Strong Asymptotics of Orthogonal Polynomials with Respect to Exponential Weights

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Abstract

We consider asymptotics of orthogonal polynomials with respect to weights $w(x)dx = e^{-Q(x)}dx$ on the real line, where $Q(x) = \sum_{k=0}^{2m} q_k x^k$, $q_{2m} > 0$, denotes a polynomial of even order with positive leading coefficient. The orthogonal polynomial problem is formulated as a Riemann-Hilbert problem following [22, 23].

We employ the steepest-descent-type method introduced in [18] and further developed in [17, 19] in order to obtain uniform Plancherel-Rotach-type asymptotics in the entire complex plane, as well as asymptotic formulae for the zeros, the leading coefficients, and the recurrence coefficients of the orthogonal polynomials. © 1999 John Wiley & Sons, Inc.

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1 Introduction and Background

Let

$$Q(x) = \sum_{k=0}^{2m} q_k x^k, \quad q_{2m} > 0,\ m > 0, \tag{1.1}$$

be a polynomial of even degree with a positive leading coefficient. We denote by $\pi_n(x,Q) = \pi_n(x) = x^n + \cdots$ the n^{th} monic orthogonal polynomial with respect to the measure

$$w(x)dx = e^{-Q(x)}dx \tag{1.2}$$

on the real line and by $p_n(x,Q) = p_n(x) = \gamma_n \pi_n(x)$, $\gamma_n > 0$, the normalized n^{th} orthogonal polynomial or simply the n^{th} orthogonal polynomial, with respect to the measure $w(x)dx$, i.e.,

$$\int_{\mathbb{R}} p_n(x) p_m(x) e^{-Q(x)} dx = \delta_{n,m}, \quad n,m \in \mathbb{N}. \tag{1.3}$$

Furthermore, we denote by $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ the coefficients of the associated three-term recurrence relations, namely,

$$x p_n(x) = b_n p_{n+1}(x) + a_n p_n(x) + b_{n-1} p_{n-1}(x), \quad n \in \mathbb{N}, \tag{1.4}$$

and denote by

$$x_{1,n} > x_{2,n} > \cdots > x_{n,n} \tag{1.5}$$

the roots of p_n. The statement of results involves the n^{th} *Mhaskar-Rakhmanov-Saff numbers* (in short, MRS numbers [34, 37]) α_n and β_n, which can be determined from the equations

$$\frac{1}{2\pi} \int_{\alpha_n}^{\beta_n} \frac{Q'(t)(t-\alpha_n)}{\sqrt{(\beta_n - t)(t-\alpha_n)}}\, dt = n, \tag{1.6}$$

$$\frac{1}{2\pi} \int_{\alpha_n}^{\beta_n} \frac{Q'(t)(\beta_n - t)}{\sqrt{(\beta_n - t)(t-\alpha_n)}}\, dt = -n, \tag{1.7}$$

and, in particular, the interval $[\alpha_n, \beta_n]$ whose width and midpoint are given by

$$c_n := \frac{\beta_n - \alpha_n}{2}, \qquad d_n := \frac{\beta_n + \alpha_n}{2}. \tag{1.8}$$

For the weights under consideration, it will be straightforward to prove the existence of the MRS numbers for sufficiently large n (see Proposition 5.2 below). Indeed, they can be expressed in a power series in $n^{-1/2m}$. We obtain

$$c_n = n^{\frac{1}{2m}} \sum_{l=0}^{\infty} c^{(l)} n^{-\frac{l}{2m}}, \quad c^{(0)} = (q_{2m} m A_m)^{-\frac{1}{2m}}, \quad c^{(1)} = 0, \tag{1.9}$$

$$d_n = \sum_{l=0}^{\infty} d^{(l)} n^{-\frac{l}{2m}}, \quad d^{(0)} = -\frac{q_{2m-1}}{2mq_{2m}}, \tag{1.10}$$

where

$$A_m := \prod_{j=1}^{m} \frac{2j-1}{2j}, \quad m \in \mathbb{N}. \tag{1.11}$$

The coefficients of the series for c_n and d_n can be computed explicitly, and we have just written down the first ones in (1.9) and (1.10) for the reader's convenience. From now on, we will assume that n is sufficiently large for (1.9) and (1.10) to hold.

The results in this paper concern the asymptotics of leading coefficients γ_n, recurrence coefficients a_n and b_n, and zeros $x_{j,n}$, as well as Plancherel-Rotach-type asymptotics for the orthogonal polynomials p_n, i.e., asymptotics for $p_n(c_n z + d_n)$ uniformly for all $z \in \mathbb{C}$. The name "Plancherel-Rotach" refers to the work [36] in which the authors prove asymptotics of this type in the classical case of Hermite polynomials.

There is a vast literature on asymptotic questions for orthogonal polynomials. Among the measures considered in this paper, the case of the Hermite weight was the first to be understood (see [36, 43]). During the last thirty years, more general classes of weights on $\mathbb{R}$ have been studied. Here the class of Freud weights played a most important role. Freud weights are of the form $e^{-Q(x)}dx$, where Q grows like a power at infinity (see [28] for a recent survey). We refer the reader to [28] for a full description of the wide variety of results that have been obtained in this field.

We will now describe briefly the results in the literature related to Theorems 2.1, 2.2, and 2.3 below. The measures considered are always of the form

$$w(x)dx = e^{-Q(x)}dx. \tag{1.12}$$

The function Q will be either a polynomial of even degree (as in our results) or a power of the form $Q(x) = |x|^\beta$, $\beta > 0$, or lie in some class of functions that grow like powers at infinity and that we will simply call Freud weights without distinguishing the different (and rather technical) additional assumptions.

1.1 Asymptotics of Leading and Recurrence Coefficients

Strong asymptotics for the leading coefficients γ_n of orthogonal polynomials have been obtained by Lubinsky and Saff [30] and by Totik [4] for even Freud weights and by Rakhmanov [38] for $Q(x) = |x|^\beta$, $\beta > 1$. These papers are concerned only with determining γ_n to leading order in n. In particular, in [30] and [4] no estimates on the rate of convergence are provided, whereas in [38] an error bound of order $O(n^{-1/3})$ is given.

In the case that Q is a polynomial of even order with positive leading coefficient, Magnus [31] showed that $b_{n-1}/c_n \to \frac{1}{2}$ and $a_n n^{-1/2m} \to 0$. The existence of an asymptotic expansion was then proved by Bauldry, Máté, and Nevai [4]. In the case of even Freud weights, Lubinsky, Mhaskar, and Saff [29] proved $b_{n-1}/c_n \to$

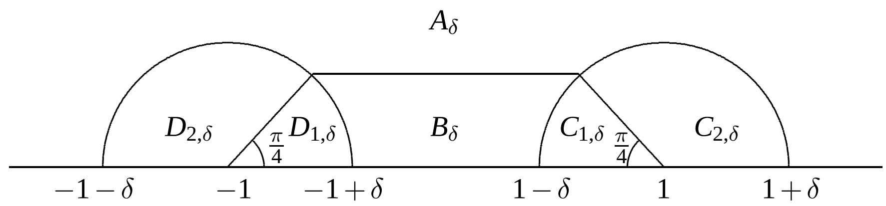

FIGURE 1.1. Different asymptotic regions for $p_n(c_n z + d_n)$ in $\overline{\mathbb{C}}_\pm$.

$\frac{1}{2}$. Note that the limit of b_{n-1}/c_n as $n \to \infty$ is related to the well-known Freud conjecture, i.e., $b_{n-1}/x_{1,n} \to \frac{1}{2}$. The two limits are equivalent by the asymptotics for the largest zero $x_{1,n}$ (see, e.g., Theorem 2.3). Note the high accuracy of the Freud conjecture as displayed in (2.11), where all the terms up to order $O(n^{-2})$ vanish.

1.2 Plancherel-Rotach Asymptotics

Because $p_n(z) = \overline{p_n(\overline{z})}$, we will only describe the asymptotics of $p_n(c_n z + d_n)$ in the closed upper half-plane $\overline{\mathbb{C}}_+$. Depending on a small parameter δ, we divide $\overline{\mathbb{C}}_+$ into six closed regions, as shown in Figure 1.1.

The asymptotic behavior of $p_n(c_n z + d_n)e^{-\frac{1}{2}Q(c_n z + d_n)}$ in the region A_δ was determined by Lubinsky and Saff [30] and Totik [44] for even Freud weights, and by Rakhmanov [38] for $Q(x) = |x|^\beta$, $\beta > 1$. In a different direction, Geronimo and Van Assche (see, e.g., [45]) imposed conditions directly on the recurrence coefficients rather than on the weight function e^{-Q} and obtained asymptotic results for the region A_δ.

In the region B_δ, results have been obtained by Nevai [35] in the case $Q(x) = x^4$, by Bauldry [2] for $Q(x) = x^4 + q(x)$, where q is an arbitrary polynomial of degree 3, by Sheen [41] for $Q(x) = x^6$, by Rakhmanov [38] for $Q(x) = |x|^\beta$, $\beta > 1$, and by Lubinsky [27] for even Freud weights. Estimates on the rate of convergence were given in [35] and [41], where the error term was shown to be of order $O(n^{-1})$, uniformly for $z \in [-1+\delta, 1-\delta]$, for any $\delta > 0$. Nevai conjectured that the same rate of convergence would apply for Freud weights. Formula (2.18) proves this conjecture in the case of Q being a polynomial of even order with positive leading coefficient.

1.3 Asymptotic Location of the Zeros

Máté, Nevai, and Totik proved in [32] that an asymptotic formula similar to (2.27) holds if the recurrence coefficients satisfy $a_n \equiv 0$, $b_{n-1} = cn^\gamma(1 + o(n^{-2/3}))$, for some positive constants c and γ. This proves (2.27) (with a weaker error term than we obtain) in the case of $Q(x) = x^{2m}$ (and for a slightly more general class of even polynomials Q, cf. [4] or Theorem 2.1 above). Asymptotics for the largest zeros also follow from recent work of Chen and Ismail [9] for convex polynomials Q. For even Freud weights it was shown in ([26]) that $x_{1,n}/c_n = 1 + O(n^{-2/3})$.

Estimates on the distance between neighboring zeros were obtained in the case of even Freud weights by Criscuolo, Della Vecchia, Lubinsky, and Mastroianni [11].

Second-order differential equations have played an important role in analyzing the asymptotic behavior of orthogonal polynomials. They have been derived by Nevai [35], Sheen [41], Bonan and Clark [8], Bauldry [3], Chen and Ismail [9] for polynomial Q, and Mhaskar [33] for even Freud weights. Although we do not use second-order differential equations for p_n to derive our asymptotic results, we can derive such differential equations from the Riemann-Hilbert formulation as indicated in Appendix C.

To prove our results stated in the following section, we apply, in Section 3, the reformulation of Fokas, Its, and Kitaev [22, 23] of the problem of orthogonal polynomials as a Riemann-Hilbert problem. In the remaining sections, we derive the asymptotics of the solution to the Riemann-Hilbert problem. We use the method of steepest descent introduced by Deift and Zhou in [18] and further developed in [19], and also by Deift, Venakides, and Zhou in [17]. Our analysis also makes use of results obtained by Deift, Kriecherbauer, and McLaughlin in [14]. To our knowledge the asymptotic behavior of p_n near the largest and smallest zeros (i.e., regions $C_{1,\delta}$, $C_{2,\delta}$, $D_{1,\delta}$, and $D_{2,\delta}$) has not been determined before except, of course, in the classical case of Hermite polynomials (see [36, 43]). Also, formulae (2.29) and (2.30), which locate $x_{k,n}$ for arbitrary ratios $\frac{k}{n} \in [0,1]$, are, we believe, new. The method presented in this paper allows explicit asymptotic expansions for the various quantities to be obtained to all orders.

In Section 2 we state our results. In Section 3 we present the reformulation of the problem for orthogonal polynomials as a Riemann-Hilbert problem (RHP). We give an overview of the calculation in Section 4. We solve the RHP in Sections 5 through 7. The main results, Theorems 2.1, 2.2, and 2.3, are proved in Section 8.

In a related paper ([16], whose results were announced in [15]) we consider asymptotics for polynomials $p_k(z;N)$, $k = 0,1,2,\dots$, orthogonal with respect to varying exponential weights $e^{-NV(x)}dx$, where V is real analytic and satisfies the growth condition $V(x)/\log(1+x^2) \to \infty$ as $|x| \to \infty$. Of particular interest are the asymptotics of $p_n(z;n)$ and $p_{n-1}(z;n)$ as $n \to \infty$. These asymptotics are the crucial ingredients in proving a variety of universality conjectures in random matrix theory in [16]. The analysis in [16] and in the present paper are based on the same approach, but there are crucial differences as discussed in Remark 4.1 below. A pedagogic discussion of our methods is given in [12] in the special case where Q is a monomial $Q(x) = x^{2m}$. In the special case where V is an even quartic polynomial, the results in [16] should be compared with [7].

The methods presented in this paper are developments of earlier work of the authors in inverse scattering theory. The first connection between inverse scattering theory and Riemann-Hilbert problems was established by Šabat [39]. The first systematic and rigorous analysis of the inverse scattering theory of first-order systems using Riemann-Hilbert techniques is due to Beals and Coifman [5].

2 Statement of Results

To make our subsequent analysis simpler, we have normalized the interval $[\alpha_n, \beta_n]$ to be $[-1,1]$ by making the linear change of variable

$$\lambda_n : \mathbb{C} \to \mathbb{C} : z \mapsto c_n z + d_n \,, \tag{2.1}$$

which takes the interval $[-1,1]$ onto $[\alpha_n, \beta_n]$, and we work with the function

$$V_n(z) := \frac{1}{n} Q(\lambda_n(z)) \,. \tag{2.2}$$

The function V_n is again a polynomial of degree $2m$ with a leading coefficient of $(mA_m)^{-1} > 0$ and all other coefficients tending to zero as n tends to ∞.

We present our results in terms of the well-known equilibrium measure μ_n (see, e.g., [40] and (4.17) below) with respect to V_n, which is defined as the unique minimizer in $M_1(\mathbb{R}) = \{\text{probability measures on } \mathbb{R}\}$ of the functional

(2.3)

$$I^{V_n} : M_1(\mathbb{R}) \to (-\infty, \infty] : \mu \mapsto \int_{\mathbb{R}^2} \log |x-y|^{-1} d\mu(x) d\mu(y) + \int_{\mathbb{R}} V_n(x) d\mu(x) \,.$$

The equilibrium measure together with the the corresponding variational problem emerge naturally in our asymptotic analysis of the Riemann-Hilbert problem (cf. Section 4). The minimizing measure is given by

$$d\mu_n(x) = \frac{1}{2\pi} \sqrt{1-x^2} h_n(x) \mathbf{1}_{[-1,1]}(x) dx \,, \tag{2.4}$$

where $\mathbf{1}_{[-1,1]}$ denotes the indicator function of the set $[-1,1]$ and h_n is a polynomial of degree $2m-2$,

$$h_n(x) = \sum_{k=0}^{2m-2} h_{n,k} x^k \,, \tag{2.5}$$

and the real coefficients $h_{n,k}$ can be expanded in a power series in $n^{-1/2m}$

$$h_{n,k} = \sum_{l=0}^{\infty} h_k^{(l)} n^{-\frac{l}{2m}} \,. \tag{2.6}$$

Again the coefficients $h_k^{(l)}$ can be computed explicitly (see (5.25) below) and the leading-order behavior is given by

$$h_{2k}^{(0)} = 2 \frac{A_{m-k-1}}{A_m} \,, \quad h_{2k+1}^{(0)} = 0 \,, \quad 0 \le k \le m-1 \,. \tag{2.7}$$

Finally, to state our first theorem, we define

$$l_n := \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-t^2}\,h_n(t)\log|t|dt - V_n(0)\,, \tag{2.8}$$

which also has an explicitly computable power series in $n^{-1/2m}$,

$$l_n = \sum_{k=0}^{\infty} l^{(k)} n^{-\frac{k}{2m}} \quad \text{with } l^{(0)} = -\frac{1}{m} - 2\log 2\,. \tag{2.9}$$

2.1 Asymptotics of Leading and Recurrence Coefficients of Orthogonal Polynomials p_n

THEOREM 2.1 *In the notation above we have*

$$\begin{aligned}\gamma_n\sqrt{\pi c_n^{2n+1}e^{nl_n}} = 1 - \frac{1}{n}\left(\frac{4h_n(1)-3h_n'(1)}{48h_n(1)^2} + \frac{4h_n(-1)+3h_n'(-1)}{48h_n(-1)^2}\right)\\ + O\left(\frac{1}{n^2}\right),\end{aligned} \tag{2.10}$$

$$\frac{b_{n-1}}{c_n} = \frac{1}{2} + O\left(\frac{1}{n^2}\right), \tag{2.11}$$

$$a_n = d_n + \frac{c_n}{2n}\left(\frac{1}{h_n(1)} - \frac{1}{h_n(-1)} + O\left(\frac{1}{n}\right)\right). \tag{2.12}$$

In all three cases there are explicit integral formulae for the error terms, all of which have an asymptotic expansion in $n^{-1/2m}$, e.g., $O(\frac{1}{n}) = \frac{1}{n}(\kappa_0 + \kappa_1 n^{-1/2m} + \cdots)$. The coefficients of these expansions can be computed via the calculus of residues by purely algebraic means.

Next we will state the Plancherel-Rotach-type asymptotics of the orthogonal polynomials p_n, i.e., the limiting behavior of the rescaled n^{th} orthogonal polynomial $p_n(\lambda_n(z))$ as n tends to infinity and $z \in \mathbb{C}$ remains fixed. We will give the leading-order behavior and produce error bounds that are uniform in the entire complex plane $\mathbb{C}$.

Remark (Notation). Throughout this paper we will denote for $\alpha \in \mathbb{R}$ the function

$$(\cdot)^\alpha : \mathbb{C}\setminus(-\infty,0] \to \mathbb{C} : z \mapsto e^{\alpha\log z}\,, \tag{2.13}$$

where log denotes the principal branch of the logarithm. In addition, we will reserve the notation $\sqrt{a}$ for nonnegative numbers a, and we always take $\sqrt{a}$ nonnegative. Thus $\sqrt{1-x^2}$, $-1 \le x \le 1$, in (2.4) is positive and equals $(1-x)^{1/2}(1+x)^{1/2}$, etc.

2.2 Plancherel-Rotach Asymptotics

We state our second theorem in terms of the function

$$\psi_n : \mathbb{C} \setminus ((-\infty,-1] \cup [1,\infty)) \to \mathbb{C} : z \mapsto \frac{1}{2\pi}(1-z)^{1/2}(1+z)^{1/2} h_n(z)\,. \tag{2.14}$$

The function ψ_n is an analytic extension of the density of μ_n on $(-1,1)$ to $\mathbb{C} \setminus ((-\infty,-1] \cup [1,\infty))$ and thus closely linked to the equilibrium measure (cf. (2.4)). We show that there exist analytic functions f_n and $\tilde{f}_n$ in a neighborhood of 1 and -1, respectively, satisfying

$$(-f_n(z))^{3/2} = -n\frac{3\pi}{2}\int_1^z \psi_n(y)dy \quad \text{for } |z-1| \text{ small, } z \notin [1,\infty)\,. \tag{2.15}$$

$$(\tilde{f}_n(z))^{3/2} = n\frac{3\pi}{2}\int_{-1}^z \psi_n(y)dy \quad \text{for } |z+1| \text{ small, } z \notin (-\infty,-1]\,. \tag{2.16}$$

(See Proposition 7.3, the remark thereafter, and (7.38)). Again each of the Taylor coefficients of $n^{-2/3} f_n$ at 1 and of $n^{-2/3}\tilde{f}_n$ at -1 can be computed explicitly as a series in $n^{-1/2m}$.

THEOREM 2.2 *There exists a* δ_0 *such that for all* $0 < \delta \le \delta_0$ *the following holds* (*see Figure* 1.1):

(i) *For* $z \in A_\delta$,

(2.17)

$$p_n(c_n z + d_n)e^{-\frac{1}{2}Q(c_n z + d_n)} = \sqrt{\frac{1}{4\pi c_n}}\exp\left(-n\pi i \int_1^z \psi_n(y)dy\right)\left(\frac{(z-1)^{1/4}}{(z+1)^{1/4}} + \frac{(z+1)^{1/4}}{(z-1)^{1/4}}\right)\left(1 + O\left(\frac{1}{n}\right)\right).$$

(ii) *For* $z \in B_\delta$,

$$\begin{aligned} &p_n(c_n z + d_n)e^{-\frac{1}{2}Q(c_n z + d_n)} \\ &\quad = \sqrt{\frac{2}{\pi c_n}}(1-z)^{-1/4}(1+z)^{-1/4} \\ &\qquad \times \left\{\cos\left(n\pi\int_1^z \psi_n(y)dy + \frac{1}{2}\arcsin z\right)\left(1 + O\left(\frac{1}{n}\right)\right)\right. \\ &\qquad\qquad \left. + \sin\left(n\pi\int_1^z \psi_n(y)dy - \frac{1}{2}\arcsin z\right) O\left(\frac{1}{n}\right)\right\}. \end{aligned} \tag{2.18}$$

(iii) *For* $z \in C_{1,\delta}$,

$$\begin{aligned} & p_n(c_n z + d_n) e^{-\frac{1}{2}Q(c_n z + d_n)} \\ &= \sqrt{\frac{1}{c_n}} \left\{ \left(\frac{(z+1)^{1/4}}{(z-1)^{1/4}} \left(f_n(z)\right)^{1/4} \operatorname{Ai}\left(f_n(z)\right) \right) \left(1 + O\left(\frac{1}{n}\right)\right) \right. \\ & \qquad \left. - \left(\frac{(z-1)^{1/4}}{(z+1)^{1/4}} \left(f_n(z)\right)^{-1/4} \operatorname{Ai}'\left(f_n(z)\right) \right) \left(1 + O\left(\frac{1}{n}\right)\right) \right\}. \end{aligned} \tag{2.19}$$

(iv) *For* $z \in C_{2,\delta}$,

$$\begin{aligned} & p_n(c_n z + d_n) e^{-\frac{1}{2}Q(c_n z + d_n)} \\ &= \sqrt{\frac{1}{c_n}} \left\{ \frac{(z+1)^{1/4}}{(z-1)^{1/4}} (f_n(z))^{1/4} \operatorname{Ai}(f_n(z)) \right. \\ & \qquad \left. - \frac{(z-1)^{1/4}}{(z+1)^{1/4}} (f_n(z))^{-1/4} \operatorname{Ai}'(f_n(z)) \right\} \\ & \quad \times \left(1 + O\left(\frac{1}{n}\right)\right). \end{aligned} \tag{2.20}$$

(v) *For* $z \in D_{1,\delta}$,

$$\begin{aligned} & p_n(c_n z + d_n) e^{-\frac{1}{2}Q(c_n z + d_n)} \\ &= (-1)^n \sqrt{\frac{1}{c_n}} \left\{ \left(\frac{(z-1)^{1/4}}{(z+1)^{1/4}} \left(-\tilde{f}_n(z)\right)^{1/4} \operatorname{Ai}(-\tilde{f}_n(z)) \right) \left(1 + O\left(\frac{1}{n}\right)\right) \right. \\ & \qquad \left. - \left(\frac{(z+1)^{1/4}}{(z-1)^{1/4}} \left(-\tilde{f}_n(z)\right)^{-1/4} \operatorname{Ai}'(-\tilde{f}_n(z)) \right) \left(1 + O\left(\frac{1}{n}\right)\right) \right\}. \end{aligned} \tag{2.21}$$

(vi) *For* $z \in D_{2,\delta}$,

$$\begin{aligned} & p_n(c_n z + d_n) e^{-\frac{1}{2}Q(c_n z + d_n)} \\ &= (-1)^n \sqrt{\frac{1}{c_n}} \left\{ \frac{(z-1)^{1/4}}{(z+1)^{1/4}} \left(-\tilde{f}_n(z)\right)^{1/4} \operatorname{Ai}(-\tilde{f}_n(z)) \right. \\ & \qquad \left. - \frac{(z+1)^{1/4}}{(z-1)^{1/4}} \left(-\tilde{f}_n(z)\right)^{-1/4} \operatorname{Ai}'(-\tilde{f}_n(z)) \right\} \left(1 + O\left(\frac{1}{n}\right)\right). \end{aligned} \tag{2.22}$$

All the error terms are uniform for $\delta \in$ *compact subsets of* $(0, \delta_0]$ *and for* $z \in X_\delta$, *where* $X \in \{A, B, C_1, C_2, D_1, D_2\}$. *There are integral formulae for the error terms from which one can extract an explicit asymptotic expansion in* $n^{-1/2m}$.

