]+
$$

$$
+\frac{1}{2\pi^2}\sum_{\substack{k,l=1\\k\ne l}}^{q}\frac{1}{l^2-k^2}\left[\frac{l}{k}\zeta_{2k-1}^{(i_1)}\zeta_{2l-1}^{(i_1)}+\zeta_{2k}^{(i_1)}\zeta_{2l}^{(i_1)}\right]\Biggr).
$$

Table 5.37: Check of formula (5.53)

$\varepsilon/(T-t)^3$	0.0459	0.0072	$7.5722\cdot 10^{-4}$	$7.5973\cdot 10^{-5}$	$7.5990\cdot 10^{-6}$
q	1	10	100	1000	10000

Then we will get

$$
\mathsf{M}\left\{\left(I_{01_{T,t}}^{*(i_1i_1)}-I_{01_{T,t}}^{*(i_1i_1)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{10_{T,t}}^{*(i_1i_1)}-I_{10_{T,t}}^{*(i_1i_1)q}\right)^2\right\}=
$$

$$
=\frac{(T-t)^4}{4}\Biggl[\frac{2}{\pi^4}\left(\frac{\pi^4}{90}-\sum_{r=1}^{q}\frac{1}{r^4}\right)+\frac{1}{\pi^4}\left(\frac{\pi^2}{6}-\sum_{r=1}^{q}\frac{1}{r^2}\right)^2+
$$

$$
+\frac{1}{\pi^4}\left(\sum_{\substack{k,l=1\\k\ne l}}^{\infty}-\sum_{\substack{k,l=1\\k\ne l}}^{q}\right)\frac{l^2+k^2}{k^2(l^2-k^2)^2}\Biggr]. \qquad (5.58)
$$

Considering (5.56) we will rewrite relation (5.58) in the following form

$$
\mathsf{M}\left\{\left(I_{01_{T,t}}^{*(i_1i_1)}-I_{01_{T,t}}^{*(i_1i_1)q}\right)^2\right\}=\mathsf{M}\left\{\left(I_{10_{T,t}}^{*(i_1i_1)}-I_{10_{T,t}}^{*(i_1i_1)q}\right)^2\right\}=
$$

$$
=\frac{(T-t)^4}{4}\Biggl[\frac{17}{240}-\frac{1}{3\pi^2}\sum_{r=1}^{q}\frac{1}{r^2}-\frac{2}{\pi^4}\sum_{r=1}^{q}\frac{1}{r^4}+
$$

$$
+\frac{1}{\pi^4}\left(\sum_{r=1}^{q}\frac{1}{r^2}\right)^2-\frac{1}{\pi^4}\sum_{\substack{k,l=1\\k\ne l}}^{q}\frac{l^2+k^2}{k^2(l^2-k^2)^2}\Biggr]. \qquad (5.59)
$$

In tables 5.37 – 5.39 we check numerically formulas (5.53) – (5.55), (5.59) for various values q. In tables 5.37 – 5.39, ε — means the right parts of mentioned formulas.

Formulas (5.56), (5.57) appear to be very interesting. Let's confirm numerically their rightness (tables 5.40, 5.41; ε_p — is an absolute deviation of multiple partial sums with the upper limit of summation p for series (5.56), (5.57) from the right parts of formulas (5.56), (5.57); convergence of multiple series is regarded here when $p_1=p_2=p\to\infty$, which is acceptable according to theorem 1).

Using the trigonometric system of functions, let's analyze approximations of multiple stochastic integrals of the following form:

$$
J_{(\lambda_k\ldots\lambda_1)s,t}^{*(i_k\ldots i_1)}=\int\limits_t^{*s}\ldots\int\limits_t^{*\tau_2}d\mathbf{w}_{\tau_1}^{(i_k)}\ldots d\mathbf{w}_{\tau_k}^{(i_1)},
$$

Table 5.38: Check of formulas (5.54), (5.55)

$4\varepsilon/(T-t)^4$	0.0540	0.0082	$8.4261\cdot 10^{-4}$	$8.4429\cdot 10^{-5}$	$8.4435\cdot 10^{-6}$
q	1	10	100	1000	10000

Table 5.39: Check of formula (5.59)

$4\varepsilon/(T-t)^4$	0.0268	0.0034	$3.3955\cdot 10^{-4}$	$3.3804\cdot 10^{-5}$	$3.3778\cdot 10^{-6}$
q	1	10	100	1000	10000

where $\mathbf{w}_{\tau_l}^{(i)}=\mathbf{f}_{\tau_l}^{(i)};\ i=1,\ldots,m$ if $\lambda_l=1$ and $\mathbf{w}_{\tau_l}^{(0)}=\tau$ if $\lambda_l=0.$

It is easy to see, that approximations $J_{(\lambda_2\lambda_1)T,t}^{*(i_2i_1)q},\ J_{(\lambda_3\lambda_2\lambda_1)T,t}^{*(i_3i_2i_1)q}$ of stochastic integrals $J_{(\lambda_2\lambda_1)T,t}^{*(i_2i_1)},\ J_{(\lambda_3\lambda_2\lambda_1)T,t}^{*(i_3i_2i_1)}$ are detected by the right parts of formulas (5.44), (5.45), where it is necessary to take $\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)};\ i_1,\ i_2,\ i_3=0,\ 1,\ldots,m.$

Since

$$\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(0)}=\begin{cases}\sqrt{T-t} & \text{if } j=0\\ 0 & \text{if } j\neq 0\end{cases},$$

then it is easy to get from (5.44) and (5.45), considering, that in these equalities $\zeta_j^{(i)}=\int\limits_t^T\phi_j(s)d\mathbf{w}_s^{(i)};\ i_1,\ i_2,\ i_3=0,\ 1,\ldots,m$, the following family of formulas:

$$J_{(10)T,t}^{(i_20)q}=\frac{1}{2}(T-t)^{\frac{3}{2}}\left[\zeta_0^{(i_2)}+\frac{\sqrt{2}}{\pi}\left(\sum_{r=1}^q\frac{1}{r}\zeta_{2r-1}^{(i_2)}+\sqrt{\alpha_q}\xi_q^{(i_2)}\right)\right],$$

$$J_{(01)T,t}^{(0i_1)q}=\frac{1}{2}(T-t)^{\frac{3}{2}}\left[\zeta_0^{(i_1)}-\frac{\sqrt{2}}{\pi}\left(\sum_{r=1}^q\frac{1}{r}\zeta_{2r-1}^{(i_1)}+\sqrt{\alpha_q}\xi_q^{(i_2)}\right)\right],$$

$$J_{(001)T,t}^{(00i_1)q}=(T-t)^{\frac{5}{2}}\left[\frac{1}{6}\zeta_0^{(i_1)}+\frac{1}{2\sqrt{2}\pi^2}\left(\sum_{r=1}^q\frac{1}{r^2}\zeta_{2r}^{(i_1)}+\sqrt{\beta_q}\mu_q^{(i_1)}\right)+\right.$$

$$\left.+\frac{1}{2\sqrt{2}\pi}\left(\sum_{r=1}^q\frac{1}{r}\zeta_{2r-1}^{(i_1)}+\sqrt{\alpha_q}\xi_q^{(i_1)}\right)\right],$$

Table 5.40: Check of formula (5.56)

ε_p	2.0294	0.3241	0.0330	0.0033	$3.2902\cdot 10^{-4}$
q	1	10	100	1000	10000

Table 5.41: Check of formula (5.57)

ε_p	10.9585	1.8836	0.1968	0.0197	0.0020
q	1	10	100	1000	10000

$$
J_{(010)T,t}^{(0i_20)q}=(T-t)^{\frac{5}{2}}\Biggl[\frac{1}{6}\zeta_0^{(i_2)}-\frac{1}{\sqrt{2}\pi^2}\Biggl(\sum_{r=1}^q\frac{1}{r^2}\zeta_{2r}^{(i_2)}+\sqrt{\beta_q}\mu_q^{(i_2)}\Biggr)\Biggr],
$$

$$
J_{(100)T,t}^{(i_300)q}=(T-t)^{\frac{5}{2}}\Biggl[\frac{1}{6}\zeta_0^{(i_3)}+\frac{1}{2\sqrt{2}\pi^2}\Biggl(\sum_{r=1}^q\frac{1}{r^2}\zeta_{2r}^{(i_3)}+\sqrt{\beta_q}\mu_q^{(i_3)}\Biggr)-
$$

$$
-\frac{1}{2\sqrt{2}\pi}\Biggl(\sum_{r=1}^q\frac{1}{r}\zeta_{2r-1}^{(i_3)}+\sqrt{\alpha_q}\xi_q^{(i_3)}\Biggr)\Biggr],
$$

$$
J_{(011)T,t}^{*(0i_2i_1)q}=(T-t)^2\Biggl(\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\xi_q^{(i_1)}\zeta_0^{(i_2)}+
$$

$$
+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\Biggl(\mu_q^{(i_1)}\zeta_0^{(i_2)}-2\mu_q^{(i_2)}\zeta_0^{(i_1)}\Biggr)+
$$

$$
+\frac{1}{2\sqrt{2}}\sum_{r=1}^q\Biggl[\frac{1}{\pi r}\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_2)}+\frac{1}{\pi^2r^2}\left\{\zeta_{2r}^{(i_1)}\zeta_0^{(i_2)}-2\zeta_{2r}^{(i_2)}\zeta_0^{(i_1)}\right\}\Biggr]+
$$

$$
+\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\ r\ne l}}^q\frac{1}{r^2-l^2}\Biggl[\zeta_{2r}^{(i_1)}\zeta_{2l}^{(i_2)}+\frac{r}{l}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_2)}\Biggr]+
$$

$$
+\sum_{r=1}^q\Biggl[\frac{1}{4\pi r}\left\{\zeta_{2r}^{(i_1)}\zeta_{2r-1}^{(i_2)}-\zeta_{2r-1}^{(i_1)}\zeta_{2r}^{(i_2)}\right\}+
$$

$$
+\frac{1}{8\pi^2r^2}\left\{3\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_2)}+\zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_2)}\right\}\Biggr]\Biggr),
$$

$$
J_{(110)T,t}^{*(i_3i_20)q}=(T-t)^2\Biggl(\frac{1}{6}\zeta_0^{(i_3)}\zeta_0^{(i_2)}-\frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\xi_q^{(i_3)}\zeta_0^{(i_2)}+
$$

$$
+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\Biggl(\mu_q^{(i_3)}\zeta_0^{(i_2)}-2\mu_q^{(i_2)}\zeta_0^{(i_3)}\Biggr)+
$$

$$
+\frac{1}{2\sqrt{2}}\sum_{r=1}^q\Biggl[-\frac{1}{\pi r}\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_2)}+\frac{1}{\pi^2r^2}\left\{-2\zeta_{2r}^{(i_2)}\zeta_0^{(i_3)}+\zeta_{2r}^{(i_3)}\zeta_0^{(i_2)}\right\}\Biggr]+
$$

$$+\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\r\ne l}}^{q}\frac{1}{l^2-r^2}\Biggl[\frac{l}{r}\zeta_{2r-1}^{(i_2)}\zeta_{2l-1}^{(i_3)}+\zeta_{2r}^{(i_2)}\zeta_{2l}^{(i_3)}\Biggr]+$$

$$+\sum_{r=1}^{q}\Biggl[\frac{1}{4\pi r}\left\{-\zeta_{2r-1}^{(i_2)}\zeta_{2r}^{(i_3)}+\zeta_{2r-1}^{(i_3)}\zeta_{2r}^{(i_2)}\right\}+$$

$$+\frac{1}{8\pi^2 r^2}\left\{3\zeta_{2r-1}^{(i_2)}\zeta_{2r-1}^{(i_3)}+\zeta_{2r}^{(i_3)}\zeta_{2r}^{(i_2)}\right\}\Biggr]\Biggr),$$

$$J_{(101)T,t}^{*(i_3 0 i_1)q}=(T-t)^2\Biggl(\frac{1}{6}\zeta_0^{(i_1)}\zeta_0^{(i_3)}+\frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\left(\xi_q^{(i_1)}\zeta_0^{(i_3)}-\xi_q^{(i_3)}\zeta_0^{(i_1)}\right)+$$

$$+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\left(\mu_q^{(i_1)}\zeta_0^{(i_3)}+\mu_q^{(i_3)}\zeta_0^{(i_1)}\right)+$$

$$+\frac{1}{2\sqrt{2}}\sum_{r=1}^{q}\Biggl[\frac{1}{\pi r}\left\{\zeta_{2r-1}^{(i_1)}\zeta_0^{(i_3)}-\zeta_{2r-1}^{(i_3)}\zeta_0^{(i_1)}\right\}+$$

$$+\frac{1}{\pi^2 r^2}\left\{\zeta_{2r}^{(i_1)}\zeta_0^{(i_3)}+\zeta_{2r}^{(i_3)}\zeta_0^{(i_1)}\right\}\Biggr]-\frac{1}{2\pi^2}\sum_{\substack{r,l=1\\r\ne l}}^{q}\frac{1}{rl}\zeta_{2r-1}^{(i_1)}\zeta_{2l-1}^{(i_3)}-$$

$$-\sum_{r=1}^{q}\frac{1}{4\pi^2 r^2}\left\{3\zeta_{2r-1}^{(i_1)}\zeta_{2r-1}^{(i_3)}+\zeta_{2r}^{(i_1)}\zeta_{2r}^{(i_3)}\right\}\Biggr).$$

5.4 Convergence with probability 1 of expansions of some specific multiple stochastic integrals

Let's address now to the convergence with probability 1. Let's analyze in detail the multiple stochastic integral of type:

$$I_{00_{T,t}}^{*(i_1 i_2)}=\frac{T-t}{2}\Biggl[\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left\{\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right\}\Biggr]. \qquad (5.60)$$

When $i_1=i_2$ from (5.60) we get the following equality:

$$I_{00_{T,t}}^{*(i_1 i_1)}=\frac{1}{2}(T-t)\left(\zeta_0^{(i_1)}\right)^2,$$

which is correct with probability 1 and may be obtained using the Ito formula.

Let's examine the case $i_1\ne i_2$. First, note the well-known fact.

Lemma 11. *If for the sequence of random values ξ_n for some $p>0$ the number series*

$$\sum_{n=1}^{\infty}\mathsf{M}\{|\xi_n|^p\}$$

converges, then the sequence ξ_n converges to zero with probubility 1.

In our specific case:

$$I_{00_{T,t}}^{*(i_1i_2)}=\frac{T-t}{2}\Biggl[\zeta_0^{(i_1)}\zeta_0^{(i_2)}+\sum_{i=1}^{n}\frac{1}{\sqrt{4i^2-1}}\left\{\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right\}\Biggr]+\xi_n,$$

$$\xi_n=\frac{T-t}{2}\sum_{i=n+1}^{\infty}\frac{1}{\sqrt{4i^2-1}}\left\{\zeta_{i-1}^{(i_1)}\zeta_i^{(i_2)}-\zeta_i^{(i_1)}\zeta_{i-1}^{(i_2)}\right\},$$

$$\mathsf{M}\{|\xi_n|^2\}=\int\limits_{[t,T]^2}R_n^2(t_1,t_2)dt_1dt_2=\frac{(T-t)^2}{2}\sum_{i=n+1}^{\infty}\frac{1}{4i^2-1},\tag{5.61}$$

$$\sum_{i=n+1}^{\infty}\frac{1}{4i^2-1}\le\int\limits_{n}^{\infty}\frac{1}{4x^2-1}dx=-\frac{1}{4}\ln\left|1-\frac{2}{2n+1}\right|\le\frac{C_0}{n};\ C_0<\infty,\tag{5.62}$$

$$\xi_n\stackrel{\rm def}{=}\sum_{(t_1,t_2)}\int\limits_t^T\int\limits_t^{t_2}R_n(t_1,t_2)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)}=$$

$$=\int\limits_{[t,T]^2}R_n(t_1,t_2)d\mathbf{f}_{t_1}^{(i_1)}d\mathbf{f}_{t_2}^{(i_2)};\,R_n(t_1,t_1)\equiv 0.$$

Therefore, taking $p=2$ in lemma 11, we may not prove convergence of ξ_n to zero with probability 1, since the series

$$\sum_{n=1}^{\infty}\mathsf{M}\{|\xi_n|^p\}$$

will be majorized by the divergent series of Dirichlet with index 1. Let's take $p=4$ and estimate $\mathsf{M}\{|\xi_n|^4\}$.

According to (1.45) when $k=2$ and $n=2$, (5.61), (5.62) there will be such constants $C,\ C_1<\infty$, that

$$\mathsf{M}\{|\xi_n|^4\}\le C\Biggl(\int\limits_{[t,T]^2}R_n^2(t_1,t_2)dt_1dt_2\Biggr)^2\le\frac{C_1}{n^2}\ \Rightarrow$$

$$\Rightarrow\sum_{n=1}^{\infty}\mathsf{M}\{|\xi_n|^4\}\le C_1\sum_{n=1}^{\infty}\frac{1}{n^2}.\tag{5.63}$$

Since the series in (5.63) converges, then according to lemma 11 $\xi_n \to 0$ when $n\to\infty$ with probability 1 $\Rightarrow I_{00_{T,t}}^{*(i_1i_2)n} \to I_{00_{T,t}}^{*(i_1i_2)}$ when $n\to\infty$ with probubility 1.

Let's analyze stochastic integrals $I_{01_{T,t}}^{*(i_1i_2)}$, $I_{10_{T,t}}^{*(i_1i_2)}$ whose expansions look as (5.7), (5.8). It is obvious, that the case $i_1\ne i_2$ is analyzed absolutely similarly to the above mentioned arguments.

When $i_1=i_2$ from expansions (5.7), (5.8) it is clear, that $\mathsf{M}\{|\xi_n|^2\}\le C_2/n^2$.

It means, that for proving of convergence with probubility 1 we may use $p=2$ in lemma 11.

Expansions (5.3)–(5.5), (5.16) for integrals $I_{0_{T,t}}^{(i_1)}$, $I_{1_{T,t}}^{(i_1)}$, $I_{2_{T,t}}^{(i_1)}$, $I_{3_{T,t}}^{(i_1)}$ are initially correct with probability 1 (they include 1, 2, 3 and 4 members of sum, correspondently). Apparently, using the proposed scheme we may prove convergence with probability 1 of multiple stochastic integrals of multiplicity $k>2$.

5.5 About the structure of functions $K(t_1,\ldots,t_k)$, used in applications

The systems of multiple stochastic integrals (7.26) – (7.29), (7.16), (7.20), are included in the stochastic Taylor expansions (unified and classical), described in chapter 7.

In the context of theorems 1–7, the systems (7.26), (7.27), (7.16) when $k=1,\ 2,\ 3,\ldots$, the systems (7.28), (7.29) when $k=1,\ 2$, as well as stochastic integrals of type (7.28) and (7.29) ($l_1=\ldots=l_k=0$) when $k=3,\ 4$ are to be of some interest.

The functions $K(t_1,\ldots,t_k)$, included in the formulation of theorem 1, for the family (5.1) look as follows:

$$K(t_1,\ldots,t_k)=\begin{cases}(t-t_k)^{l_k}\ldots(t-t_1)^{l_1},\ t_1<\ldots<t_k\\ 0,\ \text{otherwise}\end{cases};\ t_1,\ldots,t_k\in[t,T]. \tag{5.64}$$

In particular, for stochastic integrals $I_{1_{T,t}}^{(i_1)}$, $I_{2_{T,t}}^{(i_1)}$, $I_{00_{T,t}}^{(i_1i_2)}$, $I_{000_{T,t}}^{(i_1i_2i_3)}$, $I_{01_{T,t}}^{(i_1i_2)}$, $I_{10_{T,t}}^{(i_1i_2)}$, $I_{0000_{T,t}}^{(i_1\ldots i_4)}$, $I_{20_{T,t}}^{(i_1i_2)}$, $I_{11_{T,t}}^{(i_1i_2)}$, $I_{02_{T,t}}^{(i_1i_2)}$; $i_1,\ldots,i_4=1,\ldots,m$ the functions $K(t_1,\ldots,t_k)$ of type (5.64) correspondently look as follows:

$$K_1(t_1)=t-t_1,\ K_2(t_1)=(t-t_1)^2, \tag{5.65}$$

$$K_{00}(t_1,t_2)=\begin{cases}1,\ t_1<t_2\\ 0,\ \text{otherwise}\end{cases},\ K_{000}(t_1,t_2,t_3)=\begin{cases}1,\ t_1<t_2<t_3\\ 0,\ \text{otherwise}\end{cases}, \tag{5.66}$$

$$K_{01}(t_1,t_2)=\begin{cases} t-t_2, \ t_1<t_2 \\ 0, \ \text{otherwise} \end{cases}, \ K_{10}(t_1,t_2)=\begin{cases} t-t_1, \ t_1<t_2 \\ 0, \ \text{otherwise} \end{cases}, \tag{5.67}$$

$$K_{0000}(t_1,t_2)=\begin{cases} 1, \ t_1<t_2<t_3<t_4 \\ 0, \ \text{otherwise} \end{cases}, \ K_{20}(t_1,t_2)=\begin{cases} (t-t_1)^2, \ t_1<t_2 \\ 0, \ \text{otherwise} \end{cases}, \tag{5.68}$$

$$K_{11}(t_1,t_2)=\begin{cases} (t-t_1)(t-t_2), \ t_1<t_2 \\ 0, \ \text{otherwise} \end{cases}, \ K_{02}(t_1,t_2)=\begin{cases} (t-t_2)^2, \ t_1<t_2 \\ 0, \ \text{otherwise} \end{cases}, \tag{5.69}$$

where $t_1,\ldots,t_4\in[t,T]$.

It is obvious, that the most simple (with a finite number of members of sum) expansion into the Fourier series using the full orthonormal system of functions in the space $L_2([t,T])$ for the polynomial of finite degree will be its expansion according to the system of Legendre polynomials. The polynomial functions are included in functions (5.65) – (5.69) as their components, so, it is logical to expect, that the most simple expansions into multiple Fourier series for functions (5.65) – (5.69) will be their expansions into multiple Fourier-Legendre series when $l_1^2+\ldots+l_k^2>0$. If $l_1=\ldots=l_k=0$ (see functions $K_{00}(t_1,t_2)$, $K_{000}(t_1,t_2,t_3)$, $K_{0000}(t_1,\ldots,t_4)$), then we can expect, that in this case expansions of the mentioned functions into multiple Fourier series using trigonometric functions and Legendre polynomials will be of the same complexity.

