Strong Banach Property (T) for Simple Algebraic Groups of Higher Rank

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Abstract

In [Laf08, Laf09], Vincent Lafforgue proved strong Banach property (T) for SL_3 over a non archimedean local field F. In this paper, we extend his results to Sp_4 and therefore to any connected almost F-simple algebraic group with F-split rank ≥ 2. As applications, the family of expanders constructed by finite quotients of a lattice in such a group does not admit a uniform embedding in any Banach space of type > 1, and any affine isometric action of such a group, or of any cocompact lattice in it, in a Banach space of type > 1 has a fixed point.

Contents

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1 Introduction

In [Laf08, Laf09], Vincent Lafforgue proved strong Banach property (T) for SL_3 over a non archimedean local field F. In this paper, we extend his results to Sp_4 and therefore to any connected almost F-simple algebraic group with F-split rank ≥ 2. As the first application, the family of expanders constructed by finite quotients of a lattice in such a group does not admit a uniform embedding in any Banach space of type > 1. As the second application, we prove that any affine isometric action of such a group, or of any cocompact lattice in it, in a Banach space of type > 1 has a fixed point. In [BFGM], it is conjectured that any isometric affine action of a higher rank simple algebraic group over a local field and of its lattice in a uniformly convex space has a fixed point. As a consequence of the second application, we confirm this conjecture for any non archimedean local field and the corresponding cocompact lattices.

To announce the precise statements, we begin by recalling some definitions and notations from [Laf09].

Definition 1.1 *A class of Banach spaces $\mathcal{E}$ is of type > 1 if one of the following two equivalent conditions holds.*

- *i) There exist $n \in \mathbb{N}$ and $\varepsilon > 0$ such that for any Banach space $E \in \mathcal{E}$, E does not contain ℓ_1^n $(1+\varepsilon)$-isometrically;*
- *ii) There exist $p > 1$ (called the type) and $T \in \mathbb{R}_+$ such that for any $E \in \mathcal{E}$, $n \in \mathbb{N}^*$ and $x_1, ..., x_n \in E$, we have*

$$\Big(\mathop{\mathbb{E}}_{\varepsilon_i = \pm 1} \| \sum_{i=1}^{n} \varepsilon_i x_i \|_E^2 \Big)^{\frac{1}{2}} \leq T \Big(\sum_{i=1}^{n} \|x_i\|_E^p \Big)^{\frac{1}{p}}.$$

Remark 1. We say that a class of Banach spaces $\mathcal{E}$ is given by a super-property, if any Banach space F finitely representable in $\mathcal{E}$ (i.e. for any finite dimensional subspace $V \subset F$ and $\varepsilon > 0$ there exists $E \in \mathcal{E}$ which contains V $(1+\varepsilon)$-isometrically) is an element of $\mathcal{E}$. It is clear that a class of type > 1 is given by a super-property.

Remark 2. If $\mathcal{E}$ is a class of Banach spaces given by a super-property and not a class of type > 1, then $\mathcal{E}$ contains $L_1(\mu)$, where μ is any σ-finite measure. In fact, by the classification of σ-finite measures it suffices to show that ℓ_1 and $L_1(\{0,1\}^\infty)$ are elements of $\mathcal{E}$. $L_1(\{0,1\}^\infty)$ is finitely representable in ℓ_1. By condition i) in the definition, ℓ_1 is finitely representable in the class $\mathcal{E}$. Since $\mathcal{E}$ is given by a super-property, we conclude that $L_1(\{0,1\}^\infty)$ and ℓ_1 belong to $\mathcal{E}$.

Let $\mathcal{E}$ be a class of Banach spaces stable under complex conjugation and duality. Let G be a locally compact topological group. Let ℓ be a continuous length function of G. Denote $\mathcal{E}_{G,\ell}$ the set of isomorphism classes of strongly continuous representations (E, π) of G such that $E \in \mathcal{E}$ and

$$\|\pi(g)\|_{\mathcal{L}(E)} \leq e^{\ell(g)}$$

for any $g \in G$. Denote $\mathcal{C}_\ell^{\mathcal{E}}(G)$ the completion of compactly supported functions $C_c(G)$ on G with respect to the norm

$$\|f\|_{\mathcal{C}_\ell^{\mathcal{E}}(G)} = \sup_{(E,\pi)\in\mathcal{E}_{G,\ell}} \| \int f(g)\pi(g)dg\|_{\mathcal{L}(E)}.$$