Remarks. 1. Some of the expressions in Theorem 2.2 are not well-defined for all $z \in \mathbb{R}$ (see, e.g., $(z-1)^{1/4}$, $\int_1^z \psi_n(y)dy$). In these cases we always take the limiting expressions as z is approached from the upper half-plane.

2. The function arcsin is defined as the inverse function of
$$\sin : \left\{ z \in \mathbb{C} : |\operatorname{Re}(z)| < \frac{\pi}{2} \right\} \to \mathbb{C} \setminus ((-\infty,-1] \cup [1,\infty)) .$$

3. We denote by Ai the Airy function as defined in [1, 10.4]. Note that the function Ai is uniquely determined as the solution of
$$\operatorname{Ai}''(z) = z \operatorname{Ai}(z) , \tag{2.23}$$
satisfying
$$\lim_{x\to\infty} \operatorname{Ai}(x)\sqrt{4\pi}x^{1/4} \exp\left(\frac{2}{3}x^{3/2}\right) = 1 . \tag{2.24}$$

4. For $z \in \mathbb{C}_+$ the integral $\int_1^z \psi_n(y)dy$ is of the form
$$\int_1^z \psi_n(y)dy = \frac{1}{2\pi}(1-z)^{1/2}(1+z)^{1/2}H_n(z) + \frac{1}{\pi}\arcsin z - \frac{1}{2} \tag{2.25}$$
where H_n is a polynomial of degree $2m-1$ whose (real) coefficients can again be computed explicitly (cf. (5.40) below).

5. We will check explicitly that the different formulae match at the boundaries of the different regions (see (8.43)–(8.47)).

2.3 Asymptotic Location of the Zeros

In order to state our result on the location of the zeros, we denote the zeros of the Airy function Ai by
$$0 > -\iota_1 > -\iota_2 > \cdots . \tag{2.26}$$
Recall that all the zeros of Ai lie in $(-\infty,0)$, so that there exists a largest zero $-\iota_1 < 0$. Furthermore, note that $[-1,1] \ni x \mapsto \int_x^1 \psi_n(t)dt \in [0,1]$ is bijective, and we define its inverse function to be $\zeta_n : [0,1] \mapsto [-1,1]$.

THEOREM 2.3 *The zeros $x_{1,n} > x_{2,n} > \cdots > x_{n,n}$ of the n^{th} orthogonal polynomials p_n satisfy the following asymptotic formulae*:

(i) *Fix $k \in \mathbb{N}$. Then*
$$\frac{x_{k,n}-d_n}{c_n} = 1 - \left(\frac{2}{h_n(1)^2}\right)^{1/3}\frac{\iota_k}{n^{2/3}} + O\left(\frac{1}{n}\right), \qquad \text{as } n\to\infty, \tag{2.27}$$
$$\frac{x_{n-k,n}-d_n}{c_n} = -1 + \left(\frac{2}{h_n(-1)^2}\right)^{1/3}\frac{\iota_k}{n^{2/3}} + O\left(\frac{1}{n}\right), \quad \text{as } n\to\infty. \tag{2.28}$$

(ii) *There exist constants $k_0, C > 0$ such that for all $k_0 \le k \le n-k_0$ the following holds*:
$$\frac{x_{k,n}-d_n}{c_n} \in \left(\zeta_n\left(\frac{6k-1}{6n}\right), \zeta_n\left(\frac{6k-5}{6n}\right)\right) , \tag{2.29}$$

(2.30) $$\left|\frac{x_{k,n}-d_n}{c_n}-\zeta_n\left(\frac{6k-3}{6n}+\frac{1}{2\pi n}\arcsin\left(\zeta_n\left(k/n\right)\right)\right)\right|\leq\frac{C}{n^2[\alpha(1-\alpha)]^{4/3}}$$

where $\alpha := k/n$.

(iii) *There exists a constant* $C_1 > 0$ *such that*

(2.31) $$\frac{1}{C_1}<\frac{x_{k,n}-x_{k+1,n}}{c_n}[nk(n-k)]^{1/3}<C_1 \quad \textit{for all } 1\leq k\leq n-1\,.$$

Remarks. 1. Using the asymptotic expansion for the error terms in Theorem 2.2, one can, of course, approximate the k^{th} zero $x_{k,n}$ of the orthogonal polynomial p_n to arbitrary accuracy.

2. Note that the error term in (2.30) is at most of order $O(n^{-2/3})$. Furthermore, it is obvious that for any compact subset K of $(0,1)$, there exists a constant C_K such that the error term in (2.30) is bounded by C_K/n^2 as long as $\alpha = k/n \in K$.

3. In the special case $Q(x) = x^{2m}$, the results can be stated more explicitly since the Mhaskar-Rakhmanov-Saff numbers have a simple form. One verifies by a straightforward calculation that $c_n = n^{1/2m}(mA_m)^{-1/2m}$ and $d_n = 0$ solve (1.6), (1.7), and (1.8). Following the analysis in Section 5 below, one sees that $h_n(x) = \frac{2}{A_m}\sum_{k=0}^{m-1} A_{m-k-1}x^{2k}$ and $l_n = -\frac{1}{m} - 2\log 2$ are independent of n. Furthermore, one verifies that $h_n(1) = 4m$ and $h_n'(1) = \frac{16}{3}m(m-1)$ by induction on m. Finally, note that as the weight function is symmetric, the recurrence coefficients $a_n \equiv 0$. Theorem 2.1 now reads:

(2.32) $$\gamma_n\sqrt{\pi}\left[\frac{1}{2^{2m}e}\left(\frac{n}{mA_m}\right)^{1+\frac{1}{2n}}\right]^{\frac{n}{2m}}=1+\left(\frac{m-2}{24m}\right)\frac{1}{n}+O\left(\frac{1}{n^2}\right),$$

(2.33) $$(mA_m)^{\frac{1}{2m}}\frac{b_{n-1}}{n^{\frac{1}{2m}}}=\frac{1}{2}+O\left(\frac{1}{n^2}\right),$$

(2.34) $$a_n = 0\,.$$

Rather than restating Theorems 2.2 and 2.3 with all the details, we just note that, in addition to the explicit formulae for c_n, d_n, h_n, and l_n given above, the function $z \mapsto \int_1^z \psi_n(y)dy$ is also independent of n (as ψ_n is n-independent, see (2.14)) and is given, once again explicitly, by

(2.35)
$$\int_1^z \psi_n(y)dy=\frac{1}{2\pi mA_m}\left(\sum_{k=0}^{m-1}A_{m-k-1}z^{2k+1}\right)(1-z)^{1/2}(1+z)^{1/2}-\frac{1}{\pi}\arccos z\,.$$

The asymptotic expansion of the error terms in Theorems 2.1 and 2.2 are given in powers of $\frac{1}{n}$; i.e., fractional powers of $\frac{1}{n}$ do not appear in this special case.

3 The Fokas-Its-Kitaev Reformulation as a Riemann-Hilbert Problem

Our approach is based on the following determination of orthogonal polynomials in terms of a Riemann-Hilbert problem (RHP), due to Fokas, Its, and Kitaev [22, 23] (as a general reference for Riemann-Hilbert problems, see [10]; for the convenience of the reader, we present a sketch of the basic constructions in Riemann-Hilbert theory in Appendix A): For fixed n and a given weight $w(x)$, let $Y = (Y_{ij})_{1\le i,j\le 2} = Y(z) = Y(z;n,w)$ be the (unique) 2×2 matrix function with the properties

$$Y(z) \text{ is analytic in } \mathbb{C}\setminus\mathbb{R}\,, \tag{3.1}$$

$$Y_+(s) = Y_-(s)\begin{pmatrix} 1 & w(s) \\ 0 & 1 \end{pmatrix}, \quad s\in\mathbb{R}\,, \tag{3.2}$$

where

$$Y_\pm(s) \equiv \lim_{\zeta\to s} Y(\zeta)\,, \quad s\in\mathbb{R},\ \pm\mathrm{Im}\,\zeta > 0\,,$$

and

$$Y(z)\begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} = I + O\left(\frac{1}{|z|}\right) \quad \text{as } |z|\to\infty\,. \tag{3.3}$$

Then the n^{th} monic orthogonal polynomial

$$\pi_n(x) = \frac{1}{\gamma_n} p_n(x) = x^n + \cdots \tag{3.4}$$

is given by

$$\pi_n(z) = Y_{11}(z;n,w)\,. \tag{3.5}$$

Asymptotic problems for orthogonal polynomials $\pi_n(z)$ as $n\to\infty$ are converted in this way to asymptotic questions for RHPs containing a large external parameter.

The following theorem, due to Fokas, Its, and Kitaev (see [22, 23]; a specialized version also appeared in [13]), proves the existence and uniqueness of the solution of the Riemann-Hilbert problem and computes the leading coefficients and the recurrence coefficients of the orthogonal polynomials in terms of this solution:

THEOREM 3.1 *Let $w:\mathbb{R}\to\mathbb{R}_+$ denote a function with the property that $w(s)s^k$ belongs to the Sobolev space $H^1(\mathbb{R})$ for all $k\in\mathbb{N}$. Suppose, furthermore, that n is a positive integer.*

Then the Riemann-Hilbert problem (3.6)–(3.8),

$$Y:\mathbb{C}\setminus\mathbb{R}\to\mathbb{C}^{2\times 2} \textit{ is analytic}\,, \tag{3.6}$$

$$Y_+(s) = Y_-(s)\begin{pmatrix} 1 & w(s) \\ 0 & 1 \end{pmatrix} \quad \textit{for } s\in\mathbb{R}\,, \tag{3.7}$$

$$Y(z)\begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} = I + O\left(\frac{1}{|z|}\right) \quad \text{as } |z| \to \infty, \tag{3.8}$$

has a unique solution, given by

$$Y(z) = \begin{pmatrix} \pi_n(z) & \int_{\mathbb{R}} \frac{\pi_n(s)w(s)}{s-z}\frac{ds}{2\pi i} \\ -2\pi i\gamma_{n-1}^2\pi_{n-1}(z) & \int_{\mathbb{R}} \frac{-\gamma_{n-1}^2\pi_{n-1}(s)w(s)}{s-z}\, ds \end{pmatrix}, \tag{3.9}$$

where π_n denotes the n^{th} monic orthogonal polynomial with respect to the measure $w(x)dx$ on $\mathbb{R}$ and $\gamma_n > 0$ denotes the leading coefficient of the n^{th} orthogonal polynomial $p_n = \gamma_n\pi_n$. Furthermore, there exist Y_1, $Y_2 \in \mathbb{C}^{2\times 2}$ such that

$$Y(z)\begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} = I + \frac{Y_1}{z} + \frac{Y_2}{z^2} + O\left(\frac{1}{|z|^3}\right) \quad \text{as } |z| \to \infty, \tag{3.10}$$

and

$$\gamma_{n-1} = \sqrt{\frac{(Y_1)_{21}}{-2\pi i}}, \qquad \gamma_n = \sqrt{\frac{1}{-2\pi i (Y_1)_{12}}}, \tag{3.11}$$

$$a_n = (Y_1)_{11} + \frac{(Y_2)_{12}}{(Y_1)_{12}}, \qquad b_{n-1} = \sqrt{(Y_1)_{21}(Y_1)_{12}}, \tag{3.12}$$

where a_n and b_n are the recurrence coefficients associated to the orthogonal polynomials p_n (cf. 1.4*).*

Remarks. 1. The jump condition (3.7) is a shorthand notation for the following: The functions $Y|_{\mathbb{C}_\pm}$ have a continuous extension to $\overline{\mathbb{C}}_\pm$ with boundary values $Y_\pm$ satisfying relation (3.7).

2. The second column of Y in the solution (3.9) of the Riemann-Hilbert problem is the *Cauchy transform*

$$Cf(z) \equiv \int_{\mathbb{R}} \frac{f(s)}{s-z}\frac{ds}{2\pi i}, \quad z \in \mathbb{C}\setminus\mathbb{R},\ f \in L^2(\mathbb{R}), \tag{3.13}$$

of the product $w(s)$ times the first column. A brief summary of relevant properties of the Cauchy transform is given in Appendix A.

PROOF OF THEOREM 3.1: *Uniqueness.* We first note that any solution Y of the Riemann-Hilbert problem (3.6)–(3.8) satisfies $\det Y(z) = 1$ for all $z \in \mathbb{C}\setminus\mathbb{R}$. Indeed, condition (3.7) implies that $\det Y$ can be continued to an entire function. Using property (3.8), it follows from Liouville's theorem that $\det Y$ must be identically equal to 1, and so Y^{-1} is also analytic in $\mathbb{C}\setminus\mathbb{R}$. Denote by $\tilde{Y}$ a second solution of (3.6)–(3.8) and define $M := \tilde{Y}Y^{-1}$. A simple calculation shows that $M_+ = M_-$. Hence M has an extension to all of $\mathbb{C}$ as an entire function. Again condition (3.8), together with Liouville's theorem, implies that $M \equiv I$, which proves uniqueness.

Existence. We verify that the function Y given in (3.9) solves the Riemann-Hilbert problem. The following observation is immediate from the properties of the Cauchy transform stated in Appendix A. Given any entire function f with wf lying in $H^1(\mathbb{R})$, the row-vector-valued function $(f, C(wf))$ is analytic on $\mathbb{C}\setminus\mathbb{R}$

and, by property 4 of the Cauchy transform (see Appendix A), satisfies on $\mathbb{R}$ the jump condition

$$(f, C(wf))_+ = (f, C(wf))_- \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}. \tag{3.14}$$

Therefore Y satisfies (3.6) and (3.7).

We will now see that the asymptotic condition (3.8) leads to the orthogonal polynomials. Clearly (3.8) holds for the 11-entry and for the 12-entry because π_n is a monic polynomial of degree n and $-2\pi i \gamma_{n-1}^2 \pi_{n-1}(z)$ is a polynomial of degree $n-1$. In order to investigate the 12-entry, observe that for any $l \geq 0$,

$$\frac{1}{s-z} = -\sum_{k=0}^{l} \frac{s^k}{z^{k+1}} + \frac{s^{l+1}}{z^{l+1}(s-z)} \quad \text{for } s \neq z. \tag{3.15}$$

Hence

$$Y_{12}(z) = \frac{1}{z^{l+1}} \int_{\mathbb{R}} \frac{s^{l+1}\pi_n(s)w(s)}{s-z} \frac{ds}{2\pi i} - \sum_{k=0}^{l} \frac{1}{z^{k+1}} \int_{\mathbb{R}} s^k \pi_n(s) w(s) \frac{ds}{2\pi i}. \tag{3.16}$$

The last term in (3.16) is of order $O(1/|z|^{n+1})$, because we have assumed that $s^{n+1}\pi_n(s)w(s)$ lies in $H^1(\mathbb{R})$ (again use property 4 of the Cauchy transform stated in Appendix A). Hence, by orthogonality, $Y_{12}(z) = O(1/|z|^{n+1})$ as $|z|$ tends to infinity. Denote

$$\pi_n(z) = z^n + \sum_{k=0}^{n-1} t_{k,n} z^k. \tag{3.17}$$

Then, by arguments similar to the above,

$$\begin{aligned} Y(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} = & \begin{pmatrix} 1 & 0 \\ 0 & \gamma_{n-1}^2 \int s^{n-1}\pi_{n-1}(s)w(s)ds \end{pmatrix} \\ & + \frac{1}{z} \begin{pmatrix} t_{n-1,n} & -\int s^n \pi_n(s) w(s) \frac{ds}{2\pi i} \\ -2\pi i \gamma_{n-1}^2 & \gamma_{n-1}^2 \int s^n \pi_{n-1}(s) w(s) ds \end{pmatrix} \\ & + \frac{1}{z^2} \begin{pmatrix} * & -\int s^{n+1}\pi_n(s)w(s)\frac{ds}{2\pi i} \\ * & * \end{pmatrix} + O\left(\frac{1}{|z|^3}\right). \end{aligned} \tag{3.18}$$

Observe that, again by orthogonality, for any $l \geq 0$,

$$\int s^l \pi_l(s) w(s) ds = \int \pi_l(s)^2 w(s) ds = \gamma_l^{-2} \int p_l(s)^2 w(s) ds = \gamma_l^{-2} \tag{3.19}$$

and

$$\begin{aligned} \int s^{l+1}\pi_l(s)w(s)ds &= \int (s^{l+1} - \pi_{l+1}(s))\pi_l(s)w(s)ds \\ &= -t_{l,l+1} \int s^l \pi_l(s) w(s) ds = -t_{l,l+1}\gamma_l^{-2}. \end{aligned} \tag{3.20}$$

Equation (3.18) therefore simplifies to

$$
\begin{aligned}
Y(z)\begin{pmatrix} z^{-n} & 0 \\ 0 & z^{n}\end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} + \frac{1}{z}\begin{pmatrix} t_{n-1,n} & -\frac{1}{2\pi i\gamma_n^2} \\ -2\pi i\gamma_{n-1}^2 & -t_{n-1,n}\end{pmatrix} \\
&\quad + \frac{1}{z^2}\begin{pmatrix} * & \frac{t_{n,n+1}}{2\pi i\gamma_n^2} \\ * & * \end{pmatrix} + O\left(\frac{1}{|z|^3}\right).
\end{aligned}
\tag{3.21}
$$

Thus we have established property (3.8) as well as the existence of Y_1 and Y_2, satisfying (3.10) and (3.11). In order to verify (3.12), we recall that the coefficients of the recurrence relations a_n and b_n can be expressed in terms of the coefficients of the orthogonal polynomials simply by collecting terms of the same order in (1.4). We obtain

$$
a_n = t_{n-1,n} - t_{n,n+1}\,, \qquad b_{n-1} = \frac{\gamma_{n-1}}{\gamma_n}\,. \tag{3.22}
$$

Using equation (3.21), one easily verifies (3.12). □

4 Overview of the Calculation

We evaluate the solution of the Fokas-Its-Kitaev RHP for Y asymptotically by applying to it a series of transformations

$$
Y \to U \to T \to S \to R\,, \tag{4.1}
$$

each of which determines a contribution to the solution and/or simplifies the problem. The final quantity R can be expressed as a Neumann series. The solution Y follows from R and the composition of these transformations.

Successive transformations with contour deformations that produce exponentially decaying jump matrices form the basis of the steepest-descent method for oscillatory RHPs introduced by Deift and Zhou in [18] and further developed by them in [19]. A significant extension of the steepest-descent method, by Deift, Venakides, and Zhou in [17], incorporates into the method a nonlinear analogue of WKB analysis that makes asymptotics of fully nonlinear oscillations possible, thus allowing the method to be applied to a new class of problems.

Our analysis of the problem follows the framework of [17]. We briefly describe the transformations involved:

- $Y \to U$ is a rescaling.
- $U \to T$ involves the function g that is the analogue for the RHP of the phase function of linear WKB theory.
- $T \to S$ involves a factorization of the jump matrix and a deformation of the contour. Under this deformation, oscillatory terms are transformed into exponentially decaying terms, which may be neglected, and all that remains is a simple RHP on a finite interval.
- $S \to R$ involves the construction, following [19], of a parametrix for S that is particularly delicate at the endpoint of the contour.

We now describe the above steps in greater detail.

4.1 $Y \to U$

For reasons that will become clear further on (see Remark 4.1; see also Appendix B) we scale $z \mapsto \lambda_n(z) = c_n z + d_n$, where c_n and d_n are related to the MRS numbers (see (1.8) above). A simple calculation shows that

$$U(z) \equiv \begin{pmatrix} c_n^{-n} & 0 \\ 0 & c_n^n \end{pmatrix} Y\left(\lambda_n(z)\right) \tag{4.2}$$

solves the scaled RHP (4.3)–(4.5) below, where $nV_n(z) = Q(\lambda_n(z))$ as in (2.2).

$$U : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2} \text{ is analytic}, \tag{4.3}$$

$$U_+(s) = U_-(s) \begin{pmatrix} 1 & e^{-nV_n(s)} \\ 0 & 1 \end{pmatrix} \quad \text{for } s \in \mathbb{R}, \tag{4.4}$$

$$U(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} = I + O\left(\frac{1}{|z|}\right) \quad \text{as } |z| \to \infty, \tag{4.5}$$

4.2 $U \to T$

Let σ_3 denote the Pauli matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Following [17], set

$$T(z) \equiv T_n(z) = e^{-n\frac{l}{2}\sigma_3} U(z) e^{-n\left(g(z) - \frac{l}{2}\right)\sigma_3}, \tag{4.6}$$

where the constant l and the function $g(z)$ are to be determined below. The exponential factor $e^{ng(z)}$ can be viewed for RHPs as the analogue of the "fast phase" arising in the analysis of linear differential equations in the WKB limit. We require that

$$g(z) \text{ be analytic on } \mathbb{C} \setminus \mathbb{R}. \tag{4.7}$$

This insures that the new quantity $T(z)$ solves a RHP on the same contour as U. Set $g_\pm(s) \equiv g(s \pm i0)$, $s \in \mathbb{R}$. Second, we require that $g(z)$ behave as

$$g(z) \approx \log z \quad \text{as } z \to \infty \tag{4.8}$$

in order to normalize the RHP at infinity (cf. (4.11) below). A simple computation now shows that T is the unique solution of (4.9)–(4.11)

$$T : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2} \text{ is analytic}. \tag{4.9}$$

$$T_+ = T_- \begin{pmatrix} e^{-n(g_+ - g_-)} & e^{n(g_+ + g_- - V_n - l)} \\ 0 & e^{n(g_+ - g_-)} \end{pmatrix} \quad \text{on } \mathbb{R}, \tag{4.10}$$

$$T(z) = I + O\left(\frac{1}{|z|}\right) \quad \text{as } |z| \to \infty. \tag{4.11}$$

Guided by the procedure in [17], we suppose further that g satisfies the following conditions: There exists a finite closed interval $I \subset \mathbb{R}$ such that

$$g_+(s) + g_-(s) - V_n(s) - l = 0 \qquad \text{for } s \in I, \tag{4.12}$$

$$g_+(s) - g_-(s) \text{ is purely imaginary} \quad \text{for } s \in I, \tag{4.13}$$

$$\text{and } i\frac{d}{ds}(g_+(s) - g_-(s)) > 0 \qquad \text{for } s \in \overset{\circ}{I},$$

$$g_+(s) + g_-(s) - V_n(s) - l < 0 \qquad \text{for } s \in \mathbb{R} \setminus I, \tag{4.14}$$

$$e^{g_+(s) - g_-(s)} = 1 \qquad \text{for } s \in \mathbb{R} \setminus I. \tag{4.15}$$

We call conditions (4.7), (4.8), and (4.12)–(4.15) the *phase conditions* (PC) for g. The motivation for (4.12)–(4.15) will become clear below.