Note, that the given assumptions are confirmed completely (compare formulas (5.4), (5.5), (5.7), (5.8) with formulas (5.43), (5.48), (5.47), (5.46) correspondently). So usage of Legendre polynomials in the considered area is an unquestionable step forward.

Chapter 6

Other methods of strong approximation of multiple stochastic Stratonovich and Ito Integrals

This chapter is devoted to other methods of mean-square approximations of multiple stochastic integrals. For example, we examine Milstein method in comparison with method of multiple Fourier series (theorem 1), combined method, which is hybrid of method of multiple integral sums and method based on theorem 1, method of multiple integral sums. Also we make a comparison (by computational experiments) between the effectiveness of different methods of mean-square approximations of multiple stochastic integrals. We demonstrate, that system of Legendre polynomials gives decreasing of computational costs in comparison with trigonometric system of functions.

6.1 Milstein method of strong approximation of multiple stochastic integrals

6.1.1 Introduction

G.N. Milstein proposed in [23] the method of expansion of stochastic integrals based on expansion of Brownian bridge process into the trigonometric Fourier series with random coefficients.

Let's analyze the Brownian bridge process

$$\mathbf{f}_t - \frac{t}{\Delta}\mathbf{f}_\Delta,\ t \in [0, \Delta],\ \Delta > 0, \tag{6.1}$$

where $\mathbf{f}_t \in \Re^m$ — is a standard vector Wiener process with independent pomponents $\mathbf{f}_t^{(i)};\ i = 1, \ldots, m$.

Let's also analyze the componentwise expansion of process (6.1) into the trigonometric Fourier series converging in the mean-square sense

$$\mathbf{f}_t^{(i)} - \frac{t}{\Delta}\mathbf{f}_\Delta^{(i)} = \frac{1}{2}a_{i,0} + \sum_{r=1}^{\infty}\left(a_{i,r}\cos\frac{2\pi rt}{\Delta} + b_{i,r}\sin\frac{2\pi rt}{\Delta}\right), \qquad (6.2)$$

where

$$a_{i,r} = \frac{2}{\Delta}\int_0^\Delta\left(\mathbf{f}_s^{(i)} - \frac{s}{\Delta}\mathbf{f}_\Delta^{(i)}\right)\cos\frac{2\pi rs}{\Delta}ds,\ \ b_{i,r} = \frac{2}{\Delta}\int_0^\Delta\left(\mathbf{f}_s^{(i)} - \frac{s}{\Delta}\mathbf{f}_\Delta^{(i)}\right)\sin\frac{2\pi rs}{\Delta}ds;$$

$r = 0,\ 1, \ldots;\ i = 1, \ldots, m.$

It is easy to demonstrate [23], that random variables $a_{i,r}, b_{i,r}$ are Gaussian ones and they satisfy the following relations:

$$\mathsf{M}\left\{a_{i,r}b_{i,r}\right\} = \mathsf{M}\left\{a_{i,r}b_{i,k}\right\} = 0,\ \mathsf{M}\left\{a_{i,r}a_{i,k}\right\} = \mathsf{M}\left\{b_{i,r}b_{i,k}\right\} = 0,$$

$$\mathsf{M}\left\{a_{i_1,r}a_{i_2,r}\right\} = \mathsf{M}\left\{b_{i_1,r}b_{i_2,r}\right\} = 0,\ \mathsf{M}\left\{a_{i,r}^2\right\} = \mathsf{M}\left\{b_{i,r}^2\right\} = \frac{\Delta}{2\pi^2r^2},$$

where $i,\ i_1,\ i_2 = 1, \ldots, m;\ r \neq k;\ i_1 \neq i_2.$

According to (6.2) we have

$$\mathbf{f}_t^{(i)} = \mathbf{f}_\Delta^{(i)}\frac{t}{\Delta} + \frac{1}{2}a_{i,0} + \sum_{r=1}^{\infty}\left(a_{i,r}\cos\frac{2\pi rt}{\Delta} + b_{i,r}\sin\frac{2\pi rt}{\Delta}\right), \qquad (6.3)$$

where the series converges in the mean-square sense.

6.1.2 Approximation of multiple stochastic integrals of 1st and 2nd multiplicity

Using the relation (6.3), it is easy to get the following expansions [23], converging in the mean-square sense:

$$\int_0^{*t} d\mathbf{f}_t^{(i)} = \frac{t}{\Delta}\mathbf{f}_\Delta^{(i)} + \frac{1}{2}a_{i,0} + \sum_{r=1}^{\infty}\left(a_{i,r}\cos\frac{2\pi rt}{\Delta} + b_{i,r}\sin\frac{2\pi rt}{\Delta}\right), \qquad (6.4)$$

$$\int_0^{*t}\int_0^{*\tau} d\mathbf{f}_{\tau_1}^{(i)}d\tau = \frac{t^2}{2\Delta}\mathbf{f}_\Delta^{(i)} + \frac{t}{2}a_{i,0} + \frac{\Delta}{2\pi}\sum_{r=1}^{\infty}\frac{1}{r}\left\{a_{i,r}\sin\frac{2\pi rt}{\Delta} - b_{i,r}\left(\cos\frac{2\pi rt}{\Delta} - 1\right)\right\}, \qquad (6.5)$$

$$\int_0^{*t}\int_0^{*\tau} d\tau_1 d\mathbf{f}_\tau^{(i)} = t\int_0^{*t} d\mathbf{f}_t^{(i)} - \int_0^{*t}\int_0^{*\tau} d\mathbf{f}_{\tau_1}^{(i)}d\tau = \frac{t^2}{2\Delta}\mathbf{f}_\Delta^{(i)} + t\sum_{r=1}^{\infty}\left\{a_{i,r}\cos\frac{2\pi rt}{\Delta} + \right.$$

$$+b_{i,r}\sin\frac{2\pi rt}{\Delta}\Bigg\} - \frac{\Delta}{2\pi}\sum_{r=1}^{\infty}\frac{1}{r}\left\{a_{i,r}\sin\frac{2\pi rt}{\Delta} - b_{i,r}\left(\cos\frac{2\pi rt}{\Delta}-1\right)\right\}, \qquad (6.6)$$

$$\int\limits_0^{*t}\int\limits_0^{*\tau} d\mathbf{f}_{\tau_1}^{(i_1)}d\mathbf{f}_{\tau}^{(i_2)} = \frac{1}{\Delta}\mathbf{f}_{\Delta}^{(i_1)}\int\limits_0^{*t}\int\limits_0^{*\tau} d\tau_1 d\mathbf{f}_{\tau}^{(i_2)} + \frac{1}{2}a_{i_1,0}\int\limits_0^{*t} d\mathbf{f}_t^{(i_2)} +$$

$$+\frac{t\pi}{\Delta}\sum_{r=1}^{\infty} r\left(a_{i_1,r}b_{i_2,r} - b_{i_1,r}a_{i_2,r}\right) +$$

$$+\frac{1}{4}\sum_{r=1}^{\infty}\Bigg\{\left(a_{i_1,r}a_{i_2,r} - b_{i_1,r}b_{i_2,r}\right)\left(1-\cos\frac{4\pi rt}{\Delta}\right) +$$

$$+\left(a_{i_1,r}b_{i_2,r} + b_{i_1,r}a_{i_2,r}\right)\sin\frac{4\pi rt}{\Delta} +$$

$$+\frac{2}{\pi r}\mathbf{f}_{\Delta}^{(i_2)}\left(a_{i_1,r}\sin\frac{2\pi rt}{\Delta} + b_{i_1,r}\left(\cos\frac{2\pi rt}{\Delta}-1\right)\right)\Bigg\} +$$

$$+\sum_{k=1}^{\infty}\sum_{r=1(r\neq k)}^{\infty} k\Bigg\{a_{i_1,r}a_{i_2,k}\left[\frac{\cos\left(\frac{2\pi(k+r)t}{\Delta}\right)}{2(k+r)} + \frac{\cos\left(\frac{2\pi(k-r)t}{\Delta}\right)}{2(k-r)} - \frac{k}{k^2-r^2}\right] +$$

$$+a_{i_1,r}b_{i_2,k}\left[\frac{\sin\left(\frac{2\pi(k+r)t}{\Delta}\right)}{2(k+r)} + \frac{\sin\left(\frac{2\pi(k-r)t}{\Delta}\right)}{2(k-r)}\right] +$$

$$+b_{i_1,r}b_{i_2,k}\left[\frac{\cos\left(\frac{2\pi(k-r)t}{\Delta}\right)}{2(k-r)} - \frac{\cos\left(\frac{2\pi(k+r)t}{\Delta}\right)}{2(k+r)} - \frac{r}{k^2-r^2}\right] +$$

$$+\frac{\Delta}{2\pi}b_{i_1,r}a_{i_2,k}\left[\frac{\sin\left(\frac{2\pi(k+r)t}{\Delta}\right)}{2(k+r)} - \frac{\sin\left(\frac{2\pi(k-r)t}{\Delta}\right)}{2(k-r)}\right]\Bigg\}. \qquad (6.7)$$

It is necessary to pay attention to the circumstance, that the double series in (6.7) should be appreciated as a repeated, and not as a multiple (theorem 1), i.e. as a repeated limit of the sequence of double partial sums.

It is connected with the fact, that iterated substitution of expansions of Wiener processes into the multiple stochastic integral results in repeated taking of operation of passage to the limit.

Note, that the multiple series is more preferable, than the repeated one when it is presented approximately by the repeated partial sum, since the convergence of such approximations is provided with any method of jointly convergence to infinity of upper summation limits of repeated partial sum (for clearness we denote them as $p_1, \ldots, p_k$; see theorem 1). In particular, in the most simple case we may assume, that $p_1 = \ldots = p_k = p \to \infty$. At the same time the last condition in the strict sense doesn't guarantee convergence of repeated series with the same partial sum as for considered multiple series.

Hereafter, we will see, that usage of G.N. Milstein method for approximation of simple stochastic integrals of minimum 3rd multiplicity is connected with the problem described before. Note, that in [24] nevertheless the following condition is used not quite reasonable: $p_1 = p_2 = p_3 = p \to \infty$ [24] (p. 202, 204).

Assume, that in relations (6.4)–(6.7) $t = \Delta$ (at that double partial sums of repeated series in (6.7) will become zero).

As a result we will get the following expansions converging in the mean-square sense:

$$\int\limits_0^{*\Delta} d\mathbf{f}_t^{(i)} = \mathbf{f}_\Delta^{(i)}, \tag{6.8}$$

$$\int\limits_0^{*\Delta}\int\limits_0^{*\tau} d\mathbf{f}_{\tau_1}^{(i)} d\tau = \frac{1}{2}\Delta\left(\mathbf{f}_\Delta^{(i)} + a_{i,0}\right), \tag{6.9}$$

$$\int\limits_0^{*\Delta}\int\limits_0^{*\tau} d\tau_1 d\mathbf{f}_{\tau}^{(i)} = \frac{1}{2}\Delta\left(\mathbf{f}_\Delta^{(i)} - a_{i,0}\right), \tag{6.10}$$

$$\int\limits_0^{*\Delta}\int\limits_0^{*\tau} d\mathbf{f}_{\tau_1}^{(i_1)} d\mathbf{f}_{\tau}^{(i_2)} = \frac{1}{2}\mathbf{f}_\Delta^{(i_1)}\mathbf{f}_\Delta^{(i_2)} - \frac{1}{2}\left(a_{i_2,0}\mathbf{f}_\Delta^{(i_1)} - a_{i_1,0}\mathbf{f}_\Delta^{(i_2)}\right) +$$

$$+\pi\sum_{r=1}^{\infty} r\left(a_{i_1,r}b_{i_2,r} - b_{i_1,r}a_{i_2,r}\right). \tag{6.11}$$

Deriving (6.8)–(6.11) we used the relation

$$a_{i,0} = -2\sum_{r=1}^{\infty} a_{i,r}, \tag{6.12}$$

which results from (6.2) when $t = \Delta$.

The explanation, that the obtained expansions converge right to the correspondent stochastic Stratonovich integrals is given in [24].

6.1.3 Comparison with method based on multiple Fourier series

Let's compare expansions of some multiple stochastic Stratonovich integrals of 1st and 2nd multiplicity (here we mean, that integration according to Wiener processes in the multiple stochastic integrals is performed two times, maximum), obtained by G.N. Milstein method and method, based on multiple Fourier series according to trigonometric function.

We will introduce the following standard Gaussian random variables in our analysis:

$$\xi_i = \frac{\mathbf{f}_{\Delta}^{(i)}}{\sqrt{\Delta}}, \quad \rho_{i,r} = \sqrt{\frac{2}{\Delta}}\pi r a_{i,r}, \quad \eta_{i,r} = \sqrt{\frac{2}{\Delta}}\pi r b_{i,r}, \tag{6.13}$$

where $i = 1, \ldots, m;\ r = 1,\ 2, \ldots.$

Due to (6.12) we get

$$a_{i,0} = -\sqrt{2\Delta}\sum_{r=1}^{\infty}\frac{1}{\pi r}\rho_{i,r}. \tag{6.14}$$

Substituting the relations (6.13) and (6.14) into (6.8)–(6.11), we get the following converging in the mean-square sense expansions:

$$\int\limits_0^{*\Delta} d\mathbf{f}_t^{(i)} = \sqrt{\Delta}\xi_i, \tag{6.15}$$

$$\int\limits_0^{*\Delta}\int\limits_0^{*\tau} d\mathbf{f}_{\tau_1}^{(i)} d\tau = \frac{1}{2}\Delta^{\frac{3}{2}}\left(\xi_i - \frac{\sqrt{2}}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\rho_{i,r}\right), \tag{6.16}$$

$$\int\limits_0^{*\Delta}\int\limits_0^{*\tau} d\tau_1 d\mathbf{f}_{\tau}^{(i)} = \frac{1}{2}\Delta^{\frac{3}{2}}\left(\xi_i + \frac{\sqrt{2}}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\rho_{i,r}\right), \tag{6.17}$$

$$\int\limits_0^{*\Delta}\int\limits_0^{*\tau} d\mathbf{f}_{\tau_1}^{(i_1)} d\mathbf{f}_{\tau}^{(i_2)} = \frac{\Delta}{2}\Biggl[\xi_{i_1}\xi_{i_2} + \frac{1}{\pi}\sum_{r=1}^{\infty}\frac{1}{r}\Bigl(\rho_{i_1,r}\eta_{i_2,r} - \eta_{i_1,r}\rho_{i_2,r} + $$

$$+\sqrt{2}\left(\rho_{i_2,r}\xi_{i_1} - \rho_{i_1,r}\xi_{i_2}\right)\Bigr)\Biggr]. \tag{6.18}$$

Considering notations taken by us previously for multiple stochastic integrals we may write down

$$\int\limits_0^{*\Delta} d\mathbf{f}_t^{(i)} = I_{0_{\Delta,0}}^{*(i)} = J_{(1)\Delta,0}^{*(i)}, \tag{6.19}$$

$$\int\limits_0^{*\Delta}\int\limits_0^{*\tau} d\mathbf{f}_{\tau_1}^{(i)} d\tau = \Delta I_{0_{\Delta,0}}^{*(i)} + I_{1_{\Delta,0}}^{*(i)} = J_{(10)\Delta,0}^{*(i0)}, \tag{6.20}$$

$$\int\limits_0^{*\Delta}\int\limits_0^{*\tau} d\tau_1 d\mathbf{f}_\tau^{(i)} = -I_{1_{\Delta,0}}^{*(i)} = J_{(01)\Delta,0}^{*(0i)}, \qquad (6.21)$$

$$\int\limits_0^{*\Delta}\int\limits_0^{*\tau} d\mathbf{f}_{\tau_1}^{(i_1)} d\mathbf{f}_\tau^{(i_2)} = I_{00_{\Delta,0}}^{*(i_1 i_2)} = J_{(11)\Delta,0}^{*(i_1 i_2)}. \qquad (6.22)$$

Substituting the expansions of integrals $I_{0_{\Delta,0}}^{*(i)}$, $I_{1_{\Delta,0}}^{*(i)}$, $I_{00_{\Delta,0}}^{*(i_2 i_1)}$, obtained before using the method based on multiple Fourier series according to trigonometric system of functions into representations (6.19)–(6.22), with accuracy up to notations we get expansions (6.15)–(6.18). It testifies, that at least for analyzed multiple stochastic Stratonovich integrals and trigonometric system of functions, the method of G.N. Milstein and the method based on multiple Fourier series give the same result (it is an interesting fact, although it is rather expectable).

In the next section we will discuss usage of G.N. Milstein method for multiple stochastic integrals of 3rd multiplicity.

6.1.4 About problems of Milstein method in relation to multiple stochastic integrals of multiplicities above the second

We mentioned before, that technical peculiarities of the G.N. Milstein method may result in repeated series (in contradiction to multiple series taken from theorem 1) taken from the product of standard Gaussian random values. In case of the simplest stochastic integral of 2nd multiplicity, this problem was avoided as we saw in the previous section. However, the situation is not the same for the simplest stochastic integrals of 3rd multiplicity.

Let's mention the expansion of multiple stochastic Stratonovich integral of 3rd multiplicity obtained in [24] by the method of G.N. Milstein:

$$J_{(111)\Delta,0}^{*(i_1 i_2 i_3)} = \frac{1}{\Delta} J_{(1)\Delta,0}^{*(i_1)} J_{(011)\Delta,0}^{*(0 i_2 i_3)} + \frac{1}{2} a_{i_1,0} J_{(11)\Delta,0}^{*(i_2 i_3)} + \frac{1}{2\pi} b_{i_1} J_{(1)\Delta,0}^{(i_2)} J_{(1)\Delta,0}^{*(i_3)} -$$

$$-\Delta J_{(1)\Delta,0}^{*(i_2)} B_{i_1 i_3} + \Delta J_{(1)\Delta,0}^{*(i_3)} \left(\frac{1}{2} A_{i_1 i_2} - C_{i_2 i_1}\right) + \Delta^{\frac{3}{2}} D_{i_1 i_2 i_3}, \qquad (6.23)$$

where

$$J_{(011)\Delta,0}^{*(0 i_2 i_3)} = \frac{1}{6} J_{(1)\Delta,0}^{*(i_2)} J_{(1)\Delta,0}^{*(i_3)} - \frac{1}{\pi} \Delta J_{(1)\Delta,0}^{*(i_3)} b_{i_2} +$$

$$+\Delta^2 B_{i_2 i_3} - \frac{1}{4} \Delta a_{i_3,0} J_{(1)\Delta,0}^{*(i_2)} + \frac{1}{2\pi} \Delta b_{i_3} J_{(1)\Delta,0}^{*(i_2)} + \Delta^2 C_{i_2 i_3} + \frac{1}{2} \Delta^2 A_{i_2 i_3},$$

$$A_{i_2 i_3} = \frac{\pi}{\Delta}\sum_{r=1}^{\infty} r\left(a_{i_2,r}b_{i_3,r} - b_{i_2,r}a_{i_3,r}\right),$$

$$C_{i_2 i_3} = -\frac{1}{\Delta}\sum_{l=1}^{\infty}\sum_{r=1(r\neq l)}^{\infty}\frac{r}{r^2-l^2}\left(ra_{i_2,r}a_{i_3,l} + lb_{i_2,r}b_{i_3,l}\right),$$

$$B_{i_2 i_3} = \frac{1}{2\Delta}\sum_{r=1}^{\infty}\left(a_{i_2,r}a_{i_3,r} + b_{i_2,r}b_{i_3,r}\right),\ \ b_i = \sum_{r=1}^{\infty}\frac{1}{r}b_{i,r},$$

$$D_{i_1 i_2 i_3} =$$

$$-\frac{\pi}{2\Delta^{\frac{3}{2}}}\sum_{l=1}^{\infty}\sum_{r=1}^{\infty} l\Big(a_{i_2,l}\left(a_{i_3,l+r}b_{i_1,r} - a_{i_1,r}b_{i_3,l+r}\right) + b_{i_2,l}\left(a_{i_1,r}a_{i_3,r+l} + b_{i_1,r}b_{i_3,l+r}\right)\Big)$$

$$+\frac{\pi}{2\Delta^{\frac{3}{2}}}\sum_{l=1}^{\infty}\sum_{r=1}^{l-1} l\Big(a_{i_2,l}\left(a_{i_1,r}b_{i_3,l-r} + a_{i_3,l-r}b_{i_1,r}\right) - b_{i_2,l}\left(a_{i_1,r}a_{i_3,l-r} - b_{i_1,r}b_{i_3,l-r}\right)\Big)$$

$$+\frac{\pi}{2\Delta^{\frac{3}{2}}}\sum_{l=1}^{\infty}\sum_{r=l+1}^{\infty} l\Big(a_{i_2,l}\left(a_{i_3,r-l}b_{i_1,r} - a_{i_1,r}b_{i_3,r-l}\right) + b_{i_2,l}\left(a_{i_1,r}a_{i_3,r-l} + b_{i_1,r}b_{i_3,r-l}\right)\Big);$$

we met all other notations in the previous section.

From the form of expansion (6.23) and expansion of integral $J^{*(i_2 i_3)}_{(011)\Delta,0}$ we may conclude, that they include repeated series. Hereafter in the course of approximation of examined stochastic integrals in [24] it is proposed to put upper limits of summation by equal p, that is according to arguments given before is incorrectly.

We may avoid this and other problems (see introduction of this book) using the method, based on theorem 1 (theorem 4).

If we propose, that the members of expansion (6.23) coincide with the members of its analogue, obtained using theorem 1 and formulas of connection of stochastic Ito and Stratonovich integrals (this, as we saw in the previous section, is actual for the simplest stochastic integrals of first and second multiplicity), then we may replace the repeated series in (6.23) by the multiple ones, as in theorem 1, as was made formally in [24]. However, it requires separate and rather complex argumentation.