Definition 1.2 *We say that a locally compact group G has strong Banach property (T) if for any class of Banach spaces $\mathcal{E}$ of type $>$ 1, stable under complex conjugation and duality, and any continuous length function ℓ over G, there exists $s_0 > 0$ such that the following holds. For any $C > 0$ and $s_0 \geq s \geq 0$, there exists a real self-adjoint idempotent element* p *in $\mathcal{C}^{\mathcal{E}}_{C+s\ell}(G)$, such that for any representation $(E, \pi) \in \mathcal{E}_{G,C+s\ell}$, the image of $\pi(\mathrm{p})$ consists of all G-invariant vectors in E, i.e.*

$$\pi(\mathrm{p})E = E^{\pi(G)}.$$

Remark. In this definition, the condition of type > 1 cannot be replaced by a weaker condition given by a super-property because otherwise it would be satisfied only for compact groups. Indeed when G is non compact, suppose that $\mathcal{E}$ is a class of Banach spaces (stable under complex conjugation and duality) given by a super-property, and that there exists a real self-ajoint idempotent $\mathrm{p} \in C_0^{\mathcal{E}}(G)$ such that for any $(E, \pi) \in \mathcal{E}_{G,0}$ we have $\pi(\mathrm{p})E = E^{\pi(G)}$, we show that $\mathcal{E}$ is a class of Banach spaces of type > 1. If not, by remark 2 below definition 1.1, $\mathcal{E}$ must contain $L^1(G)$. Note that for any $(E_1, \pi_1), (E_2, \pi_2) \in \mathcal{E}_{G,0}$, any surjective morphism $E_1 \to E_2$ in the category $\mathcal{E}_{G,0}$ induces a surjective morphism from $E_1^G = \pi_1(\mathrm{p})E_1$ to $E_2^G = \pi_2(\mathrm{p})E_2$. Now consider the morphism from $L^1(G)$ (with the left regular representation of G) to $\mathbb{C}$ (with the trivial action of G) by integration on G. Since G is non compact, there is no non zero G-invariant integrable function on G, therefore $L^1(G)^G = \{0\}$. However, $\mathbb{C}^G = \mathbb{C}$, and this is a contradiction to that $L^1(G)^G \to \mathbb{C}^G$ must be a surjective morphism. Therefore, $\mathcal{E}$ must be a class of type > 1 (see the second remark below definition 0.2 in [Laf09]).

Let F be a non archimedean local field. The purpose of this paper is to prove the following theorem.

Theorem 1.3 *Any connected almost F-simple algebraic group with F-split rank ≥ 2 has strong Banach property (T).*

Remark. This result cannot be extended to any almost F-simple algebraic group with F-split rank $= 1$ because they do not even have Kazhdan's property (T).

The following definition corresponds to the special case of isometric actions.

Definition 1.4 *We say that a locally compact group G has Banach property (T) if for any class of Banach spaces $\mathcal{E}$ of type > 1 stable under complex conjugation and duality, there exists a real self-adjoint idempotent element* p *in $\mathcal{C}_0^{\mathcal{E}}(G)$, such that for any representation $(E, \pi) \in \mathcal{E}_{G,0}$, the image of $\pi(\mathrm{p})$ consists of all G-invariant vectors in E.*

Remark. If a locally compact group G has (strong) Banach property (T) with $\mathrm{p} \in \mathcal{C}^{\mathcal{E}}_{C+s\ell}(G)$ being the corresponding idempotent, there always exist $\mathrm{p}_n \in C_c(G)$ of integral 1, such that p_n converges to p in $\mathcal{C}^{\mathcal{E}}_{C+s\ell}(G)$. In fact, let $\check{\mathrm{p}}_n \in C_c(G)$ be any sequence such that $\check{\mathrm{p}}_n \to \mathrm{p}$. Let $s_n = \int_G \check{\mathrm{p}}_n(g)dg$. Then

$$\begin{aligned}\|\mathrm{p} - s_n \mathrm{p}\|_{\mathcal{C}^{\mathcal{E}}_{C+s\ell}(G)} &= \|\mathrm{p}^2 - \check{\mathrm{p}}_n \mathrm{p}\|_{\mathcal{C}^{\mathcal{E}}_{C+s\ell}(G)} \\ &\leq \|\mathrm{p} - \check{\mathrm{p}}_n\|_{\mathcal{C}^{\mathcal{E}}_{C+s\ell}(G)} \|\mathrm{p}\|_{\mathcal{C}^{\mathcal{E}}_{C+s\ell}(G)},\end{aligned}$$

and hence $|1 - s_n| \leq \|\mathrm{p} - \check{\mathrm{p}}_n\|_{\mathcal{C}^{\mathcal{E}}_{C+s\ell}(G)} \to 1$ when $n \to \infty$. Therefore, $s_n \neq 0$ for big enough n and $\mathrm{p}_n = \check{\mathrm{p}}_n / s_n$ has integral 1 and tends to p.