It is a remarkable piece of luck (cf. [17]) that the phase conditions can be expressed simply in terms of the variational conditions for a well-known minimization problem in logarithmic potential theory. Indeed, set

$$g(z) = \int \log(z - x)\, d\mu_{V_n}(x), \tag{4.16}$$

where the measure μ_{V_n} is the unique minimizer (see [40]) of

$$E = \inf_{\mu \in M_1(\mathbb{R})} \left[\int_{\mathbb{R}^2} \log|x - y|^{-1} d\mu(x) d\mu(y) + \int_{\mathbb{R}} V_n(x) d\mu(x) \right], \tag{4.17}$$

where $M_1(\mathbb{R})$ denotes the space of all probability measures on $\mathbb{R}$. The measure μ_{V_n} is called the *equilibrium measure* and is characterized by the Euler-Lagrange equations (cf. [40]; see also [14]): There exists a real number l (the Lagrange multiplier) such that

$$2\int \log|x - y| d\mu(y) - V_n(x) - l \le 0 \quad \text{for } x \in \mathbb{R}, \tag{4.18}$$

$$2\int \log|x - y| d\mu(y) - V_n(x) - l = 0 \quad \text{for } x \in \operatorname{supp}(\mu). \tag{4.19}$$

For measures $\mu \in M_1(\mathbb{R})$ that are supported in the single interval $\operatorname{supp}(\mu) = I$, it is easy to see that

$$\begin{aligned} g(z) = \int \log(z - x) d\mu(x) \text{ solves (PC)}& \\ \iff \mu \text{ solves the variational conditions (4.18) and (4.19).}& \end{aligned}$$

Remark 4.1. Phase conditions (4.7), (4.8), and (4.12)–(4.15) are specific to the problem at hand. More generally (see [17]), one can consider g-functions that satisfy conditions (4.12)–(4.14) and an appropriate modification of (4.15) in the case of a finite union of disjoint intervals $I = \bigcup I_j$. Such a general situation arises in the related paper ([16]; see also [15]). However, in the present paper it turns out that for n sufficiently large, the support of the equilibrium measure μ_{V_n} is a single interval; in fact, the scaling using the MRS numbers (see (1.6) and (1.7) above) is chosen precisely to insure that the interval $I = [-1,1]$.

Remark 4.2. In Appendix B we present another approach to the g-function (and the rescaling) that is closer in spirit to standard constructions in the theory of orthogonal polynomials. It is based on the well-known connection between the asymptotic distribution of zeros of orthogonal polynomials and the equilibrium measure. A g-function also appears in the recent work of Ercolani, Levermore, and Zhang [20, 21] on the zero dispersion limit of the KdV equation via Lax-Levermore theory [25].

4.3 $T \to S$

The significance of conditions (4.12)–(4.15) is the following: Conditions (4.12) and (4.13) lead to an upper-triangular jump matrix

$$v_T = \begin{pmatrix} e^{-n(g_+-g_-)} & 1 \\ 0 & e^{n(g_+-g_-)} \end{pmatrix} \quad \text{on } [-1,1]\,,$$

and (4.14) and (4.15) yield a jump matrix

$$v_T = \begin{pmatrix} 1 & e^{n(g_+(s)+g_-(s)-V_n(s)-l)} \\ 0 & 1 \end{pmatrix} = I + o(1) \quad \text{on } \mathbb{R}\setminus[-1,1] \text{ as } n\to\infty\,.$$

The oscillatory terms $e^{\pm n(g_+-g_-)}$ can be transformed into exponentially decaying terms as follows: Factor

$$v_T = \begin{pmatrix} 1 & 0 \\ e^{n(g_+-g_-)} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-n(g_+-g_-)} & 1 \end{pmatrix} \equiv v_- v_0 v_+\,. \tag{4.20}$$

Since $g_+ - g_- = 2g_+ - V_n - l = -2g_- + V_n + l$ on $[-1,1]$, $g_+ - g_-$ has an analytic continuation above and below $[-1,1]$, and by the Cauchy-Riemann condition (4.13) the real part of $(g_+ - g_-)(z)$ is positive above $(-1,1)$ and negative below $(-1,1)$. Hence $v_+(z)$ (respectively, $v_-(z)$) has an analytic continuation above (respectively, below) $(-1,1)$ that converges exponentially to the identity as $n\to\infty$.

The factorization (4.20) suggests the following deformation of the RHP for T. Let $\Sigma_S = \bigcup_{i=1}^5 \Sigma_i$ be the oriented contour in Figure 4.1. Let

(4.21)

$$S(z) \equiv T(z) \quad \text{for } z \text{ outside the lens-shaped region,}$$

(4.22)

$$S(z) \equiv T(z) v_+^{-1}(z) = T(z) \begin{pmatrix} 1 & 0 \\ -e^{-n(g_+(z)-g_-(z))} & 1 \end{pmatrix} \quad \text{in the upper lens region,}$$

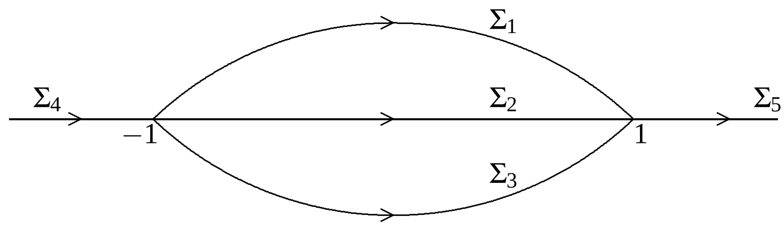

FIGURE 4.1. The contour Σ_S.

(4.23)
$$S(z) \equiv T(z)v_-(z) = T(z)\begin{pmatrix} 1 & 0 \\ e^{n(g_+(z)-g_-(z))} & 1 \end{pmatrix} \qquad \text{in the lower lens region.}$$

Then S satisfies

$$S : \mathbb{C} \setminus \Sigma_S \to \mathbb{C}^{2\times 2} \text{ is analytic}\,. \tag{4.24}$$

$$S_+(s) = S_-(s)v_S(s)\,, \quad s \in \Sigma_S\,, \tag{4.25}$$

$$S(z) = I + O\left(\frac{1}{|z|}\right) \text{ as } |z| \to \infty\,, \tag{4.26}$$

where $v_S(s) = v_i(s)$ for $s \in \Sigma_i$, $1 \le i \le 5$, and $v_1 = v_+$, $v_2 = v_0$, $v_3 = v_-$, $v_4 = v_T$, $v_5 = v_T$.

4.4 $S \to R$

The preceding definitions and calculations have the following result: *The jump matrix v_S converges to the identity matrix everywhere on Σ_S except on $\Sigma_2 = [-1,1]$.* The above RHP for S is clearly equivalent to the original RHP (3.1)–(3.3) in the sense that the solution of the one problem implies the solution of the other, and vice versa. This suggests that for large n the solution of (4.24)–(4.26) should be close to the solution of the following limiting RHP:

$$S^{(\infty)} : \mathbb{C} \setminus [-1,1] \to \mathbb{C}^{2\times 2} \text{ is analytic}\,, \tag{4.27}$$

$$S_+^{(\infty)}(s) = S_-^{(\infty)}(s)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{for } s \in [-1,1]\,, \tag{4.28}$$

$$S^{(\infty)}(z) = I + O\left(\frac{1}{|z|}\right) \quad \text{as } |z| \to \infty\,. \tag{4.29}$$

The RHP (4.27)–(4.29) can be solved explicitly by diagonalizing the jump matrix and hence reducing it to two scalar RHPs. The solution is given by

$$S^{(\infty)}(z) = \frac{1}{2}\begin{pmatrix} a(z) + a(z)^{-1} & i(a(z)^{-1} - a(z)) \\ i(a(z) - a(z)^{-1}) & a(z) + a(z)^{-1} \end{pmatrix} \tag{4.30}$$

with

$$a(z)=\frac{(z-1)^{1/4}}{(z+1)^{1/4}}. \tag{4.31}$$

However, to prove rigorously that indeed $S\to S^{(\infty)}$ as $n\to\infty$, numerous technicalities arise. The origin of these difficulties can be seen by defining

$$M:=S(S^{(\infty)})^{-1}. \tag{4.32}$$

Denote by v_∞ the jump matrix for $S^{(\infty)}$. The matrix-valued function M then satisfies an RHP of the form

$$M_+=M_-\left(S_-^{(\infty)}v_S v_\infty^{-1}\left(S_-^{(\infty)}\right)^{-1}\right)\quad\text{on } \Sigma_S, \tag{4.33}$$

$$M(z)=I+O\left(\frac{1}{|z|}\right)\quad\text{as } |z|\to\infty. \tag{4.34}$$

In order to prove that the solution M of (4.33)–(4.34) is close to the identity, and hence $S\approx S^{(\infty)}$, one needs to know (see (A.4)–(A.9) below) that the jump matrix for M is close to the identity in the L_2 *and* in the L_∞ sense. Due to the fourth-root singularity of $S^{(\infty)}$ at ±1, however, we see that the jump matrix for M is unbounded. In order to remedy this difficulty, one needs to construct an explicit solution P of the RHP (4.24)–(4.26) in small neighborhoods of ±1 that matches $S^{(\infty)}$ at the boundary of these neighborhoods up to order $o(1)$. Such an explicit local solution P is constructed in terms of Airy functions (see Section 7.1 below). Finally, we define the parametrix S_{par} for S by $S_{\text{par}}=P$ in neighborhoods of ±1 and by $S_{\text{par}}=S^{(\infty)}$ elsewhere. Then $R\equiv SS_{\text{par}}^{-1}$ solves a RHP with jumps of the form $I+o(1)$ and can be computed to any order by a Neumann series.

5 The *g*-Function and the Equilibrium Measure

5.1 Rescaling

The transformation $Y\to U$ (see (4.2)) leads to the following relations:

PROPOSITION 5.1 *Let Y and U be the solutions of the RHPs* (3.6)–(3.8) *and* (4.3)–(4.5)*, respectively. Furthermore, let Y_1, Y_2, U_1, and U_2 be given according to* (3.10)*. Then*

$$Y_1=c_n\begin{pmatrix}c_n^n & 0\\ 0 & c_n^{-n}\end{pmatrix}U_1\begin{pmatrix}c_n^{-n} & 0\\ 0 & c_n^{n}\end{pmatrix}+\begin{pmatrix}-nd_n & 0\\ 0 & nd_n\end{pmatrix}, \tag{5.1}$$

$$\begin{aligned}Y_2={}&c_n^2\begin{pmatrix}c_n^n & 0\\ 0 & c_n^{-n}\end{pmatrix}U_2\begin{pmatrix}c_n^{-n} & 0\\ 0 & c_n^{n}\end{pmatrix}\\ &+c_n\begin{pmatrix}c_n^n & 0\\ 0 & c_n^{-n}\end{pmatrix}U_1\begin{pmatrix}c_n^{-n} & 0\\ 0 & c_n^{n}\end{pmatrix}\begin{pmatrix}d_n(1-n) & 0\\ 0 & d_n(1+n)\end{pmatrix}\\ &+\begin{pmatrix}\frac{n(n-1)d_n^2}{2} & 0\\ 0 & \frac{n(n+1)d_n^2}{2}\end{pmatrix}.\end{aligned} \tag{5.2}$$

The proof is a simple calculation.

5.2 MRS Numbers and the Construction of the Function $g_n(z)$ and of the Equilibrium Measure Through the Solution of the Variational Problem

We first construct the MRS numbers for sufficiently large n and then solve the variational problem showing that the support of the equilibrium measure is indeed given by the interval [-1,1] after the corresponding rescaling. Our solution of the variational problem follows [14].

Construction of the MRS Numbers

PROPOSITION 5.2 *There exist $n_1 \in \mathbb{N}$ and sequences $(\alpha^{(k)})_{k\in\mathbb{N}}$ and $(\beta^{(k)})_{k\in\mathbb{N}}$ such that*

$$\alpha_n = n^{\frac{1}{2m}} \left(\sum_{k=0}^{\infty} \alpha^{(k)} n^{-\frac{k}{2m}} \right) \tag{5.3}$$

and

$$\beta_n = n^{\frac{1}{2m}} \left(\sum_{k=0}^{\infty} \beta^{(k)} n^{-\frac{k}{2m}} \right) \tag{5.4}$$

converge for all $n \geq n_1$ and satisfy the conditions

$$\frac{1}{2\pi} \int_{\alpha_n}^{\beta_n} \frac{Q'(t)(t-\alpha_n)}{\sqrt{(\beta_n - t)(t-\alpha_n)}}\, dt = n \tag{5.5}$$

and

$$\frac{1}{2\pi} \int_{\alpha_n}^{\beta_n} \frac{Q'(t)(\beta_n - t)}{\sqrt{(\beta_n - t)(t-\alpha_n)}}\, dt = -n\,. \tag{5.6}$$

The coefficients $\alpha^{(k)}$ and $\beta^{(k)}$ can be expressed explicitly in terms of the coefficients $q_0, \dots, q_{2m}$ of the polynomial Q. For example,

$$\beta^{(0)} = -\alpha^{(0)} = (q_{2m} m A_m)^{-\frac{1}{2m}}\,, \tag{5.7}$$

$$\beta^{(1)} = \alpha^{(1)} = -\frac{q_{2m-1}}{2m q_{2m}}\,, \tag{5.8}$$

where A_m are the numerical coefficients defined in (1.11)*.*

PROOF: We first introduce a few auxiliary functions. For α, β, ε, $x \in \mathbb{R}$, define

$$Q(x,\varepsilon) := \sum_{k=0}^{2m} q_k \varepsilon^{2m-k} x^k \quad \left(= \varepsilon^{2m} Q\left(\frac{x}{\varepsilon}\right)\right), \tag{5.9}$$

$$V(x,\alpha,\beta,\varepsilon) := Q\left(\frac{\beta-\alpha}{2}x + \frac{\alpha+\beta}{2}, \varepsilon\right), \tag{5.10}$$

$$G(\alpha,\beta,\varepsilon) := \begin{pmatrix} \int_{-1}^{1} \frac{V'(y,\alpha,\beta,\varepsilon)}{\sqrt{1-y^2}}\,dy \\ \int_{-1}^{1} \frac{yV'(y,\alpha,\beta,\varepsilon)}{\sqrt{1-y^2}}\,dy \end{pmatrix}, \tag{5.11}$$

where we denote by V' the derivative with respect to the first variable. It is easy to see that $\alpha < \beta$ solve (5.5) and (5.6) for some $n \geq 1$ if and only if

$$G\left(\alpha n^{-\frac{1}{2m}}, \beta n^{-\frac{1}{2m}}, n^{-\frac{1}{2m}}\right) = \begin{pmatrix} 0 \\ 2\pi \end{pmatrix}. \tag{5.12}$$

It suffices therefore to find an $\varepsilon_0 > 0$ and real analytic functions α, β: $(-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ satisfying

$$G(\alpha(\varepsilon), \beta(\varepsilon), \varepsilon) = \begin{pmatrix} 0 \\ 2\pi \end{pmatrix} \quad \text{for all } \varepsilon \in (-\varepsilon_0, \varepsilon_0)\,. \tag{5.13}$$

Indeed, choosing then $\alpha_n := n^{1/2m}\alpha(n^{-1/2m})$ and $\beta_n := n^{1/2m}\beta(n^{-1/2m})$ for $n > (1/\varepsilon_0)^{2m}$ yields a solution of (5.5) and (5.6) of the form (5.3) and (5.4). The existence of the real analytic functions α and β is guaranteed by the implicit function theorem, since one easily checks that

$$G\big(\alpha^{(0)}, \beta^{(0)}, 0\big) = \begin{pmatrix} 0 \\ 2\pi \end{pmatrix} \quad \text{(cf. (5.7))} \tag{5.14}$$

and

$$D_{(\alpha,\beta)} G\big(\alpha^{(0)}, \beta^{(0)}, 0\big) = \frac{2m\pi}{\beta^{(0)}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{5.15}$$

Finally, it is easy to see that the integrals involved in computing the derivatives of G at $(\alpha^{(0)}, \beta^{(0)}, 0)$ can be evaluated explicitly, and by an inductive procedure the coefficients $\alpha^{(k)}$ and $\beta^{(k)}$ can be determined. □

Using the expressions obtained for α_n and β_n, one obtains immediately (cf. (1.8)–(1.10)) the coefficients of the polynomials for the rescaled field V_n (2.2): For $n \geq n_1$

$$V_n(x) = \sum_{k=0}^{2m} \upsilon_{n,k} x^k \quad \text{with } \upsilon_{n,k} = \sum_{l=0}^{\infty} \upsilon_k^{(l)} n^{-\frac{l}{2m}}\,. \tag{5.16}$$

Furthermore,

$$\upsilon_{n,2m} = \frac{1}{mA_m} + O\left(n^{-\frac{2}{2m}}\right), \tag{5.17}$$

$$\upsilon_{n,k} = O\left(n^{\frac{k}{2m}-1}\right) \quad \text{for } 0 \leq k \leq 2m-1\,. \tag{5.18}$$

The Solution of the Variational Problem

In order to determine the equilibrium measure for the external field V_n, we proceed as in [14]. Set

$$r(z) \equiv (z-1)^{1/2}(z+1)^{1/2} \qquad \text{for } z \in \mathbb{C} \setminus [-1,1], \tag{5.19}$$

$$F_n(z) \equiv \frac{r(z)}{\pi i} \int_{-1}^{1} \frac{-V_n'(y)}{r_+(y)(y-z)} \frac{dy}{2\pi i} \quad \text{for } z \in \mathbb{C} \setminus [-1,1], \tag{5.20}$$

$$h_n(z) \equiv \frac{1}{2\pi i} \oint_{\Gamma_z} \frac{V_n'(y)}{r(y)(y-z)}\, dy \qquad \text{for } z \in \mathbb{C}, \tag{5.21}$$

where Γ_z denotes any simple, closed contour that is oriented counterclockwise and contains $[-1,1] \cup \{z\}$ in its interior. A standard residue calculation shows that

$$F_n(z) = -\frac{V_n'(z)}{2\pi i} + \frac{r(z)}{2\pi i} h_n(z) \quad \text{for all } z \in \mathbb{C} \setminus [-1,1]. \tag{5.22}$$

PROPOSITION 5.3 *Given the notation above, there exists $n_2 \geq n_1$ such that for all $n \geq n_2$ the equilibrium measure with respect to the external field V_n has support $[-1,1]$ and its density is given by*

$$d\mu_n(x) = \frac{1}{2\pi} \sqrt{1-x^2}\, h_n(x) dx \quad \textit{for } -1 \leq x \leq 1. \tag{5.23}$$

Furthermore, h_n is a polynomial of degree $2m-2$ whose (real) coefficients can be computed explicitly, and there exists a constant $h_0 > 0$ such that $h_n(x) > h_0$ for all $n \geq n_2$ and $x \in \mathbb{R}$.

PROOF: We start by proving the claims related to h_n. We evaluate (5.21) by calculating the residue of the integrand at infinity. For example, for any $k \in \mathbb{N}$,

$$\frac{1}{2\pi i} \oint_{\Gamma_z} \frac{y^k}{r(y)(y-z)}\, dy = \sum_{j=0}^{\left[\frac{k-1}{2}\right]} A_j z^{k-1-2j}. \tag{5.24}$$

In the notation of (5.16), we obtain

$$h_n(z) = \sum_{k=0}^{2m-2} h_{n,k} z^k, \quad h_{n,k} = \sum_{j=0}^{\left[\frac{2m-2-k}{2}\right]} A_j (k+2+2j) \upsilon_{n,k+2+2j}, \tag{5.25}$$

yielding that for each $0 \leq k \leq 2m-2$, the coefficient $h_{n,k}$ can be expressed in a series $h_{n,k} = \sum_{l=0}^{\infty} h_k^{(l)} n^{-l/2m}$, and, by (5.17), the leading-order behavior is given by

$$h_{2k}^{(0)} = 2\frac{A_{m-k-1}}{A_m}, \quad h_{2k+1}^{(0)} = 0, \quad \text{for } 0 \leq k \leq m-1. \tag{5.26}$$

This, in turn, implies that there exists a $n_2 \geq n_1$ and a constant $h_0 > 0$ such that $h_n(x) > h_0$ for all $n \geq n_2$ and all $x \in \mathbb{R}$.

In the notation of (5.10), we can express

$$V_n(x) = V\left(x, n^{-\frac{1}{2m}}\alpha_n, n^{-\frac{1}{2m}}\beta_n, n^{-\frac{1}{2m}}\right). \tag{5.27}$$

From the proof of Proposition 5.2 we conclude

$$G\left(n^{-\frac{1}{2m}}\alpha_n, n^{-\frac{1}{2m}}\beta_n, n^{-\frac{1}{2m}}\right) = \begin{pmatrix} \int_{-1}^{1} \frac{V_n'(y)}{\sqrt{1-y^2}}\,dy \\ \int_{-1}^{1} \frac{yV_n'(y)}{\sqrt{1-y^2}}\,dy \end{pmatrix} = \begin{pmatrix} 0 \\ 2\pi \end{pmatrix}. \tag{5.28}$$

The first row of (5.28) together with (5.20) implies that $F_n(z) = O(1/z)$ as z tends to infinity. From the representation of F_n in (5.22), it is then obvious that the $F_n|_{\mathbb{C}_\pm}$ lie in the Hardy spaces $H^2(\mathbb{C}_\pm)$. Set $\hat{\psi}_n := \mathrm{Re}(F_n)_+ \in L^2(\mathbb{R})$. Since h_n is a strictly positive function for $n \geq n_2$, we conclude from (5.22) that $\operatorname{supp}\hat{\psi}_n = [-1,1]$. Furthermore, it follows from (5.20) that $\overline{F_n(\overline{z})} = -F_n(z)$. Then a standard argument in harmonic analysis (see, e.g., [42]) leads to

$$F_n(z) = \int_{\mathbb{R}} \frac{\hat{\psi}_n(x)}{x-z}\,\frac{dx}{\pi i} \quad \text{and} \tag{5.29}$$

$$(F_n)_\pm(x) = \pm\hat{\psi}_n(x) + i(H\hat{\psi}_n)(x), \tag{5.30}$$

where

$$H : L^2(\mathbb{R}) \to L^2(\mathbb{R}) : f \mapsto \frac{1}{\pi}\,\mathrm{PV}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy \tag{5.31}$$

denotes the Hilbert transform. We will now prove that $\hat{\psi}_n\,dx$ is indeed the equilibrium measure with respect to the field V_n. The representation of $\hat{\psi}_n$ as given in (5.23) follows easily from (5.22), and therefore it is clear that $\hat{\psi}_n$ is nonnegative. In order to see that $\hat{\psi}_n$ is also a probability measure, we use the two representations for F_n, namely, (5.20) and (5.29). The first formula together with the second row of (5.28) yields that $F_n(z) = -1/\pi i z + O(z^{-2})$, whereas it follows from (5.29) that $F_n(z) = -1/\pi i z \int_{-1}^{1}\hat{\psi}_n(x)dx + O(z^{-2})$: Thus $\int_{-1}^{1}\hat{\psi}_n(x)dx = 1$. Finally, we have to show that $\hat{\psi}_n$ satisfies the variational conditions (4.18)–(4.19). To that end we conclude from (5.30) and (5.22) that for $x \in \mathbb{R}$

$$\begin{aligned} &\frac{d}{dx}\left(2\int \log|x-y|\hat{\psi}_n(y)dy - V_n(x)\right) \\ &\quad = 2\pi(H\hat{\psi}_n)(x) - V_n'(x) \\ &\quad = \mathrm{Re}\left(-2\pi i(F_n)_+(x) - V_n'(x)\right) = -\mathrm{Re}(r_+(x)h_n(x)). \end{aligned} \tag{5.32}$$

The fact that h_n is positive on the real line implies that $2\int\log|x-y|\hat{\psi}_n(y)dy - V_n(x)$ is constant on $[-1,1]$, increasing for $x < -1$, and decreasing for $x > 1$, and thus $\hat{\psi}_n(x)dx$ satisfies the Euler-Lagrange equations (4.18)–(4.19). □

Remark. Note that the density of the equilibrium measure is denoted by $\hat{\psi}_n(x)$ in order to distinguish it from $\psi_n(z)$, which was defined in (2.14) as the analytic extension of $\hat{\psi}_n|_{(-1,1)}$ to $\mathbb{C}\setminus((-\infty,-1]\cup[1,\infty))$.