6.2 Representation of multiple stochastic Ito integrals using Hermite polynomials

In the previous sections of this chapter we analyzed the general theory of approximation of multiple stochastic Ito and Stratonovich integrals. However, in some particular cases we may get exact expressions for multiple stochastic Ito and Stratonovich integrals in the form of polynomials of finite degrees from one standard Gaussian random variable. This and next sections will be devoted to this question. The results described in them may be met, for example, in [29].

Let's analyze the family of constructing polynomials $H_n(x,y);\ n = 0,\ 1,\ldots$ of type:

$$H_n(x,y) = \frac{d^n}{d\alpha^n} e^{\alpha x - \frac{1}{2}\alpha^2 y}\Big|_{\alpha=0}.$$

It is well known, that polynomials $H_n(x,y)$ are connected with Hermit polynomials $h_n(x)$ by the formula $H_n(x,y) = \left(\frac{y}{2}\right)^{\frac{n}{2}} h_n\left(\frac{x}{\sqrt{2y}}\right)$, where $h_n(x)$ is Hermit polynomial.

Using the recurrent formulas

$$\frac{dh_n}{dz}(z) = 2nh_{n-1}(z);\ n = 1,\ 2,\ldots,$$

$$h_n(z) = 2zh_{n-1}(z) - 2(n-1)h_{n-2}(z);\ n = 2,\ 3,\ldots,$$

it is easy to get the following recurrent relations for polynomials $H_n(x,y)$:

$$\frac{\partial H_n}{\partial x}(x,y) = nH_{n-1}(x,y);\ n = 1,\ 2,\ldots, \tag{6.24}$$

$$\frac{\partial H_n}{\partial y}(x,y) = \frac{n}{2y}H_n(x,y) - \frac{nx}{2y}H_{n-1}(x,y);\ n = 1,\ 2,\ldots, \tag{6.25}$$

$$\frac{\partial H_n}{\partial y}(x,y) = -\frac{n(n-1)}{2}H_{n-2}(x,y);\ n = 2,\ 3,\ldots \tag{6.26}$$

It follows from (6.24) – (6.26), that

$$\frac{\partial H_n}{\partial y}(x,y) + \frac{1}{2}\frac{\partial^2 H_n}{\partial x^2}(x,y) = 0;\ n = 2,\ 3,\ldots. \tag{6.27}$$

Using the Ito formula with probability 1 we have

$$H_n(f_t,t) - H_n(0,0) = \int\limits_0^t \frac{\partial H_n}{\partial x}(f_s,s)df_s + \int\limits_0^t \left(\frac{\partial H_n}{\partial y}(f_s,s) + \frac{1}{2}\frac{\partial^2 H_n}{\partial x^2}(f_s,s)\right)ds, \tag{6.28}$$

where $f_t \in \Re^1$ — is a standard Wiener process.

According to (6.27) and $H_n(0,0) = 0;\ n = 2,\ 3,\ldots$ from (6.28) with probubilty 1 we get

$$H_n(f_t,t) = \int\limits_0^t nH_{n-1}(f_s,s)df_s;\ n = 2,\ 3,\ldots$$

Hereafter, in accordance with induction it is easy to get the following relation with probability 1:

$$I_t^{(n)} \stackrel{\rm def}{=} \int\limits_0^t \ldots \int\limits_0^{t_2} df_{t_1} \ldots df_{t_n} = \frac{1}{n!} H_n(f_t, t);\ n = 1,\ 2, \ldots \qquad (6.29)$$

Let's examine one of extensions [29] from formula (6.29):

$$J_t^{(n)} \stackrel{\rm def}{=} \int\limits_0^t \psi_{t_n} \ldots \int\limits_0^{t_2} \psi_{t_1} df_{t_1} \ldots df_{t_n} = \frac{1}{n!} H_n(\delta_t, \Delta_t);\ n = 1,\ 2, \ldots, \qquad (6.30)$$

where

$$\delta_t \stackrel{\rm def}{=} \int\limits_0^t \psi_s df_s;\ \Delta_t \stackrel{\rm def}{=} \int\limits_0^t \psi_s^2 ds;$$

ψ_t — is non-anticipating stochastic process, which satisfy the conditions of existing of Ito stochastic integral in the mean-square sense (see sect. 7.1).

It is easy to check, that first eight formulas from the family (6.30) look as follows

$$J_t^{(1)} = \frac{1}{1!}\delta_t,\ J_t^{(2)} = \frac{1}{2!}\left(\delta_t^2 - \Delta_t\right),\ J_t^{(3)} = \frac{1}{3!}\left(\delta_t^3 - 3\delta_t\Delta_t\right),$$

$$J_t^{(4)} = \frac{1}{4!}\left(\delta_t^4 - 6\delta_t^2\Delta_t + 3\Delta_t^2\right),\ J_t^{(5)} = \frac{1}{5!}\left(\delta_t^5 - 10\delta_t^3\Delta_t + 15\delta_t\Delta_t^2\right),$$

$$J_t^{(6)} = \frac{1}{6!}\left(\delta_t^6 - 15\delta_t^4\Delta_t + 45\delta_t^2\Delta_t^2 - 15\Delta_t^3\right),$$

$$J_t^{(7)} = \frac{1}{7!}\left(\delta_t^7 - 21\delta_t^5\Delta_t + 105\delta_t^3\Delta_t^2 - 105\delta_t\Delta_t^3\right),$$

$$J_t^{(8)} = \frac{1}{8!}\left(\delta_t^8 - 28\delta_t^6\Delta_t + 210\delta_t^4\Delta_t^2 - 420\delta_t^2\Delta_t^3 + 105\Delta_t^4\right)$$

with probubility 1.

6.3 One formula for multiple stochastic Stratonovich integrals

Let's prove with probability 1 the following relation for multiple Stratonovich integrals [24]

$$I_t^{*(n)} = \frac{1}{n!} f_t^n,\ I_t^{*(n)} \stackrel{\rm def}{=} \int\limits_0^{*t} \ldots \int\limits_0^{*t_2} df_{t_1} \ldots df_{t_n} \text{ w. p. 1.} \qquad (6.31)$$

At first, we will examine the case $n = 2$. Using theorem 11 we obtain:

$$I_t^{*(2)} = I_t^{(2)} + \frac{1}{2}\int\limits_0^t dt_1 \text{ w. p. 1.} \tag{6.32}$$

From the relation (6.29) when $n = 2$ follows, that with probubility 1

$$I_t^{(2)} = \frac{1}{2}f_t^2 - \frac{1}{2}\int\limits_0^t dt_1. \tag{6.33}$$

Substituting (6.33) into (6.32), with probability 1 we have $I_t^{*(2)} = f_t^2/2!$. So, formula (6.31) is correct when $n = 2$.

Assume, that the formula (6.31) is correct when $n = k$, i.e. with probability 1: $I_t^{*(k)} = f_t^k/k!$, and examine $\int\limits_0^{*t} I_\tau^{*(k)} df_\tau \stackrel{\text{def}}{=} I_t^{*(k+1)}$.

From sect.7.2 and inductive hypotheses with probability 1 we get

$$I_t^{*(k+1)} = \int\limits_0^t \frac{f_\tau^k}{k!} df_\tau + \frac{1}{2}\int\limits_0^t \frac{f_\tau^{k-1}}{(k-1)!} d\tau. \tag{6.34}$$

Let's introduce a stochastic process ξ_t of type $\xi_t = f_t^{k+1}/(k+1)!$ and will find its stochastic differential using the Ito formula:

$$d\xi_t = \frac{1}{2}\frac{f_t^{k-1}}{(k-1)!}dt + \frac{f_t^k}{k!}df_t. \tag{6.35}$$

Since $\xi_0 = 0$, then from (6.34) and (6.35) it follows, that $I_t^{*(k+1)} = f_t^{k+1}/(k+1)!$ with probability 1. So, relation (6.31) is proven in accordance with induction.

It is easy to see, that formula (6.31) admits the following extension:

$$J_t^{*(n)} = \frac{1}{n!}\delta_t^n, \tag{6.36}$$

where $J_t^{*(n)} \stackrel{\text{def}}{=} \int\limits_0^{*t} \psi(t_n) \ldots \int\limits_0^{*t_2} \psi(t_1) df_{t_1} \ldots df_{t_n}$; $\delta_t = \int\limits_0^t \psi(s) df_s$, a $\psi(s) : [0,t] \to \Re^1$ — is some continuously differentiated function.

6.4 Usage of multiple integral sums for approximation of multiple stochastic Ito integrals

We noted in the introduction, that considering the modern state of question about approximation of multiple stochastic integrals, the method analyzed further is unlikely of any practical value. However, we will analyze it in order to get the overall picture.

Note, that in several works (see, for example, [23]) it was proposed to use various variants of integral sums for approximation of multiple stochastic integrals. In this section we will analyze one of the simplest modifications of the method of integral sums.

Let the functions $\psi_l(\tau);\ l=1,\ldots,k$ satisfy to Lipschitz conditions at the interval $[t,T]$ with constants C_l:

$$|\psi_l(\tau_1)-\psi_l(\tau_2)|\le C_l|\tau_1-\tau_2| \text{ for all } \tau_1,\tau_2\in[t,T]. \tag{6.37}$$

Then, according to lemma 1 with probability 1 the following equality is reasonable

$$J[\psi^{(k)}]_{T,t}=\underset{N\to\infty}{\text{l.i.m.}}\sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{j_2-1}\prod_{l=1}^{k}\psi_l(\tau_{j_l})\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}.$$

Here the sense of formula (1.8) notations is kept.

We will represent the approximation of multiple stochastic Ito integral $J[\psi^{(k)}]_{T,t}$ in the following form

$$J[\psi^{(k)}]_{T,t}^{N}=\sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{j_2-1}\prod_{l=1}^{k}\psi_l(\tau_{j_l})\Delta\mathbf{w}_{\tau_{j_l}}^{(i_l)}. \tag{6.38}$$

Relation (6.38) may be rewritten in the following form:

$$J[\psi^{(k)}]_{T,t}^{N}=\sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{j_2-1}\prod_{l=1}^{k}(\Delta\tau_{j_l})^{\frac{1}{2}}\psi_l(\tau_{j_l})\mathbf{u}_{j_l}^{(i_l)}, \tag{6.39}$$

where $\mathbf{u}_j^{(i)}\stackrel{\text{def}}{=}(\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_j}^{(i)})/(\Delta\tau_j)^{\frac{1}{2}};\ i=1,\ldots,m$ — are independent when $i\ne 0$ and various j standard Gaussian random values: $\mathbf{u}_j^{(0)}=(\Delta\tau_j)^{\frac{1}{2}}$.

Assume, that

$$\tau_j=t+j\Delta;\ j=0,1,\ldots,N;\ \tau_N=T,\ \Delta>0. \tag{6.40}$$

Then the formula (6.39) will be as follows:

$$J[\psi^{(k)}]_{T,t}^{N}=\Delta^{\frac{k}{2}}\sum_{j_k=0}^{N-1}\ldots\sum_{j_1=0}^{j_2-1}\prod_{l=1}^{k}\psi_l(t+j_l\Delta)\mathbf{u}_{j_l}^{(i_l)}, \tag{6.41}$$

where $\mathbf{u}_j^{(i)}\stackrel{\text{def}}{=}(\mathbf{w}_{t+(j+1)\Delta}^{(i)}-\mathbf{w}_{t+j\Delta}^{(i)})/\sqrt{\Delta};\ i=1,\ldots,m.$

Lemma 12. *Assume, that functions $\psi_l(\tau)$; $l = 1, \ldots k$ satisfy to Lipschitz condition* (6.37), *and $\{\tau_j\}_{j=0}^{N-1}$ — is a partition of interval $[t,T]$ of type* (6.40).

Then, for sufficiently small value $T-t$ there exists such constant $H_k < \infty$, that:

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{N}\right)^2\right\} \leq \frac{H_k(T-t)^2}{N}.$$

Proof. It is easy to see, that in case of sufficiently small value $T-t$, there exists such constant C_k, that

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^{N}\right)^2\right\} \leq C_k \mathsf{M}\left\{\left(J[\psi^{(2)}]_{T,t} - J[\psi^{(2)}]_{T,t}^{N}\right)^2\right\};$$

$$J[\psi^{(2)}]_{T,t} - J[\psi^{(2)}]_{T,t}^{N} = \sum_{j=1}^{3} S_j^N,$$

where

$$S_1^N = \sum_{j_1=0}^{N-1} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \psi_2(t_2) \int\limits_{\tau_{j_1}}^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} d\mathbf{w}_{t_2}^{(i_2)};$$

$$S_2^N = \sum_{j_1=0}^{N-1} \int\limits_{\tau_{j_1}}^{\tau_{j_1+1}} \left(\psi_2(t_2) - \psi_2(\tau_{j_1})\right) d\mathbf{w}_{t_2}^{(i_2)} \sum_{j_2=0}^{j_1-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)};$$

$$S_3^N = \sum_{j_1=0}^{N-1} \psi_2(\tau_{j_1}) \Delta\mathbf{w}_{\tau_{j_1}}^{(i_2)} \sum_{j_2=0}^{j_1-1} \int\limits_{\tau_{j_2}}^{\tau_{j_2+1}} \left(\psi_1(t_1) - \psi_1(\tau_{j_2})\right) d\mathbf{w}_{t_1}^{(i_1)}.$$

Therefore, according to Minkowsky inequality we have:

$$\left(\mathsf{M}\left\{\left(J[\psi^{(2)}]_{T,t} - J[\psi^{(2)}]_{T,t}^{N}\right)^2\right\}\right)^{\frac{1}{2}} \leq \sum_{j=1}^{3} \left(\mathsf{M}\left\{\left(S_j^N\right)^2\right\}\right)^{\frac{1}{2}}.$$

Using moment features of stochastic integrals (see chapter 7) let's estimate values $\mathsf{M}\left\{\left(S_i^N\right)^2\right\}$; $i = 1, 2, 3$. To do it let's examine four cases.

The case 1. $i_1,\ i_2 \neq 0$:

$$\mathsf{M}\left\{\left(S_1^N\right)^2\right\} \leq \tfrac{\Delta}{2}(T-t) \sup_{s\in[t,T]} \left\{\psi_2^2(s)\psi_1^2(s)\right\},$$
$$\mathsf{M}\left\{\left(S_2^N\right)^2\right\} \leq \tfrac{\Delta^2}{6}(T-t)^2 \left(C_2\right)^2 \sup_{s\in[t,T]} \left\{\psi_1^2(s)\right\},$$
$$\mathsf{M}\left\{\left(S_3^N\right)^2\right\} \leq \tfrac{\Delta^2}{6}(T-t)^2 \left(C_1\right)^2 \sup_{s\in[t,T]} \left\{\psi_2^2(s)\right\}.$$

The case 2. $i_1 \neq 0,\ i_2 = 0:$

$$\mathsf{M}\left\{\left(S_1^N\right)^2\right\} \le \frac{\Delta}{2}(T-t)^2 \sup_{s\in[t,T]}\left\{\psi_2^2(s)\psi_1^2(s)\right\},$$
$$\mathsf{M}\left\{\left(S_2^N\right)^2\right\} \le \frac{\Delta^2}{3}(T-t)^3 \left(C_2\right)^2 \sup_{s\in[t,T]}\left\{\psi_1^2(s)\right\},$$
$$\mathsf{M}\left\{\left(S_3^N\right)^2\right\} \le \frac{\Delta^2}{3}(T-t)^3 \left(C_1\right)^2 \sup_{s\in[t,T]}\left\{\psi_2^2(s)\right\}.$$

The case 3. $i_2 \neq 0,\ i_1 = 0:$

$$\mathsf{M}\left\{\left(S_1^N\right)^2\right\} \le \frac{\Delta^2}{3}(T-t) \sup_{s\in[t,T]}\left\{\psi_2^2(s)\psi_1^2(s)\right\},$$
$$\mathsf{M}\left\{\left(S_2^N\right)^2\right\} \le \frac{\Delta^2}{3}(T-t)^3 \left(C_2\right)^2 \sup_{s\in[t,T]}\left\{\psi_1^2(s)\right\},$$
$$\mathsf{M}\left\{\left(S_3^N\right)^2\right\} \le \frac{\Delta^2}{8}(T-t)^3 \left(C_1\right)^2 \sup_{s\in[t,T]}\left\{\psi_2^2(s)\right\}.$$

The case 4. $i_1 = i_2 = 0:$

$$\mathsf{M}\left\{\left(S_1^N\right)^2\right\} \le \frac{\Delta^2}{4}(T-t)^2 \sup_{s\in[t,T]}\left\{\psi_2^2(s)\psi_1^2(s)\right\},$$
$$\mathsf{M}\left\{\left(S_2^N\right)^2\right\} \le \frac{\Delta^2}{4}(T-t)^4 \left(C_2\right)^2 \sup_{s\in[t,T]}\left\{\psi_1^2(s)\right\},$$
$$\mathsf{M}\left\{\left(S_3^N\right)^2\right\} \le \frac{\Delta^2}{16}(T-t)^4 \left(C_1\right)^2 \sup_{s\in[t,T]}\left\{\psi_2^2(s)\right\}.$$

According to obtained estimations and condition (6.37) we have

$$\mathsf{M}\left\{\left(J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t}^N\right)^2\right\} \le H_k(T-t)\Delta = \frac{H_k(T-t)^2}{N},$$

where $H_k < \infty$. The lemma is proven. □

It is easy to check, that the following relation is correct:

$$\mathsf{M}\left\{\left(I_{00_{T,t}}^{(i_2 i_1)} - I_{00_{T,t}}^{(i_2 i_1)N}\right)^2\right\} = \frac{(T-t)^2}{2N}, \tag{6.42}$$

where $i_1, i_2 = 1, \ldots, m$ and $I_{00_{T,t}}^{(i_2 i_1)N}$ — is the approximation of stochastic integral $I_{00_{T,t}}^{(i_2 i_1)}$ from the family (5.1), obtained according to the formula (6.41).

Note, that the method, based on multiple integral sums, converges in the mean-square sense significantly slower, than the method based on the multiple Fourier series (see. (6.42), (5.18), (5.49) and table 6.1, 6.4).

Table 6.1: Values q_{trig}, q_{pol}, T^*_{trig}, T^*_{pol}.

$T-t$	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
q_{trig}	3	4	7	14	27	53
q_{pol}	5	9	17	33	65	129
T^*_{trig}, sec	4	5	7	10	16	30
T^*_{pol}, sec	3	4	7	13	23	45

6.5 Comparison of effectiveness of Fourier-Legendre series, trigonometric Fourier series and integral sums for approximation of stochastic integrals

In this section we will compare effectiveness of usage of polynomial and trigonometric functions in the course of approximation of multiple stochastic integrals. In addition, we will compare effectiveness of usage of methods, based on multiple Fourier series and multiple integral sums.

Let's examine stochastic integrals $I^{(1)}_{0_{T,t}}$, $I^{(21)}_{00_{T,t}}$, which may be met, for example, while realizing the strong numerical procedure of order of accuracy 1.0 for stochastic differential Ito equation [23], [24], [46]. We will approximate them firstly using the trigonometric system of functions (formulas (5.42), (5.44)), and then using Legendre polynomials (formulas (5.3), (5.6)).

The number $q=q_{\text{trig}}$ in the first case we will select from the condition

$$\frac{(T-t)^2}{2\pi^2}\biggl(\frac{\pi^2}{6}-\sum_{r=1}^{q_{\text{trig}}}\frac{1}{r^2}\biggr)\le\varepsilon, \tag{6.43}$$

and number $q=q_{\text{pol}}$ in the second case we will be selected from the condition:

$$\frac{(T-t)^2}{2}\biggl(\frac{1}{2}-\sum_{i=1}^{q_{\text{pol}}}\frac{1}{4i^2-1}\biggr)\le\varepsilon, \tag{6.44}$$

where q_{trig} and q_{pol} — are minimal natural numbers, satisfying to the conditions (6.43) and (6.44) correspondently.

In table 6.1 the values q_{trig}, q_{pol} when $\varepsilon=(T-t)^3$, $T-t=2^{-j}$; $j=5,6,\ldots,10$ are given. The values T^*_{trig}, T^*_{pol} correspond to the computer time, consumed on 200 independent numerical modellings of integrals $I^{(1)}_{0_{T,t}}$, $I^{(21)}_{00_{T,t}}$ according to formulas (5.42), (5.44) when $q=q_{\text{trig}}$ and according to formulas (5.3), (5.6) when $q=q_{\text{pol}}$. At the same time, each fixed modeling according to formulas (5.42), (5.44) and (5.3), (5.6) correspond to the same realization of the sequence of independent standard Gaussian random variables.

Note, that formula (5.44) was used here without a sum member

$$\frac{1}{2}(T-t)\frac{\sqrt{2}}{\pi}\sqrt{\alpha_q}\left(\xi_q^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\xi_q^{(i_2)}\right)$$

which requires the certain computer time for its numerical modeling.

From the results given in table 6.1, it is clear, that when $T-t>2^{-7}$, the polynomial system is a little bit better, than the trigonometric one according to consumption of computer time. However, even when $T-t\le 2^{-8}$, usage of the trigonometric system provides an insignificant advantage.

This picture changes cardinally when analyzing combinations of more complicated stochastic integrals.

Let's analyze stochastic integrals

$$I_{0_{T,t}}^{(i_1)},\ I_{1_{T,t}}^{(i_1)},\ I_{00_{T,t}}^{(i_2 i_1)},\ I_{000_{T,t}}^{(i_3 i_2 i_1)};\ i_1,i_2,i_3=1,\ldots,m. \tag{6.45}$$

which may be met, for example, while realizing the strong numerical procedure of order of accuracy 1.5 for stochastic differential Ito equation [23], [24], [46].

Let's present the numerical result, which provides a possibility to see, that modeling the set of stochastic integrals (which is necessary for realization of strong numerical method of order of accuracy 1.5 for the stochastic differential Ito equations [23], [24], [46]) with using of the polynomial system of functions provides the advantage in computer time in more than 2 times in comparison with the trigonometric system of functions, at least, in case of not very small $T-t$ (note, that in this section we will also analyze more general situation in which the polynomial system of functions provides the advantage in three times in comparison with the trigonometric one within the limits of the considered question).