With the remark above and the same argument as in theorem 5.4 in [Laf09], we obtain the following theorem 1.5 on application to expanders.

We say that a family of graphes $\{(X_i, d_i)\}_{i \geq 1}$ is embedded uniformly in a Banach space E, if there exist a function $\rho : \mathbb{N} \to \mathbb{R}_+$ that tends to infinity at infinity and 1-Lipschitz maps $f_i : X_i \to E$ such that

$$\|f_i(x) - f_i(y)\|_E \geq \rho(d_i(x, y))$$

for any $i \in \mathbb{N}$ and $x, y \in X_i$.

Let Γ be a discrete group with Banach property (T). Let $(\Gamma_i)_{i \in \mathbb{N}}$ be a family of subgroups of Γ such that $|\Gamma/\Gamma_i|$ tends to infinity. Let S a finite symmetric system of generators of Γ which contains 1. For any $i \geq 0$, $X_i = \Gamma/\Gamma_i$ is endowed with a graph structure associated to S and we denote by d_i the associated metric. As Γ has the usual property (T), X_i forms a family of expanders.

Theorem 1.5 *Let Γ be any discrete group with Banach property (T). Then the family of expanders (X_i, d_i) constructed above does not admit a uniform embedding in any Banach space of type > 1.*

Since strong Banach property clearly implies Banach property (T), and Banach property (T) is inherited by lattices (proposition 5.3 in [Laf09]), when Γ is a lattice of a connected almost F-simple algebraic groups of F-split rank ≥ 2, we see that the family of expanders constructed above does not admit a uniform embedding in any Banach space of type > 1.

We recall that it is still unknown whether or not such a family of expanders (or in fact any family of expanders) admits a uniform embedding in a Banach of finite cotype (see [Laf09], [Pis10] and [MN]).

We turn to application to fixed-point property. As a consequence of proposition 5.6 in [Laf09], we immediately obtain the following proposition, confirming conjecure 1.6 in [BFGM] for any simple algebraic group of higher rank over a non archimedean local field and its cocompact lattice.

Proposition 1.6 *Let G be a connected almost F-simple algebraic group with F-split rank ≥ 2, or a cocompact lattice of such a group. Then any affine isometric action of G on a Banach space of type > 1 has a fixed point.*

Remark 1. This result cannot be strengthened to affine isometric actions for a larger class of Banach spaces defined by a super-property. If so, first of all by remark 2 below definition 1.1 this class must contain all L^1 spaces and their closed subspaces. Denote $d\mu$ the Haar measure on G, and $L^1_i(G)$ the space of functions $f \in L^1(G)$ such that $\int_G f(g)d\mu(g) = i, i = 0, 1$. Then $L^1_1(G)$ is an affine Banach space with $L^1_0(G)$ as the underlying Banach space. Let G act on $L^1_1(G)$ by left translation. It is an affine isometric action of G without fixed point, since G is not compact.

Remark 2. As pointed out by Mikael de la Salle and the editor, let us mention that it is shown in [BGM] that fixed point property for all L^1 spaces is a characterization of Kazhdan's property (T) for locally compact topological groups.

This paper will be part of my PhD thesis in Université Paris Diderot- Paris 7. I would like to thank my thesis adviser Vincent Lafforgue for his encouragement and guidance, and very helpful discussions about this paper. I also thank Yanqi Qiu for the discussion of type of a Banach space.

Here is how the paper is organized. In section 2, we review the theorem of strong Banach property (T) for SL_3 in [Laf09] and announce the corresponding theorem 2.3 for Sp_4. In section 3, we prove theorem 2.3 when $\text{char}(F) \neq 2$ by constructing matrices for Sp_4 and adapting the arguments in [Laf09]. In section 4, we prove theorem 2.3 when $\text{char}(F) = 2$ by constructing new matrices for the local estimate of the move $(0,2)$ and establishing the existence of two limits in the spherical proposition. In section 5, we adapt a well known argument [DK, Vas, Wang] and extend the results of SL_3 and Sp_4 to any almost F-simple algebraic groups with F-split rank ≥ 2.