Set

$$g_n(z) \equiv \int_{-1}^{1} \hat{\psi}_n(t)\log(z-t)\,dt\,, \quad z \in \mathbb{C}\setminus(-\infty,1]\,, \tag{5.33}$$

$$\xi_n(z) \equiv -2\pi i\int_{1}^{z} \psi_n(y)\,dy\,, \qquad z \in \mathbb{C}\setminus((-\infty,-1]\cup[1,\infty)) \tag{5.34}$$

(cf. (2.14)), and recall the definition of the constant in the Euler-Lagrange equation given in (2.8)

$$l_n = \frac{1}{\pi}\int_{-1}^{1} \sqrt{1-t^2}\,h_n(t)\log|t|\,dt - V_n(0)\,. \tag{5.35}$$

PROPOSITION 5.4 *Let $n_2 \in \mathbb{N}$ be given as in Proposition* 5.3*. Then there exists a $\delta_1 > 0$ such that for all $n \geq n_2$ the following holds*:

(i) *g_n is analytic and $g_n|_{\mathbb{C}_\pm}$ have continuous extensions to $\overline{\mathbb{C}}_\pm$.*

(ii) *The map $z \mapsto e^{ng_n(z)}$ possesses an analytic continuation to $\mathbb{C}\setminus[-1,1]$ and*

$$e^{ng_n(z)}z^{-n} = 1 + O(1/z) \quad \textit{as } z \to \infty\,. \tag{5.36}$$

(iii)

$$(g_n)_+(x) - (g_n)_-(x) = \begin{cases} 2\pi i & \textit{for } x \leq -1 \\ \xi_n(x) & \textit{for } |x| < 1 \\ 0 & \textit{for } x \geq 1\,. \end{cases} \tag{5.37}$$

(iv)

$$-V_n(x) + (g_n)_+(x) + (g_n)_-(x) - l_n = \begin{cases} (\xi_n)_+(x) - 2\pi i & \textit{for } x \leq -1 \\ 0 & \textit{for } |x| < 1 \\ (\xi_n)_+(x) & \textit{for } x \geq 1\,. \end{cases} \tag{5.38}$$

Furthermore, $\mathrm{Re}(\xi_n)_+(x) < -\sqrt{2}h_0(|x|-1)^{3/2}$ *for all $|x| > 1$ (cf. Proposition* 5.3*).*

(v) *The function ξ_n can be computed explicitly,*

$$\xi_n(z) = -iH_n(z)(1-z)^{1/2}(1+z)^{1/2} - 2i\arcsin z + i\pi\,, \tag{5.39}$$

$$H_n(z) = \sum_{k=0}^{2m-1}\left(\sum_{l=0}^{\left[m-\frac{k+1}{2}\right]} A_l v_{n,k+2l+1}\right) z^k\,. \tag{5.40}$$

Furthermore,

$$\mathrm{Re}\,\xi_n(z) > 0 \quad \textit{for } 0 < \mathrm{Im}\,z < \delta_1,\ -1 < \mathrm{Re}\,z < 1\,, \tag{5.41}$$

$$\mathrm{Re}\,\xi_n(z) < 0 \quad \textit{for } -\delta_1 < \ \mathit{Im}\,z < 0,\ -1 < \mathrm{Re}\,z < 1\,. \tag{5.42}$$

(vi) *The constant l_n can be expressed in a convergent series $l_n = \sum_{k=0}^{\infty} l^{(k)} n^{-k/2m}$ with leading coefficient $l^{(0)} = -1/m - 2\log 2$. More generally,*

$$l_n = -\sum_{k=0}^{m} A_k \upsilon_{n,2k} - 2\log 2\,. \tag{5.43}$$

PROOF: 1. For $z_1, z_2 \in \mathbb{C}_+$ one can show using (5.29) that

$$g_n(z_2) - g_n(z_1) = -\pi i \int_{z_1}^{z_2} F_n(y)\,dy\,. \tag{5.44}$$

It follows from (5.22) and (5.29) that F_n is a bounded function. Consequently, g_n is uniformly Lipschitz-continuous in the upper half-plane and can therefore be continued to $\overline{\mathbb{C}}_+$. The same argument holds for the lower half-plane. Furthermore,

$$(g_n)_{\pm}(x) = \int_{-1}^{1} \hat{\psi}_n(t) \log|x-t|\,dt \pm \pi i \int_{x}^{1} \hat{\psi}_n(t)\,dt \quad \text{for } x \in \mathbb{R}\,. \tag{5.45}$$

2. We observe from (5.45) and the fact that $\hat{\psi}_n\,dx$ is a probability measure with support $[-1,1]$ that the jump $(g_n)_+ - (g_n)_-$ vanishes on $(1,\infty)$ and equals $2\pi i$ on $(-\infty,-1)$. Hence $e^{ng_n(z)}$ can be continued analytically outside $[-1,1]$. From

$$\log(z-t) = \log z - \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{t}{z}\right)^k \tag{5.46}$$

for all $t \in [-1,1]$, $z \in \mathbb{C} \setminus (-\infty,0]$ with$|z| \geq 2$, it follows easily that

$$g_n(z) = \log z + O(1/|z|)\,, \tag{5.47}$$

which in turn proves (5.36).
3. This follows immediately from (5.45) and the definition of ξ_n in (5.34).
4. Equation (5.45) implies

$$(g_n)_+(x) + (g_n)_-(x) - V_n(x) = 2\int_{-1}^{1} \hat{\psi}_n(t) \log|x-t|\,dt - V_n(x) \tag{5.48}$$

for all $x \in \mathbb{R}$. As $\hat{\psi}_n$ satisfies the Euler-Lagrange equations (4.18)–(4.19), equation (5.38) is clear for $|x| < 1$. The other two cases follow from the calculation in (5.32). The proof of the estimate on $\mathrm{Re}(\xi_n)_+(x)$ follows easily from Proposition 5.3 and from (5.34).
5. Recall that h_n is a polynomial of degree $2m-2$ (cf. Proposition 5.3), and therefore by elementary calculations there exists a polynomial H_n of degree $2m-1$ and a constant Λ_n such that

$$\begin{aligned} \xi_n(z) &= -i\int_1^z (1-y)^{1/2}(1+y)^{1/2} h_n(y)\,dy \\ &= -i\big(H_n(z)(1-z)^{1/2}(1+z)^{1/2} + \Lambda_n(\arcsin z - \pi/2)\big)\,. \end{aligned} \tag{5.49}$$

One can determine the coefficients of H_n and Λ_n inductively. However, we find it more convenient to proceed as follows: From (5.38), (5.47), and (5.49) one remarks that

$$l_n = \lim_{x\to\infty}\left(-V_n(x) + 2\log x + H_n(x)\sqrt{x^2-1} - \Lambda_n \log(x+\sqrt{x^2-1})\right). \tag{5.50}$$

Clearly, the limit can only exist if $\Lambda_n = 2$ and

$$\lim_{x\to\infty}\left(H_n(x) - \frac{V_n(x)}{x\sqrt{1-x^{-2}}}\right) = 0. \tag{5.51}$$

Expanding $(1-x^{-2})^{-1/2}$ immediately leads to (5.40).

We next prove (5.41) and (5.42). As ξ_n is purely imaginary on $(-1,1)$, we only need to show that there exists a suitable neighborhood of the interval $(-1,1)$ on which the imaginary part of the derivative of ξ_n is negative for all $n \geq n_2$. This, however, follows from the uniform positivity of h_n as stated in Proposition 5.3.

6. In order to evaluate formula (5.50), we observe that

$$\lim_{x\to\infty}\left(2\log x - 2\log(x+\sqrt{x^2-1})\right) = -2\log 2 \tag{5.52}$$

and that for x large

$$H_n(x)\sqrt{x^2-1} = \left(\sum_{k=0}^{2m-1}\sum_{l=0}^{\left[m-\frac{k+1}{2}\right]} A_l v_{n,k+2l+1} x^{k+1}\right)\left(\sum_{j=0}^{\infty} B_j x^{-2j}\right), \tag{5.53}$$

where B_j denote the coefficients of the expansion of $(1-x)^{1/2}$ at $x=0$. Because the A_j denote the Taylor coefficients of $(1-x)^{-1/2}$ at $x=0$, we have for every $r \geq 1$ that $\sum_{j=0}^{r} A_j B_{r-j} = 0$. A straightforward calculation then shows that the constant term of $-V_n(x) + H_n(x)\sqrt{x^2-1}$ is given by $-\sum_{k=0}^{m} A_k v_{n,2k}$. $\square$

5.3 The Transformed RHP for T

In this section we state the RHP for the matrix T.

THEOREM 5.5 *Let $U : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{2\times 2}$; recall the definition of the Pauli matrix*

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and let g_n, l_n, and ξ_n be as calculated above. Furthermore, set

$$T(z) \equiv e^{-n\frac{l_n}{2}\sigma_3} U(z) e^{-n\left(g_n(z)-\frac{l_n}{2}\right)\sigma_3} \quad \text{for } z \in \mathbb{C}\setminus\mathbb{R}. \tag{5.54}$$

The function U solves the Riemann-Hilbert problem (4.3)–(4.5) *if and only if T solves* (5.55)–(5.57).

$$T : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{2\times 2} \text{ is analytic}. \tag{5.55}$$

$$T_+(s) = T_-(s) \begin{cases} \begin{pmatrix} e^{-n\xi_n(s)} & 1 \\ 0 & e^{n\xi_n(s)} \end{pmatrix} & \textit{for } -1 < s < 1\,, \\ \begin{pmatrix} 1 & e^{n(\xi_n)_+(s)} \\ 0 & 1 \end{pmatrix} & \textit{for } |s| \geq 1\,. \end{cases} \tag{5.56}$$

$$T(z) = I + O\left(\frac{1}{|z|}\right) \quad \textit{as } |z| \to \infty\,. \tag{5.57}$$

PROOF: Employing (i), (ii), (iii), and (iv) of Proposition 5.4, it is a straightforward calculation to show that any solution U of (4.3)–(4.5) leads to a solution T of (5.55)–(5.57), and vice versa. □

Note that the jump matrix for T is oscillatory in $(-1,1)$ and converges exponentially fast (as $n \to \infty$) to the identity outside $[-1,1]$.

6 Steepest Descent: Jump Matrix Factorization and Contour Deformation

The basic idea behind the steepest-descent method is to deform the contour so that the rapidly oscillating terms become exponentially decaying. Note that by Proposition 5.4(v) the entries of the jump matrix for T have analytic continuations into a neighborhood of $(-1,1)$, where the 11-entry $e^{-n\xi_n}$ decays exponentially (as $n \to \infty$) in the upper half-plane and grows in the lower half-plane, while the 22-entry $e^{n\xi_n}$ has the opposite behavior. However, we can split the 11-entry and the 22-entry by the following factorization of the jump matrix:

$$\begin{pmatrix} e^{-n\xi_n} & 1 \\ 0 & e^{n\xi_n} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{n\xi_n} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-n\xi_n} & 1 \end{pmatrix} \equiv v_- v_0 v_+\,. \tag{6.1}$$

We now deform the Riemann-Hilbert problem in the sense of [17, 18, 19] to an equivalent Riemann-Hilbert problem for a 2×2–matrix-valued function S on the oriented contour Σ_S shown in Figure 6.1, where

$$S(z) \equiv T(z) \quad \text{for } z \text{ outside the lens-shaped region}\,, \tag{6.2}$$

$$S(z) \equiv T(z) v_+^{-1}(z) = T(z) \begin{pmatrix} 1 & 0 \\ -e^{-n\xi_n(z)} & 1 \end{pmatrix} \quad \text{in the upper lens region}\,, \tag{6.3}$$

$$S(z) \equiv T(z) v_-(z) = T(z) \begin{pmatrix} 1 & 0 \\ e^{n\xi_n(z)} & 1 \end{pmatrix} \quad \text{in the lower lens region}\,. \tag{6.4}$$

The precise shape of the lens will be determined in the beginning of Section 7.1 (see Figures 7.4 and 7.5). Of course, it will be contained in the region where (5.41) (respectively, (5.42)) hold.

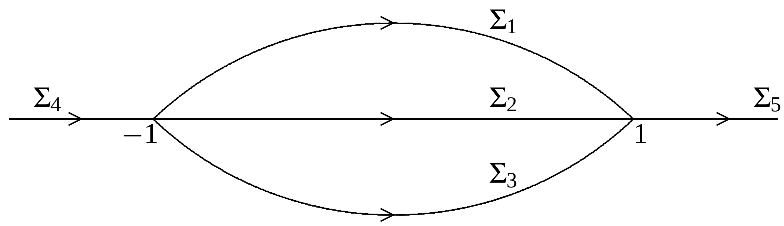

FIGURE 6.1. The contour Σ_S.

We decompose the contour Σ_S by $\Sigma_S \equiv \bigcup_{j=1}^{5} \Sigma_j$ according to Figure 6.1. Furthermore, let

$$v_1(s) \equiv \begin{pmatrix} 1 & 0 \\ e^{-n\xi_n(s)} & 1 \end{pmatrix} \quad \text{for } s \in \Sigma_1\,, \tag{6.5}$$

$$v_2(s) \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{for } s \in \Sigma_2\,, \tag{6.6}$$

$$v_3(s) \equiv \begin{pmatrix} 1 & 0 \\ e^{n\xi_n(s)} & 1 \end{pmatrix} \quad \text{for } s \in \Sigma_3\,, \tag{6.7}$$

$$v_4(s) \equiv \begin{pmatrix} 1 & e^{n(\xi_n)_+(s)} \\ 0 & 1 \end{pmatrix} \quad \text{for } s \in \Sigma_4\,, \tag{6.8}$$

$$v_5(s) \equiv \begin{pmatrix} 1 & e^{n(\xi_n)_+(s)} \\ 0 & 1 \end{pmatrix} \quad \text{for } s \in \Sigma_5\,. \tag{6.9}$$

LEMMA 6.1 *Let $T : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$, and define S according to* (6.2)–(6.4). *Then T solves the Riemann-Hilbert problem* (5.55)–(5.57) *if and only if S solves* (6.10)–(6.12) *below*:

$$S : \mathbb{C} \setminus \Sigma_S \to \mathbb{C}^{2\times 2} \quad \textit{is analytic}\,. \tag{6.10}$$

$$S_+(s) = S_-(s) v_j(s) \quad \textit{for } s \in \Sigma_j,\ 1 \le j \le 5\,, \tag{6.11}$$

$$S(z) = I + O\left(\frac{1}{|z|}\right) \quad \textit{as } |z| \to \infty\,. \tag{6.12}$$

PROOF: The proof is straightforward, and we will only verify the jump conditions on Σ_1 through Σ_3:

$$\begin{aligned}
&\Sigma_1 : S_+ = T_+ = T_- = S_- v_+ = S_- v_1\,,\\
&\Sigma_2 : S_+ = T_+ (v_+)^{-1} = T_-(v_- v_0 v_+)(v_+^{-1}) = S_- v_0 = S_- v_2\,,\\
&\Sigma_3 : S_+ = T_+ v_- = T_- v_- = S_- v_- = S_- v_3\,.
\end{aligned}$$

□

Remark. Again, the jump condition (6.11) is to be understood in the sense that S is continuous in each component of $\mathbb{C}\setminus\Sigma_S$ up to the boundary with boundary values $S_\pm$ satisfying (6.11) on Σ_S (cf. the remark after Theorem 3.1).

Note that the Riemann-Hilbert problem for S (6.10)–(6.12), which is equivalent to the Riemann-Hilbert problem (3.6)–(3.8) for U, has jump matrices that converge exponentially fast (as $n\to\infty$) to the identity except on the interval $(-1,1)$, where the jump matrix is a constant. At this point it becomes clear how to construct a parametrix, i.e., an approximate solution for the Riemann-Hilbert problem. In a first step we simply neglect those jump conditions that tend to the identity. We consider the following Riemann-Hilbert problem:

$$N:\mathbb{C}\setminus[-1,1]\to\mathbb{C}^{2\times 2}\qquad\text{is analytic}\,,\tag{6.13}$$

$$N_+(s)=N_-(s)\begin{pmatrix}0&1\\-1&0\end{pmatrix}\qquad\text{for } s\in[-1,1]\,,\tag{6.14}$$

$$N(z)=I+O\left(\frac{1}{|z|}\right)\qquad\text{as } |z|\to\infty\,.\tag{6.15}$$

The Riemann-Hilbert problem (6.13)–(6.15) can be solved explicitly by diagonalizing the jump matrix and hence reducing it to two scalar Riemann-Hilbert problems. One obtains the (unique) solution

$$\begin{aligned}N(z)&=\frac{1}{2}\begin{pmatrix}1&1\\i&-i\end{pmatrix}\begin{pmatrix}a(z)&0\\0&a(z)^{-1}\end{pmatrix}\begin{pmatrix}1&-i\\1&i\end{pmatrix}\\&=\frac{1}{2}\begin{pmatrix}a(z)+a(z)^{-1}&i(a(z)^{-1}-a(z))\\i(a(z)-a(z)^{-1})&a(z)+a(z)^{-1}\end{pmatrix},\end{aligned}\tag{6.16}$$

with

$$a(z)=\frac{(z-1)^{1/4}}{(z+1)^{1/4}}\,.\tag{6.17}$$

Observe that a is analytic on $\mathbb{C}\setminus[-1,1]$.

Remark. Note that N as defined in (6.16) does not solve the Riemann-Hilbert problem (6.13)–(6.15) in the sense described above (see the remark after Theorem 3.1) as $N|_{\mathbb{C}_\pm}$ cannot be extended continuously to $\overline{\mathbb{C}}_\pm$. However, $N(\cdot\pm i\varepsilon)$ converge in $L^2_{\mathrm{loc}}(\mathbb{R})$ as $\varepsilon\searrow 0$ to functions $N_\pm$ in $L^2([-1,1])$ that satisfy (6.14) almost everywhere on $[-1,1]$. It is not difficult to show that N is the unique solution of (6.13)–(6.15), where (6.14) is interpreted in this L^2_{loc} sense. This implies, in particular, that there is no solution of (6.13)–(6.15) with continuous boundary values. The fact that N is a solution of a (generalized) RHP is not used in the paper.

As explained in Section 4, we need the parametrix to be uniformly bounded in order to prove that there is a solution of the full Riemann-Hilbert problem close to the parametrix. We see that N does not satisfy this requirement for a parametrix as there are $\frac{1}{4}$-root singularities at the endpoints of the interval $[-1,1]$ (cf. Section 4).

We will therefore introduce a different parametrix close to the endpoints of the support of the equilibrium measure ± 1 in the next section.

7 Solution of the RHP for S

7.1 Analysis at the Endpoints of the Equilibrium Measure

In this section we will provide a parametrix for the Riemann-Hilbert problem (6.10)–(6.12) near the endpoints of the equilibrium measure. Let us first consider the right endpoint. By a parametrix near 1, we mean a solution of the following Riemann-Hilbert problem: Denote $U_\delta := \{z \in \mathbb{C} : |z-1| < \delta\}$ for $1 > \delta > 0$.

$$P : U_\delta \setminus \Sigma_S \to \mathbb{C}^{2\times 2} \quad \text{is analytic}, \tag{7.1}$$

$$P_+(s) = P_-(s) v_j(s) \quad \text{for } s \in \Sigma_S \cap U_\delta,\ j \in \{1,2,3,5\}, \tag{7.2}$$

$$P(z) \approx N(z) \quad \text{for } |z-1| = \delta. \tag{7.3}$$

This Riemann-Hilbert problem is not fully determined in two ways. First, recall that we have not given a precise definition of the contour Σ_S yet (cf. below (6.4)), and we will choose Σ_S as part of the construction of the parametrix. Second, the more usual asymptotic condition that normalizes the solution at ∞ to ensure uniqueness is replaced by the approximate matching condition (7.3). We are led to a unique determination of the parametrix in the construction below by satisfying (7.3) in an "optimal" way.

The general way to construct such a parametrix is by using the vanishing lemma [16]. However, in the generic case where the equilibrium measure vanishes as a square root at the endpoints of its support, we can express the parametrix in terms of Airy functions. As these functions appear in the asymptotic analysis of orthogonal polynomials in many different situations (see, e.g., [7, 9, 32, 36]), we choose to proceed in the latter way; we devote the remainder of this subsection to constructing an explicit solution of (7.1) and (7.2) using Airy functions that will satisfy (7.3) up to an error of order $O(\frac{1}{n})$.

We begin with an observation that allows us to transform to constant jump matrices. Define

$$\varphi_n : \mathbb{C} \setminus \mathbb{R},\ z \mapsto \begin{cases} -\frac{1}{2}\xi_n(z) & \text{for } \operatorname{Im} z > 0, \\ \frac{1}{2}\xi_n(z) & \text{for } \operatorname{Im} z < 0. \end{cases} \tag{7.4}$$

One concludes easily from the definition of ξ_n in (5.34) (see also (2.14)) that $(\varphi_n)_+(x) = (\varphi_n)_-(x)$ for $x > 1$ and $(\varphi_n)_+(x) = (\varphi_n)_-(x) - 2\pi i$ for $x < -1$. Hence the function $e^{n\varphi_n}$ possesses an analytic continuation to $\mathbb{C} \setminus [-1,1]$. A straightforward calculation now yields

$$v_j(s) = e^{-n(\varphi_n)_-(s)\sigma_3} w_j e^{n(\varphi_n)_+(s)\sigma_3} \quad \text{for } 1 \le j \le 5, \tag{7.5}$$

where

$$w_1 = w_3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad w_4 = w_5 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{7.6}$$

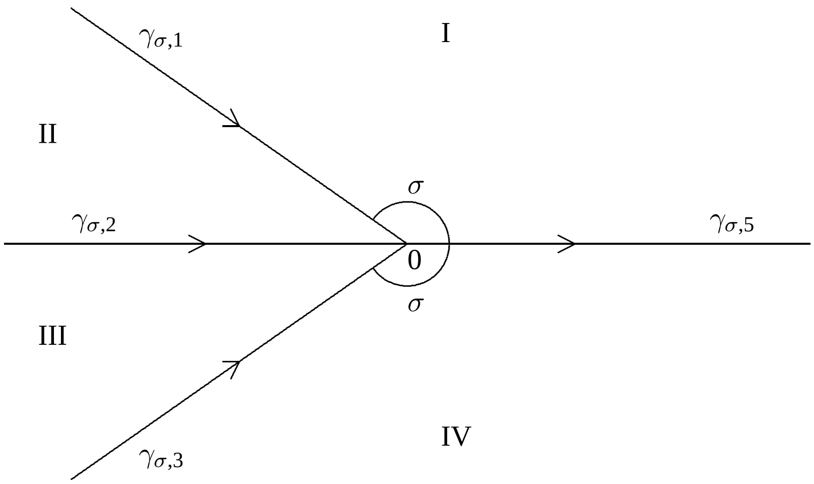

FIGURE 7.1. The contour γ_σ with opening angle 2σ.

Note that if P satisfies (7.2), then $Pe^{-n\varphi_n(z)\sigma_3}$ satisfies the corresponding jump relations where v_j is replaced by the constant matrix w_j.

We now introduce the auxiliary contour $\gamma_\sigma = \bigcup_{j=1}^{3}\gamma_{\sigma,j} \cup \gamma_{\sigma,5}$, which depends on the parameter $\sigma \in (\frac{\pi}{3}, \pi)$ as displayed in Figure 7.1 below and divides the complex plane into four regions, I, II, III, and IV.