At first, let's examine the simplified set of stochastic integrals $I_{00_{T,t}}^{(21)}$, $I_{000_{T,t}}^{(321)}$.

In case of polynomial system of functions we will be looking for numbers q,q_1 in the approximations $I_{00_{T,t}}^{(21)q}$, $I_{000_{T,t}}^{(321)q_1}$, defined according to formulas (5.6), (5.28), on the basis of following conditions:

$$\frac{(T-t)^2}{2}\left(\frac{1}{2}-\sum_{i=1}^{q}\frac{1}{4i^2-1}\right)\le (T-t)^4, \tag{6.46}$$

$$(T-t)^3\left(\frac{1}{6}-\sum_{i,j,k=0}^{q_1}\frac{\left(C_{kji}\right)^2}{(T-t)^3}\right)\le (T-t)^4, \tag{6.47}$$

Table 6.2: Values q, q_1, T^*_{100}, $\tilde{T}^*_{100}$ (polynomial system)

$T-t$	0.08222	0.05020	0.02310	0.01956
q	19	51	235	328
q_1	1	2	5	6
T^*_{100}, sec	5	12	52	73
$\tilde{T}^*_{100}$, sec	13.5	36	181	225

Table 6.3: Values q, q_1, T^*_{100}, $\tilde{T}^*_{100}$ (trigonometric system)

$T-t$	0.08222	0.05020	0.02310	0.01956
q	8	21	96	133
q_1	1	1	3	4
T^*_{100}, sec	12	24.5	105	148
$\tilde{T}^*_{100}$, sec	44	88	411	660

where

$$C_{kji} = \frac{\sqrt{(2i+1)(2j+1)(2k+1)}}{8}(T-t)^{3/2}\bar{C}_{kji};$$

$$\bar{C}_{kji} = \int\limits_{-1}^{1} P_k(z)\int\limits_{-1}^{z} P_j(y)\int\limits_{-1}^{y} P_i(x)dxdydz;$$

$P_i(x)$ — is a Legendre polynomial.

In case of trigonometric system of functions we will use formulas (5.44), (5.45) when $i_3=3$, $i_2=2$, $i_1=1$, and we will be looking for numbers $q,\ q_1$ from the conditions:

$$\mathsf{M}\left\{\left(I_{00_{T,t}}^{(i_2i_1)} - I_{00_{T,t}}^{(i_2i_1)q}\right)^2\right\} = \frac{(T-t)^2}{2\pi^2}\left(\frac{\pi^2}{6} - \sum_{r=1}^{q}\frac{1}{r^2}\right) \le \varepsilon, \tag{6.48}$$

$$\mathsf{M}\left\{\left(I_{000_{T,t}}^{(i_3i_2i_1)} - I_{000_{T,t}}^{(i_3i_2i_1)q,q_1}\right)^2\right\} = (T-t)^3\left\{\frac{4}{45} - \frac{1}{4\pi^2}\sum_{r=1}^{q_1}\frac{1}{r^2} - \right.$$

$$\left. -\frac{55}{32\pi^4}\sum_{r=1}^{q_1}\frac{1}{r^4} - \frac{1}{4\pi^4}\sum_{\substack{r,l=1\\ r\ne l}}^{q_1}\frac{5l^4+4r^4-3r^2l^2}{r^2l^2\left(r^2-l^2\right)^2}\right\} \le \varepsilon. \tag{6.49}$$

In table 6.2 we can see minimal values of numbers q, q_1, satisfying the conditions (6.46), (6.47) for various values $T-t$. In table 6.3 we can see the values of the same numbers for conditions (6.48), (6.49) when $\varepsilon = (T-t)^4$.

Let's provide 100 independent numerical modellings for various values $T-t$ of set of stochastic integrals $I_{00_{T,t}}^{(21)}$, $I_{000_{T,t}}^{(321)}$, defined using formulas (5.6), (5.28), obtained using polynomial system of functions.

In table 6.2 we can see the values of computer time T_{100}^{*}, consumed for solving of this task with various values $T-t$.

Let's repeat this numerical experiment for approximations (5.44), (5.45) when $i_3=3$, $i_2=2$, $i_1=1$ which were obtained using trigonometric system of functions. Its results are inserted in table 6.3.

Let's note, that hereinafter in this section formulas (5.44) and (5.45) were used without members

$$\frac{1}{2}(T-t)\frac{\sqrt{2}}{\pi}\sqrt{\alpha_q}\left(\xi_q^{(i_1)}\zeta_0^{(i_2)}-\zeta_0^{(i_1)}\xi_q^{(i_2)}\right)$$

and

$$(T-t)^{\frac{3}{2}}\Biggl(\frac{1}{2\sqrt{2}\pi}\sqrt{\alpha_q}\left(\xi_q^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-\xi_q^{(i_3)}\zeta_0^{(i_2)}\zeta_0^{(i_1)}\right)+$$
$$+\frac{1}{2\sqrt{2}\pi^2}\sqrt{\beta_q}\Bigl(\mu_q^{(i_1)}\zeta_0^{(i_2)}\zeta_0^{(i_3)}-$$
$$-2\mu_q^{(i_2)}\zeta_0^{(i_1)}\zeta_0^{(i_3)}+\mu_q^{(i_3)}\zeta_0^{(i_1)}\zeta_0^{(i_2)}\Bigr)\Biggr)$$

correspondently.

It support numerical results specified in tables 6.2 and 6.3.

Comparing obtained results we come to the conclusion, that within the limits of numerical experiment when modelling correspondent collection of stochastic integrals, the polynomial system of functions gives the advantage in two times in computer time in comparison with the trigonometric system of functions.

Let's extract approximations $I_{000_{T,t}}^{(123)q}$, defined by (5.29) in an explicit form when $q=1,2,5,6$, taking into account their practical importance (in theorem 4 the formulas given below correspond to approximations $I_{000_{T,t}}^{*(i_1i_2i_3)q}$ in case of any possible $i_1,i_2,i_3=1,\ldots,m$ and $q=1,2,5,6$ when replacing their upper index (123) by index $(i_1i_2i_3)$ and upper indexes $(1),(2),(3)$ by indexes $(i_1),(i_2),(i_3)$ correspondently):

$$I_{000_{T,t}}^{(123)1}=(T-t)^{\frac{3}{2}}\Biggl[\left(\left(\frac{1}{6}\zeta_0^{(1)}-\frac{1}{4\sqrt{3}}\zeta_1^{(1)}\right)\zeta_0^{(2)}+\frac{1}{20}\zeta_1^{(1)}\zeta_1^{(2)}\right)\zeta_0^{(3)}+$$
$$+\left(\left(\frac{1}{4\sqrt{3}}\zeta_0^{(1)}-\frac{1}{10}\zeta_1^{(1)}\right)\zeta_0^{(2)}+\frac{1}{20}\zeta_0^{(1)}\zeta_1^{(2)}\right)\zeta_1^{(3)}\Biggr],$$

$$I_{000_{T,t}}^{(123)2}=(T-t)^{\frac{3}{2}}\Biggl[\Biggl\{\left(\frac{1}{6}\zeta_0^{(1)}-\frac{1}{4\sqrt{3}}\zeta_1^{(1)}+\frac{1}{12\sqrt{5}}\zeta_2^{(1)}\right)\zeta_0^{(2)}+$$

$$+\left(\frac{1}{20}\zeta_1^{(1)}-\frac{1}{4\sqrt{15}}\zeta_2^{(1)}\right)\zeta_1^{(2)}+\left(-\frac{1}{6\sqrt{5}}\zeta_0^{(1)}+\frac{1}{4\sqrt{15}}\zeta_1^{(1)}+\frac{1}{84}\zeta_2^{(1)}\right)\zeta_2^{(2)}\Biggr\}\zeta_0^{(3)}+$$

$$+\Biggl\{\left(\frac{1}{4\sqrt{3}}\zeta_0^{(1)}-\frac{1}{10}\zeta_1^{(1)}\right)\zeta_0^{(2)}+\left(\frac{1}{20}\zeta_0^{(1)}-\frac{1}{14\sqrt{5}}\zeta_2^{(1)}\right)\zeta_1^{(2)}+$$

$$+\left(-\frac{1}{4\sqrt{15}}\zeta_0^{(1)}+\frac{1}{7\sqrt{5}}\zeta_1^{(1)}\right)\zeta_2^{(2)}\Biggr\}\zeta_1^{(3)}+$$

$$+\Biggl\{\left(\frac{1}{12\sqrt{5}}\zeta_0^{(1)}-\frac{1}{42}\zeta_2^{(1)}\right)\zeta_0^{(2)}+\left(\frac{1}{4\sqrt{15}}\zeta_0^{(1)}-\frac{1}{14\sqrt{5}}\zeta_1^{(1)}\right)\zeta_1^{(2)}+$$

$$+\frac{1}{84}\zeta_0^{(1)}\zeta_2^{(2)}\Biggr\}\zeta_2^{(3)}\Biggr],$$

$$I_{000_{T,t}}^{(123)5}=I_{000_{T,t}}^{(123)2}+(T-t)^{\frac{3}{2}}\Biggl[\Biggl\{\frac{1}{10\sqrt{21}}\zeta_3^{(1)}\zeta_1^{(2)}+\left(-\frac{1}{4\sqrt{35}}\zeta_3^{(1)}+\frac{1}{28\sqrt{5}}\zeta_4^{(1)}\right)\zeta_2^{(2)}+$$

$$+\left(-\frac{\sqrt{3}}{20\sqrt{7}}\zeta_1^{(1)}+\frac{1}{4\sqrt{35}}\zeta_2^{(1)}+\frac{1}{180}\zeta_3^{(1)}-\frac{1}{12\sqrt{7}}\zeta_4^{(1)}+\frac{1}{9\sqrt{77}}\zeta_5^{(1)}\right)\zeta_3^{(2)}+$$

$$+\left(-\frac{1}{21\sqrt{5}}\zeta_2^{(1)}+\frac{1}{12\sqrt{7}}\zeta_3^{(1)}+\frac{1}{308}\zeta_4^{(1)}-\frac{1}{12\sqrt{11}}\zeta_5^{(1)}\right)\zeta_4^{(2)}+$$

$$+\left(-\frac{5}{36\sqrt{77}}\zeta_3^{(1)}+\frac{1}{12\sqrt{11}}\zeta_4^{(1)}+\frac{1}{468}\zeta_5^{(1)}\right)\zeta_5^{(2)}\Biggr\}\zeta_0^{(3)}+$$

$$+\Biggl\{\frac{1}{20\sqrt{21}}\zeta_3^{(1)}\zeta_0^{(2)}+\frac{1}{140}\zeta_4^{(1)}\zeta_1^{(2)}+\left(-\frac{1}{12\sqrt{5}}\zeta_3^{(1)}+\frac{1}{14\sqrt{165}}\zeta_5^{(1)}\right)\zeta_2^{(2)}+$$

$$+\left(\frac{\sqrt{3}}{20\sqrt{7}}\zeta_0^{(1)}+\frac{1}{3\sqrt{105}}\zeta_2^{(1)}-\frac{19}{220\sqrt{21}}\zeta_4^{(1)}\right)\zeta_3^{(2)}+$$

$$+\left(-\frac{1}{70}\zeta_1^{(1)}+\frac{23}{220\sqrt{21}}\zeta_3^{(1)}-\frac{8}{91\sqrt{33}}\zeta_5^{(1)}\right)\zeta_4^{(2)}+$$

$$+\left(-\frac{\sqrt{5}}{42\sqrt{33}}\zeta_2^{(1)}+\frac{37}{364\sqrt{33}}\zeta_4^{(1)}\right)\zeta_5^{(2)}\Biggr\}\zeta_1^{(3)}+$$

$$+\Biggl\{\left(\frac{1}{84\sqrt{5}}\zeta_2^{(1)}-\frac{1}{154}\zeta_4^{(1)}\right)\zeta_0^{(2)}+\left(\frac{1}{180}\zeta_1^{(1)}-\frac{1}{55\sqrt{21}}\zeta_3^{(1)}-\frac{5\sqrt{3}}{364}\zeta_5^{(1)}\right)\zeta_1^{(2)}+$$

$$+\left(\frac{1}{28\sqrt{5}}\zeta_0^{(1)}-\frac{1}{132}\zeta_2^{(1)}+\frac{1}{286\sqrt{5}}\zeta_4^{(1)}\right)\zeta_2^{(2)}+$$

$$
+\left(\frac{1}{12\sqrt{7}}\zeta_0^{(1)}-\frac{19}{220\sqrt{21}}\zeta_1^{(1)}+\frac{2}{2145}\zeta_3^{(1)}+\frac{1}{156\sqrt{77}}\zeta_5^{(1)}\right)\zeta_3^{(2)}+
$$

$$
+\left(\frac{1}{308}\zeta_0^{(1)}-\frac{1}{572\sqrt{5}}\zeta_2^{(1)}\right)\zeta_4^{(2)}+
$$

$$
+\left(-\frac{1}{12\sqrt{11}}\zeta_0^{(1)}+\frac{37}{364\sqrt{33}}\zeta_1^{(1)}-\frac{1}{78\sqrt{77}}\zeta_3^{(1)}-\frac{1}{18564}\zeta_5^{(1)}\right)\zeta_5^{(2)}\Biggr\}\zeta_4^{(3)}+
$$

$$
+\Biggl\{\left(\frac{1}{36\sqrt{77}}\zeta_3^{(1)}-\frac{1}{234}\zeta_5^{(1)}\right)\zeta_0^{(2)}+\left(\frac{1}{21\sqrt{165}}\zeta_2^{(1)}-\frac{5}{364\sqrt{33}}\zeta_4^{(1)}\right)\zeta_1^{(2)}+
$$

$$
+\left(\frac{1}{14\sqrt{165}}\zeta_1^{(1)}-\frac{37}{468\sqrt{385}}\zeta_3^{(1)}+\frac{2}{819\sqrt{5}}\zeta_5^{(1)}\right)\zeta_2^{(2)}+
$$

$$
+\left(\frac{1}{9\sqrt{77}}\zeta_0^{(1)}-\frac{29}{234\sqrt{385}}\zeta_2^{(1)}+\frac{1}{156\sqrt{77}}\zeta_4^{(1)}\right)\zeta_3^{(2)}+
$$

$$
+\left(\frac{1}{12\sqrt{11}}\zeta_0^{(1)}-\frac{8}{91\sqrt{33}}\zeta_1^{(1)}+\frac{1}{156\sqrt{77}}\zeta_3^{(1)}+\frac{1}{9282}\zeta_5^{(1)}\right)\zeta_4^{(2)}+
$$

$$
+\left(\frac{1}{468}\zeta_0^{(1)}-\frac{1}{819\sqrt{5}}\zeta_2^{(1)}-\frac{1}{18564}\zeta_4^{(1)}\right)\zeta_5^{(2)}\Biggr\}\zeta_5^{(3)}+
$$

$$
+\Biggl\{\left(\frac{1}{20\sqrt{21}}\zeta_1^{(1)}-\frac{1}{90}\zeta_3^{(1)}+\frac{1}{36\sqrt{77}}\zeta_5^{(1)}\right)\zeta_0^{(2)}+
$$

$$
+\left(\frac{1}{10\sqrt{21}}\zeta_0^{(1)}-\frac{1}{12\sqrt{105}}\zeta_2^{(1)}-\frac{1}{55\sqrt{21}}\zeta_4^{(1)}\right)\zeta_1^{(2)}+
$$

$$
+\left(\frac{1}{4\sqrt{35}}\zeta_0^{(1)}-\frac{1}{12\sqrt{5}}\zeta_1^{(1)}+\frac{1}{186\sqrt{5}}\zeta_3^{(1)}-\frac{37}{468\sqrt{385}}\zeta_5^{(1)}\right)\zeta_2^{(2)}+
$$

$$
+\left(\frac{1}{180}\zeta_0^{(1)}-\frac{1}{372\sqrt{5}}\zeta_2^{(1)}+\frac{2}{2145}\zeta_4^{(1)}\right)\zeta_3^{(2)}+
$$

$$
+\left(-\frac{1}{12\sqrt{7}}\zeta_0^{(1)}+\frac{23}{220\sqrt{21}}\zeta_1^{(1)}-\frac{4}{2145}\zeta_3^{(1)}+\frac{1}{156\sqrt{77}}\zeta_5^{(1)}\right)\zeta_4^{(2)}+
$$

$$
+\left(-\frac{5}{36\sqrt{77}}\zeta_0^{(1)}+\frac{19\sqrt{5}}{468\sqrt{77}}\zeta_2^{(1)}-\frac{1}{78\sqrt{77}}\zeta_4^{(1)}\right)\zeta_5^{(2)}\Biggr\}\zeta_3^{(3)}+
$$

$$
+\Biggl\{\frac{1}{84\sqrt{5}}\zeta_4^{(1)}\zeta_0^{(2)}+\left(-\frac{1}{12\sqrt{105}}\zeta_3^{(1)}+\frac{1}{21\sqrt{165}}\zeta_5^{(1)}\right)\zeta_1^{(2)}-\frac{1}{132}\zeta_4^{(1)}\zeta_2^{(2)}+
$$

$$
+\left(-\frac{1}{4\sqrt{35}}\zeta_0^{(1)}+\frac{1}{3\sqrt{105}}\zeta_1^{(1)}-\frac{1}{372\sqrt{5}}\zeta_3^{(1)}-\frac{29}{234\sqrt{385}}\zeta_5^{(1)}\right)\zeta_3^{(2)}
$$

$$
+\left(-\frac{1}{21\sqrt{5}}\zeta_0^{(1)}+\frac{1}{66}\zeta_2^{(1)}-\frac{1}{572\sqrt{5}}\zeta_4^{(1)}\right)\zeta_4^{(2)}+
$$

$$+\left(-\frac{\sqrt{5}}{42\sqrt{33}}\zeta_1^{(1)}+\frac{19\sqrt{5}}{468\sqrt{77}}\zeta_3^{(1)}-\frac{1}{819\sqrt{5}}\zeta_5^{(1)}\right)\zeta_5^{(2)}\Biggr\}\zeta_2^{(3)}\Biggr],$$

$$I_{000_{T,t}}^{(123)6}=I_{000_{T,t}}^{(123)5}+(T-t)^{\frac{3}{2}}\Biggl[\Biggl\{\left(\frac{1}{132\sqrt{13}}\zeta_4^{(1)}-\frac{1}{330}\zeta_6^{(1)}\right)\zeta_0^{(2)}+$$

$$+\left(\frac{5}{132\sqrt{273}}\zeta_3^{(1)}-\frac{1}{30\sqrt{429}}\zeta_5^{(1)}\right)\zeta_1^{(2)}+$$

$$+\left(\frac{5}{462\sqrt{13}}\zeta_2^{(1)}-\frac{19}{924\sqrt{65}}\zeta_4^{(1)}+\frac{1}{561\sqrt{5}}\zeta_6^{(1)}\right)\zeta_2^{(2)}+$$

$$+\left(\frac{5}{66\sqrt{273}}\zeta_1^{(1)}-\frac{5}{396\sqrt{13}}\zeta_3^{(1)}+\frac{\sqrt{143}}{8415\sqrt{7}}\zeta_5^{(1)}\right)\zeta_3^{(2)}+$$

$$+\left(\frac{5}{132\sqrt{13}}\zeta_0^{(1)}-\frac{2\sqrt{5}}{231\sqrt{13}}\zeta_2^{(1)}+\frac{1}{476\sqrt{13}}\zeta_4^{(1)}+\frac{1}{9690}\zeta_6^{(1)}\right)\zeta_4^{(2)}+$$

$$+\left(\frac{1}{4\sqrt{143}}\zeta_0^{(1)}-\frac{4}{15\sqrt{429}}\zeta_1^{(1)}+\frac{1}{510\sqrt{1001}}\zeta_3^{(1)}+\frac{1}{5814\sqrt{13}}\zeta_5^{(1)}\right)\zeta_5^{(2)}+$$

$$+\left(\frac{1}{660}\zeta_0^{(1)}+\frac{1}{1122\sqrt{5}}\zeta_2^{(1)}-\frac{1}{19380}\zeta_4^{(1)}\right)\zeta_6^{(2)}\Biggr\}\zeta_6^{(3)}+$$

$$+\Biggl\{\left(-\frac{1}{22\sqrt{13}}\zeta_4^{(1)}+\frac{1}{4\sqrt{143}}\zeta_5^{(1)}+\frac{1}{660}\zeta_6^{(1)}\right)\zeta_6^{(2)}+$$

$$+\left(\frac{5}{132\sqrt{13}}\zeta_4^{(2)}-\frac{1}{4\sqrt{143}}\zeta_5^{(2)}\right)\zeta_6^{(1)}\Biggr\}\zeta_0^{(3)}+$$

$$+\Biggl\{\left(-\frac{5}{44\sqrt{273}}\zeta_1^{(1)}+\frac{5}{198\sqrt{13}}\zeta_3^{(1)}-\frac{113}{3060\sqrt{1001}}\zeta_5^{(1)}\right)\zeta_6^{(2)}+$$

$$+\left(\frac{5}{2132\sqrt{273}}\zeta_1^{(2)}-\frac{5}{396\sqrt{13}}\zeta_3^{(2)}+\frac{61}{3060\sqrt{1001}}\zeta_5^{(2)}\right)\zeta_6^{(1)}\Biggr\}\zeta_3^{(3)}+$$

$$+\Biggl\{\left(-\frac{1}{22\sqrt{13}}\zeta_0^{(1)}+\frac{59}{924\sqrt{65}}\zeta_2^{(1)}-\frac{1}{238\sqrt{13}}\zeta_4^{(1)}-\frac{1}{19380}\zeta_6^{(1)}\right)\zeta_6^{(2)}+$$

$$+\left(\frac{1}{132\sqrt{13}}\zeta_0^{(2)}-\frac{19}{924\sqrt{65}}\zeta_2^{(2)}+\frac{1}{476\sqrt{13}}\zeta_4^{(2)}\right)\zeta_6^{(1)}\Biggr\}\zeta_4^{(3)}+$$

$$+\Biggl\{\left(-\frac{5}{44\sqrt{273}}\zeta_3^{(1)}+\frac{\sqrt{3}}{10\sqrt{143}}\zeta_5^{(1)}\right)\zeta_6^{(2)}+$$

$$+\left(\frac{5}{66\sqrt{273}}\zeta_3^{(2)}-\frac{4}{15\sqrt{429}}\zeta_5^{(2)}\right)\zeta_6^{(1)}\Biggr\}\zeta_1^{(3)}+$$

$$+\left\{\left(-\frac{5}{231\sqrt{13}}\zeta_2^{(1)}+\frac{59}{924\sqrt{65}}\zeta_4^{(1)}-\frac{1}{1122\sqrt{5}}\zeta_6^{(1)}\right)\zeta_6^{(2)}+\right.$$

$$\left.+\left(\frac{5}{462\sqrt{13}}\zeta_2^{(2)}-\frac{2\sqrt{5}}{231\sqrt{13}}\zeta_4^{(2)}\right)\zeta_6^{(1)}\right\}\zeta_2^{(3)}+$$

$$+\left\{\left(-\frac{1}{4\sqrt{143}}\zeta_0^{(1)}+\frac{\sqrt{3}}{10\sqrt{143}}\zeta_1^{(1)}-\frac{113}{3060\sqrt{1001}}\zeta_3^{(1)}-\frac{1}{2907\sqrt{13}}\zeta_5^{(1)}\right)\zeta_6^{(2)}+\right.$$

$$\left.\left.+\left(\frac{1}{30\sqrt{429}}\zeta_1^{(2)}+\frac{\sqrt{143}}{8415\sqrt{7}}\zeta_3^{(2)}+\frac{1}{5814\sqrt{13}}\zeta_5^{(2)}\right)\zeta_6^{(1)}\right\}\zeta_5^{(3)}\right].$$

Let's demonstrate, that in some situations the advantage of polynomial system of functions in relation to the trigonometric one in terms of computer time for modelling of collections of multiple stochastic integrals turns to be more impressive.