2 Strong Banach property (T) for $Sp_4(F)$

Let $\mathcal{E}$ be any class of Banach spaces of type > 1, stable under complex conjugation and duality. Let F be a non archimedean local field, $\mathcal{O}$ the ring of integers of F, π one of its uniformizer, $\mathbb{F}$ the residue field, and q the cardinality of $\mathbb{F}$, i.e. $q = \frac{1}{|\pi|}$. The following proposition from [Laf09] (corollary 2.3) introduces parameters $\alpha > 0$ and $h \in \mathbb{N}^*$ for the class $\mathcal{E}$.

Proposition 2.1 *There exist $\alpha > 0$ and $h \in \mathbb{N}^*$ such that for any $E \in \mathcal{E}$ we have*

$$\|T_{\mathcal{O}/\pi^h\mathcal{O}} \otimes 1_E\| \leq e^{-\alpha},$$

where $T_{\mathcal{O}/\pi^h\mathcal{O}} \otimes 1_E \in \mathcal{L}\big(\ell^2(\mathcal{O}/\pi^h\mathcal{O}, E), \ell^2(\widehat{\mathcal{O}/\pi^h\mathcal{O}}, E)\big)$ is defined by

$$\big(T_{\mathcal{O}/\pi^h\mathcal{O}} \otimes 1_E\big)(f)(\chi) = \mathop{\mathbb{E}}_{a\in\mathcal{O}/\pi^h\mathcal{O}} \chi(a) f(a),$$

for any $\chi \in \widehat{\mathcal{O}/\pi^h\mathcal{O}}$ and $f \in \ell^2(\mathcal{O}/\pi^h\mathcal{O}, E)$.

It is proved in [Laf09] that $SL_3(F)$ has strong Banach property (T).

Theorem 2.2 *(Theorem 4.1 of [Laf09]) Let $G = SL_3(F)$, and ℓ be the length function on G defined by*

$$\ell\Big(k(\pi^{\frac{i+2j}{3}}\begin{pmatrix}\pi^{-i-j} & & \\ & \pi^{-j} & \\ & & 1\end{pmatrix})k'\Big) = i+j,$$

for any $k, k' \in SL_3(\mathcal{O})$ and $i, j \geq 0$ with $i - j \in 3\mathbb{Z}$. Let $\beta \in [0, \frac{\alpha}{3h})$. There exist $t, C' > 0$ such that for any $C \in \mathbb{R}_+$, there exists a real and self-adjoint idempotent element $\mathrm{p} \in \mathcal{C}^{\mathcal{E}}_{C+\beta\ell}(G)$ such that

- *(i) for any representation $(E,\pi) \in \mathcal{E}_{G,C+\beta\ell}$, the image of $\pi(\mathrm{p})$ is the subspace of E consisting of all G-invariant vectors,*
- *(ii) there exists a sequence $\mathrm{p}_n \in C_c(G)$, such that $\int_G |\mathrm{p}_n(g)|dg \leq 1$, p_n has support in $\{g \in G, \ell(g) \leq n\}$, and*

$$\|\mathrm{p} - \mathrm{p}_n\|_{\mathcal{C}^{\mathcal{E}}_{C+\beta\ell}(G)} \leq C' e^{2C-tn}.$$

Now we turn to Sp_4. Let $G = Sp_4(F)$, which is the group of 4×4 matrices g over F such that ${}^t gJg = J$ where J is the skew-symmetric matrix,

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Let $K = Sp_4(\mathcal{O})$ (i.e. the subgroup in $Sp_4(F)$ whose matrix elements are in $\mathcal{O}$). For any $(i,j) \in \mathbb{Z}^2$ let

$$D(i,j) = \begin{pmatrix} \pi^{-i} & & & \\ & \pi^{-j} & & \\ & & \pi^{j} & \\ & & & \pi^{i} \end{pmatrix}.$$

By $\|g\|$ we denote the norm of the operator $g \in \mathrm{End}(F^4)$ w.r.t. the standard norm on F^4, i.e. $\|g\| = \max_{1 \le \alpha,\beta \le 4} |g_{\alpha\beta}|$. Similarly, denote $\|\Lambda^2 g\|$ the biggest norm of all 2×2 minors of $g \in G$, which is the norm of $\Lambda^2 g \in \mathrm{End}(\Lambda^2 F^4)$ w.r.t. the standard norm on $\Lambda^2 F^4$. Let $\Lambda = \{(i,j) \in \mathbb{N}^2, i \ge j\}$. Any element in G has the form $kD(i,j)k'$ for some $(i,j) \in \Lambda$ and $k,k' \in K$. For such a $g = kD(i,j)k' \in G$, we have $\|g\| = q^i$ and $\|\Lambda^2 g\| = q^{i+j}$, and this gives a bijection from $K\backslash G/K$ to Λ by $g \mapsto (i,j)$, which is the inverse of $(i,j) \mapsto KD(i,j)K$. Let ℓ be the length function of G defined by $\ell(kD(i,j)k') = i + j$, for any $k, k' \in K$ and $(i,j) \in \Lambda$.