Prescribing the jump matrix w_j on $\gamma_{\sigma,j}$ ($j = 1,2,3,5$), we arrive at the Riemann-Hilbert problem (7.10) and (7.11) below, for which we can write down an explicit solution in terms of Airy functions (cf. [19]). Denote

$$\omega := e^{\frac{2\pi i}{3}} \tag{7.7}$$

and define

$$\Psi^\sigma : \mathbb{C} \setminus \gamma_\sigma \to \mathbb{C}^{2\times 2}, \tag{7.8}$$

$$\Psi^\sigma(\zeta) = \begin{cases} \begin{pmatrix} \mathrm{Ai}(\zeta) & \mathrm{Ai}(\omega^2\zeta) \\ \mathrm{Ai}'(\zeta) & \omega^2\,\mathrm{Ai}'(\omega^2\zeta) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3} & \text{for } \zeta \in \mathrm{I}, \\ \begin{pmatrix} \mathrm{Ai}(\zeta) & \mathrm{Ai}(\omega^2\zeta) \\ \mathrm{Ai}'(\zeta) & \omega^2\,\mathrm{Ai}'(\omega^2\zeta) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} & \text{for } \zeta \in \mathrm{II}, \\ \begin{pmatrix} \mathrm{Ai}(\zeta) & -\omega^2\,\mathrm{Ai}(\omega\zeta) \\ \mathrm{Ai}'(\zeta) & -\mathrm{Ai}'(\omega\zeta) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \text{for } \zeta \in \mathrm{III}, \\ \begin{pmatrix} \mathrm{Ai}(\zeta) & -\omega^2\,\mathrm{Ai}(\omega\zeta) \\ \mathrm{Ai}'(\zeta) & -\mathrm{Ai}'(\omega\zeta) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3} & \text{for } \zeta \in \mathrm{IV}. \end{cases} \tag{7.9}$$

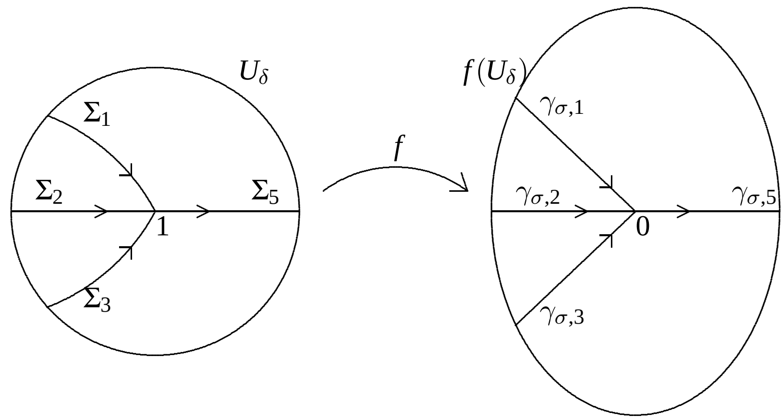

FIGURE 7.2. Map between the contours Σ_S and γ_σ.

LEMMA 7.1 *For all* $\sigma \in (\frac{\pi}{3}, \pi)$ *the function* Ψ^σ *satisfies the following two conditions*:

$$\Psi^\sigma : \mathbb{C} \setminus \gamma_\sigma \to \mathbb{C}^{2\times 2} \text{ is analytic} \quad \text{and} \tag{7.10}$$

$$(\Psi^\sigma)_+(s) = (\Psi^\sigma)_-(s) w_j \quad \text{for } s \in \gamma_{\sigma,j},\ j = 1,2,3,5\,. \tag{7.11}$$

PROOF: The proof just uses the following identity:

$$\operatorname{Ai}(\zeta) + \omega \operatorname{Ai}(\omega\zeta) + \omega^2 \operatorname{Ai}(\omega^2\zeta) = 0 \quad \text{for all } z \in \mathbb{C}\,. \tag{7.12}$$

□

Suppose now that we have a biholomorphic map $f : U_\delta \to f(U_\delta) \subset \mathbb{C}$ with $f(1) = 0$ such that the contour $\Sigma_S \cap U_\delta$ is mapped onto $\gamma_\sigma \cap f(U_\delta)$ (see Figure 7.2). Then, obviously, $\Psi^\sigma(f(z))e^{n\varphi_n(z)\sigma_3}$ will solve (7.1) and (7.2). As we will see below, the asymptotics for the Airy function together with condition (7.3) determine the function f (and hence the contour Σ_S). Finally, we observe that the jump relation for $\Psi^\sigma(f(z))e^{n\varphi_n(z)\sigma_3}$ is not changed under multiplication by an analytic matrix-valued function from the left. This provides additional degrees of freedom that enable us to satisfy (7.3) up to order $O(\frac{1}{n})$. We summarize with the following:

PROPOSITION 7.2 *For any biholomorphic map* $f : U_\delta \to f(U_\delta)$ *satisfying* $f(\Sigma_j \cap U_\delta) = \gamma_{\sigma,j} \cap f(U_\delta)$ *for* $j = 1,2,3,5$ (*and preserving orientation*) *and any analytic map* $E : U_\delta \to \mathbb{C}^{2\times 2}$, *the matrix-valued function*

$$U_\delta \setminus \Sigma_S \ni z \mapsto E(z)\Psi^\sigma(f(z))e^{n\varphi_n(z)\sigma_3} \in \mathbb{C}^{2\times 2} \tag{7.13}$$

satisfies conditions (7.1) *and* (7.2).

We now choose the parameters E and f of Proposition 7.2 in order to satisfy the matching condition (7.3) in an optimal way. To this end, observe first that we have to compensate for the (nonanalytic) term $e^{n\varphi_n}$ in (7.13). The asymptotic expansion for Ai (cf. [1, 10.4.59]) shows that f has to be chosen to satisfy

$$\frac{2}{3}(f(z))^{3/2} = n\varphi_n(z)\,. \tag{7.14}$$

We shall now demonstrate that for each $n \geq n_2$ (cf. Proposition 5.3) there exists such a biholomorphic function $f(z) = f_n(z)$.

PROPOSITION 7.3 *Let n_2 be defined as in Proposition* 5.3. *There exists a $\delta_2 > 0$ such that for all $n \geq n_2$ there are biholomorphic functions $\phi_n : U_{\delta_2} \to \phi_n(U_{\delta_2}) \subset \mathbb{C}$ and*

(i) $\phi_n(U_{\delta_2} \cap \mathbb{R}) = \phi_n(U_{\delta_2}) \cap \mathbb{R},\ \phi_n(U_{\delta_2} \cap \mathbb{C}_\pm) = \phi_n(U_{\delta_2}) \cap \mathbb{C}_\pm$.
(ii) $\phi_n(z)^{3/2} = \frac{3}{2}\varphi_n(z)$ *for all* $z \in U_{\delta_2} \setminus (-\infty, 1]$.
(iii) *There exists a constant $c_0 > 0$ such that for all $z \in U_{\delta_2}$ and all $n \geq n_2$ the derivative of ϕ_n can be estimated by $c_0 < |\phi_n'(z)| < 1/c_0$ and $|\arg \phi_n'(z)| < \pi/15$.*

Remark. In order to satisfy (7.14), we choose $f_n = n^{2/3}\phi_n$. Note that Proposition 7.3 also implies (2.15) (see (7.4) and (5.34)).

PROOF: We define ϕ_n explicitly. From (7.4), (5.34), and (2.14) it follows that

$$\frac{3}{2}\varphi_n'(z) = \left[\frac{3}{4}h_n(z)(z+1)^{1/2}\right](z-1)^{1/2} \tag{7.15}$$

for all $z \in U_{\delta=2} \setminus (-1,1]$. The function $\frac{3}{4}h_n(z)(z+1)^{1/2}$ is analytic in $U_{\delta=2}$ and hence there exists a power series

$$\frac{3}{4}h_n(z)(z+1)^{1/2} = \sum_{k=0}^{\infty} r_{n,k}(z-1)^k\,. \tag{7.16}$$

Define

$$\hat{\phi}_n(z) \equiv \sum_{k=0}^{\infty} \frac{2r_{n,k}}{3+2k}(z-1)^k \quad \text{for } |z-1| < 2\,. \tag{7.17}$$

A short calculation yields

$$\frac{3}{2}\varphi_n(z) = (z-1)^{3/2}\hat{\phi}_n(z) \quad \text{for all } z \in U_{\delta=2} \setminus (-1,1]\,. \tag{7.18}$$

We observe that

$$r_{n,0} > \frac{3}{4}h_0\sqrt{2} > 0 \quad \text{for all } n \geq n_2 \text{ (cf. Proposition 5.3)} \tag{7.19}$$

and by Cauchy's integral formula

$$\sup_{k \geq 0, n \geq n_2} |r_{n,k}| < \sup_{|z-1|=1, n \geq n_2} \left|\frac{3}{4}h_n(z)(z+1)^{1/2}\right| < \infty\,. \tag{7.20}$$

Therefore there exists a $\delta > 0$ such that $\operatorname{Re}\hat{\phi}_n(z) > 0$ for all $z \in U_\delta$ and for all $n \geq n_2$. Hence

$$\phi_n(z) \equiv (z-1)\left(\hat{\phi}_n(z)\right)^{2/3} \quad \text{for } z \in U_\delta \tag{7.21}$$

is an analytic function. The injectivity of ϕ_n on a possibly smaller (but n independent) domain U_{δ_2} follows, for example, from the uniform boundedness (in n and z) of the second derivatives of ϕ_n together with an n-independent lower bound on $|\phi_n'(1)|$, both of which are consequences of (7.19) and (7.20). Similarly, one shows property (iii). Property (i) follows from the fact that ϕ_n and consequently also its inverse ϕ_n^{-1} are real on the real axis. Finally, (7.18) together with property (i) allows us to conclude (ii). ☐

The only freedom left in the choice of the parametrix is the analytic matrix-valued function E (cf. Proposition 7.2). Keeping condition (7.3) in mind, we seek an analytic approximation to

$$N(z)\left[\Psi^\sigma(f_n(z))e^{n\varphi_n(z)\sigma_3}\right]^{-1}. \tag{7.22}$$

This leads to the following definitions: For $n \geq n_2$ and $\sigma \in (\frac{\pi}{3}, \pi)$, set

$$E_n(z) \equiv \sqrt{\pi}e^{\frac{\pi i}{6}}\begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix}\begin{pmatrix} (f_n(z))^{1/4}a(z)^{-1} & 0 \\ 0 & (f_n(z))^{-1/4}a(z) \end{pmatrix} \tag{7.23}$$

for $z \in U_{\delta_2}$, and

$$P_n(z) \equiv E_n(z)\Psi^\sigma(f_n(z))e^{n\varphi_n(z)\sigma_3} \quad \text{for } z \in U_{\delta_2} \setminus f_n^{-1}(\gamma_\sigma). \tag{7.24}$$

Remarks. 1. Recall the definition of $a(z) = \frac{(z-1)^{1/4}}{(z+1)^{1/4}}$ (cf. (6.17)) and note that

$$(f_n(z))^{1/4}a(z)^{-1} = n^{1/6}(\hat{\phi}_n(z))^{1/6}(z+1)^{1/4}. \tag{7.25}$$

Therefore E_n indeed represents an analytic function in U_{δ_2}.

2. We suppress the dependence of P_n on the parameter σ in the notation.

We now summarize the properties of P_n. To that end, we introduce coefficients s_k and t_k, which are taken from the asymptotic expansion of the Airy function at ∞ (cf. [1, 10.4.59, 10.4.61]).

$$\operatorname{Ai}(\zeta) \sim \frac{1}{2\sqrt{\pi}}\zeta^{-\frac{1}{4}}e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}\sum_{k=0}^{\infty}(-1)^k s_k\left(\frac{2}{3}\zeta^{\frac{3}{2}}\right)^{-k} \quad \text{for } |\arg\zeta| < \pi, \tag{7.26}$$

$$\operatorname{Ai}'(\zeta) \sim -\frac{1}{2\sqrt{\pi}}\zeta^{\frac{1}{4}}e^{-\frac{2}{3}\zeta^{\frac{3}{2}}}\sum_{k=0}^{\infty}(-1)^k t_k\left(\frac{2}{3}\zeta^{\frac{3}{2}}\right)^{-k} \quad \text{for } |\arg\zeta| < \pi, \tag{7.27}$$

where

$$s_0 = t_0 = 1, \quad s_k = \frac{\Gamma(3k+\frac{1}{2})}{54^k k!\,\Gamma(k+\frac{1}{2})}, \quad t_k = -\frac{6k+1}{6k-1}s_k, \quad \text{for } k \geq 1, \tag{7.28}$$

and Γ denotes the (factorial) gamma function.

LEMMA 7.4 *Let n_2 and δ_2 be given as in Proposition* 7.3 *and let $n \geq n_2$, $\sigma \in (\frac{\pi}{3}, \pi)$. Suppose further that $\Sigma_S \cap U_{\delta_2} = f_n^{-1}(\gamma_\sigma) \cap U_{\delta_2}$. Then the matrix-valued function P_n (given by* (7.24)) *satisfies* (7.1) *and* (7.2). *In addition, for $0 < |z-1| < \delta_2$, there exists an asymptotic expansion for $P_n(z)N^{-1}(z)$ in powers of u^{-1}, $u \equiv n\varphi_n(z)$, which is given by*

$$\begin{aligned}&\frac{1}{2}\sum_{k=0}^{\infty}\begin{pmatrix} s_{2k}+t_{2k} & i(s_{2k}-t_{2k}) \\ -i(s_{2k}-t_{2k}) & s_{2k}+t_{2k}\end{pmatrix}u^{-2k} \\ &\quad + \frac{1}{2}\sum_{k=0}^{\infty}\begin{pmatrix} -(s_{2k+1}a^{-2}+t_{2k+1}a^2) & i(s_{2k+1}a^{-2}-t_{2k+1}a^2) \\ i(s_{2k+1}a^{-2}-t_{2k+1}a^2) & s_{2k+1}a^{-2}+t_{2k+1}a^2\end{pmatrix}u^{-2k-1}.\end{aligned} \tag{7.29}$$

PROOF: Recall that by Proposition 7.2 we only have to verify the asymptotic expansion for P_nN^{-1}, which in turn follows from the following observation: For $\sigma \in (\frac{\pi}{3}, \pi)$, $\zeta \in \mathbb{C} \setminus \gamma_\sigma$, and $u \equiv \frac{2}{3}\zeta^{3/2}$ the asymptotic formulae for Airy functions (cf. (7.26) and (7.27)) yield the uniform expansion

$$\Psi^\sigma(\zeta)e^{u\sigma_3} \sim \frac{e^{\frac{\pi i}{12}}}{2\sqrt{\pi}}\begin{pmatrix}\zeta^{-1/4} & 0 \\ 0 & \zeta^{1/4}\end{pmatrix}\sum_{k=0}^{\infty}\begin{pmatrix}(-1)^k s_k & s_k \\ -(-1)^k t_k & t_k\end{pmatrix}e^{-\frac{\pi i}{4}\sigma_3}u^{-k}. \tag{7.30}$$

□

Remark. Note that the expansion (7.29) is uniform in σ and z if σ and z are chosen to lie in compact subsets of $(\frac{\pi}{3}, \pi)$, $\{0 < |z-1| \leq \delta_2\}$, respectively.

The following result is immediate from (7.29):

COROLLARY 7.5 *Let $0 < \varepsilon \leq \delta_2$ and let K be a compact subset of $(\frac{\pi}{3}, \pi)$. Then there exists a constant $C > 0$ such that for all $\varepsilon \leq |z-1| \leq \delta_2$, $\sigma \in K$, and $n \geq n_2$*

$$\left| P_n(z)N^{-1}(z) - I - \frac{1}{n}\frac{\begin{pmatrix} 2z-12 & i(12z-2) \\ i(12z-2) & -2z+12\end{pmatrix}}{96(z+1)^{1/2}(z-1)^2\hat{\phi}_n(z)}\right| \leq \frac{C}{n^2}. \tag{7.31}$$

Similar calculations also produce the parametrix at -1. We summarize: For $n \geq n_2$, we define

$$\tilde{\varphi}_n(z) \equiv \begin{cases}\varphi_n(z) + \pi i = \pi i \int_{-1}^{z}\psi_n(y)dy & \text{for } \operatorname{Im} z > 0, \\ \varphi_n(z) - \pi i = -\pi i \int_{-1}^{z}\psi_n(y)dy & \text{for } \operatorname{Im} z < 0,\end{cases} \tag{7.32}$$

$$\tilde{U}_\delta \equiv \{z \in \mathbb{C} : |z+1| < \delta\}, \tag{7.33}$$

$$\frac{3}{4}h_n(z)(1-z)^{1/2} = \sum_{k=0}^{\infty}\tilde{r}_{n,k}(z+1)^k, \tag{7.34}$$

$$\hat{\tilde{\phi}}_n(z) \equiv \sum_{k=0}^{\infty}\frac{2\tilde{r}_{n,k}}{3+2k}(z+1)^k \quad \text{for } |z+1| < 2, \tag{7.35}$$

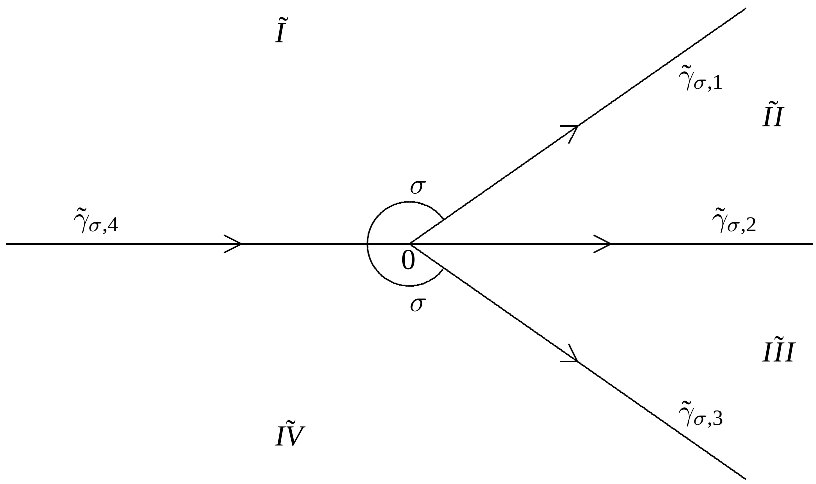

FIGURE 7.3. The contour $\tilde{\gamma}_\sigma$.

$$\tilde{\phi}_n(z) \equiv (z+1)\left(\hat{\tilde{\phi}}_n(z)\right)^{2/3} \quad \text{for } z \in \tilde{U}_{\tilde{\delta}_2} \text{ with suitable } \tilde{\delta}_2 > 0\,, \tag{7.36}$$

$$\tilde{f}_n(z) \equiv n^{2/3}\tilde{\phi}_n(z) \qquad \text{for } z \in \tilde{U}_{\tilde{\delta}_2}\,. \tag{7.37}$$

Then (2.16) holds, or, equivalently,

$$\frac{2}{3}(-\tilde{f}_n(z))^{3/2} = n\tilde{\varphi}_n(z)\,. \tag{7.38}$$

Furthermore, for $\sigma \in (\frac{\pi}{3}, \pi)$ we introduce the contour $\tilde{\gamma}_\sigma = \bigcup_{j=1}^4 \tilde{\gamma}_{\sigma,j}$ with $\tilde{\gamma}_{\sigma,1} = -\gamma_{\sigma,3}$, $\tilde{\gamma}_{\sigma,2} = -\gamma_{\sigma,2}$, $\tilde{\gamma}_{\sigma,3} = -\gamma_{\sigma,1}$, $\tilde{\gamma}_{\sigma,4} = -\gamma_{\sigma,5}$ with orientation as shown in Figure 7.3 (cf. Figure 7.1).

Denote

$$\tilde{\Psi}^\sigma(\zeta) \equiv \sigma_3\Psi^\sigma(-\zeta)\sigma_3 \quad \text{for } \zeta \in \mathbb{C}\setminus\tilde{\gamma}_\sigma\,, \tag{7.39}$$

(7.40)

$$\tilde{E}_n(z) \equiv \sqrt{\pi}e^{\frac{\pi i}{6}}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}\begin{pmatrix}(-\tilde{f}_n(z))^{1/4}a(z) & 0\\ 0 & (-\tilde{f}_n(z))^{-1/4}a(z)^{-1}\end{pmatrix} \quad \text{for } z \in \tilde{U}_{\tilde{\delta}_2}\,,$$

$$P_n(z) \equiv \tilde{E}_n(z)\tilde{\Psi}^\sigma(\tilde{f}_n(z))e^{n\tilde{\varphi}_n(z)\sigma_3} \quad \text{for } z \in \tilde{U}_{\tilde{\delta}_2}\setminus\tilde{f}_n^{-1}(\tilde{\gamma}_\sigma)\,. \tag{7.41}$$

Propositions 7.2 and 7.3 and Lemmata 7.1 and 7.4 hold mutatis mutandis. In particular, under the assumption that $\Sigma_S \cap \tilde{U}_{\tilde{\delta}_2} = \tilde{f}_n^{-1}(\tilde{\gamma}_\sigma) \cap \tilde{U}_{\tilde{\delta}_2}$, one obtains

$$P_n : \tilde{U}_{\tilde{\delta}_2}\setminus\Sigma_S \to \mathbb{C}^{2\times 2} \text{ is analytic}, \tag{7.42}$$

$$(P_n)_+(s) = (P_n)_-(s)v_j(s) \quad \text{for } s \in \Sigma_S \cap U_{\tilde{\delta}_2},\ j \in \{1,2,3,4\}, \tag{7.43}$$

$$P_n(z)N^{-1}(z) \sim \frac{1}{2}\sum_{k=0}^{\infty}\begin{pmatrix} s_{2k}+t_{2k} & -i(s_{2k}-t_{2k}) \\ i(s_{2k}-t_{2k}) & s_{2k}+t_{2k}\end{pmatrix}\tilde{u}^{-2k} + \frac{1}{2}\sum_{k=0}^{\infty}\begin{pmatrix} -(s_{2k+1}a^2+t_{2k+1}a^{-2}) & -i(s_{2k+1}a^2-t_{2k+1}a^{-2}) \\ -i(s_{2k+1}a^2-t_{2k+1}a^{-2}) & s_{2k+1}a^2+t_{2k+1}a^{-2}\end{pmatrix}\tilde{u}^{-2k-1}, \tag{7.44}$$

where the asymptotic variable $\tilde{u}$ is given by $\tilde{u} \equiv n\tilde{\varphi}_n(z)$. Furthermore,

$$\left| P_n(z)N^{-1}(z) - I - \frac{1}{n}\frac{\begin{pmatrix} -2z-12 & i(12z+2) \\ i(12z+2) & 2z+12\end{pmatrix}}{96(1-z)^{1/2}(z+1)^2\hat{\tilde{\phi}}_n(z)}\right| \le \frac{C}{n^2}, \tag{7.45}$$

where the constant C can be chosen independently of z and σ as long as $|z+1|$ lies in a compact subset of $(0,\tilde{\delta}_2]$ and σ lies in a compact subset of $(\frac{\pi}{3},\pi)$.

7.2 The Parametrix S_{par} and the Solution of the RHP for R

In this section we will define a parametrix S_{par} for the Riemann-Hilbert problem (6.10)–(6.12). We then state the Riemann-Hilbert problem for $R := SS_{\text{par}}^{-1}$ and show that its jump matrix is of the form $I + O(1/n)$ in the L_2 sense *and* in the L_∞ sense. Therefore we can express R in terms of a Neumann series.

As motivated in Section 4, the parametrix S_{par} for the Riemann-Hilbert problem (6.10)–(6.12) will be given by P_n near the endpoints of the equilibrium measure and by N everywhere else. We will now make this precise.

First, we will define the contour Σ_S. Keeping the hypothesis of Lemma 7.8 in mind (i.e., $\Sigma_S \cap U_\delta = f_n^{-1}(\gamma_\sigma) \cap U_\delta$ and similarly $\Sigma_S \cap \tilde{U}_\delta = \tilde{f}_n^{-1}(\tilde{\gamma}_\sigma) \cap \tilde{U}_\delta$), we construct Σ_S depending on the parameters δ, n, and α (a new parameter replacing σ). Fix $n \ge n_2$ (cf. Proposition 5.3) and set $\delta_0 := \min\{\delta_1, \delta_2, \tilde{\delta}_2\}$ (cf. Propositions 5.4 and 7.3 and equivalence (7.36)). Choose $\delta \in (0,\delta_0)$ and $\alpha \in (\frac{2\pi}{3}, \frac{5\pi}{6})$. From Proposition 7.3(iii) it is clear that there exists a $\sigma = \sigma(n,\alpha,\delta) \in (\frac{3\pi}{5}, \frac{9\pi}{10})$ such that $f_n^{-1}(\gamma_{\sigma,1}) \cap \partial U_\delta = \{1+\delta e^{i\alpha}\}$. By the symmetry $\overline{f_n(z)} \equiv f_n(\bar{z})$, it then follows that $f_n^{-1}(\gamma_{\sigma,3}) \cap \partial U_\delta = \{1+\delta e^{-i\alpha}\}$ (see Figure 7.4).