The fact is, that when solving practical tasks we often have to model several stochastic integrals of the same type taken for various combinations of upper indexes at each step of integration. In this case it is useful to reduce the total number of modeled integrals using the following relations

$$I_{00_{T,t}}^{(i_1 i_2)}+I_{00_{T,t}}^{(i_2 i_1)}=I_{0_{T,t}}^{(i_1)}I_{0_{T,t}}^{(i_2)}\ \text{w. p. 1},$$

$$I_{000_{T,t}}^{(i_1 i_2 i_3)}+I_{000_{T,t}}^{(i_1 i_3 i_2)}+I_{000_{T,t}}^{(i_2 i_1 i_3)}+I_{000_{T,t}}^{(i_2 i_3 i_1)}+I_{000_{T,t}}^{(i_3 i_2 i_1)}+I_{000_{T,t}}^{(i_3 i_1 i_2)}=I_{0_{T,t}}^{(i_1)}I_{0_{T,t}}^{(i_2)}I_{0_{T,t}}^{(i_3)}\ \text{w. p. 1},$$

where i_1, i_2, i_3 — are different; $i_1, i_2, i_3 \in \{1,\ldots,m\}$.

In accordance with mentioned, let's analyze the following collection of stochastic integrals:

$$I_{0_{T,t}}^{(i)},\ I_{1_{T,t}}^{(i)},\ I_{00_{T,t}}^{(12)},\ I_{00_{T,t}}^{(13)},\ I_{00_{T,t}}^{(23)},\ I_{000_{T,t}}^{(123)},\ I_{000_{T,t}}^{(132)},\ I_{000_{T,t}}^{(213)},\ I_{000_{T,t}}^{(231)},\ I_{000_{T,t}}^{(312)}, \quad (6.50)$$

where $i = 1,\ 2,\ 3$.

Let's make independently 100 models for various values $T-t$ of the set of stochastic integrals (6.50) using formulas (5.3), (5.4), (5.6), (5.28). In table 6.2 we can see values of $\tilde{T}_{100}^*$, consumed for solution of this task with various values $T-t$. Let's repeat this numerical experiment using approximations (5.42)–(5.45). Its results are inserted in table 6.3.

Comparing the obtained numerical results we can note, that in this case the polynomial system of functions gives advantage in 3 times it terms of computer time when modelling the collection of multiple stochastic integrals.

Note, that generally speaking the set (6.45) includes m^3+m^2+2m of various multiple stochastic integrals. When $m>3$ the number m^3+m^2+2m may turn out to be significantly bigger, than in case (6.50) (in (6.50) $m=3$ and the multiple stochastic integrals with two coincidental upper indexes from three ones are not considered) and as the author suppose, the advantage of polynomial system of functions will be even more essential. We may expect the same effect when analyzing more complicate collections of multiple stochastic integrals than in (6.45), which are necessary for building more accurate strong numerical methods for stochastic differential Ito equations [23], [24], [46].

Apparently, the mentioned tendency is connected with the fact, that the polynomial system of functions has a significant advantage over the trigonometric system of functions for approximation of multiple stochastic integrals where not all weight functions of type $\psi(\tau)\equiv(t-\tau)^l;\ l=0,\ 1,\ 2,\ldots$ identically equal to 1, that corresponds to $l\ge 1$ in the given representation. In order to understand it is enough to compare formulas (5.4), (5.5), (5.7), (5.8), obtained using Legendre polynomials with their analogues (5.43), (5.48), (5.47), (5.46), obtained using trigonometric system of functions.

Finally, we will demonstrate, that according to computational costs on modeling of collection of multiple stochastic integrals, the method based on multiple Fourier series is signficiantly better, than the method based on multiple integral sums.

Let's examine approximations of multiple stochastic Ito integrals, obtained using the method based on multiple integral sums:

$$I_{0_{T,t}}^{(1)q}=\sqrt{\Delta}\sum_{j=0}^{q-1}\xi_j^{(1)}, \tag{6.51}$$

$$I_{00_{T,t}}^{(21)q}=\Delta\sum_{j=0}^{q-1}\xi_j^{(2)}\sum_{i=0}^{j-1}\xi_i^{(1)}, \tag{6.52}$$

where $\xi_j^{(i)}=(\mathbf{f}_{t+(j+1)\Delta}^{(i)}-\mathbf{f}_{t+j\Delta}^{(i)})/\sqrt{\Delta};\ i=1,2$ — are independent standard Gaussian random values; $\Delta=(T-t)/q$; $I_{00_{T,t}}^{(21)q}$, $I_{0_{T,t}}^{(1)q}$ — are approximations of integrals $I_{00_{T,t}}^{(21)}$, $I_{0_{T,t}}^{(1)}$.

Let's choose number q, included in (6.51), (6.52) from the condition

$$\mathsf{M}\left\{\left(I_{00_{T,t}}^{(21)}-I_{00_{T,t}}^{(21)q}\right)^2\right\}=\frac{(T-t)^2}{2q}\le (T-t)^3.$$

Let's make 200 independent numerical modellings of the collection of stochastic integrals $I_{00_{T,t}}^{(21)}$, $I_{0_{T,t}}^{(1)}$ using formulas (6.51), (6.52) when $T-t=$

Table 6.4: Values q and $T^*_{\rm sum}$ (method of integral sums).

$T-t$	2^{-5}	2^{-6}	2^{-7}
q	16	32	64
$T^*_{\rm sum}$, sec	26	93	391

2^{-j}; $j=5,6,7$. In table 6.4 we can see the time $T^*_{\rm sum}$, which was necessary for performing of this task.

Comparing tables 6.1 and 6.4 we come to conclusion, that the method, based on multiple integral sums even when $T-t=2^{-7}$ is more than 50 times worse in terms of computer time for modelling the collection of stochastic integrals $I_{00_{T,t}}^{(21)}$, $I_{0_{T,t}}^{(1)}$, than the method based on multiple Fourier series.

Not difficult to see, that this effect will be essentally bigger, if we consider multiple stochastic integrals of multiplicity 3, 4, ... or choose value $T-t$ smaller.

Demonstrated numerical experiments provides a possibility to get sketchy idea about "good" and "bad" numerical methods, but we can see rather well-defined picture.

6.6 Multiple stochastic integrals as solutions of systems of linear stochastic differential equations

G.N. Milstein [23] proposed an approach to numerical modelling of multiple stochastic integrals, based on their representation in the form of systems of linear stochastic differential equations. Let's analyze this approach using the following collection of multiple stochastic Ito integrals as an example:

$$I_{0_{s,t}}^{(i_1)}=\int\limits_t^s d\mathbf{f}_\tau^{(i_1)},\ \ I_{00_{s,t}}^{(i_1i_2)}=\int\limits_t^s\int\limits_t^\tau d\mathbf{f}_\theta^{(i_1)}d\mathbf{f}_\tau^{(i_2)}, \qquad (6.53)$$

where $i_1,i_2=1,\ldots,m$; $0\le t<s\le T$; $\mathbf{f}_\tau^{(i)}$; $i=1,\ldots,m$ — are independent standard Wiener processes.

Then we have the following representation:

$$d\begin{pmatrix} I_{0_{s,t}}^{(i_1)}\\ I_{00_{s,t}}^{(i_1i_2)}\end{pmatrix}=\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix} I_{0_{s,t}}^{(i_1)}\\ I_{00_{s,t}}^{(i_1i_2)}\end{pmatrix}d\mathbf{f}_s^{(i_2)}+\begin{pmatrix}1&0\\0&0\end{pmatrix}d\begin{pmatrix}\mathbf{f}_s^{(i_1)}\\ \mathbf{f}_s^{(i_2)}\end{pmatrix}. \qquad (6.54)$$

It is well-known [23], [24], that solution of the system (6.54) may be represented in the following integral form:

$$\begin{pmatrix} I_{0_{s,t}}^{(i_1)} \\ I_{00_{s,t}}^{(i_1 i_2)} \end{pmatrix} = \int\limits_t^s e^{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\left(\mathbf{f}_s^{(i_2)} - \mathbf{f}_\theta^{(i_2)}\right)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} d \begin{pmatrix} \mathbf{f}_\theta^{(i_1)} \\ \mathbf{f}_\theta^{(i_2)} \end{pmatrix}, \qquad (6.55)$$

where e^A — is a matrix exponent: $e^A \stackrel{\rm def}{=} \sum\limits_{k=0}^{\infty} A^k/k!$; A — is a square matrix; $A^0 \stackrel{\rm def}{=} I$ — is a unity matrix.

Numerical modeling of the right part of (6.55) is unlikely simpler task than the jointly numerical modeling of the collection of stochastic integrals (6.53). We have to perform numerical modeling of the collection (6.53) within the limits of this approach by numerical integration of the system of linear stochastic differential equations (6.54). This procedure may be realized using the Euler method [23], [24]. Note, that the expressions of more accurate numerical methods for the system (6.54) [23], [24], [46] contain multiple stochastic integrals (6.53) and therefore they useless in our situation.

Assume, that $\{\tau_j\}_{j=0}^N$ — is a partition of the interval $[t, s]$, for which $\tau_j = t + j\Delta$; $j = 0, 1, \ldots, N$; $\tau_N = s$. Let's write down the Euler method [23], [24] for the system of linear stochastic differential equations (6.54):

$$\begin{pmatrix} \mathbf{y}_{p+1}^{(i_1)} \\ \mathbf{y}_{p+1}^{(i_1 i_2)} \end{pmatrix} = \begin{pmatrix} \mathbf{y}_p^{(i_1)} \\ \mathbf{y}_p^{(i_1 i_2)} \end{pmatrix} + \begin{pmatrix} \Delta \mathbf{f}_{\tau_p}^{(i_1)} \\ \mathbf{y}_p^{(i_1)} \Delta \mathbf{f}_{\tau_p}^{(i_2)} \end{pmatrix}, \mathbf{y}_0^{(i_1)} = 0, \mathbf{y}_0^{(i_1 i_2)} = 0, \qquad (6.56)$$

where $\mathbf{y}_{\tau_p}^{(i_1)} \stackrel{\rm def}{=} \mathbf{y}_p^{(i_1)}$; $\mathbf{y}_{\tau_p}^{(i_1 i_2)} \stackrel{\rm def}{=} \mathbf{y}_p^{(i_1 i_2)}$ — are approximations of multiple stochastic integrals $I_{0_{\tau_p,t}}^{(i_1)}, I_{00_{\tau_p,t}}^{(i_1 i_2)}$, obtained using the numerical scheme (6.56); $\Delta \mathbf{f}_{\tau_p}^{(i)} = \mathbf{f}_{\tau_{p+1}}^{(i)} - \mathbf{f}_{\tau_p}^{(i)}$; $i = 1, \ldots, m$.

Iterating the expression (6.56), we have

$$\mathbf{y}_N^{(i_1)} = \sum_{l=0}^{N-1} \Delta \mathbf{f}_{\tau_l}^{(i_1)}, \ \mathbf{y}_N^{(i_1 i_2)} = \sum_{q=0}^{N-1} \sum_{l=0}^{q-1} \Delta \mathbf{f}_{\tau_l}^{(i_1)} \Delta \mathbf{f}_{\tau_q}^{(i_2)}, \qquad (6.57)$$

where $\sum\limits_{\emptyset} \stackrel{\rm def}{=} 0$.

The formulas (6.57) are the formulas for approximations of multiple stochastic integrals (6.53), obtained using the method, based on multiple integral sums.

Consequently, the effectiveness of methods of approximation of multiple stochastic integrals based on multiple integral sums and numerical integration of systems of linear stochastic differential equations using the Euler method turns out to be similar.

6.7 Combined method of approximation of multiple stochastic integrals

In this section we build the "hybrid" of methods of approximation of multiple stochastic integrals, based on multiple Fourier series (theorem 1) and multiple integral sums (hereinafter referred to as the combined method). It appears, that when storing the required relation of one method impact on the other, we may achieve some advantages over the "clean" usage of the method, based on the multiple Fourier series.

Namely, it is explored, that the combined method of approximation of multiple stochastic integrals provides a possibility to diminish significantly the whole number of coefficients of multiple Fourier series which are necessary for approximation of the considered multiple stochastic integral. However, in this connection the computational costs for approximation of the mentioned stochastic integral slightly increase.

6.7.1 Basic relations

Using the property of additivity of the stochastic Ito integral we may write:

$$I_{0_{T,t}}^{(i_1)}=\sqrt{\Delta}\sum_{k=0}^{N-1}\zeta_{0,k}^{(i_1)}\ \text{w. p. 1}, \tag{6.58}$$

$$I_{1_{T,t}}^{(i_1)}=\sum_{k=0}^{N-1}\left(I_{1_{\tau_{k+1},\tau_k}}^{(i_1)}-\Delta^{3/2}k\zeta_{0,k}^{(i_1)}\right)\ \text{w. p. 1}, \tag{6.59}$$

$$I_{00_{T,t}}^{(i_1i_2)}=\Delta\sum_{k=0}^{N-1}\sum_{l=0}^{k-1}\zeta_{0,k}^{(i_2)}\zeta_{0,l}^{(i_1)}+\sum_{k=0}^{N-1}I_{00_{\tau_{k+1},\tau_k}}^{(i_1i_2)}\ \text{w. p. 1}, \tag{6.60}$$

$$I_{000_{T,t}}^{(i_1i_2i_3)}=\Delta^{3/2}\sum_{k=0}^{N-1}\sum_{l=0}^{k-1}\sum_{q=0}^{l-1}\zeta_{0,k}^{(i_3)}\zeta_{0,l}^{(i_2)}\zeta_{0,q}^{(i_1)}+$$

$$+\sqrt{\Delta}\sum_{k=0}^{N-1}\sum_{l=0}^{k-1}\left(\zeta_{0,k}^{(i_3)}I_{00_{\tau_{l+1},\tau_l}}^{(i_1i_2)}+\zeta_{0,l}^{(i_1)}I_{00_{\tau_{k+1},\tau_k}}^{(i_2i_3)}\right)+\sum_{k=0}^{N-1}I_{000_{\tau_{k+1},\tau_k}}^{(i_1i_2i_3)}\ \text{w. p. 1}; \tag{6.61}$$

$i_1,\ldots,i_k=1,\ldots,m$; $T-t=N\Delta$; $\tau_k=t+k\Delta$;

$$\zeta_{0,k}^{(i)}\stackrel{\rm def}{=}\Delta^{-1/2}\int\limits_{\tau_k}^{\tau_{k+1}}d\mathbf{w}_s^{(i)};$$

$k=0,1,\ldots,N-1$; $N<\infty$; the sum according to empty set is equals to zero.

In the formulas mentioned above we examined stochastic Ito integrals from the family (5.1).

Substituting the relation

$$I_{1_{\tau_{k+1},\tau_k}}^{(i_1)}=-\frac{\Delta^{3/2}}{2}\left(\zeta_{0,k}^{(i_1)}+\frac{1}{\sqrt{3}}\zeta_{1,k}^{(i_1)}\right)\ \text{w. p. 1},$$

into (6.59), where $\zeta_{0,k}^{(i_1)}$, $\zeta_{1,k}^{(i_1)}$ — are independent standard Gaussian random values, we get:

$$I_{1_{T,t}}^{(i_1)}=-\Delta^{3/2}\sum_{k=0}^{N-1}\left(\left(\frac{1}{2}+k\right)\zeta_{0,k}^{(i_1)}+\frac{1}{2\sqrt{3}}\zeta_{1,k}^{(i_1)}\right)\ \text{w. p. 1.} \qquad (6.62)$$

Let's approximate, using the method of multiple Fourier series according to the Legendre polynomials, the following multiple stochastic integrals $I_{00_{\tau_{k+1},\tau_k}}^{(i_1i_2)}$, $I_{00_{\tau_{k+1},\tau_k}}^{(i_2i_3)}$, $I_{000_{\tau_{k+1},\tau_k}}^{(i_1i_2i_3)}$, included in the right parts of (6.60), (6.61).

As a result we get

$$I_{00_{T,t}}^{(i_1i_2)N,q}=\Delta\sum_{k=0}^{N-1}\sum_{l=0}^{k-1}\zeta_{0,k}^{(i_2)}\zeta_{0,l}^{(i_1)}+\sum_{k=0}^{N-1}I_{00_{\tau_{k+1},\tau_k}}^{(i_1i_2)q}, \qquad (6.63)$$

$$I_{000_{T,t}}^{(i_1i_2i_3)N,q_1,q_2}=\Delta^{3/2}\sum_{k=0}^{N-1}\sum_{l=0}^{k-1}\sum_{q=0}^{l-1}\zeta_{0,k}^{(i_3)}\zeta_{0,l}^{(i_2)}\zeta_{0,q}^{(i_1)}+$$

$$+\sqrt{\Delta}\sum_{k=0}^{N-1}\sum_{l=0}^{k-1}\left(\zeta_{0,k}^{(i_3)}I_{00_{\tau_{l+1},\tau_l}}^{(i_1i_2)q_1}+\zeta_{0,l}^{(i_1)}I_{00_{\tau_{k+1},\tau_k}}^{(i_2i_3)q_1}\right)+\sum_{k=0}^{N-1}I_{000_{\tau_{k+1},\tau_k}}^{(i_1i_2i_3)q_2}, \qquad (6.64)$$

where approximations $I_{00_{\tau_{k+1},\tau_k}}^{(i_1i_2)q}$, $I_{000_{\tau_{k+1},\tau_k}}^{(i_1i_2i_3)q_2}$ are obtained using the method of multiple Fourier series (theorem 1) according to Legendre polynomials.

In particular, when $N=2$, the formulas (6.58), (6.62)-(6.64) will look as follows :

$$I_{0_{T,t}}^{(i_1)}=\sqrt{\Delta}\left(\zeta_{0,0}^{(i_1)}+\zeta_{0,1}^{(i_1)}\right)\ \text{w. p. 1}, \qquad (6.65)$$

$$I_{1_{T,t}}^{(i_1)}=-\Delta^{3/2}\left(\frac{1}{2}\zeta_{0,0}^{(i_1)}+\frac{3}{2}\zeta_{0,1}^{(i_1)}+\frac{1}{2\sqrt{3}}\left(\zeta_{1,0}^{(i_1)}+\zeta_{1,1}^{(i_1)}\right)\right)\ \text{w. p. 1}, \qquad (6.66)$$

$$I_{00_{T,t}}^{(i_1i_2)2,q}=\Delta\left(\zeta_{0,1}^{(i_2)}\zeta_{0,0}^{(i_1)}+I_{00_{\tau_1,\tau_0}}^{(i_1i_2)q}+I_{00_{\tau_2,\tau_1}}^{(i_1i_2)q}\right), \qquad (6.67)$$

$$I_{000_{T,t}}^{(i_1i_2i_3)2,q_1,q_2}=\sqrt{\Delta}\left(\zeta_{0,1}^{(i_3)}I_{00_{\tau_1,\tau_0}}^{(i_1i_2)q_1}+\zeta_{0,0}^{(i_1)}I_{00_{\tau_2,\tau_1}}^{(i_2i_3)q_1}\right)+I_{000_{\tau_1,\tau_0}}^{(i_1i_2i_3)q_2}+I_{000_{\tau_2,\tau_1}}^{(i_1i_2i_3)q_2}, \qquad (6.68)$$

where $\Delta=(T-t)/2$; $\tau_k=t+k\Delta$; $k=0,1,2$.

Note, that if $N=1$, then (6.58), (6.62)-(6.64) transfer to the formulas for numerical modeling of mentioned stochastic integrals using the method of multiple Fourier series. So, we may claim, that the method of multiple Fourier series is a particular case of combined method when $N = 1$.

Note, that later we will demonstrate, that modelling of multiple stochastic integrals $I_{0_{T,t}}^{(i_1)}$, $I_{1_{T,t}}^{(i_1)}$, $I_{00_{T,t}}^{(i_1 i_2)}$, $I_{000_{T,t}}^{(i_1 i_2 i_3)}$ using formulas (6.65) – (6.68) results in abrupt decrease of the total number of Fourier coefficients, which are necessary for approximation of these integrals using the method, based on theorem 1.

At the same time, each right part of formulas (6.67), (6.68) include two approximations of multiple stochastic integrals of 2nd and 3rd multiplicity, and each one of them must be obtained using the method based on theorem 1. Obviously it results in increasing of calculation costs for approximation.