We will prove the following theorem with the argument used in [Laf09] for the proof of theorem 2.2 (note that the statement is the same except for the range of β).

Theorem 2.3 *Let α and h be as in proposition 2.1, and $\beta \in [0, \frac{\alpha}{8h})$. There exist $t, C' > 0$ such that for any $C \in \mathbb{R}_+$, there exists a real and self-adjoint idempotent element $\mathrm{p} \in \mathcal{C}^{\mathcal{E}}_{C+\beta\ell}(G)$ such that*

- *(i) for any representation $(E,\pi) \in \mathcal{E}_{G,C+\beta\ell}$, the image of $\pi(\mathrm{p})$ is the subspace of E consisting of all G-invariant vectors,*
- *(ii) there exists a sequence $\mathrm{p}_n \in C_c(G)$, such that $\int_G |\mathrm{p}_n(g)| dg \le 1$, p_n has support in $\{g \in G, \ell(g) \le n\}$, and*

$$\|\mathrm{p} - \mathrm{p}_n\|_{\mathcal{C}^{\mathcal{E}}_{C+\beta\ell}(G)} \le C' e^{2C - tn}.$$

3 Proof of theorem 2.3 when char$(F) \neq 2$

This section is dedicated to the proof of theorem 2.3 when the characteristic of F is different from 2. We will first reduce the theorem to two propositions on matrix coefficients, and then prove them by a zig-zag argument in the Weyl chamber with two local estimates of the matrix coefficients.

Most of the claims in this section are only true when $\mathrm{char}(F) \neq 2$, but some are still valid in characteristic 2 and will be used in the next section for the proof in characteristic 2.

When $\mathrm{char}F \neq 2$, we denote v_0 the valuation of $2 \in \mathcal{O}$. For any $a \in \mathbb{R}$, denote $\lfloor a \rfloor$ (resp. $\lceil a \rceil$) the biggest (resp. smallest) integer $\leq a$ (resp. $\geq a$).

Let (E, π) be any continuous representation of G of a Banach space E, (V, τ) any irreducible unitary representation of K. For fixed $\xi \in E$ and $\eta \in V \otimes E^*$, we denote $c(g) = \langle \eta, \pi(g)\xi \rangle \in V$ for any $g \in G$. By abuse of notation we write

$$c(i, j) = \langle \eta, \pi(D(i, j))\xi \rangle.$$

The following is the proposition on spherical matrix coefficients, which will be used to construct the idempotent element p in theorem 2.3.

Proposition 3.1 *Suppose that char$(F) \neq 2$. Let α be as in proposition 2.1, $\beta \in [0, \frac{\alpha}{4h})$. There exists $C' > 0$, such that the following holds. Let $C \in \mathbb{R}_+^*$, (E, π) any element in $\mathcal{E}_{G,C+\beta\ell}$, and $\xi \in E$, $\eta \in E^*$ any K-invariant vectors of norm 1. There exists $c_\infty \in \mathbb{C}$, such that for any $i \geq j \geq 0$,*

$$|c(i, j) - c_\infty| \leq C' e^{2C - (\frac{\alpha}{2h} - 2\beta)i}.$$

Next we turn to the proposition on non spherical matrix coeffients.

Proposition 3.2 *Suppose that char$(F) \neq 2$. Let α be as in proposition 2.1, $\beta \in [0, \frac{\alpha}{4h})$, and (V, τ) a non trivial irreducible unitary representation of K. There exists $C' > 0$, such that the following holds. Let $C \in \mathbb{R}_+^*$, (E, π) any element in $\mathcal{E}_{G,C+\beta\ell}$, and $\xi \in E$, $\eta \in V \otimes E^*$ (endowed with the ℓ^2 norm with respect to some fixed basis of V) any K-invariant vectors of norm 1. We have for any $i \geq j \geq 0$,*