An analogue construction at -1 leads to Figure 7.5, where the contour within the circle at 1 is given as the inverse f_n image of $\gamma_{\sigma(n,\alpha,\delta)}$ and, correspondingly, the contour within the circle at -1 is the inverse $\tilde{f}_n$ image of $\tilde{\gamma}_{\tilde{\sigma}(n,\alpha,\delta)}$. Define $\Sigma_S \equiv \bigcup_{i=1}^{5}\Sigma_i$, where $\Sigma_1 \equiv \Sigma_1' \cup \Sigma_1'' \cup \Sigma_1'''$, and so on. Note that the circles Σ_6 and Σ_7 are *not* contained in Σ_S. Furthermore, define $\Sigma_R := \Sigma_S \cup \Sigma_6 \cup \Sigma_7$ and for $z \in \mathbb{C}\setminus\Sigma_R$, let

$$S_{\text{par}}(z) := \begin{cases} P_n(z) & \text{if } |z-1| < \delta \text{ or } |z+1| < \delta, \\ N(z) & \text{otherwise}. \end{cases} \tag{7.46}$$

The following proposition can be easily verified:

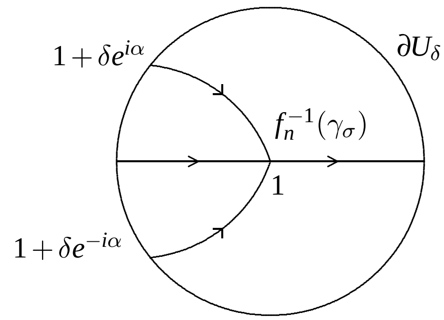

FIGURE 7.4. The contour Σ_S near 1 depending on α, δ, and n.

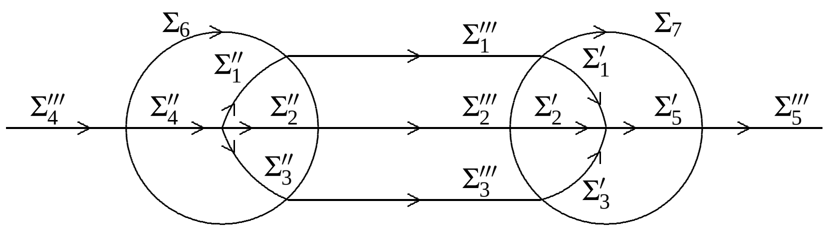

FIGURE 7.5. The contour Σ_S together with Σ_6 and Σ_7.

PROPOSITION 7.6 *Let $n \geq n_2$, $0 < \delta \leq \delta_0$, $\alpha \in (\frac{2\pi}{3}, \frac{5\pi}{6})$, and $S : \mathbb{C} \setminus \Sigma_S \to \mathbb{C}^{2\times 2}$ be a matrix-valued function. Define*

$$R(z) \equiv S(z) S_{\mathrm{par}}^{-1}(z) \quad \textit{for } z \in \mathbb{C} \setminus \Sigma_R \,. \tag{7.47}$$

S is a solution of the Riemann-Hilbert problem (6.10)–(6.12) *if and only if R solves* (7.48)–(7.50) *below*:

$$R : \mathbb{C} \setminus \Sigma_R \to \mathbb{C}^{2\times 2} \textit{ is analytic}, \tag{7.48}$$

$$R_+(s) = R_-(s) \begin{cases} N(s) v_i(s) N^{-1}(s) & \textit{for } s \in \Sigma_i''', i = 1,3,4,5 \\ P_n(s) N^{-1}(s) & \textit{for } s \in \Sigma_6 \cup \Sigma_7 \\ I & \textit{inside the circles and on } \Sigma_2''', \end{cases} \tag{7.49}$$

$$R(z) = I + O\left(\frac{1}{|z|}\right) \quad \textit{as } |z| \to \infty \,. \tag{7.50}$$

Note that any solution of (7.48)–(7.50) will have jumps only on the smaller contour $\hat{\Sigma}_R = \Sigma''' \cup \Sigma_6 \cup \Sigma_7$, where $\Sigma''' = \Sigma_1''' \cup \Sigma_3''' \cup \Sigma_4''' \cup \Sigma_5'''$ (see Figure 7.6).

We now give estimates for the jump matrix v_R on $\hat{\Sigma}_R$. Set

$$\Delta_R := v_R - I \,. \tag{7.51}$$

PROPOSITION 7.7 *Let K be a compact subset of $(0, \delta_0]$. There exist positive constants $c_1, C_1 > 0$ such that for all $n \geq n_2$, $\delta \in K$, and $\alpha \in (\frac{2\pi}{3}, \frac{5\pi}{6})$, the following*

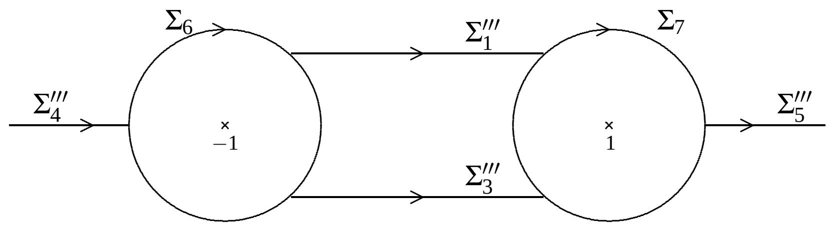

FIGURE 7.6. The contour $\hat{\Sigma}_R$.

estimates hold:

$$\|\Delta_R\|_{L^1(\Sigma_6\cup\Sigma_7)} + \|\Delta_R\|_{L^2(\Sigma_6\cup\Sigma_7)} + \|\Delta_R\|_{L^\infty(\Sigma_6\cup\Sigma_7)} \le \frac{C_1}{n}, \tag{7.52}$$

$$\|\Delta_R\|_{L^1(\Sigma''')} + \|\Delta_R\|_{L^2(\Sigma''')} + \|\Delta_R\|_{L^\infty(\Sigma''')} \le C_1 e^{-nc_1}. \tag{7.53}$$

Furthermore, Δ_R has an analytic continuation to, say, a $\frac{\delta}{100}$ neighborhood of Σ''' with

$$|\Delta_R(z)| \le e^{-nc_1|z|} \quad \textit{for } \operatorname{dist}(z,\Sigma''') \le \frac{\delta}{100}. \tag{7.54}$$

PROOF: The estimate (7.52) is immediate from (7.31) and (7.45). In order to prove (7.53), note first that N and N^{-1} are uniformly bounded outside the circles U_δ and $\tilde{U}_\delta$ for $\delta \in K$. The claim then follows from (5.41), (5.42), (6.5), and (6.7) on Σ_1''' and Σ_3''' and from (6.8), (6.9), and Proposition 5.4(iv) on Σ_4''' and Σ_5'''. Using the additional information on the polynomials h_n provided by (5.25) and (5.26), the pointwise estimate (7.54) is immediate (by decreasing δ_0 if necessary). □

Proposition 7.7 implies, in particular, that

$$C_{\Delta_R}(f) \equiv C_-(f\Delta_R) \quad \text{for } f \in L_2\big(\hat{\Sigma}_R;\mathbb{C}^{2\times 2}\big) \tag{7.55}$$

defines a bounded linear operator from $L_2(\hat{\Sigma}_R;\mathbb{C}^{2\times 2})$ into itself with operator norm $\|C_{\Delta_R}\| = O\left(\frac{1}{n}\right)$. Thus $\mathbf{1} - C_{\Delta_R}$ can be inverted by a Neumann series for n sufficiently large. We define

$$\mu_R := (\mathbf{1} - C_{\Delta_R})^{-1}(C_-\Delta_R) \in L_2(\hat{\Sigma}_R). \tag{7.56}$$

We are now able to present an explicit formula for R.

THEOREM 7.8 *Let $\delta \in (0,\delta_0]$ and $\alpha \in (\frac{2\pi}{3},\frac{5\pi}{6})$. For n sufficiently large, the unique solution of the Riemann-Hilbert problem* (7.48)–(7.50) *is given by*

$$R = I + C(\Delta_R + \mu_R\Delta_R). \tag{7.57}$$

PROOF: Let R denote the unique solution of (7.48)–(7.50). We first prove that in $\mathbb{C}\setminus\hat{\Sigma}_R$

$$R = I + C(R_-\Delta_R), \tag{7.58}$$

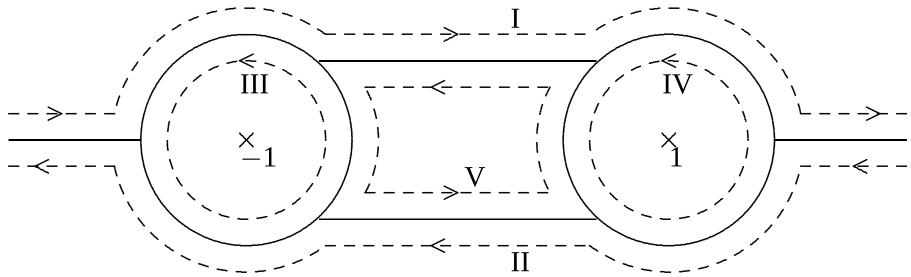

FIGURE 7.7

which is equivalent to the Plemelj relation

$$R - I = C((R_+ - I) - (R_- - I)) \tag{7.59}$$

(cf. (7.51)). Although this formula is well-known for smooth contours without self-intersections, we shall verify it directly for the contour $\hat{\Sigma}_R$. It is easy to see that for $z \in \mathbb{C} \setminus \hat{\Sigma}_R$ the term $C((R_+ - I) - (R_- - I))(z)$ is equivalent to the sum of five integrals, each of which is the integral of

$$\frac{1}{2\pi i} \frac{R(s) - I}{s - z}$$

along the positively oriented boundary of one of the five components of $\mathbb{C} \setminus \hat{\Sigma}_R$ (see Figure 7.7). Together with (7.50), relation (7.59) and consequently (7.58) follow from Cauchy's theorem.

From (7.58) we conclude that

$$R_- - I = C_{\Delta_R}(R_- - I) + C_-(\Delta_R)\,. \tag{7.60}$$

Since $\mathbf{1} - C_{\Delta_R}$ is invertible on $L_2(\hat{\Sigma}_R)$, we conclude

$$R_- = I + (\mathbf{1} - C_{\Delta_R})^{-1}(C_- \Delta_R) = I + \mu_R\,. \tag{7.61}$$

By (7.58) this proves the claim. ☐

We now deduce a useful corollary of Theorem 7.8, which states the uniform boundedness of R.

COROLLARY 7.9 *For every compact subset $K \subset (0, \delta_0]$, there exist $n_0 \geq n_2$ and $C > 0$ such that the unique solution R of* (7.48)–(7.50) *satisfies*

$$|R(z)| \leq C \quad \textit{for all } z \in \mathbb{C} \setminus \hat{\Sigma}_R\,,\ \alpha \in \left(\frac{2\pi}{3}, \frac{5\pi}{6}\right),\ \delta \in K\,,\ n \geq n_0\,. \tag{7.62}$$

PROOF: Proposition 7.7 implies that

$$\|\Delta_R + \mu_R \Delta_R\|_{L^2(\hat{\Sigma}_R)} = O\left(\frac{1}{n}\right) \tag{7.63}$$

uniformly in α and δ. Clearly, by (7.57) there exists a uniform bound on $|R|$ for all $z \in \mathbb{C}$ satisfying, say, $\operatorname{dist}(z, \hat{\Sigma}_R) \geq \delta/100$. In order to obtain a uniform bound

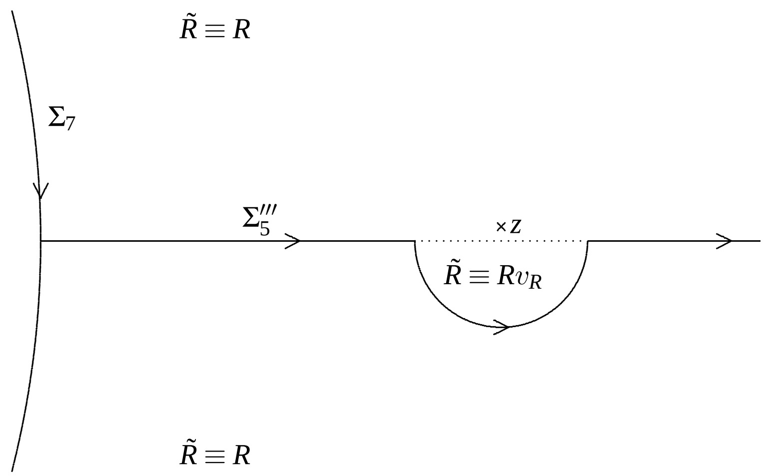

FIGURE 7.8. Contour deformation.

for all $z \in \mathbb{C} \setminus \hat{\Sigma}_R$, we use a contour deformation argument. Suppose, for example, that z is close to Σ_5'''. Define $\tilde{R}$ as shown in Figure 7.8, where the radius of the half-circle is chosen to be $\delta/100$. Then $\tilde{R}$ satisfies the same Riemann-Hilbert problem (7.48)–(7.50) as R except that the dotted contour is to be replaced by the half-circle and the jump at the half-circle is given by the analytic continuation of v_R, which exists by Proposition 7.7. By (7.54) the estimates (7.52) and (7.53) also hold for the deformed Riemann-Hilbert problem, and the analogue of Theorem 7.8 holds as well. Thus the boundedness of $\tilde{R}(z)$ follows now as above, since $\operatorname{dist}(z, \hat{\Sigma}_{\tilde{R}}) > \delta/100$. This completes the proof as $R(z) = \tilde{R}(z)$ by definition. Finally, note that for z close to $\Sigma_6 \cup \Sigma_7$, the contour deformation can also be achieved by varying the parameters δ and α. $\square$

The following theorem demonstrates how one can extract an explicitly computable asymptotic expansion for R from (7.57):

THEOREM 7.10 *For every compact subset $K \subset (0, \delta_0]$, there exists $n_0 \geq n_2$ such that for all $n \geq n_0$, $\delta \in K$, and $\alpha \in (\frac{2\pi}{3}, \frac{5\pi}{6})$ the unique solution R of* (7.48)–(7.50) *has an asymptotic expansion in $n^{-1/2m}$ of the form*

$$R(z) \sim I + \frac{1}{n} \sum_{k=0}^{\infty} r_k(z) n^{-\frac{k}{2m}}. \tag{7.64}$$

Here $r_k(z)$ are bounded functions that are analytic in $\mathbb{C} \setminus (\Sigma_6 \cup \Sigma_7)$ and can be computed explicitly. The expansion is uniform in z and in the parameters α and δ.

PROOF: Fix $\alpha \in (\frac{2\pi}{3}, \frac{5\pi}{6})$, $\delta \in K$, and let $l \in \mathbb{N}$. We define

$$\Delta_l(s) := \begin{cases} \sum_{j=1}^{l} \beta_{j,n}(s) \frac{1}{n^j} & \text{for } s \in \Sigma_6 \cup \Sigma_7 \\ 0 & \text{for } s \in \hat{\Sigma}_R \setminus (\Sigma_6 \cup \Sigma_7), \end{cases} \tag{7.65}$$

where (cf. (7.29) and (7.43))

(7.66)

$$\beta_{2k,n}(z) := \begin{cases} \dfrac{1}{2}\begin{pmatrix} s_{2k}+t_{2k} & i(s_{2k}-t_{2k}) \\ -i(s_{2k}-t_{2k}) & s_{2k}+t_{2k} \end{pmatrix} \varphi_n(z)^{-2k} & \text{for } z \in U_{\delta_0} \setminus \{1\} \\ \dfrac{1}{2}\begin{pmatrix} s_{2k}+t_{2k} & -i(s_{2k}-t_{2k}) \\ i(s_{2k}-t_{2k}) & s_{2k}+t_{2k} \end{pmatrix} \tilde{\varphi}_n(z)^{-2k} & \text{for } z \in \tilde{U}_{\delta_0} \setminus \{-1\}, \end{cases}$$

and

(7.67)

$$\beta_{2k+1,n}(z) := \begin{cases} \dfrac{1}{2a(z)^2\varphi_n(z)^{2k+1}} \begin{pmatrix} -(s_{2k+1}+t_{2k+1}a(z)^4) & i(s_{2k+1}-t_{2k+1}a(z)^4) \\ i(s_{2k+1}-t_{2k+1}a(z)^4) & s_{2k+1}+t_{2k+1}a(z)^4 \end{pmatrix} \\ \quad \text{for } z \in U_{\delta_0} \setminus \{1\}, \\ \dfrac{a(z)^2}{2\tilde{\varphi}_n(z)^{2k+1}} \begin{pmatrix} -(s_{2k+1}+t_{2k+1}a(z)^{-4}) & -i(s_{2k+1}-t_{2k+1}a(z)^{-4}) \\ -i(s_{2k+1}-t_{2k+1}a(z)^{-4}) & s_{2k+1}+t_{2k+1}a(z)^{-4} \end{pmatrix} \\ \quad \text{for } z \in \tilde{U}_{\delta_0} \setminus \{-1\}. \end{cases}$$

Note that $\beta_{j,n}$ is meromorphic in $U_{\delta_0} \cup \tilde{U}_{\delta_0}$ with poles of order $(3j+1)/2$ at ± 1 (see (6.17) and (7.18)). Following (7.55) we define

$$C_{\Delta_l}(f) \equiv C_-(f\Delta_l) \quad \text{for } f \in L_2(\Sigma_6 \cup \Sigma_7). \tag{7.68}$$

Finally, set

$$\mu_l := \sum_{j=0}^{l} (C_{\Delta_l})^j (C_-\Delta_l), \tag{7.69}$$

$$R_l := I + C(\Delta_l + \mu_l \Delta_l). \tag{7.70}$$

We will show below that

$$R(z) = R_l(z) + O\left(\frac{1}{n^{l+1}}\right), \tag{7.71}$$

where the error term is uniform in α, δ, and z. Note that the Cauchy operators used for the definition of μ_l and R_l correspond to a contour consisting of the two circles Σ_6 and Σ_7 only. Furthermore, μ_l has an analytic continuation inside the circles and Δ_l has a meromorphic continuation inside the circles with poles of order $(3l+1)/2$ at ± 1. Therefore one can compute R_l explicitly using the calculus of residues. The asymptotic expansion (7.64) follows then immediately from the expansion for φ_n in powers of $n^{-1/2m}$ (cf. (7.66), (7.67), (7.4), (7.32), Proposition 5.4(v), (5.34), and (5.16)).

It remains to prove (7.71). We first consider the case that the distance between z and the circles $\Sigma_6 \cup \Sigma_7$ is bigger than $\delta/100$. Denote by $\tilde{C}$ the Cauchy operator associated with the contour $\Sigma_6 \cup \Sigma_7$. We estimate

$$\begin{aligned}|R(z) - R_l(z)| \leq{}& |C(\Delta_R + \mu_R \Delta_R)(z) - \tilde{C}(\Delta_R + \mu_R \Delta_R)(z)| \\ &+ |\tilde{C}\left[(\Delta_R + \mu_R \Delta_R) - (\Delta_l + \mu_l \Delta_l)\right](z)| \,. \end{aligned} \tag{7.72}$$

Using the assumption that $\operatorname{dist}(z, \Sigma_6 \cup \Sigma_7) > \delta/100$, one can prove (7.71) by showing that for each $l \in \mathbb{N}$ there exist a constant $C_l > 0$ such that for all $\delta \in K$, $\alpha \in (2\pi/3, 5\pi/6)$, and $z \in \mathbb{C} \setminus \hat{\Sigma}_R$ with $\operatorname{dist}(z, \Sigma_6 \cup \Sigma_7) > \delta/100$, the following two inequalities hold:

$$|C(\Delta_R + \mu_R \Delta_R)(z) - \tilde{C}(\Delta_R + \mu_R \Delta_R)(z)| \leq \frac{C_l}{n^{l+1}}\,, \tag{7.73}$$

$$\|(\Delta_R + \mu_R \Delta_R) - (\Delta_l + \mu_l \Delta_l)\|_{L^2(\Sigma_6 \cup \Sigma_7)} \leq \frac{C_l}{n^{l+1}}\,. \tag{7.74}$$

We first prove (7.73). Recall the definition of $\Sigma''' \equiv \Sigma_1''' \cup \Sigma_3''' \cup \Sigma_4''' \cup \Sigma_5'''$ (see (7.50)). By (7.61) and (7.51), inequality (7.73) is equivalent to

$$\left| \frac{1}{2\pi i} \int_{\Sigma'''} \frac{R_+(s) - R_-(s)}{s - z}\, ds \right| \leq \frac{C_l}{n^{l+1}}\,. \tag{7.75}$$

Note that $R_+ - R_-$ has an analytic extension to a $\delta/100$-neighborhood of Σ''' given by $R v_R^{-1} \Delta_R$ on the $+$ side and by $R\Delta_R$ on the $-$ side of Σ'''. Since $\operatorname{dist}(z, \Sigma_6 \cup \Sigma_7) > \delta/100$, we can always deform the contour Σ''' of the integral (7.75) to a contour Σ_z''' lying in a $\delta/100$-neighborhood of Σ''' such that $\operatorname{dist}(z, \Sigma_z''') > \delta/100$ and

$$\int_{\Sigma'''} \frac{R_+(s) - R_-(s)}{s - z}\, ds = \int_{\Sigma_z'''} \frac{R_+(s) - R_-(s)}{s - z}\, ds\,. \tag{7.76}$$

Because R is uniformly bounded (cf. Corollary 7.9), inequality (7.75) now follows from (7.54). (In fact, the bound in (7.75) is exponentially small in n.)

Now we verify (7.74). The definition of Δ_l together with Lemma 7.4, (7.44), and (7.53) yield that

$$\|\Delta_R - \Delta_l\|_{L^1(\hat{\Sigma}_R)} + \|\Delta_R - \Delta_l\|_{L^2(\hat{\Sigma}_R)} + \|\Delta_R - \Delta_l\|_{L^\infty(\hat{\Sigma}_R)} = O\left(\frac{1}{n^{l+1}}\right). \tag{7.77}$$

Writing $\mu_R \Delta_R - \mu_l \Delta_l = (\mu_R - \mu_l)\Delta_R + \mu_l(\Delta_R - \Delta_l)$ and using (7.52), (7.65), and (7.69), it suffices to show that $\|\mu_R - \mu_l\|_{L^2(\Sigma_6 \cup \Sigma_7)} = O(1/n^{l+1})$. By the definition of μ_R and μ_l (see (7.56) and (7.69)), this follows from

$$\|C_- \Delta_R - C_- \Delta_l\|_{L^2(\hat{\Sigma}_R)} = O\left(\frac{1}{n^{l+1}}\right) \tag{7.78}$$

and from the following estimates of operator norms:

$$\|(\mathbf{1} - C_{\Delta_R})^{-1} - (\mathbf{1} - C_{\Delta_l})^{-1}\| = O\left(\frac{1}{n^{l+1}}\right), \tag{7.79}$$

$$\|(\mathbf{1} - C_{\Delta_l})^{-1} - \sum_{j=0}^{l} (C_{\Delta_l})^j\| = O\left(\frac{1}{n^{l+1}}\right). \tag{7.80}$$

Estimates (7.78) and (7.79) are immediate consequences of (7.77), and (7.80) follows from $\|C_{\Delta_l}\| = O(\frac{1}{n})$. This completes the proof of (7.71) in the case that $\operatorname{dist}(z, \Sigma_6 \cup \Sigma_7) > \delta/100$.

In order to prove (7.71) for arbitrary $z \in \mathbb{C} \setminus \hat{\Sigma}_R$, we note as in (the final sentence of) the proof of Corollary 7.9 that one can always achieve z to be bounded away from $\Sigma_6 \cup \Sigma_7$ by at least $\delta/100$ by modifying the parameters δ and α without changing the value of $R(z)$. Using Cauchy's theorem, it is not then difficult to see that the value of $R_l(z)$ is also unchanged under the modification of parameters. $\square$

8 Proof of the Main Theorems

8.1 Proof of Theorem 2.1

Recall from Theorem 3.1 the formulae for the leading coefficients γ_n (see (3.11)) and the recurrence coefficients a_n and b_n (see (3.12)). The solution Y of the Riemann-Hilbert problem (3.6)–(3.8) with $w(s) = e^{-Q(s)}$ is connected to the solution R of the Riemann-Hilbert problem (7.48)–(7.50) by a series of transformations

$$Y \to U \to T \to S \to R \tag{8.1}$$

(see Propositions 5.1 and 7.6 and Lemmas 5.5 and 6.1). We will now unfold these relations.