6.7.2 Calculation of mean-square error

Let's calculate mean-square errors of approximations, defined using the formulas (6.63), (6.64). We have

$$\varepsilon_{N,q} \stackrel{\rm def}{=} {\sf M}\left\{\left(I_{00_{T,t}}^{(i_1 i_2)} - I_{00_{T,t}}^{(i_1 i_2)N,q}\right)^2\right\} = \sum_{k=0}^{N-1} {\sf M}\left\{\left(I_{00_{\tau_{k+1},\tau_k}}^{(i_1 i_2)} - I_{00_{\tau_{k+1},\tau_k}}^{(i_1 i_2)q}\right)^2\right\} =$$

$$= N\frac{\Delta^2}{2}\left(\frac{1}{2} - \sum_{l=1}^{q}\frac{1}{4l^2-1}\right) = \frac{(T-t)^2}{2N}\left(\frac{1}{2} - \sum_{l=1}^{q}\frac{1}{4l^2-1}\right); \qquad (6.69)$$

$$\varepsilon_{N,q_1,q_2} \stackrel{\rm def}{=} {\sf M}\left\{\left(I_{000_{T,t}}^{(i_1 i_2 i_3)} - I_{000_{T,t}}^{(i_1 i_2 i_3)N,q_1,q_2}\right)^2\right\} =$$

$$= {\sf M}\Biggl\{\Biggl(\sum_{k=0}^{N-1}\Biggl(\sqrt{\Delta}\sum_{l=0}^{k-1}\Biggl(\zeta_{0,k}^{(i_3)}\left(I_{00_{\tau_{l+1},\tau_l}}^{(i_1 i_2)} - I_{00_{\tau_{l+1},\tau_l}}^{(i_1 i_2)q_1}\right) +$$

$$+\zeta_{0,l}^{(i_1)}\left(I_{00_{\tau_{k+1},\tau_k}}^{(i_2 i_3)} - I_{00_{\tau_{k+1},\tau_k}}^{(i_2 i_3)q_1}\right)\Biggr)\Biggr) + I_{000_{\tau_{k+1},\tau_k}}^{(i_1 i_2 i_3)} - I_{000_{\tau_{k+1},\tau_k}}^{(i_1 i_2 i_3)q_2}\Biggr)\Biggr)^2\Biggr\} =$$

$$= \sum_{k=0}^{N-1}{\sf M}\Biggl\{\Biggl(\sqrt{\Delta}\sum_{l=0}^{k-1}\Biggl(\zeta_{0,k}^{(i_3)}\left(I_{00_{\tau_{l+1},\tau_l}}^{(i_1 i_2)} - I_{00_{\tau_{l+1},\tau_l}}^{(i_1 i_2)q_1}\right) +$$

$$+\zeta_{0,l}^{(i_1)}\left(I_{00_{\tau_{k+1},\tau_k}}^{(i_2 i_3)} - I_{00_{\tau_{k+1},\tau_k}}^{(i_2 i_3)q_1}\right)\Biggr)\Biggr) + I_{000_{\tau_{k+1},\tau_k}}^{(i_1 i_2 i_3)} - I_{000_{\tau_{k+1},\tau_k}}^{(i_1 i_2 i_3)q_2}\Biggr)^2\Biggr\} =$$

$$= \sum_{k=0}^{N-1}\Biggl(\Delta{\sf M}\Biggl\{\Biggl(\zeta_{0,k}^{(i_3)}\sum_{l=0}^{k-1}\left(I_{00_{\tau_{l+1},\tau_l}}^{(i_1 i_2)} - I_{00_{\tau_{l+1},\tau_l}}^{(i_1 i_2)q_1}\right)\Biggr)^2\Biggr\} +$$

$$+\Delta\mathsf{M}\left\{\left(\left(I_{00_{\tau_{k+1},\tau_k}}^{(i_2i_3)}-I_{00_{\tau_{k+1},\tau_k}}^{(i_2i_3)q_1}\right)\sum_{l=0}^{k-1}\zeta_{0,l}^{(i_1)}\right)^2\right\}+\delta_{k,q_2}^{(i_1i_2i_3)}\Biggr)=$$

$$=\sum_{k=0}^{N-1}\Biggl(\Delta\sum_{l=0}^{k-1}\mathsf{M}\left\{\left(I_{00_{\tau_{l+1},\tau_l}}^{(i_1i_2)}-I_{00_{\tau_{l+1},\tau_l}}^{(i_1i_2)q_1}\right)^2\right\}+$$

$$+k\Delta\mathsf{M}\left\{\left(I_{00_{\tau_{k+1},\tau_k}}^{(i_2i_3)}-I_{00_{\tau_{k+1},\tau_k}}^{(i_2i_3)q_1}\right)^2\right\}+\delta_{k,q_2}^{(i_1i_2i_3)}\Biggr)=$$

$$=\sum_{k=0}^{N-1}\left(2k\Delta\mathsf{M}\left\{\left(I_{00_{\tau_{k+1},\tau_k}}^{(i_1i_2)}-I_{00_{\tau_{k+1},\tau_k}}^{(i_1i_2)q_1}\right)^2\right\}+\delta_{k,q_2}^{(i_1i_2i_3)}\right)=$$

$$=\sum_{k=0}^{N-1}\left(2k\Delta\frac{\Delta^2}{2}\left(\frac{1}{2}-\sum_{l=1}^{q_1}\frac{1}{4l^2-1}\right)+\delta_{k,q_2}^{(i_1i_2i_3)}\right)=$$

$$=\Delta^3\frac{N(N-1)}{2}\left(\frac{1}{2}-\sum_{l=1}^{q_1}\frac{1}{4l^2-1}\right)+\sum_{k=0}^{N-1}\delta_{k,q_2}^{(i_1i_2i_3)}=$$

$$=\frac{1}{2}(T-t)^3\left(\frac{1}{N}-\frac{1}{N^2}\right)\left(\frac{1}{2}-\sum_{l=1}^{q_1}\frac{1}{4l^2-1}\right)+\sum_{k=0}^{N-1}\delta_{k,q_2}^{(i_1i_2i_3)},\qquad(6.70)$$

where

$$\delta_{k,q_2}^{(i_1i_2i_3)}=\mathsf{M}\left\{\left(I_{000_{\tau_{k+1},\tau_k}}^{(i_1i_2i_3)}-I_{000_{\tau_{k+1},\tau_k}}^{(i_1i_2i_3)q_2}\right)^2\right\};$$

$i_1\ne i_2$ in (6.69) and not all indexes i_1,i_2,i_3 in (6.70) are the same (otherwise there are exact relationships for modeling of integrals $I_{00_{T,t}}^{(i_1i_2)}$, $I_{000_{T,t}}^{(i_1i_2i_3)}$).

For definiteness, assume, that i_1,i_2,i_3 in (6.70) are different. Then

$$\delta_{k,q_2}^{(i_1i_2i_3)}=\Delta^3\left(\frac{1}{6}-\sum_{i,j,l=0}^{q_2}\frac{C_{lji}^2}{\Delta^3}\right),\qquad(6.71)$$

where

$$C_{lji}=\frac{\sqrt{(2i+1)(2j+1)(2l+1)}}{8}\Delta^{3/2}\bar{C}_{lji},$$

$$\bar{C}_{lji}=\int\limits_{-1}^{1}P_l(z)\int\limits_{-1}^{z}P_j(y)\int\limits_{-1}^{y}P_i(x)dxdydz;$$

$P_i(x)$ — is a Legendre polynomial.

Substituting (6.71) into (6.70) we get

$$\varepsilon_{N,q_1,q_2}=\frac{1}{2}(T-t)^3\left(\frac{1}{N}-\frac{1}{N^2}\right)\left(\frac{1}{2}-\sum_{l=1}^{q_1}\frac{1}{4l^2-1}\right)+$$

Table 6.5: $T - t = 0.1$.

N	q	q_1	q_2	M
1	13	–	1	21
2	6	0	0	7
3	4	0	0	5

Table 6.6: $T - t = 0.05$.

N	q	q_1	q_2	M
1	50	–	2	77
2	25	2	0	26
3	17	1	0	18

$$+\frac{(T-t)^3}{N^2}\left(\frac{1}{6}-\sum_{i,j,l=0}^{q_2}\frac{(2i+1)(2j+1)(2l+1)}{64}\bar{C}_{lji}^2\right). \qquad (6.72)$$

Note, that when $N = 1$ formulas (6.69), (6.72) pass into the corresponding formulas intended for mean-square errors of approximations of the integrals $I_{00_{T,t}}^{(i_1 i_2)}$, $I_{000_{T,t}}^{(i_1 i_2 i_3)}$, obtained using the method of multiple Fourier series (theorem 1) according to Legendre polynomials.

6.7.3 Numerical experiments

Let's analyze modelling the integrals $I_{0_{T,t}}^{(i_1)}$, $I_{00_{T,t}}^{(i_1 i_2)}$. To do it we may use relations (6.58), (6.63). At that, the mean-square error of approximation of the integral $I_{00_{T,t}}^{(i_1 i_2)}$ is defined by the formula (6.69) using Legendre polynomials. Let's calculate the value $\varepsilon_{N,q}$ for various N и q :

$$\varepsilon_{3,2} \approx 0.0167(T-t)^2, \ \varepsilon_{2,3} \approx 0.0179(T-t)^2, \qquad (6.73)$$

$$\varepsilon_{1,6} \approx 0.0192(T-t)^2. \qquad (6.74)$$

Note, that the combined method (formulas (6.73)) requires calculation of significantly smaller number of Fourier coefficients, than the method of multiple Fourier series (formula (6.74)).

Assume, that the mean-square error of approximation of the stochastic integrals $I_{00_{T,t}}^{(i_1 i_2)}$, $I_{000_{T,t}}^{(i_1 i_2 i_3)}$ is equals to $(T-t)^4$.

In tables 6.5–6.7 we can see the values N, q, q_1, q_2, which satisfy the system of inequalities:

$$\begin{cases} \varepsilon_{N,q} \le (T-t)^4 \\ \varepsilon_{N,q_1,q_2} \le (T-t)^4 \end{cases} \qquad (6.75)$$

Table 6.7: $T - t = 0.02$.

N	q	q_1	q_2	M
1	312	–	6	655
2	156	4	2	183
3	104	6	0	105

and the total number M of Fourier coefficients, which are nesessary for approximation of the integrals $I_{00_{T,t}}^{(i_1 i_2)}$, $I_{000_{T,t}}^{(i_1 i_2 i_3)}$ when $T - t = 0.1, 0.05, 0.02$ (numbers q, q_1, q_2 were taken in such a manner, that number M were the smallest one).

From tables 6.5–6.7 it is clear, that the combined method with small N ($N = 2$) provides a possibility to decrease significantly the total number of Fourier coefficients, which are necessary for approximation of the integrals $I_{00_{T,t}}^{(i_1 i_2)}$, $I_{000_{T,t}}^{(i_1 i_2 i_3)}$ in comparison with the method of multiple Fourier series ($N = 1$). However, as we noted before, as a result the computation costs of approximation are increased. The approximation accuracy of stochastic integrals for the combined method and the method of multiple Fourier series was taken similar and equaled to $(T - t)^4$.

Chapter 7

Stochastic integrals and stochastic differential equations

7.1 Stochastic Ito integral

Assume, that $(\Omega, \mathrm{F}, \mathsf{P})$ — is a fixed probubility space and $f_t;\ t \in [0,T]$ — is standard Wiener process, defined at $(\Omega, \mathrm{F}, \mathsf{P})$. Let's analyze the collection of σ-algebras $\{\mathrm{F}_t,\ t \in [0,T]\}$, defined at $(\Omega, \mathrm{F}, \mathsf{P})$ and connected with the process f_t in such a way, that:

1. $\mathrm{F}_s \subset \mathrm{F}_t \subset \mathrm{F}$ for $s < t$;

2. Process f_t is F_t-measurable for all $t \in [0,T]$;

3. Process $f_{t+\Delta} - f_\Delta$ for all $\Delta \geq 0,\ t > 0$ is independent with the events of σ-algebra F_Δ.

Let's analyze the class $\mathrm{M}_2([0,T])$ of functions $\xi:\ [0,T] \times \Omega \to \Re^1$, which satisfy the to conditions:

1. The function $\xi(t,\omega)$ is a measurable in accordance with the collection of variables (t,ω);

2. The function $\xi(t,\omega)$ is F_t-measurable for all $t \in [0,T]$ and $\xi(\tau,\omega)$ independent with increments $f_{t+\Delta} - f_\Delta$ for $\Delta \geq \tau,\ t > 0$;

3. $\int\limits_0^T \mathsf{M}\left\{(\xi(t,\omega))^2\right\} dt < \infty$;

4. $\mathsf{M}\left\{(\xi(t,\omega))^2\right\} < \infty$ for all $t \in [0,T]$.

For any partition $\{\tau_j\}_{j=0}^N$ of the interval $[0,T]$ such, that, $0 = \tau_0 < \tau_1 < \ldots < \tau_N = T$, we will define the sequense of step functions $\xi^{(N)}(t,\omega)$: $\xi^{(N)}(t,\omega) = \xi(\tau_j^{(N)},\omega)$ with probubility 1 for $t \in [\tau_j^{(N)}, \tau_{j+1}^{(N)})$, where $j = 0,\ 1, \ldots, N-1$; $N = 1,\ 2, \ldots$.

Let's define the stochastic Ito integral for $\xi_t \in \mathrm{M}_2([0,T])$ as the following mean-square limit

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{j=0}^{N-1} \xi^{(N)}(\tau_j^{(N)},\omega)\left(f(\tau_{j+1}^{(N)},\omega) - f(\tau_j^{(N)},\omega)\right) \stackrel{\text{def}}{=} \int_0^T \xi_\tau df_\tau, \qquad (7.1)$$

where $\xi^{(N)}(t,\omega)$ — is any step function, which converges to the function $\xi(t,\omega)$ in the following sense

$$\lim_{N\to\infty} \int_0^T \mathsf{M}\left\{\left|\xi^{(N)}(t,\omega) - \xi(t,\omega)\right|^2\right\} dt = 0. \qquad (7.2)$$

It is well known [2], that the stochastic Ito integral exists, doesn't depend on the selecting sequence $\xi^{(N)}(t,\omega)$ and has the following properties:

1. $\mathsf{M}\left\{\int_0^T \xi_\tau df_\tau\right\} = 0$;
2. $\mathsf{M}\left\{\left(\int_0^T \xi_\tau df_\tau\right)^2\right\} = \int_0^T \mathsf{M}\{\xi_\tau^2\}d\tau$;
3. $\int_0^T (\alpha\xi_\tau + \beta\eta_\tau)\, df_\tau = \alpha\int_0^T \xi_\tau df_\tau + \beta\int_0^T \eta_\tau df_\tau$ w. p. 1;
4. $\mathsf{M}\left\{\int_0^T \xi_\tau df_\tau \int_0^T \eta_\tau df_\tau\right\} = \int_0^T \mathsf{M}\left\{\xi_\tau\eta_\tau\right\} d\tau$.

Also

$$\int_0^T \xi_\tau \mathbf{1}_{[t_0,t_1]}(\tau) df_\tau \stackrel{\text{def}}{=} \int_{t_0}^{t_1} \xi_\tau df_\tau,$$

where $\mathbf{1}_{[t_0,t_1]}(\tau) = 1$ for $\tau \in [t_0, t_1]$ and $\mathbf{1}_{[t_0,t_1]}(\tau) = 0$ otherwise.

Using feature 3 for $\xi_\tau\mathbf{1}_{[t_0,t]}(\tau) = \xi_\tau\mathbf{1}_{[t_0,t_1]}(\tau) + \xi_\tau\mathbf{1}_{[t_1,t]}(\tau)$, $\tau \neq t_1$, we get

$$\int_{t_0}^{t_1} \xi_s df_s + \int_{t_1}^{t} \xi_s df_s = \int_{t_0}^{t} \xi_s df_s \text{ w. p. 1},$$

where $0 \le t_0 \le t_1 \le t \le T$.

Let's define the stochastic integral for $\xi \in \mathrm{M}_2([0,T])$ as the following mean-square limit

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{j=0}^{N-1} \xi^{(N)}\left(\tau_j^{(N)},\omega\right)\left(\tau_{j+1}^{(N)} - \tau_j^{(N)}\right) \stackrel{\text{def}}{=} \int_0^T \xi_\tau d\tau,$$

where $\xi^{(N)}(t,\omega)$ — is any step function from the class $\mathrm{M}_2([0,T])$, which converges in the sense of relation (7.2) to the function $\xi(t,\omega)$.

Let's analyze the well-known features of the stochastic integral $\int\limits_0^T \xi_\tau d\tau$:

1. $\mathsf{M}\left\{\int\limits_0^T \xi_\tau d\tau\right\} = \int\limits_0^T \mathsf{M}\left\{\xi_\tau\right\} d\tau;$

2. $\mathsf{M}\left\{\left(\int\limits_0^T \xi_\tau d\tau\right)^2\right\} \le T\int\limits_0^T \mathsf{M}\left\{\xi_\tau^2\right\} d\tau;$

3. $\int\limits_0^T (\alpha\xi_\tau + \beta\eta_\tau)\, d\tau = \alpha\int\limits_0^T \xi_\tau d\tau + \beta\int\limits_0^T \eta_\tau d\tau$ w. p. 1 $\forall\alpha,\ \beta \in \Re^1$.

The property of additivity may be analyzed also as for the stochastic Ito integral.

Note, also, that [2]

$$\mathsf{M}\left\{\left|\int\limits_{t_0}^t \xi_\tau df_\tau\right|^{2n}\right\} \le (t-t_0)^{n-1}\left(n(2n-1)\right)^n \int\limits_{t_0}^t \mathsf{M}\left\{|\xi_\tau|^{2n}\right\} d\tau, \tag{7.3}$$

$$\mathsf{M}\left\{\left|\int\limits_{t_0}^t \xi_\tau d\tau\right|^{2n}\right\} \le (t-t_0)^{2n-1} \int\limits_{t_0}^t \mathsf{M}\left\{|\xi_\tau|^{2n}\right\} d\tau, \tag{7.4}$$

where $(\xi_\tau)^n \in \mathrm{M}_2([t_0, t])$.

7.2 Stochastic Stratonovich integral

Let's examine the class $Q_{2m}([t,T])$ of Ito processes $\eta_\tau \in \Re^1$; $\tau \in [t,T]$ such, that

$$\eta_\tau = \eta_t + \int\limits_t^\tau a_s ds + \int\limits_t^\tau b_s df_s, \tag{7.5}$$

where $f_s \in \Re^1$ is F_s-measurable for all $s \in [t,T]$ standard Wiener process and

1. $a_s^m,\ b_s^m \in \mathrm{M}_2([t,T])$.

2. For all $s,\ \tau \in [t,T]$ and some positive constants $C,\ \gamma < \infty$: $\mathsf{M}\{|b_s - b_\tau|^4\} \le C|s-\tau|^\gamma$.

Assume, that $C_2(\Re^1, [t,T])$ — is a space of functions $F(x,\tau) : \Re^1 \times [t,T] \to \Re^1$, which continuously differentiated two times using variable x, and these derivatives are bounded uniformly for $x \in \Re^1$, $\tau \in [t,T]$.

Let's define the stochastic Stratonovich integral for the process $F(\eta_\tau,\tau)$; $\tau \in [t,T]$ $(F(x,\tau) \in C_2(\Re^1,[t,T]))$ as the following mean-square limit

$$\underset{N\to\infty}{\text{l.i.m}} \sum_{j=0}^{N-1} F\left(\frac{1}{2}\left(\eta_{\tau_j^{(N)}} + \eta_{\tau_{j+1}^{(N)}}\right), \tau_j^{(N)}\right)\left(f_{\tau_{j+1}^{(N)}} - f_{\tau_j^{(N)}}\right) \stackrel{\text{def}}{=} \int\limits_t^{*T} F(\eta_\tau,\tau) df_\tau, \tag{7.6}$$

where the sense of formula (7.1) notations is kept.

It is easy to demonstrate, that if $\eta_\tau \in Q_8([t,T])$, $F(\eta_\tau,\tau) \in \mathrm{M}_2([t,T])$, where $F(x,\tau) \in C_2(\Re^1,[t,T])$, then the following relation between the stochastic Stratonovich and Ito integrals is reasonable

$$\int\limits_t^{*T} F(\eta_\tau,\tau)df_\tau = \int\limits_t^T F(\eta_\tau,\tau)df_\tau + \frac{1}{2}\int\limits_t^T \frac{\partial F}{\partial x}(\eta_\tau,\tau)b_\tau d\tau \ \text{w. p. 1.} \qquad (7.7)$$

If the Wiener processes in (7.5) and (7.7) are independent, then with probubility 1

$$\int\limits_t^{*T} F(\eta_\tau,\tau)df_\tau = \int\limits_t^T F(\eta_\tau,\tau)df_\tau.$$

Also, if $\eta_\tau^{(i)} \in Q_4([t,T]);\ i=1,\ 2$, then

$$\int\limits_t^{*T} \eta_\tau^{(i)}d\mathbf{f}_\tau^{(j)} = \int\limits_t^T \eta_\tau^{(i)}d\mathbf{f}_\tau^{(j)} + \frac{1}{2}\mathbf{1}_{\{i=j\}}\int\limits_t^T b_\tau d\tau \ \text{w. p.1},$$

where the process $\eta_\tau^{(i)}$ looks as follows

$$\eta_\tau^{(i)} = \eta_t^{(i)} + \int\limits_t^\tau a_s ds + \int\limits_t^\tau b_s d\mathbf{f}_s^{(i)}.$$

Here $\mathbf{f}_\tau^{(j)} \in \Re^1;\ j=1,\ 2$ — are independent standard Wiener processes; $\mathbf{1}_A$ — is an indicator of the set A.