For large $|z|$

$$U(z) = e^{n\frac{l_n}{2}\sigma_3} R(z) N(z) e^{n(g_n(z) - \frac{l_n}{2})\sigma_3}. \tag{8.2}$$

Clearly, U has an expansion in powers of $\frac{1}{z}$ of type (3.10), and $N(z)$ and $e^{ng_n(z)\sigma_3}$ have asymptotic expansions in powers of $\frac{1}{z}$ by explicit calculations. It follows from (8.2) that $R(z)$ also has such an expansion. Using (7.57), (6.16), (6.17), (5.33), and (5.46), we derive

$$R(z) = I + \frac{R_1}{z} + \frac{R_2}{z^2} + O\left(\frac{1}{|z|^3}\right), \tag{8.3}$$

where

$$R_1 = -\frac{1}{2\pi i} \int_{\hat{\Sigma}_R} \Delta_R(y) + \mu_R(y) \Delta_R(y) dy, \tag{8.4}$$

$$R_2 = -\frac{1}{2\pi i} \int_{\hat{\Sigma}_R} y(\Delta_R(y) + \mu_R(y) \Delta_R(y)) dy, \tag{8.5}$$

and

$$N(z) = I + \frac{N_1}{z} + \frac{N_2}{z^2} + O\left(\frac{1}{|z|^3}\right), \tag{8.6}$$

where

$$N_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad N_2 = \frac{1}{8} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{8.7}$$

and

$$e^{ng_n(z)\sigma_3}\begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} = I + \frac{G_1}{z} + \frac{G_2}{z^2} + O\left(\frac{1}{|z|^3}\right), \tag{8.8}$$

where

$$G_1 = \begin{pmatrix} -n\int_{-1}^1 t\psi_n(t)dt & 0 \\ 0 & n\int_{-1}^1 t\psi_n(t)dt \end{pmatrix}, \qquad G_2 = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}. \tag{8.9}$$

From Proposition 5.1, we learn

$$\begin{aligned} Y_1 = c_n &\begin{pmatrix} c_n^n & 0 \\ 0 & c_n^{-n} \end{pmatrix} e^{n\frac{l_n}{2}\sigma_3}(R_1 + N_1 + G_1)e^{-n\frac{l_n}{2}\sigma_3}\begin{pmatrix} c_n^{-n} & 0 \\ 0 & c_n^n \end{pmatrix} \\ &+ \begin{pmatrix} -nd_n & 0 \\ 0 & nd_n \end{pmatrix}, \end{aligned} \tag{8.10}$$

$$\begin{aligned} Y_2 = c_n^2 &\begin{pmatrix} c_n^n & 0 \\ 0 & c_n^{-n} \end{pmatrix} e^{n\frac{l_n}{2}\sigma_3} Z e^{-n\frac{l_n}{2}\sigma_3}\begin{pmatrix} c_n^{-n} & 0 \\ 0 & c_n^n \end{pmatrix} \\ &+ \left(Y_1 - \begin{pmatrix} -nd_n & 0 \\ 0 & nd_n \end{pmatrix}\right)\begin{pmatrix} (1-n)d_n & 0 \\ 0 & (1+n)d_n \end{pmatrix} \\ &+ \begin{pmatrix} \frac{n(n-1)}{2}d_n^2 & 0 \\ 0 & \frac{n(n+1)}{2}d_n^2 \end{pmatrix}, \end{aligned} \tag{8.11}$$

$$Z = R_2 + N_2 + G_2 + R_1N_1 + R_1G_1 + N_1G_1. \tag{8.12}$$

We are interested in the following entries of Y_1 and Y_2:

$$(Y_1)_{11} = c_n\left((R_1)_{11} - n\int_{-1}^1 t\psi_n(t)dt\right) - n\,d_n, \tag{8.13}$$

$$(Y_1)_{12} = c_n^{2n+1}e^{nl_n}\left((R_1)_{12} + \frac{i}{2}\right), \tag{8.14}$$

$$(Y_1)_{21} = c_n^{1-2n}e^{-nl_n}\left((R_1)_{21} - \frac{i}{2}\right), \tag{8.15}$$

$$\begin{aligned} (Y_2)_{12} = c_n^{2n+2}e^{nl_n}&\left[(R_2)_{12} + \frac{i}{2}(R_1)_{11} + \left(n\int_{-1}^1 t\psi_n(t)dt\right)\left((R_1)_{12} + \frac{i}{2}\right)\right] \\ &+ (Y_1)_{12}d_n(1+n). \end{aligned} \tag{8.16}$$

We apply formulae (3.11) and (3.12) from Theorem 3.1 and obtain

$$\gamma_n = \left(c_n^{2n+1}e^{nl_n}\pi\,(1 - 2i(R_1)_{12})\right)^{-1/2}, \tag{8.17}$$

$$b_{n-1} = c_n\left(\frac{1}{4} + \frac{i}{2}\left((R_1)_{21} - (R_1)_{12}\right) + (R_1)_{12}(R_1)_{21}\right)^{1/2}, \tag{8.18}$$

$$a_n = d_n + \frac{c_n}{1 - 2i(R_1)_{12}}\left[2(R_1)_{11} - 2i\left((R_1)_{11}(R_1)_{12} + (R_2)_{12}\right)\right]. \tag{8.19}$$

We conclude the proof of Theorem 2.1 by determining R_1 and $(R_2)_{12}$ up to order $O(n^{-2})$. Note that by Theorem 7.10 one can compute all coefficients of the asymptotic expansions of R_1 and R_2 explicitly. Recall the definition of Δ_l in (7.65). From (7.77) we learn that $\|\Delta_R - \Delta_{l=1}\|_{L^1(\hat{\Sigma}_R)} = O(n^{-2})$. Moreover, Proposition 7.7 implies that $\|\Delta_R\|_{L^1(\hat{\Sigma}_R)} = O(n^{-1})$, and it is a consequence of (7.61) and Theorem 7.10 that $\|\mu_R\|_{L^\infty(\hat{\Sigma}_R)} = \|R_- - I\|_{L^\infty(\hat{\Sigma}_R)} = O(n^{-1})$. Thus

$$\|(\Delta_R + \mu_R\Delta_R) - \Delta_1\|_{L^1(\hat{\Sigma}_R)} = O\left(\frac{1}{n^2}\right) \tag{8.20}$$

Using estimate (7.54), one can prove in addition that

$$\int_{\hat{\Sigma}_R} |y||(\Delta_R(y) + \mu_R(y)\Delta_R(y)) - \Delta_1(y)|dy = O\left(\frac{1}{n^2}\right). \tag{8.21}$$

From (8.4) and (8.5), it then follows that

$$R_1 = -\frac{1}{2\pi i}\int_{\Sigma_6\cup\Sigma_7} \Delta_1(y)dy + O\left(\frac{1}{n^2}\right), \tag{8.22}$$

$$R_2 = -\frac{1}{2\pi i}\int_{\Sigma_6\cup\Sigma_7} y\Delta_1(y)dy + O\left(\frac{1}{n^2}\right). \tag{8.23}$$

These formulae can be evaluated using Cauchy's theorem, and we obtain

$$\begin{aligned} R_1 = \frac{1}{48n}\Bigg\{ &\frac{1}{h_n(1)}\begin{pmatrix} 3 & 4i \\ 4i & -3 \end{pmatrix} + \frac{h_n'(1)}{h_n(1)^2}\begin{pmatrix} 3 & -3i \\ -3i & -3 \end{pmatrix} \\ &+ \frac{1}{h_n(-1)}\begin{pmatrix} -3 & 4i \\ 4i & 3 \end{pmatrix} + \frac{h_n'(-1)}{h_n(-1)^2}\begin{pmatrix} 3 & 3i \\ 3i & -3 \end{pmatrix}\Bigg\} + O\left(\frac{1}{n^2}\right), \end{aligned} \tag{8.24}$$

$$(R_2)_{12} = \frac{i}{16n}\left(\frac{3h_n(1) - h_n'(1)}{h_n(1)^2} - \frac{3h_n(-1) + h_n'(-1)}{h_n(-1)^2}\right) + O\left(\frac{1}{n^2}\right). \tag{8.25}$$

8.2 Proof of Theorem 2.2

Recall that p_n denotes the normalized n^{th} orthogonal polynomial with respect to the measure $e^{-Q(x)}dx$. We will now derive the asymptotics for p_n in the variables $c_n z + d_n$. From (3.9), (4.2), and (8.17) we learn that

$$p_n(c_n z + d_n) = \gamma_n c_n^n U_{11}(z) = \left[\pi c_n e^{nl_n}\left(1 - 2i(R_1)_{12}\right)\right]^{-\frac{1}{2}} U_{11}(z). \tag{8.26}$$

We now determine $U_{11}(z)$ in the different regions by relating R and U as in the proof of Theorem 2.1 above (cf. (8.1)).

Region A_δ

In this region we obtain formula (8.2) by Proposition 7.6 and Lemmas 6.1 and 5.5, i.e.,

$$U(z) = e^{n\frac{l_n}{2}\sigma_3}R(z)N(z)e^{n(g_n(z)-\frac{l_n}{2})\sigma_3}. \tag{8.27}$$

Therefore

$$
\begin{aligned}
U_{11}(z) &= e^{ng_n(z)} N_{11}(z) \left[1 + (R_{11}(z) - 1) + R_{12}(z) \frac{N_{21}(z)}{N_{11}(z)}\right] \\
&= e^{ng_n(z)} N_{11}(z) \left[1 + O\left(\frac{1}{n}\right)\right].
\end{aligned} \tag{8.28}
$$

The last line can be derived from Lemma 7.10 together with the fact that $\frac{a-a^{-1}}{a+a^{-1}}$ is uniformly bounded on A_δ (see (6.17) for a definition of a). Furthermore, note that

$$
g_n(z) = \frac{1}{2}(V_n(z) + l_n + \xi_n(z)) \quad \text{for all } z \in \overline{\mathbb{C}}_\pm . \tag{8.29}
$$

This relation follows for $z \in [1, \infty)$ from (5.38) and can be extended to all of $\overline{\mathbb{C}}_\pm$ by analyticity. Definitions (2.1) and (2.2) imply

$$
nV_n(z) = Q(c_n z + d_n) \tag{8.30}
$$

and hence

$$
U_{11}(z) = e^{\frac{1}{2}Q(c_n z + d_n)} e^{\frac{n l_n}{2}} e^{\frac{n \xi_n(z)}{2}} N_{11}(z) \left[1 + O\left(\frac{1}{n}\right)\right]. \tag{8.31}
$$

Relation (2.17) now follows readily from (8.26), (5.34), (6.16), and (6.17).

Region B_δ

Here U is represented by

$$
U(z) = e^{n \frac{l_n}{2} \sigma_3} R(z) N(z) \begin{pmatrix} 1 & 0 \\ e^{-n\xi_n(z)} & 1 \end{pmatrix} e^{n(g_n(z) - \frac{l_n}{2})\sigma_3}. \tag{8.32}
$$

Let arcsin be defined as an analytic function on $\mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$ as described in remark 2 after the statement of Theorem 2.2. Then the following relations hold:

$$
a(z) + \frac{1}{a(z)} = \frac{\sqrt{2}}{(1-z)^{\frac{1}{4}}(1+z)^{\frac{1}{4}}} e^{-\frac{i}{2}\arcsin z} \quad \text{for } z \in \overline{\mathbb{C}}_+ , \tag{8.33}
$$

$$
-ia(z) + \frac{i}{a(z)} = \frac{\sqrt{2}}{(1-z)^{\frac{1}{4}}(1+z)^{\frac{1}{4}}} e^{\frac{i}{2}\arcsin z} \quad \text{for } z \in \overline{\mathbb{C}}_+ . \tag{8.34}
$$

Employing equation (8.29), we find for $z \in B_\delta$

$$
\begin{aligned}
&N(z) \begin{pmatrix} 1 \\ e^{-n\xi_n(z)} \end{pmatrix} e^{ng_n(z)\sigma_3} = \\
&e^{\frac{n}{2}(V_n(z) + l_n)} \frac{\sqrt{2}}{(1-z)^{\frac{1}{4}}(1+z)^{\frac{1}{4}}} \begin{pmatrix} \cos\left(\frac{in}{2}\xi_n(z) + \frac{1}{2}\arcsin z\right) \\ i\sin\left(\frac{in}{2}\xi_n(z) - \frac{1}{2}\arcsin z\right) \end{pmatrix},
\end{aligned} \tag{8.35}
$$

and therefore

$$U_{11}(z)=e^{\frac{n}{2}(V_n(z)+l_n)}\frac{\sqrt{2}}{(1-z)^{\frac14}(1+z)^{\frac14}} \times\left[R_{11}(z)\cos\left(\frac{in}{2}\xi_n(z)+\frac12\arcsin z\right)+iR_{12}(z)\sin\left(\frac{in}{2}\xi_n(z)-\frac12\arcsin z\right)\right]. \tag{8.36}$$

Theorem 7.10 together with (8.26) and (8.30) yield (2.18).

Regions $C_{1\delta}$ and $C_{2\delta}$

By definition of the parametrix P_n near $z=1$, the set $C_{1,\delta}\cup C_{2,\delta}$ consists of two regions I′ and II′ (which do not necessarily agree with regions $C_{1,\delta}$ and $C_{2,\delta}$) such that

$$U(z)=e^{n\frac{l_n}{2}\sigma_3}R(z)P_n(z)e^{n(g_n(z)-\frac{l_n}{2})\sigma_3}, \qquad z\in \mathrm{I}', \tag{8.37}$$

$$U(z)=e^{n\frac{l_n}{2}\sigma_3}R(z)P_n(z)\begin{pmatrix}1&0\\ e^{-n\xi_n(z)}&1\end{pmatrix}e^{n(g_n(z)-\frac{l_n}{2})\sigma_3}, \quad z\in \mathrm{II}'. \tag{8.38}$$

Using the definition for P in (7.24) together with (7.4), (7.9), and (8.29), we obtain the following formula for *all* $z\in C_{1,\delta}\cup C_{2,\delta}$:

$$\begin{pmatrix}U_{11}(z)\\ U_{21}(z)\end{pmatrix}=e^{n\frac{l_n}{2}\sigma_3}R(z)E_n(z)\begin{pmatrix}\mathrm{Ai}(f_n(z))\\ \mathrm{Ai}'(f_n(z))\end{pmatrix}e^{-\frac{\pi i}{6}+\frac{nV_n(z)}{2}}, \tag{8.39}$$

where E_n is as defined in (7.23). We are led to

$$U_{11}(z)=e^{\frac{n}{2}(V_n(z)+l_n)}\sqrt{\pi}\left\{R_{11}(z)\left[\frac{f_n(z)^{1/4}\mathrm{Ai}(f_n(z))}{a(z)}-\frac{a(z)\,\mathrm{Ai}'(f_n(z))}{f_n(z)^{1/4}}\right] -iR_{12}(z)\left[\frac{f_n(z)^{1/4}\mathrm{Ai}(f_n(z))}{a(z)}+\frac{a(z)\,\mathrm{Ai}'(f_n(z))}{f_n(z)^{1/4}}\right]\right\}. \tag{8.40}$$

This immediately implies the estimate in the region $C_{1,\delta}$ (cf. Theorem 7.10, (8.26), (8.30)). For $z\in C_{2,\delta}$ we can do a little better. First note that by Proposition 7.3(iii) that $f_n(z)$ lies in $S:=\{y\in\mathbb{C}:0\le \arg y\le \frac{4\pi}{5}\}$ for $z\in C_{2,\delta}$. Furthermore,

$$\sup_{y\in S}\left|\frac{y^{-1/4}\mathrm{Ai}'(y)}{y^{1/4}\mathrm{Ai}(y)}\right|<\infty. \tag{8.41}$$

This follows from the asymptotic formulae for Ai and Ai′ (cf. [1, 10.4.59, 10.4.61]) and from the fact that the Airy function has no zeros in S. Therefore by choosing δ_0 sufficiently small we can achieve

$$\sup_{z\in C_{2,\delta}}\left|\frac{a(z)\,f_n(z)^{-1/4}\mathrm{Ai}'(f_n(z))}{a(z)^{-1}f_n(z)^{1/4}\mathrm{Ai}(f_n(z))}\right|\le\frac12. \tag{8.42}$$

This proves (2.20).

Regions $D_{1,\delta}$ and $D_{2,\delta}$

The proof is the same as for regions $C_{1,\delta}$ and $C_{2,\delta}$.

We refer the reader to the proof of Theorem 7.10 for obtaining an explicit asymptotic expansion for the error terms.

Remark. Varying the parameter δ in the construction, one sees immediately that the asymptotic formulae for the different regions A, B, C, and D match at the boundaries. Nevertheless, we will now verify the matching of the leading-order terms from the asymptotic formulae in Theorem 2.2.

Regions A_δ and B_δ

From (8.33) it follows that

$$\left(\frac{(z-1)^{1/4}}{(z+1)^{1/4}}+\frac{(z+1)^{1/4}}{(z-1)^{1/4}}\right)\exp\left(-n\pi i\int_1^z\psi_n(y)dy\right)= \frac{\sqrt{2}}{(1-z)^{1/4}(1+z)^{1/4}}\exp\left(-n\pi i\int_1^z\psi_n(y)dy-\frac{i}{2}\arcsin z\right). \tag{8.43}$$

Furthermore, note that $\mathrm{Re}(-\pi i\int_1^z\psi_n(y)dy)=\mathrm{Re}(\xi_n(z)/2)>0$ for all z on the boundary between the regions A_δ and B_δ (cf. (5.34) and (5.41)). Thus, for some $c>0$,

$$\exp\left(-n\pi i\int_1^z\psi_n(y)dy-\frac{i}{2}\arcsin z\right)= 2\cos\left(n\pi\int_1^z\psi_n(y)dy+\frac{1}{2}\arcsin z\right)\left(1+O(e^{-cn})\right). \tag{8.44}$$

Regions A_δ and $C_{2,\delta}$

Proposition 7.3 implies that $0\le\arg(f_n(z))\le\frac{5\pi}{6}$. Therefore we can apply the asymptotic formulae for the Airy function and its derivative as shown in (7.26) and (7.27). Finally, we use the relations

$$\frac{2}{3}f_n(z)^{\frac{3}{2}}=n\varphi_n(z)=-\frac{n\xi_n(z)}{2} \tag{8.45}$$

(cf. (7.14) and (7.4)) and obtain

$$\begin{aligned} &a(z)^{-1}f_n(z)^{\frac14}\,\mathrm{Ai}(f_n(z))-a(z)f_n(z)^{-\frac14}\,\mathrm{Ai}'(f_n(z))\sim\\ &\frac{1}{2\sqrt{\pi}}\left(a(z)^{-1}+a(z)\right)\exp\left(-n\pi i\int_1^z\psi_n(y)dy\right). \end{aligned} \tag{8.46}$$

Regions B_δ and $C_{1,\delta}$

Again from Proposition 7.3 we learn that $\pi \geq \arg(f_n(z)) \geq \frac{2\pi}{3}$. Using the asymptotics for the Airy function and its derivative as given in [1, 10.4.60,62], together with (8.33) and (8.34), we obtain

(8.47)
$$\begin{aligned}
&a(z)^{-1} f_n(z)^{\frac{1}{4}} \operatorname{Ai}(f_n(z)) - a(z) f_n(z)^{-\frac{1}{4}} \operatorname{Ai}'(f_n(z)) \\
&\quad \sim \frac{1}{\sqrt{\pi}} \left[a(z)^{-1} e^{\frac{i\pi}{4}} \sin\left(\frac{n\xi_n(z)}{2i} + \frac{\pi}{4}\right) + a(z) e^{-\frac{i\pi}{4}} \cos\left(\frac{n\xi_n(z)}{2i} + \frac{\pi}{4}\right) \right] \\
&\quad = \frac{1}{2\sqrt{\pi}} \left[\exp\left(\frac{n\xi_n(z)}{2}\right)(a(z)^{-1} + a(z)) + \exp\left(-\frac{n\xi_n(z)}{2}\right)(ia(z)^{-1} - ia(z)) \right] \\
&\quad = \sqrt{\frac{2}{\pi}} \frac{1}{(1-z)^{\frac{1}{4}}(1+z)^{\frac{1}{4}}} \cos\left(n\pi \int_1^z \psi_n(y)dy + \frac{1}{2}\arcsin z\right).
\end{aligned}$$

8.3 Proof of Theorem 2.3

In order to locate the zeros of p_n, we proceed in two steps. We first construct a decreasing sequence $1 > y_{0,n} > \cdots > y_{n,n} > -1$ such that the signs of $p_n(c_n y_{k,n} + d_n)$ alternate in k. As p_n has exactly n zeros, one concludes that $y_{k-1,n} > \frac{x_{k,n}-d_n}{c_n} > y_{k,n}$ for $1 \leq k \leq n$. The second step then consists in finding a zero of p_n in each of the intervals $(c_n y_{k,n} + d_n, c_n y_{k-1,n} + d_n)$.

We begin by extending (2.18) to intervals $(-1 + \frac{L}{n^{2/3}}, 1 - \frac{L}{n^{2/3}})$. More precisely, we claim that there exist positive constants L and C_0 such that

$$\begin{aligned}
(8.48) \qquad &\Bigg| \sqrt{\frac{\pi c_n}{2}} (1-y^2)^{\frac{1}{4}} p_n(c_n y + d_n) e^{-\frac{1}{2}Q(c_n y + d_n)} \\
&\qquad - \cos\left(n\pi \int_1^y \psi_n(t)dt + \frac{1}{2}\arcsin y\right) \Bigg| < \frac{C_0}{n(1-y^2)^{3/2}}
\end{aligned}$$

for all $-1 + \frac{L}{n^{2/3}} < y < 1 - \frac{L}{n^{2/3}}$. By (2.18) we only need to investigate those y that are close to ± 1. Here one uses (2.19) or (2.21)), respectively, together with the asymptotic expansion for the Airy function (see [1, 10.4.60,62]) to estimate the left-hand side of (8.48) by $C(n^{-1} + |n\pi \int_1^y \psi_n(t)dt|^{-1})$ for some $C > 0$ (cf. (8.47)). The claim now follows from Proposition 5.3.

Step 1

In order to specify the sequence $y_{k,n}$, we introduce some notation. Recall that all the zeros $-\iota_k$ of the Airy function Ai and the zeros $-\omega_k$ of its derivative Ai′ are located in $(-\infty, 0)$ and interlace

$$(8.49) \qquad 0 > -\omega_1 > -\iota_1 > -\omega_2 > -\iota_2 > \cdots \to -\infty$$

(cf. (2.26)). Let L and C_0 be the constants determined above (see (8.48)) and denote $\tilde{C} := \sup_n \{h_n(1), h_n(-1)\} < \infty$ (see (5.25) and (5.26)). We choose an integer $k_0 \geq 9$

to satisfy the following four conditions:

$$\frac{2\tilde{C}C_0}{\omega_{k_0}^{3/2}} < 1/10\,, \tag{8.50}$$

$$\frac{1}{2}\left(\frac{2}{\tilde{C}^2}\right)^{1/3}\omega_{k_0} > L\,, \tag{8.51}$$

$$\left|\frac{\iota_k}{\left(\frac{3\pi}{8}(4k-1)\right)^{2/3}} - 1\right| < \frac{1}{k^2} \qquad \text{for all } k \ge k_0\,, \tag{8.52}$$

$$\left|\frac{\omega_k}{\left(\frac{3\pi}{8}(4k-3)\right)^{2/3}} - 1\right| < \frac{1}{k^2} \qquad \text{for all } k \ge k_0\,. \tag{8.53}$$

The existence of such a number k_0 can be easily deduced from the asymptotic expansion for ι_k and ω_k (see, e.g., [1, 10.4.94,95,105]). Finally, recall the definition of ζ_n as the inverse function of $\int_x^1 \psi_n(t)dt$ (see below (2.26)). We set

$$y_{k,n} := \begin{cases} \zeta_n\left(\frac{2}{3\pi n}\omega_{k+1}^{3/2}\right) & \text{for } 0 \le k < k_0\,, \\ \zeta_n\left(\frac{k}{n}\right) & \text{for } k_0 \le k \le n-k_0\,, \\ \zeta_n\left(1-\frac{2}{3\pi n}\omega_{n-k+1}^{3/2}\right) & \text{for } n-k_0 < k \le n\,, \end{cases} \tag{8.54}$$

where n is chosen sufficiently large so that all $y_{k,n}$ are well-defined. Note that the behavior of $\zeta_n(z)$ for z near 0 or 1 can be deduced from (2.14),

$$\zeta_n(z) = 1 - \left(\frac{3\pi}{\sqrt{2}h_n(1)}\right)^{2/3} z^{2/3} + O\left(z^{4/3}\right) \quad \text{for } 0 < z \text{ small} \tag{8.55}$$

and

(8.56)

$$\zeta_n(z) = -1 + \left(\frac{3\pi}{\sqrt{2}h_n(-1)}\right)^{2/3}(1-z)^{2/3} + O\left((1-z)^{4/3}\right) \quad \text{for } 0 < 1-z \text{ small}\,.$$

Hence we can assume that

$$1 - \frac{1}{2}\left(\frac{2}{h_n(1)^2}\right)^{1/3}\omega_{k_0}\frac{1}{n^{2/3}} > y_{k_0-1,n} > 1-\delta_0\,, \tag{8.57}$$

$$-1 + \frac{1}{2}\left(\frac{2}{h_n(-1)^2}\right)^{1/3}\omega_{k_0}\frac{1}{n^{2/3}} < y_{n-k_0+1,n} < -1+\delta_0\,, \tag{8.58}$$

by choosing n sufficiently large.

We verify now that the sequence $y_{k,n}$ has the properties described above. Monotonicity follows from the monotonicity of ζ_n and from condition (8.53). We now show that

$$\operatorname{sgn}[p_n(c_n y_{k,n} + d_n)] = (-1)^k\,. \tag{8.59}$$

CASE 1. $0 \le k < k_0$: Since $|1-y_{k,n}| < \delta_0$, we may use (2.19). From (2.15) and from the definition of ζ_n, we see that $f_n(y_{k,n}) = -\omega_{k+1}$ and hence we learn from (2.19) that

$$\operatorname{sgn}[p_n(c_n y_{k,n} + d_n)] = \operatorname{sgn}[\mathrm{Ai}(-\omega_{k+1})] = (-1)^k . \tag{8.60}$$

CASE 2. $k_0 \le k \le n - k_0$: From (8.57), (8.58), and (8.51) it follows that $-1 + \frac{L}{n^{2/3}} < y_{k,n} < 1 - \frac{L}{n^{2/3}}$, and estimate (8.48) can be applied. Condition (8.50) then implies that

$$\frac{C_0}{n(1-y_{k,n}^2)^{3/2}} < \frac{1}{10} . \tag{8.61}$$

Finally, observe that

$$-k\pi - \frac{\pi}{4} < n\pi \int_1^{y_{k,n}} \psi_n(t)dt + \frac{1}{2}\arcsin y_{k,n} < -k\pi + \frac{\pi}{4} , \tag{8.62}$$

which completes the proof of Case 2. Furthermore, it is clear that these estimates also imply (2.29).

CASE 3. $n - k_0 < k \le n$: This case is similar to Case 1.

Step 2

In order to locate the zeros of p_n in each of the intervals $(c_n y_{k,n} + d_n, c_n y_{k-1,n} + d_n)$, we use the following basic fact, which is an immediate consequence of the intermediate value theorem.

PROPOSITION 8.1 *Let $I \subset \mathbb{R}$ be an interval, $q \in C^1(I)$, $r \in C^0(I)$, satisfying*

$$|q'(t)| \ge c , \quad |r(t)| \le b , \quad \textit{for all } t \in I , \tag{8.63}$$

for some $b, c \in \mathbb{R}$. Suppose, furthermore, that there exists a $t_0 \in I$ such that

$$\left(t_0 - \frac{|q(t_0)| + 2b}{c}, t_0 + \frac{|q(t_0)| + 2b}{c} \right) \subset I .$$

Then there exists a $t_1 \in I$ satisfying

$$q(t_1) + r(t_1) = 0 , \tag{8.64}$$

$$|t_0 - t_1| \le \frac{|q(t_0)| + 2b}{c} . \tag{8.65}$$

We now prove statement (i) of Theorem 2.3. Fix $k \in \mathbb{N}$. Clearly, $y_{k,n} \to 1$ as $n \to \infty$. We show that for n sufficiently large

$$-\omega_{k+1} < -\iota_k < -\omega_k , \tag{8.66}$$

$$f_n(y_{k,n}) < -\iota_k < f_n(y_{k-1,n}) . \tag{8.67}$$

The first estimates (8.66) are immediate from (8.49). To see (8.67), note that for sufficiently large n,

$$f_n(y_{k,n}) = \begin{cases} -\omega_{k+1} & \text{if } 0 \le k < k_0 , \\ -\left(\frac{3\pi k}{2}\right)^{2/3} & \text{if } k \ge k_0 , \end{cases} \tag{8.68}$$

by (2.15), and (8.67) follows from (8.49) and (8.52). Because $f_n(y_{k,n})$ is independent of n, we can find a closed interval $-\iota_k \in J_k$ that is contained in the open intervals $(-\omega_{k+1}, -\omega_k)$ and $(f_n(y_{k,n}),\ f_n(y_{k-1,n}))$. Recall the definition of $f_n = n^{2/3}\phi_n$ and the properties of ϕ_n as stated in Proposition 7.3 (and in the remark after Proposition 7.3). It follows that there exists a $c_0 > 0$ such that

$$c_0 n^{2/3} < f_n'(x) < \frac{n^{2/3}}{c_0} \quad \text{for all } 1-\delta_0 < x < 1\,. \tag{8.69}$$

Finally, we learn from (2.19) that for all $1-\delta_0 < t < 1$, the expression $p_n(c_n t + d)$ equals zero if and only if

$$\mathrm{Ai}(f_n(t)) - \left(\frac{1-t}{1+t}\right)^{1/2} (-f_n(t))^{-1/2}\,\mathrm{Ai}'(f_n(t)) + O\left(\frac{1}{n}\right) = 0\,, \tag{8.70}$$

where the $O(\frac{1}{n})$ term is a short notation for some continuous (in fact, analytic) function (cf. (8.40), theorem 7.10). We apply Proposition 8.1 to (8.70) with $I := f_n^{-1}(J_k)$, $q(t) := \mathrm{Ai}(f_n(t))$, r consisting of the remaining terms in (8.70) and $t_0 := f_n^{-1}(-\iota_k)$. Observe that $q(t_0) = 0$, $|q'(t)| > c_0\tilde{c}_0 n^{2/3}$, with $\tilde{c}_0 := \inf\{|\,\mathrm{Ai}'(s)| : s \in J_k\} > 0$ and $|r(t)| = O(\frac{1}{n^{1/3}})$ as $t = 1 - O(\frac{1}{n^{2/3}})$ for all $t \in I$. We conclude that $\frac{|q(t_0)|+2b}{c} = O(\frac{1}{n})$. Since the length of the interval I is of size $n^{-2/3}$, it is clear that the hypothesis of Proposition 8.1 will be satisfied for sufficiently large n, and we find that $p_n(c_n t_1 + d_n) = 0$ for some $t_1 \in I$ satisfying $|t_1 - t_0| = O(\frac{1}{n})$. Note, furthermore, that $I \subset (y_{k,n}, y_{k-1,n})$ by the choice of the interval J_k above, and thus we have found the k^{th} zero $x_{k,n}$ of the polynomial p_n. So far we have proved

$$\frac{x_{k,n} - d_n}{c_n} - f_n^{-1}(-\iota_k) = O\left(\frac{1}{n}\right). \tag{8.71}$$

Finally, (2.27) follows from inverting

$$f_n(x) = n^{2/3}(x-1)\left[\left(\frac{h_n(1)^2}{2}\right)^{1/3} + O(|x-1|)\right], \quad 1-\delta_0 < x < 1\,, \tag{8.72}$$

which in turn is a consequence of (7.21), (7.17), and (7.16).

The proof of (2.28) is analogous to the proof of (2.27), and we do not repeat it here.

Recalling (8.48) and the choice of k_0 made above, we can prove (2.30) by finding a solution of

$$\cos\left(n\pi \int_1^y \psi_n(t)\,dt + \frac{1}{2}\arcsin y\right) + r_n(y) = 0 \tag{8.73}$$

for $y \in (\zeta_n(\frac{6k-1}{6n}), \zeta_n(\frac{6k-5}{6n}))$ (cf. (2.29)), where r_n is a continuous function satisfying

$$|r_n(y)| \le \min\left(\frac{1}{10}, \frac{C_0}{n(1-y^2)^{3/2}}\right) \tag{8.74}$$

(see (8.48) and (8.61)). Again we apply Proposition 8.1 to obtain estimates on the zero in (8.73), choosing $I := (\zeta_n(\frac{6k-1}{6n}), \zeta_n(\frac{6k-5}{6n}))$, $r := r_n$, and $t_0 := \zeta_n(\frac{6k-3}{6n} + \frac{1}{2\pi n} \arcsin(\zeta_n(\frac{k}{n})))$. For the derivation of (2.30), we also use the fact that there exists a constant $c > 0$ such that

$$c < \frac{1-\zeta_n(\alpha)^2}{[\alpha(1-\alpha)]^{2/3}} < \frac{1}{c} \quad \text{for all } n \ge n_0,\ 0 < \alpha < 1, \tag{8.75}$$

which is an immediate consequence of (8.55) and (8.56).

Claim (iii) of Theorem 2.3 follows from (i) for $1 \le k < k_0$ or $n - k_0 \le k \le n-1$. For $k_0 \le k < n - k_0$ we apply (2.29) to obtain bounds on the distance of two consecutive zeros. Estimates for the derivative of ζ_n follow from Proposition 5.3 and from (8.75).

Appendix A: Properties of the Cauchy Transform and RHPs Normalized at ∞

We begin by listing some properties of the Cauchy transform C on the line,

$$Cf(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(s)}{s-z}\, ds,$$

(see, e.g., [42]). Let $f \in L^2(\mathbb{R})$. Then

1. Cf is analytic in $\mathbb{C} \setminus \mathbb{R}$.
2. $Cf(\cdot \pm i\varepsilon)$ converge to $(Cf)_\pm$ in $L^2(\mathbb{R})$ as $\varepsilon \searrow 0$.
3. The operators $C_\pm f \equiv (Cf)_\pm$ are bounded operators from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ and $C_+ - C_- = \mathbf{1}$.
4. The operators $C_\pm$ are bounded from the Sobolev space $H^1(\mathbb{R})$ to itself because they commute with the derivative. Moreover (see, e.g., [6]), if $f \in H^1(\mathbb{R})$, then Cf is bounded and uniformly $\frac{1}{2}$-Hölder-continuous in $\mathbb{C}_+$ and in $\mathbb{C}_-$. In particular, Cf has a continuous extension to $\overline{\mathbb{C}}_+$ and $\overline{\mathbb{C}}_-$, and the values at the boundary are given by $(C_\pm f)(z)$, respectively.

Properties 1, 2, and 3 above for the Cauchy transform, suitably interpreted, are true for more general oriented contours $\Sigma \ne \mathbb{R}$. In particular, these properties are true for contours Σ with the following properties:

(i) Σ is a finite union of simple smooth arcs $\gamma_j = \{z_j(t) : 0 \le t \le 1\}$ in the Riemann sphere.

(ii) The arcs and line segments in Σ intersect at most at a finite number of points and all intersections are transversal.

In particular, 1, 2, and 3 are true for contours Σ_S, Σ_R, and $\hat{\Sigma}_R$ in the text (see Figures 6.1, 7.5, and 7.6).

If $s + i\gamma(s)$, $-\delta \le s \le \delta$, is a piecewise parameterization of two arcs in a neighborhood of a point of intersection $0 + i\gamma(0) = 0$ (see Figure A.1), then 2 should be

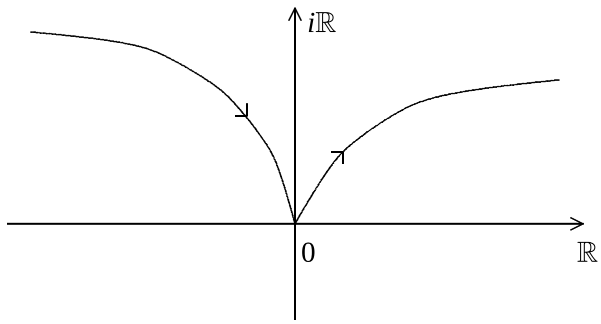

FIGURE A.1

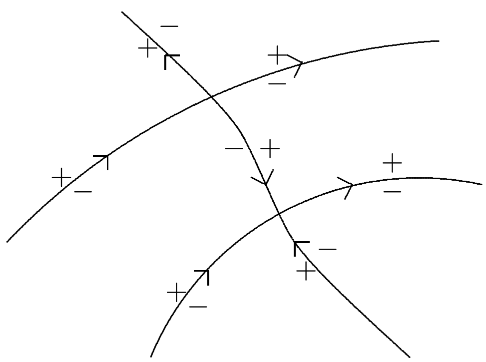

FIGURE A.2

interpreted to mean

$$\lim_{\varepsilon\searrow 0}\int_{-\delta}^{\delta}|Cf(s+i(1+\varepsilon)\gamma(s))-C_{+}f(s+i\gamma(s))|^{2}\sqrt{1+\gamma'(s)^{2}}\,ds=0\,, \tag{A.1}$$

etc.

If all the arcs of Σ are straight lines, properties 1, 2, and 3 can be read off from standard estimates of the Cauchy transform (see, e.g., [6, p. 88]). The proof of properties 1 through 3 for contours satisfying (i) and (ii) can be obtained from the straight-line case by approximating all of the segments locally by straight lines (see, e.g., [24]). For self-intersecting contours, property 4 requires additional assumptions on f that can be found in [46].

Basic references for RHPs are [10] and [24]. Let Σ be an oriented curve in the plane satisfying properties (i) and (ii) above. By convention, the $+$ side (respectively, $-$ side) of an arc in Σ lies to the left (respectively, right) as one traverses the arc in the direction of the orientation (see, e.g., Figure A.2).

Let $\Sigma_0 = \Sigma\setminus\{\text{points of self-intersection}\}$ and v be a smooth map from $\Sigma_0 \to \mathrm{Gl}(k,\mathbb{C})$ for some k. If Σ is unbounded, we require that $v(z)\to I$ as $z\to\infty$ along Σ. The RHP (Σ, v) consists of the following (see, e.g., [10]): Establish the existence and uniqueness of a $k \times k$–matrix-valued function $Y(z)$ (the *solution* of the RHP (Σ, v)) such that

$$\begin{cases} Y(z) & \text{is analytic } \mathbb{C} \setminus \Sigma, \\ Y_+(s) = Y_-(s)v(s) & \text{for } s \in \Sigma_0, \\ Y(z) \to I & \text{as } z \to \infty. \end{cases} \tag{A.2}$$

Here $Y_\pm(z) = \lim_{z' \to z} Y(z')$ where $z' \in \pm$ side of Σ. The precise sense in which these boundary values are attained, and also the precise sense in which $Y(z) \to I$ as $z \to \infty$, are technical matters that should be specified for any given RHP (Σ, v). In this paper, by a solution Y of an RHP (Σ, v), we always mean (in this connection, cf. the remark at the end of Section 6) that

(A.3) $Y(z)$ is analytic in $\mathbb{C} \setminus \Sigma$ and continuous up to the boundary (including the points in $\Sigma \setminus \Sigma_0$) in each component. The jump relation $Y_+(z) = Y_-(z)v(z)$ is taken in the sense of continuous boundary values, and $Y(z) \to I$ as $z \to \infty$ means $Y(z) = I + O(\frac{1}{|z|})$ uniformly as $z \to \infty$ in $\mathbb{C} \setminus \Sigma$.

Given (Σ,v), the existence of Y under appropriate technical assumptions on Σ and v is, in general, a subtle and difficult question. However, for the RHP (3.1)–(3.3) (which is nonregular at infinity), and hence for all RHPs obtained by deforming (3.1)–(3.3) (see, e.g., (4.3)–(4.5)), we will prove the existence of a solution directly by construction (see Theorem 3.1); uniqueness, as we see, is a simple matter.

The solution of an RHP (Σ, v) can be expressed in terms of the solution of an associated singular integral equation on Σ (see (A.6) and (A.7) below) as follows: Let $C_\pm$ be the Cauchy operators on Σ.

Let

$$v = b_-^{-1} b_+ \equiv (I - w_-)^{-1}(I + w_+) \tag{A.4}$$

be any factorization of v. We assume $b_\pm$, and hence $w_\pm$, are smooth on Σ_0, and if Σ is unbounded, we assume $b_\pm(z) \to I$ as $z \to \infty$ along Σ. Define the operator

$$C_w(f) \equiv C_+(f w_-) + C_-(f w_+). \tag{A.5}$$

By the above discussion, if $w_\pm \in L^\infty(\Sigma, |dz|)$, then C_w is bounded from $L^2(\Sigma, |dz|) \to L^2(\Sigma, |dz|)$. Suppose that the equation

$$(\mathbf{1} - C_w)\mu = I \quad \text{on } \Sigma \tag{A.6}$$

has a solution $\mu \in I + L^2(\Sigma)$, or more precisely, suppose $\mu - I \in L^2(\Sigma)$ solves

$$(\mathbf{1} - C_w)(\mu - I) = C_w I = C_+(w_-) + C_-(w_+), \tag{A.7}$$

which is a well-defined equation in $L^2(\Sigma)$ provided that $w_\pm \in L^\infty \cap L^2(\Sigma, |dz|)$. Then the solution of the RHP (A.2) is given by (see [10])

$$Y(z) = I + \int_\Sigma \frac{\mu(s)(w_+(s) + w_-(s))}{s - z} \frac{ds}{2\pi i}, \quad z \notin \Sigma. \tag{A.8}$$

Indeed, for a.e. $z \in \Sigma$, from (A.6) and from property 3 of the Cauchy transform stated above,

$$\begin{aligned} Y_+(z) &= I + C_+(\mu(w_+ + w_-)) = I + C_w(\mu) + (C_+ - C_-)(\mu w_+) \\ &= \mu + \mu w_+ = \mu(z) b_+(z)\,, \end{aligned} \tag{A.9}$$

and similarly $Y_-(z) = \mu(z)b_-(z)$, so that $Y_+(z) = Y_-(z)b_-^{-1}(z)b_+(z) = Y_-(z)v(z)$ a.e. on Σ. Under appropriate regularity assumptions on Σ and v, one then shows that $Y(z)$ solves the RHP (Σ, v) in the sense of (A.3).

Appendix B: The Hermite Case

In this appendix we use well-known facts in the theory of orthogonal polynomials to motivate the introduction of the g-function of Section 4 in the case of Hermite polynomials.

Here the weight function is given by $w(x)dx = e^{-x^2}\,dx$. As before, let $x_{1,n} > \cdots > x_{n,n}$ denote the roots of $\pi_n(z)$. From (3.5),

$$Y_{11}(z;n,w) = \pi_n(z) = \prod_{j=1}^{n}(z - x_{j,n}) = \exp\left[n\left(\frac{1}{n}\sum_{j=1}^{n}\log(z - x_{j,n})\right)\right]. \tag{B.1}$$

It is a well-known fact that under the appropriate scaling $x_{j,n} \mapsto \hat{x}_{j,n} \equiv x_{j,n}/\sqrt{2n}$, the normalized counting measure $\nu_n := \frac{1}{n}\sum_{j=1}^{n}\delta_{\hat{x}_{j,n}}$ (where δ_a denotes the Dirac measure concentrated at a) converges weakly to a measure ν that has support $[-1,1]$, $d\nu(x) = \frac{2}{\pi}\sqrt{1-x^2}\,dx$. Similar facts are true for a very general class of orthogonal polynomials (see [34, 37, 40]). Hence from (B.1) we expect that

$$Y_{11}(\sqrt{2n}z;n,w) \approx (2n)^{n/2}e^{n(\int \log(z-x)d\nu(x))} = (2n)^{n/2}e^{ng(z)} \tag{B.2}$$

where $g(z) = \int \log(z-s)d\nu(s)$. This motivates

1. the rescaling $Y \to U$ (here $\pm\sqrt{2n}$ are the MRS numbers) and
2. the introduction of the phase factor e^{-ng} in the transformation $U \to T$

in Section 4.

Direct integration yields

$$g(z) = z^2 - \frac{1}{2} - \log 2 - 2\int_1^z (t-1)^{1/2}(t+1)^{1/2}\,dt\,, \tag{B.3}$$

and it is easy to verify directly the phase conditions (4.7), (4.8), and (4.12)–(4.15) with $l = -1 - 2\log 2$.

Appendix C: A Second-Order Differential Equation for the Orthogonal Polynomials

Denote by Y the solution of the Riemann-Hilbert problem (3.6)–(3.8) with respect to the weight function $w(s) = e^{-Q(s)}$, where Q is a polynomial of even degree

$2m$ and with positive leading coefficient. Set

$$\tilde{Y}(z) := Y(z)\begin{pmatrix} e^{-\frac{Q(z)}{2}} & 0 \\ 0 & e^{\frac{Q(z)}{2}} \end{pmatrix}, \qquad V(z) := Y(z)\begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix}. \tag{C.1}$$

One easily verifies that

$$\tilde{Y}_+(s) = \tilde{Y}_-(s)\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \left(\frac{d\tilde{Y}}{dz}\right)_+(s) = \left(\frac{d\tilde{Y}}{dz}\right)_-(s)\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{C.2}$$

for $s \in \mathbb{R}$. Therefore $H(z) := \frac{d\tilde{Y}}{dz}(z)\tilde{Y}^{-1}(z)$ is an entire matrix-valued function, and we find

$$\frac{dV}{dz}V^{-1} = H(z) + \left(Q'(z) - \frac{n}{z}\right)V\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}V^{-1}. \tag{C.3}$$

From (3.9) and (C.1),

$$V(z) = I + \frac{Y_1}{z} + \cdots + \frac{Y_{2m-1}}{z^{2m-1}} + O\left(\frac{1}{|z|^{2m}}\right), \quad z \to \infty \ \text{(cf. (3.10))}, \tag{C.4}$$

and

$$\frac{dV}{dz}(z)V^{-1}(z) = O\left(\frac{1}{|z|^2}\right), \quad z \to \infty. \tag{C.5}$$

Expanding (C.3) at $z = \infty$, we learn from (C.4) and (C.5) that the entire function H is a polynomial of degree $2m-1$, which can then be determined explicitly from Q' and from the matrices $Y_1, \ldots, Y_{2m-1}$ (see (C.4)). Finally, we arrive at a differential equation for Y, using (C.1) and the definition of H,

$$\frac{dY}{dz}(z) = H(z)Y(z) + Q'(z)Y(z)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{C.6}$$

The first column of (C.6) reads

$$\begin{pmatrix} \pi_n'(z) \\ \gamma_{n-1}\pi_{n-1}'(z) \end{pmatrix} = \begin{pmatrix} H_{11}(z) + Q'(z) & H_{12}(z) \\ H_{21}(z) & H_{22}(z) + Q'(z) \end{pmatrix}\begin{pmatrix} \pi_n(z) \\ \gamma_{n-1}\pi_{n-1}(z) \end{pmatrix}. \tag{C.7}$$

Standard manipulations now lead to a linear second-order differential equation for π_n. The coefficients of this differential equation are regular except at the zeros of H_{12}. As remarked in Section 1, differential equations for the orthogonal polynomials have been derived in [3, 8, 9, 33, 35, 41].

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