7.3 Ito formula

Assume, that $(\Omega,\mathrm{F},\mathsf{P})$ — is a fixed probubility space, and $\mathbf{f}_t \in \Re^m$ is F_t-measurable for all $t \in [0,T]$ vector Wiener process with independent components $\mathbf{f}_t^{(i)};\ i=1,\ldots,m$. Assume, that the stochastic processes $\mathbf{a}_s^{(i)}$ and $B_s^{(ij)}$, $i=1,\ldots,n$; $j=1,\ldots,m$ are such, that $\mathbf{a}_s^{(i)},\ B_s^{(ij)} \in \mathrm{M}_2([0,T])$ for all $i=1,\ldots,n$; $j=1,\ldots,m$.

Let's analyze the vector Ito process $\mathbf{x}_t \in \Re^n, t \in [0,T]$ of type

$$\mathbf{x}_t = \mathbf{x}_s + \int\limits_s^t \mathbf{a}_\tau d\tau + \int\limits_s^t B_\tau d\mathbf{f}_\tau \ \text{w. p. 1,} \qquad (7.8)$$

where $0 \le s \le t \le T$.

Assume, that the function $R(\mathbf{x},t):\ \Re^n\times[0,T]\to\Re^1$ has continuous partial derivatives

$$\frac{\partial R}{\partial t},\ \frac{\partial R}{\partial \mathbf{x}^{(i)}},\ \frac{\partial^2 R}{\partial \mathbf{x}^{(i)}\partial \mathbf{x}^{(j)}};$$

$i,j=1,\ldots,n.$

Within the limits of examined assumptions for all s,t such, that $0\le s\le t\le T$, the following Ito formula takes place with probability 1 [2]:

$$R(\mathbf{x}_t,t)=R(\mathbf{x}_s,s)+\int\limits_s^t\Biggl(\frac{\partial R}{\partial t}(\mathbf{x}_\tau,\tau)+\sum_{i=1}^n\mathbf{a}_\tau^{(i)}\frac{\partial R}{\partial \mathbf{x}^{(i)}}(\mathbf{x}_\tau,\tau)+$$

$$+\frac{1}{2}\sum_{j=1}^m\sum_{i,k=1}^n B_\tau^{(ij)}B_\tau^{(kj)}\frac{\partial^2 R}{\partial \mathbf{x}^{(i)}\partial \mathbf{x}^{(k)}}(\mathbf{x}_\tau,\tau)\Biggr)d\tau+\sum_{j=1}^m\sum_{i=1}^n\int\limits_s^t B_\tau^{(ij)}\frac{\partial R}{\partial \mathbf{x}^{(i)}}(\mathbf{x}_\tau,\tau)d\mathbf{f}_\tau^{(j)}.$$

7.4 Stochastic differential Ito equation

Assume, that $(\Omega,\mathrm{F},\mathsf{P})$ — is a fixed probubility space and $\mathbf{f}_t\in\Re^m$ — is F_t-measurable for all $t\in[0,T]$ vector Wiener process with independent components $\mathbf{f}_t^{(i)};\ i=1,\ldots,m.$

Let's analyze the following stochastic differential Ito equation:

$$\mathbf{x}_t=\mathbf{x}_0+\int\limits_0^t\mathbf{a}(\mathbf{x}_\tau,\tau)d\tau+\int\limits_0^t B(\mathbf{x}_\tau,\tau)d\mathbf{f}_\tau,\ \mathbf{x}_0=\mathbf{x}(0,\omega),\qquad(7.9)$$

where the stochastic process $\mathbf{x}_t\in\Re^n$ — is a solution of equation (7.9); $\mathbf{a}:\Re^n\times[0,T]\to\Re^n$, $B:\Re^n\times[0,T]\to\Re^{n\times m}$; $\mathbf{x}_0$ — is a vector initial condition; $\mathbf{x}_0$ and $\mathbf{f}_t-\mathbf{f}_0$ — are independent when $t>0$.

The stochastic process $\mathbf{x}_t\in\Re^n$ is called as a strong solution (hereinafter referred to as solution) of stochastic differential Ito equation (7.9), if any component of $\mathbf{x}_t$ is F_t-measurable for all $t\in[0,T]$, integrals in right part of (7.9) exist and the equality (7.9) is executed for all $t\in[0,T]$ with probubility 1.

It is well known [2], that there is a unique (in the sense of stochastic equivalence) continuous with probubility 1 solution of stochastic differential Ito equation, if following 3 conditions are met:

1. The functions $\mathbf{a}(\mathbf{x},t),B_k(\mathbf{x},t):\Re^n\times[0,T]\to\Re^n;\ k=1,\ldots,m$ are measurable according to collection of variables $(\mathbf{x},t)\in\Re^n\times[0,T]$; $B_k(\mathbf{x}_\tau,\tau)$ is k-th colomn of matrix $B(\mathbf{x}_\tau,\tau)$;

2. For all $\mathbf{x},\ \mathbf{y} \in \Re^n$ there is such constant $K < \infty$, that

$$|\mathbf{a}(\mathbf{x},t) - \mathbf{a}(\mathbf{y},t)| + \sum_{k=1}^{m} |B_k(\mathbf{x},t) - B_k(\mathbf{y},t)| \leq K|\mathbf{x} - \mathbf{y}|,$$

$$|\mathbf{a}(\mathbf{x},t)|^2 + \sum_{k=1}^{m} |B_k(\mathbf{x},t)|^2 \leq K^2\left(1 + |\mathbf{x}|^2\right);$$

3. The random value $\mathbf{x}_0$ is F_0-measurable and $\mathsf{M}\left\{|\mathbf{x}_0|^2\right\} < \infty$.

7.5 Stochastic integral according to martingale

Assume, that $(\Omega, \mathrm{F}, \mathsf{P})$ — is a fixed probubility space and $\{\mathrm{F}_t, t \in [0, t]\}$ — is a non-decreasing collection of σ-algebras, defined at $(\Omega, \mathrm{F}, \mathsf{P})$. Assume, that $M_t,\ t \in [0,T]$ is F_t-measurable for all $t \in [0,T]$ martingale, which satisfies the condition $\mathsf{M}\{|M_t|\} < \infty$ and for all $t \in [0,T]$ there is a F_t-measurable and non-negative with probubility 1 stochastic process $\rho_t,\ t \in [0,T]$ such, that

$$\mathsf{M}\{(M_s - M_t)^2|\mathrm{F}_t\} = \mathsf{M}\left\{\int\limits_t^s \rho_\tau d\tau|\mathrm{F}_t\right\} \text{ w. p. 1},$$

where $0 \leq t < s \leq T$.

Let's analyze the class $H_2(\rho, [0,T])$ of stochastic processes $\varphi_t,\ t \in [0,T]$, which are F_t-measurable for all $t \in [0,T]$ and satisfies the condition $\mathsf{M}\left\{\int\limits_0^T \varphi_t^2 \rho_t dt\right\} < \infty$.

Let's analyze the partition $\{\tau_j\}_{j=0}^N$ of the interval $[0,T]$ for which

$$0 = \tau_0^{(N)} < \tau_1^{(N)} < \ldots < \tau_N^{(N)} = T,\ \Delta_N = \max_{0 \leq j \leq N-1} \left|\tau_{j+1}^{(N)} - \tau_j^{(N)}\right| \to 0$$

when $N \to \infty$.

Let's define sequence of step functions $\varphi_t^{(N)}$ such, that: $\varphi_t^{(N)} = \varphi_{\tau_j^{(N)}}$ w. p. 1 when $t \in [\tau_j^{(N)}, \tau_{j+1}^{(N)})$; $j = 0,\ 1, \ldots, N-1$; $N = 1,\ 2, \ldots$.

Let's define the stochastic integral according to martingale from the process $\varphi_t \in H_2(\rho, [0,T])$ as the following mean-square limit

$$\underset{N\to\infty}{\text{l.i.m.}} \sum_{j=0}^{N-1} \varphi_{\tau_j^{(N)}}^{(N)} \left(M_{\tau_{j+1}^{(N)}} - M_{\tau_j^{(N)}}\right) \stackrel{\text{def}}{=} \int\limits_0^T \varphi_t dM_t, \tag{7.10}$$

where $\varphi_t^{(N)}$ — is every sequence of step functions from the class $H_2(\rho,[0,T])$, which converges to φ_t in the following sense

$$\underset{N\to\infty}{\text{l.i.m.}}\ \mathsf{M}\left\{\int_0^T \left(\varphi_t-\varphi_t^{(N)}\right)^2\rho_t dt\right\}=0.$$

It is well known [2], that the stochastic integral $\int_0^T \varphi_t dM_t$ exists, it doesn't depend on the selection of sequence $\varphi_t^{(N)}$ and satisfies to the following conditions with probability 1:

1. $\mathsf{M}\left\{\int_0^T \varphi_t dM_t|\mathrm{F}_0\right\}=0$;
2. $\mathsf{M}\left\{\left|\int_0^T \varphi_t dM_t\right|^2|\mathrm{F}_0\right\}=\mathsf{M}\left\{\int_0^T \varphi_t^2\rho_t dt|\mathrm{F}_0\right\}$;
3. $\int_0^T(\alpha\varphi_t+\beta\psi_t)dM_t=\alpha\int_0^T \varphi_t dM_t+\beta\int_0^T \psi_t dM_t$;
4. $\mathsf{M}\left\{\int_0^T \varphi_t dM_t\int_0^T \psi_t dM_t|\mathrm{F}_0\right\}=\mathsf{M}\left\{\int_0^T \varphi_t\psi_t\rho_t dt|\mathrm{F}_0\right\}$.

7.6 Stochastic integral according to Poisson random measure

Let's examine the Poisson random measure in the space $[0,T]\times\mathbf{Y}$ ($\Re^n \stackrel{\text{def}}{=} \mathbf{Y}$). We will denote the values of this measure at the set $\Delta\times A$ ($\Delta\subseteq[0,T]$, $A\subset\mathbf{Y}$) as $\nu(\Delta,A)$. Let's assume, that $\mathsf{M}\{\nu(\Delta,A)\}=|\Delta|\Pi(A)$, where $|\Delta|$ — is a Lebesgue measure of Δ, $\Pi(A)$ — is a measure on σ-algebra B of Borel sets $\mathbf{Y}$, and B_0 — is a subalgebra of B, consisting of sets $A\subset\mathrm{B}$, which are satisfies to the condition $\Pi(A)<\infty$.

Let's analyze the martingale measure $\tilde{\nu}(\Delta,A)=\nu(\Delta,A)-|\Delta|\Pi(A)$.

Assume, that $(\Omega,\mathsf{F},\mathsf{P})$ — is a fixed probubility space and $\{\mathrm{F}_t,\ t\in[0,T]\}$ is a non-decreasing family of σ-algebras $\mathrm{F}_t\subset\mathrm{F}$.

Assume, that:

1. Random values $\nu([0,t),A)$ — are F_t-measurable for all $A\subseteq\mathrm{B}_0$;
2. The random values $\nu([t,t+h),A)$, $A\subseteq\mathrm{B}_0$, $h>0$, doesn't depend on σ-algebra F_t.

Let's define the class $H_l(\Pi,[0,T])$ of random functions $\varphi:[0,T]\times\mathbf{Y}\times\Omega\to\Re^1$, which for all $t\in[0,T]$, $\mathbf{y}\in\mathbf{Y}$ are F_t-measurable and satisfy to the following condition

$$\int_0^T\int_{\mathbf{Y}}\mathsf{M}\{|\varphi(t,\mathbf{y})|^l\}\Pi(d\mathbf{y})dt<\infty.$$

Let's analyze the partition $\{\tau_j\}_{j=0}^{N}$ of the interval $[0,T]$, which satisfies the same conditions as in the definition of stochastic Ito integral.

For $\varphi(t,\mathbf{y}) \in H_2(\Pi,[0,T])$ let's define the stochastic integral according to martingale Poisson measure as the following mean-square limit [2]:

$$\int_0^T\int_{\mathbf{Y}} \varphi(t,\mathbf{y})\tilde{\nu}(dt,d\mathbf{y}) \stackrel{\text{def}}{=} \underset{N\to\infty}{\text{l.i.m.}} \int_0^T\int_{\mathbf{Y}} \varphi^{(N)}(t,\mathbf{y})\tilde{\nu}(dt,d\mathbf{y}), \qquad (7.11)$$

where $\varphi^{(N)}(t,\mathbf{y})$ — is any sequense of step functions from the class $H_2(\Pi,[0,T])$ such, that

$$\lim_{N\to\infty} \int_0^T\int_{\mathbf{Y}} \mathsf{M}\{|\varphi(t,\mathbf{y}) - \varphi^{(N)}(t,\mathbf{y})|^2\}\Pi(d\mathbf{y})dt \to 0.$$

It is well known [2], that the stochastic integral (7.11) exists, it doesn't depend on selection of the sequence $\varphi^{(N)}(t,\mathbf{y})$ and it satisfies with probability 1 to the following conditions:

1. $\mathsf{M}\left\{\int_0^T\int_{\mathbf{Y}} \varphi(t,\mathbf{y})\tilde{\nu}(dt,d\mathbf{y})|\mathrm{F}_0\right\} = 0;$
2. $\int_0^T\int_{\mathbf{Y}}(\alpha\varphi_1(t,\mathbf{y}) + \beta\varphi_2(t,\mathbf{y}))\tilde{\nu}(dt,d\mathbf{y}) = \alpha\int_0^T\int_{\mathbf{Y}} \varphi_1(t,\mathbf{y})\tilde{\nu}(dt,d\mathbf{y}) +$
$$+\beta\int_0^T\int_{\mathbf{Y}} \varphi_2(t,\mathbf{y})\tilde{\nu}(dt,d\mathbf{y});$$
3. $\mathsf{M}\left\{\left|\int_0^T\int_{\mathbf{Y}} \varphi(t,\mathbf{y})\tilde{\nu}(dt,d\mathbf{y})\right|^2|\mathrm{F}_0\right\} = \int_0^T\int_{\mathbf{Y}} \mathsf{M}\left\{|\varphi(t,\mathbf{y})|^2|\mathrm{F}_0\right\}\Pi(d\mathbf{y})dt,$

where α, β — are some constants; $\varphi_1(t,\mathbf{y})$, $\varphi_2(t,\mathbf{y})$, $\varphi(t,\mathbf{y})$ from the class $H_2(\Pi,[0,T])$.

The stochastic integral

$$\int_0^T\int_{\mathbf{Y}} \varphi(t,\mathbf{y})\nu(dt,d\mathbf{y})$$

according to Poisson measure will be defined as follows

$$\int_0^T\int_{\mathbf{Y}} \varphi(t,\mathbf{y})\nu(dt,d\mathbf{y}) = \int_0^T\int_{\mathbf{Y}} \varphi(t,\mathbf{y})\tilde{\nu}(dt,d\mathbf{y}) + \int_0^T\int_{\mathbf{Y}} \varphi(t,\mathbf{y})\Pi(d\mathbf{y})dt,$$

where we propose, that the right part of the last relation exists.

7.7 Moment estimations for stochastic integrals according to Poisson measures

According to the Ito formula for the Ito process with jump componet with probubility 1 we get [2]:

$$(z_t)^r = \int_0^t \int_{\mathbf{Y}} \left((z_{\tau-} + \gamma(\tau, \mathbf{y}))^r - (z_{\tau-})^r \right) \nu(d\tau, d\mathbf{y}), \tag{7.12}$$

where

$$z_t = \int_0^t \int_{\mathbf{Y}} \gamma(\tau, \mathbf{y}) \nu(d\tau, d\mathbf{y}).$$

We suppose, that the function $\gamma(\tau, \mathbf{y})$ satisfies to well-known conditions of right part (7.12) existence.

Let's analyze [2] the useful estimation of moments of the stochastic integral according to Poisson measure:

$$a_r(T) \leq \max_{j \in \{r,1\}} \left\{ \left(\int_0^T \int_{\mathbf{Y}} \left(\left((b_r(\tau, \mathbf{y}))^{\frac{1}{r}} + 1 \right)^r - 1 \right) \Pi(d\mathbf{y}) d\tau \right)^j \right\}, \tag{7.13}$$

where

$$a_r(t) = \sup_{0 \leq \tau \leq t} \mathsf{M} \left\{ |z_\tau|^r \right\}, \ b_r(\tau, \mathbf{y}) = \mathsf{M} \left\{ |\gamma(\tau, \mathbf{y})|^r \right\}.$$

We suppose, that right part of (7.13) exists.

Since $\tilde{\nu}(dt, d\mathbf{y}) = \nu(dt, d\mathbf{y}) - \Pi(d\mathbf{y})dt$, then according to Minkowskiy inequality

$$\left(\mathsf{M} \left\{ |\tilde{z}_t|^{2r} \right\} \right)^{\frac{1}{2r}} \leq \left(\mathsf{M} \left\{ |z_t|^{2r} \right\} \right)^{\frac{1}{2r}} + \left(\mathsf{M} \left\{ |\hat{z}_t|^{2r} \right\} \right)^{\frac{1}{2r}}, \tag{7.14}$$

where

$$\hat{z}_t \stackrel{\text{def}}{=} \int_0^t \int_{\mathbf{Y}} \gamma(\tau, \mathbf{y}) \Pi(d\mathbf{y}) d\tau; \ \tilde{z}_t = \int_0^t \int_{\mathbf{Y}} \gamma(\tau, \mathbf{y}) \tilde{\nu}(d\tau, d\mathbf{y}).$$

The value $\mathsf{M}\{|\hat{z}_\tau|^{2r}\}$ may be estimated using the inequality (7.4):

$$\mathsf{M} \left\{ |\hat{z}_t|^{2r} \right\} \leq t^{2r-1} \int_0^t \mathsf{M} \left\{ \left| \int_{\mathbf{Y}} \varphi(\tau, \mathbf{y}) \Pi(d\mathbf{y}) \right|^{2r} \right\} d\tau,$$

where we suppose, that

$$\int_0^t \mathsf{M} \left\{ \left| \int_{\mathbf{Y}} \gamma(\tau, \mathbf{y}) \Pi(d\mathbf{y}) \right|^{2r} \right\} d\tau < \infty.$$

7.8 Taylor-Ito and Taylor-Stratonovich expansions

Assume, that L — is a set of functions $R(\mathbf{x},s) : \Re^n \times [0,T] \to \Re^1$, which for all $t \in [0,T]$, $\mathbf{x} \in \Re^n$ has continuous partial derivatives:

$$\frac{\partial R}{\partial t}(\mathbf{x},t),\ \frac{\partial R}{\partial \mathbf{x}^{(i)}}(\mathbf{x},t),\ \frac{\partial^2 R}{\partial \mathbf{x}^{(i)}\partial \mathbf{x}^{(j)}}(\mathbf{x},t);\ i,j=1,2,\ldots,n$$

and G_0 — is a set of functions $R(\mathbf{x},s) : \Re^n \times [0,T] \to \Re^1$, which for all $t \in [0,T]$, $\mathbf{x} \in \Re^n$ has continous derivatives:

$$\frac{\partial R}{\partial \mathbf{x}^{(i)}}(\mathbf{x},t);\ i=1,2,\ldots,n.$$

Let's define at the sets L and G_0 the following operators

$$LR(\mathbf{x},t) = \frac{\partial R}{\partial t}(\mathbf{x},t) + \sum_{i=1}^{n} \mathbf{a}^{(i)}(\mathbf{x},t)\frac{\partial R}{\partial \mathbf{x}^{(i)}}(\mathbf{x},t) +$$

$$+\frac{1}{2}\sum_{j=1}^{m}\sum_{l,i=1}^{n} B^{(lj)}(\mathbf{x},t)B^{(ij)}(\mathbf{x},t)\frac{\partial^2 R}{\partial \mathbf{x}^{(l)}\partial \mathbf{x}^{(i)}}(\mathbf{x},t),$$

$$G_0^{(i)}R(\mathbf{x},t) = \sum_{j=1}^{n} B^{(ji)}(\mathbf{x},t)\frac{\partial R}{\partial \mathbf{x}^{(j)}}(\mathbf{x},t);\ i=1,\ldots,m.$$

Let's examine the stochastic process $\eta_s = R(\mathbf{x}_s,s)$, where $R(\mathbf{x},s) : \Re^n \times [0,T] \to \Re^1$ or $\Re^n \times [0,T] \to \Re^n$ and $\mathbf{x}_t$ — is a solution of stochastic differential Ito equation (7.9).

Assume, that sufficient conditions for existence of solution of the stochastic differential Ito equation (7.9) fulfilled and $R(\mathbf{x},t) \in \mathrm{L}$; $LR(\mathbf{x}_t,t)$, $G_0^{(i)}R(\mathbf{x}_t,t) \in \mathrm{M}_2([0,T])$; $i=1,\ldots,m$.

Then according to the Ito formula for all $s,t \in [0,T]$ such, that $s \geq t$ with probability 1

$$R(\mathbf{x}_s,s) = R(\mathbf{x}_t,t) + \int_t^s LR(\mathbf{x}_\tau,\tau)d\tau + \sum_{i=1}^{m}\int_t^s G_0^{(i)}R(\mathbf{x}_\tau,\tau)d\mathbf{f}_\tau^{(i)}. \qquad (7.15)$$

Stochastic Taylor formula (Taylor-Ito expansion) may be obtained by iterated usage of the formula (7.15) for the stochastic process $R(\mathbf{x}_s,s)$.

Let's denote

$$\mathrm{G}_{rk} = \{(\lambda_k,\ldots,\lambda_1) : r+1 \leq 2k - \lambda_1 - \ldots - \lambda_k \leq 2r;$$

$$\lambda_l = 1 \text{ or } \lambda_l = 0; l=1,\ldots,k\},$$

$$\mathrm{E}_{qk} = \{(\lambda_k, \ldots, \lambda_1) : \ 2k - \lambda_1 - \ldots - \lambda_k = q; \\ \lambda_l = 1 \text{ or } \lambda_l = 0; l = 1, \ldots, k\},$$

$$\mathrm{M}_k = \{(\lambda_k, \ldots, \lambda_1) : \lambda_l = 1 \text{ or } \lambda_l = 0; l = 1, \ldots, k\}.$$

$$J^{(i_k \ldots i_1)}_{(\lambda_k \ldots \lambda_1)s,t} = \int\limits_t^s \ldots \int\limits_t^{\tau_2} d\mathbf{w}^{(i_k)}_{\tau_1} \ldots d\mathbf{w}^{(i_1)}_{\tau_k} \text{ if } k \geq 1, \tag{7.16}$$

$J^{(i_0 \ldots i_1)}_{(\lambda_0 \ldots \lambda_1)s,t} \stackrel{\text{def}}{=} 1$, $\mathbf{w}^{(i)}_\tau = \mathbf{f}^{(i)}_\tau$ when $i = 1, \ldots, m$ and $\mathbf{w}^{(0)}_\tau = \tau$.

Assume, that functions $\mathbf{a}(\mathbf{x}, t) : \ \Re^n \times [0, T] \to \Re^n$, $B(\mathbf{x}, t) : \ \Re^n \times [0, T] \to \Re^{n \times m}$, $R(\mathbf{x}, t) : \ \Re^n \times [0, T] \to \Re^1$ have smooth partial derivatives of any fixed order.

Then for all s, t such that $s > t$ with probubility 1 [24]:

$$R(\mathbf{x}_s, s) = R(\mathbf{x}_t, t) +$$

$$+ \sum_{k=1}^{r} \sum_{(\lambda_k, \ldots, \lambda_1) \in \mathrm{M}_k} \sum_{i_1 = \lambda_1}^{m\lambda_1} \ldots \sum_{i_k = \lambda_k}^{m\lambda_k} Q^{(i_k)}_{\lambda_l} \ldots Q^{(i_1)}_{\lambda_1} R(\mathbf{x}_t, t) \cdot J^{(i_k \ldots i_1)}_{(\lambda_k \ldots \lambda_1)s,t} +$$

$$+ D_{r+1_{s,t}}, \tag{7.17}$$

where $D_{r+1_{s,t}}$ — is a remainder term in the integral form [24], the right part of (7.17) exists in the mean-square sense and $\lambda_l = 1$ or $\lambda_l = 0$; $Q^{(i_l)}_{\lambda_l} = L$ and $i_l = 0$ if $\lambda_l = 0$; $Q^{(i_l)}_{\lambda_l} = G^{(i_l)}_0$ and $i_l = 1, \ldots, m$ if $\lambda_l = 1$; $l = 1, \ldots, N$.

If we put in order the members of Taylor-Ito expansion according to order of vanishing in the mean-square sense when $s \to t$, then with probability 1:

$$R(\mathbf{x}_s, s) = R(\mathbf{x}_t, t) +$$

$$+ \sum_{q,k=1}^{r} \sum_{(\lambda_k, \ldots, \lambda_1) \in \mathrm{E}_{qk}} \sum_{i_1 = \lambda_1}^{m\lambda_1} \ldots \sum_{i_k = \lambda_k}^{m\lambda_k} Q^{(i_k)}_{\lambda_l} \ldots Q^{(i_1)}_{\lambda_1} R(\mathbf{x}_t, t) \cdot J^{(i_k \ldots i_1)}_{(\lambda_k \ldots \lambda_1)s,t} +$$

$$+ H_{r+1_{s,t}}, \tag{7.18}$$

where

$$H_{r+1_{s,t}} = \sum_{k=1}^{r} \sum_{(\lambda_k, \ldots, \lambda_1) \in \mathrm{G}_{rk}} \sum_{i_1 = \lambda_1}^{m\lambda_1} \ldots \sum_{i_k = \lambda_k}^{m\lambda_k} Q^{(i_k)}_{\lambda_l} \ldots Q^{(i_1)}_{\lambda_1} R(\mathbf{x}_t, t) \cdot J^{(i_k \ldots i_1)}_{(\lambda_k \ldots \lambda_1)s,t} +$$

$$+ D_{r+1_{s,t}}.$$

Using standard relations between stochastic Stratonovich and Ito integrals we may rewrite Taylor-Ito expansion using the terms of multiple stochastic Stratonovich integrals [22], [24]. In this case stochastic Taylor formula is called as Taylor-Stratonovich expansion.

Assume, that the functions $\mathbf{a}(\mathbf{x},t):\ \Re^n\times[0,T]\to\Re^n$, $B(\mathbf{x},t):\ \Re^n\times[0,T]\to\Re^{n\times m}$, $R(\mathbf{x},t):\ \Re^n\times[0,T]\to\Re^1$ have smooth partial derivatives of any fixed order.

Then for all s,t such that $s>t$ with probability 1 [22], [24]:

$$R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+$$

$$+\sum_{k=1}^{r}\sum_{(\lambda_k,\ldots,\lambda_1)\in \mathrm{M}_k}\sum_{i_1=\lambda_1}^{m\lambda_1}\ldots\sum_{i_k=\lambda_k}^{m\lambda_k}D_{\lambda_l}^{(i_k)}\ldots D_{\lambda_1}^{(i_1)}R(\mathbf{x}_t,t)\cdot J^{*(i_k\ldots i_1)}_{(\lambda_k\ldots\lambda_1)s,t}+$$

$$+D_{r+1_{s,t}}, \tag{7.19}$$

where $D_{r+1_{s,t}}$ is a remainder term in the integral form [22], [24], the right part of (7.19) exists in the mean-square sense and also $\lambda_l=1$ or $\lambda_l=0$; $D_{\lambda_l}^{(i_l)}=L-\frac{1}{2}\sum\limits_{j=1}^{m}G_0^{(j)}G_0^{(j)}$ and $i_l=0$ when $\lambda_l=0$; $D_{\lambda_l}^{(i_l)}=G_0^{(i_l)}$ and $i_l=1,\ldots,m$ when $\lambda_l=1$; $l=1,\ldots,N$;

$$J^{*(i_k\ldots i_1)}_{(\lambda_k\ldots\lambda_1)s,t}=\int\limits_t^{*s}\ldots\int\limits_t^{*\tau_2}d\mathbf{w}_{\tau_1}^{(i_k)}\ldots d\mathbf{w}_{\tau_k}^{(i_1)}\ \text{if}\ k\geq 1, \tag{7.20}$$

$J^{*(i_0\ldots i_1)}_{(\lambda_0\ldots\lambda_1)s,t}\stackrel{\text{def}}{=}1$, $\mathbf{w}_\tau^{(i)}=\mathbf{f}_\tau^{(i)}$ when $i=1,\ldots,m$ and $\mathbf{w}_\tau^{(0)}=\tau$.

If we put in order the members of Taylor-Stratonovich expansion according to order of vanishing in the mean-square sense when $s\to t$, then with probability 1:

$$R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+$$

$$+\sum_{q,k=1}^{r}\sum_{(\lambda_k,\ldots,\lambda_1)\in \mathrm{E}_{qk}}\sum_{i_1=\lambda_1}^{m\lambda_1}\ldots\sum_{i_k=\lambda_k}^{m\lambda_k}D_{\lambda_l}^{(i_k)}\ldots D_{\lambda_1}^{(i_1)}R(\mathbf{x}_t,t)\cdot J^{*(i_k\ldots i_1)}_{(\lambda_k\ldots\lambda_1)s,t}+$$

$$+H_{r+1_{s,t}}, \tag{7.21}$$

where

$$H_{r+1_{s,t}}=\sum_{k=1}^{r}\sum_{(\lambda_k,\ldots,\lambda_1)\in \mathrm{G}_{rk}}\sum_{i_1=\lambda_1}^{m\lambda_1}\ldots\sum_{i_k=\lambda_k}^{m\lambda_k}D_{\lambda_l}^{(i_k)}\ldots D_{\lambda_1}^{(i_1)}R(\mathbf{x}_t,t)\cdot J^{*(i_k\ldots i_1)}_{(\lambda_k\ldots\lambda_1)s,t}+$$

$$+D_{r+1_{s,t}}.$$

7.9 The unified Taylor-Ito and Taylor-Stratonovich expansions

Let's analyze the stochastic differential Ito equation (7.9) and propose, that the conditions for the existence of its solution are met.

Assume, that functions $\mathbf{a}(\mathbf{x},t):\ \Re^n\times[0,T]\to\Re^n$, $B(\mathbf{x},t):\ \Re^n\times[0,T]\to\Re^{n\times m}$, $R(\mathbf{x},t):\ \Re^n\times[0,T]\to\Re^1$ have smooth partial derivatives of any fixed order.

By the iterated usage of the Ito formula for the process $R(\mathbf{x}_t,t)$, where $\mathbf{x}_t$ — is a solution of the equation (7.9), and special transformations, based on replacement of integration order in the multiple stochastic Ito integrals [31], [32], [42], [46] in [42]–[46], [48] the following unified Taylor-Ito and Taylor-Stratonovich expansions were obtained (while obtaining the unified Taylor-Stratonovich expansions we also used standard relations between multiple stochastic Ito and Stratonovich integrals [43], [46]):

$$R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+\sum_{q=1}^{r}\sum_{(k,j,l_1,\ldots,l_k)\in\mathrm{A}_q}\frac{(s-t)^j}{j!}\times$$

$$\times\sum_{i_1,\ldots,i_k=1}^{m}L^jG_{l_1}^{(i_1)}\ldots G_{l_k}^{(i_k)}R(\mathbf{x}_t,t)J_{l_1\ldots l_{k\,s,t}}^{(i_1\ldots i_k)}+D_{r+1_{s,t}} \qquad (7.22)$$

(the first unified Taylor-Ito expansion),

$$R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+\sum_{q=1}^{r}\sum_{(k,j,l_1,\ldots,l_k)\in\mathrm{A}_q}\frac{(s-t)^j}{j!}\times$$

$$\times\sum_{i_1,\ldots,i_k=1}^{m}G_{l_1}^{(i_1)}\ldots G_{l_k}^{(i_k)}L^jR(\mathbf{x}_t,t)I_{l_1\ldots l_{k\,s,t}}^{(i_1\ldots i_k)}+D_{r+1_{s,t}} \qquad (7.23)$$

(the second unified Taylor-Ito expansion),

$$R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+\sum_{q=1}^{r}\sum_{(k,j,l_1,\ldots,l_k)\in\mathrm{A}_q}\frac{(s-t)^j}{j!}\times$$

$$\times\sum_{i_1,\ldots,i_k=1}^{m}\bar{G}_{l_1}^{(i_1)}\ldots \bar{G}_{l_k}^{(i_k)}\bar{L}^jR(\mathbf{x}_t,t)I_{l_1\ldots l_{k\,s,t}}^{*(i_1\ldots i_k)}+D_{r+1_{s,t}} \qquad (7.24)$$

(the first unified Taylor-Stratonovich expansion),

$$R(\mathbf{x}_s,s)=R(\mathbf{x}_t,t)+\sum_{q=1}^{r}\sum_{(k,j,l_1,\ldots,l_k)\in\mathrm{A}_q}\frac{(s-t)^j}{j!}\times$$

$$\times \sum_{i_1,\ldots,i_k=1}^{m} \bar{L}^j \bar{G}_{l_1}^{(i_1)} \ldots \bar{G}_{l_k}^{(i_k)} R(\mathbf{x}_t, t) J_{l_1 \ldots l_{k\,s,t}}^{*(i_1 \ldots i_k)} + D_{r+1_{s,t}} \tag{7.25}$$

(the second unified Taylor-Stratonovich expansion),

where

$$I_{l_1 \ldots l_{k\,s,t}}^{(i_1 \ldots i_k)} = \int_t^s (t-t_k)^{l_k} \ldots \int_t^{t_2} (t-t_1)^{l_1} d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)}, \tag{7.26}$$

$$J_{l_1 \ldots l_{k\,s,t}}^{(i_1 \ldots i_k)} = \int_t^s (s-t_k)^{l_k} \ldots \int_t^{t_2} (s-t_1)^{l_1} d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)}, \tag{7.27}$$

$$I_{l_1 \ldots l_{k\,s,t}}^{*(i_1 \ldots i_k)} = \int_t^{*s} (t-t_k)^{l_k} \ldots \int_t^{*t_2} (t-t_1)^{l_1} d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)}, \tag{7.28}$$

$$J_{l_1 \ldots l_{k\,s,t}}^{*(i_1 \ldots i_k)} = \int_t^{*s} (s-t_k)^{l_k} \ldots \int_t^{*t_2} (s-t_1)^{l_1} d\mathbf{f}_{t_1}^{(i_1)} \ldots d\mathbf{f}_{t_k}^{(i_k)}; \tag{7.29}$$

$l_1, \ldots, l_k = 0,\ 1, \ldots;\ k = 1,\ 2, \ldots;\ i_1, \ldots, i_k = 1, \ldots, m$; these integrals when $k = 0$ are set equal to 1;

$$\mathrm{A}_q = \left\{(k, j, l_1, \ldots, l_k) : k + j + \sum_{p=1}^{k} l_p = q;\ k, j, l_1, \ldots, l_k = 0, 1, \ldots\right\},$$

$$L^j \stackrel{\text{def}}{=} \begin{cases} \underbrace{L \ldots L}_{j} & \text{when } j = 1, 2, \ldots \\ \cdot & \text{when } j = 0 \end{cases}, \quad \bar{L}^j \stackrel{\text{def}}{=} \begin{cases} \underbrace{\bar{L} \ldots \bar{L}}_{j} & \text{when } j = 1, 2, \ldots \\ \cdot & \text{when } j = 0 \end{cases},$$

$$G_p^{(i)} \stackrel{\text{def}}{=} \frac{1}{p}\left(G_{p-1}^{(i)} L - L G_{p-1}^{(i)}\right),\ \bar{G}_p^{(i)} \stackrel{\text{def}}{=} \frac{1}{p}\left(\bar{G}_{p-1}^{(i)} \bar{L} - \bar{L} \bar{G}_{p-1}^{(i)}\right);$$

$p = 1, 2, \ldots; i = 1, \ldots, m$; $\bar{L} = L - \frac{1}{2}\sum_{i=1}^{m} G_0^{(i)} G_0^{(i)},\ \bar{G}_0^{(i)} = G_0^{(i)}$; the operators L and $G_0^{(i)}$; $i = 1, \ldots, m$ are defined in previous section; $D_{r+1_{s,t}}$ — is a remainer term in the integral form (see [42]–[46], [48]).

In [42]–[46], [48] the unified Taylor-Ito and Taylor-Stratonovich expansions well-ordered according to the orders of vanishing in the mean-square sense when $t \to s$ were also examined.

In this case summation in the unified Taylor-Ito and Taylor-Stratonovich expansions is performed using the sets

$$\mathrm{D}_q = \{(k, j, l_1, \ldots, l_k) : k + 2(j + l_1 + \ldots + l_k) = q;\ k, j, l_1, \ldots, l_k = 0, 1, \ldots\},$$

instead of sets

$$\mathrm{A}_q = \left\{(k, j, l_1, \ldots, l_k) : k + j + \sum_{p=1}^{k} l_p = q;\ k, j, l_1, \ldots, l_k = 0, 1, \ldots\right\}.$$

Note, that the truncated unified Taylor-Ito and Taylor-Stratonovich expansions contain the less number of various multiple stochastic integrals (moreover, their major part will have less multiplicity) in comparison with classic Taylor-Ito and Taylor-Stratonovich expansions [22].

It is easy to note, that the stochastic integrals from the family (7.16) are connected by the linear relations. The same may be noted for the family (7.20).

However, the stochastic integrals from the families (7.26)–(7.29) can't be connected by linear relations. Therefore we call the families (7.26)–(7.29) as *stochastic basises.*

Let's name the numbers $\text{rank}_{\text{A}}(r)$ and $\text{rank}_{\text{D}}(r)$ of various multiple stochastic integrals which are included in the families (7.26)–(7.29) as the *ranks of stochastic bases*, when summation in the stochastic expansions is performed using the sets $\text{A}_q;\ q=1,\ldots,r$ and $\text{D}_q;\ q=1,\ldots,r$ correspondently; here r — is a fixed natural number.

At the beginning, let's analyze several examples.

Assume, that summation in the unified Taylor-Ito and Taylor-Stratonovich expansions is performed using the sets

$$\text{D}_q=\{(k,j,l_1,\ldots,l_k): k+2(j+l_1+\ldots+l_k)=q;\ k,j,l_1,\ldots,l_k=0,1,\ldots\}.$$

It is easy to see, that the truncated unified Taylor-Ito expansion, where summation is performed using sets D_q when $r=3$ includes 4 ($\text{rank}_{\text{D}}(3)=4$) various multiple stochastic integrals: $I_{0_{s,t}}^{(i_1)},\ I_{00_{s,t}}^{(i_2 i_1)},\ I_{1_{s,t}}^{(i_1)},\ I_{000_{s,t}}^{(i_3 i_2 i_1)}$. The same truncated classic Taylor-Ito expansion [24] contains 5 various multiple stochastic integrals: $J_{(1)s,t}^{(i_1)},\ J_{(11)s,t}^{(i_2 i_1)},\ J_{(10)s,t}^{(i_2)},\ J_{(01)s,t}^{(i_1)},\ J_{(111)s,t}^{(i_3 i_2 i_1)}$.

For $r=4$ we have 7 ($\text{rank}_{\text{D}}(4)=7$) integrals: $I_{0_{s,t}}^{(i_1)},\ I_{00_{s,t}}^{(i_2 i_1)},\ I_{1_{s,t}}^{(i_1)},\ I_{000_{s,t}}^{(i_3 i_2 i_1)},$ $I_{01_{s,t}}^{(i_2 i_1)},\ I_{10_{s,t}}^{(i_2 i_1)},\ I_{0000_{s,t}}^{(i_4 i_3 i_2 i_1)}$ against 9 stochastic integrals: $J_{(1)s,t}^{(i_1)},\ J_{(11)s,t}^{(i_2 i_1)},\ J_{(10)s,t}^{(i_2)},$ $J_{(01)s,t}^{(i_1)},\ J_{(111)s,t}^{(i_3 i_2 i_1)},\ J_{(101)s,t}^{(i_3 i_1)},\ J_{(110)s,t}^{(i_3 i_2)},\ J_{(011)s,t}^{(i_2 i_1)},\ J_{(1111)s,t}^{(i_4 i_3 i_2 i_1)}$. For $r=5$ ($\text{rank}_{\text{D}}(5)=12$) we get 12 integrals against 17 integrals and for $r=6$ and $r=7$ we have 20 against 29 and 33 against 50 correspondently.

We will get the same results when compare the unified Taylor-Stratonovich expansions with their classical analogues [24].

Note, that summation according to sets D_q is usually used while constructing strong numerical methods (built according to the mean-square criterion of convergence) for stochastic differential Ito equations [23], [24], [46].

Summation according to sets A_q is usually used when building weak numerical methods (built in accordance with the weak criterion of convergence) for stochastic differential Ito equations [23], [24].

Table 7.1: Numbers $\text{rank}_{\text{A}}(r)$, $n_{\text{M}}(r)$, $f(r) = n_{\text{M}}(r)/\text{rank}_{\text{A}}(r)$

r	1	2	3	4	5	6	7	8	9	10
$\text{rank}_{\text{A}}(r)$	1	3	7	15	31	63	127	255	511	1023
$n_{\text{M}}(r)$	1	4	11	26	57	120	247	502	1013	2036
$f(r)$	1	1.3333	1.5714	1.7333	1.8387	1.9048	1.9449	1.9686	1.9824	1.9902

Table 7.2: Numbers $\text{rank}_{\text{D}}(r)$, $n_{\text{E}}(r)$, $g(r) = n_{\text{E}}(r)/\text{rank}_{\text{D}}(r)$

r	1	2	3	4	5	6	7	8	9	10
$\text{rank}_{\text{D}}(r)$	1	2	4	7	12	20	33	54	88	143
$n_{\text{E}}(r)$	1	2	5	9	17	29	50	83	138	261
$g(r)$	1	1	1.2500	1.2857	1.4167	1.4500	1.5152	1.5370	1.5682	1.8252

For example, $\text{rank}_{\text{A}}(4) = 15$, while the total number of various multiple stochastic integrals, included in the classic Taylor-Ito expansions [24] when $r = 4$, equals to 26.

It is easy to check, that $\text{rank}_{\text{A}}(r) = 2^r - 1$ [46].

Let's denote the total number of various multiple stochastic integrals included in the classic Taylor-Ito expansion (7.17) by $n_{\text{M}}(r)$, where summation is performed using the set $\bigcup\limits_{k=1}^{r} \text{M}_k$.

We can demonstrate [46], that $n_{\text{M}}(r) = 2(2^r - 1) - r$.

It means, that $\lim\limits_{r\to\infty} n_{\text{M}}(r)/\text{rank}_{\text{A}}(r) = 2$.

In table 7.1 we can see numbers $\text{rank}_{\text{A}}(r)$, $n_{\text{M}}(r)$, $f(r) = n_{\text{M}}(r)/\text{rank}_{\text{A}}(r)$ for various values r.

In [46] it was proven, that

$$\text{rank}_{\text{D}}(r) = \begin{bmatrix} \sum\limits_{s=0}^{r-1} \sum\limits_{l=s}^{\frac{r-1}{2}+[\frac{s}{2}]} C_l^s & \text{when } r = 1,\ 3,\ 5, \ldots \\ \sum\limits_{s=0}^{r-1} \sum\limits_{l=s}^{\frac{r}{2}-1+[\frac{s+1}{2}]} C_l^s & \text{when } r = 2,\ 4,\ 6, \ldots \end{bmatrix},$$

where $[x]$ — is an integer part of number x; C_n^m — is a binomial coefficient.

Using $n_{\text{E}}(r)$ let's denote the number of various multiple stochastic integrals included in the classic Taylor-Ito expansion (7.18) (W.Wagner, E.Platen) where summation is performed using the set $\bigcup\limits_{q,k=1}^{r} \text{E}_{qk}$.

In [46] it is proven, that

$$n_{\text{E}}(r) = \sum_{s=1}^{r} \sum_{l=0}^{[\frac{r-s}{2}]} C^s_{[\frac{r-s}{2}]+s-l}, \tag{7.30}$$

where $[x]$ — is an integer part of number x; C_n^m — is a binomial coefficient.

In table 7.2 we can see numbers $\mathrm{rank}_{\mathrm{D}}(r)$, $n_{\mathrm{E}}(r)$, $g(r) = n_{\mathrm{E}}(r)/\mathrm{rank}_{\mathrm{D}}(r)$ for various values r.

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