$$\|c(i, j)\|_V \leq C' e^{2C - (\frac{\alpha}{2h} - 2\beta)i}.$$

Proof of theorem 2.3 when char$(F) \neq 2$ assuming proposition 3.1 and 3.2: Denote $e_g \in \mathcal{C}^{\mathcal{E}}_{C+\beta\ell}(G), \forall g \in G$, the limit of $\frac{\chi_{E_n}}{vol(E_n)} \in C_c(G)$ for some descending Borel subsets E_n satisfying $\cap_n E_n = \{g\}$. For any $(\pi, E) \in \mathcal{E}_{G,C+\beta\ell}$, by strong continuity we have $\pi(e_g)\xi = \pi(g)\xi, \forall \xi \in E$. Let $P_g = e_K e_g e_K$, where $e_K = \int_K e_k dk$ and dk is the Haar measure on K such that K has volume 1. As a consequence of proposition 3.1 we see that the limit $\mathrm{p} = \lim_{\ell(g)\to\infty} P_g$ exists in $\mathcal{C}^{\mathcal{E}}_{C+\beta\ell}(G)$. It is a real and self-adjoint element because $\bar{P}_g = P_g$, and $P_g^* = P_{g^{-1}}$. Moreover for any $k \in K$ and $g, g' \in G$ we have $\ell(gkg') \geq \ell(g') - \ell(g^{-1})$, which gives

$$e_K e_g \mathrm{p} = \lim_{\ell(g')\to\infty} e_K \int_K P_{gkg'} dk e_K = \mathrm{p}, \tag{1}$$

and therefore $\mathrm{p}^2 = \mathrm{p}$.

On the other hand, for any non trivial irreducible representation (V, τ) of K, denote $e_K^V = n \int_K \overline{\mathrm{Tr}(\tau(k))} e_k dk \in \mathcal{C}^{\mathcal{E}}_{C+\beta\ell}(G)$, where $n = \dim V$. For any $(E, \pi) \in \mathcal{E}_{G,C+\beta\ell}$, denote $\pi^* : G \to \mathcal{L}(E^*)$ the contragredient representation of π, i.e. $\pi^*(g) = {}^t\pi(g^{-1})$, then $\pi^*(e_K^V)E^*$ is the subspace of vectors in E^* whose K-type is V. For any $\xi \in \pi^*(e_K^V)E^*$ there exist K-invariant vectors $\eta_i \in V^* \otimes E^*$ and vectors $v_i \in V, 1 \leq i \leq n$, such that $\xi = \sum_{i=1}^n \langle \eta_i, v_i \rangle$. By applying proposition 3.2 to V^* and E we have $e_K^V e_g e_K \to 0$ in $\mathcal{C}^{\mathcal{E}}_{C+\beta\ell(g)}$ when $\ell(g) \to \infty$, and therefore

$$e_K^V e_g \mathrm{p} = 0. \tag{2}$$

Note that any vector $z \in E$ satisfying $\pi(e_K^V)z = 0$ for any irreducible representation V of K must be the zero vector (since $\pi(f)z = 0$ for any class function $f \in C(K)$, i.e. continuous function invariant under the conjugate action of K). Now for any $x \in E$ apply this to $z = \pi(e_g \mathrm{p} - \mathrm{p})x$, and in view of (1) and (2), we have

$$\pi(e_g \mathrm{p}) = \pi(\mathrm{p}).$$

Therefore $\pi(\mathrm{p})E$ is the subspace of G-invariant vectors in E.

Finally we complete the proof by taking $\mathrm{p}_n = P_{D(n,0)}$ and $t = \frac{\alpha}{2h} - 2\beta$. □

Now we turn to the proof of proposition 3.1 on spherical matrix coefficients, which is based on two local estimates on spherical matrix coefficients corresponding to the move $(0,1)$ and $(1,-1)$ in the Weyl chamber.

Lemma 3.3 *Suppose char$(F) \neq 2$. Let α be as in proposition 2.1. Let $\beta \in [0, \frac{\alpha}{2h})$. Then there exists $C' > 0$, such that for any $C \in \mathbb{R}_+^*$, any $(E, \pi) \in \mathcal{E}_{G,C+\beta\ell}$, and any K-invariant vectors $\xi \in E$, $\eta \in E^*$ of norm 1, and any $(i, j) \in \Lambda$ with $i - j \geq v_0 + 1$, we have*

$$|c(i,j) - c(i,j+1)| \leq C' e^{2C - (\frac{\alpha}{h} - 2\beta)i + \frac{\alpha}{h}j},$$

where C' is a constant depending on q, h, v_0, α, β.

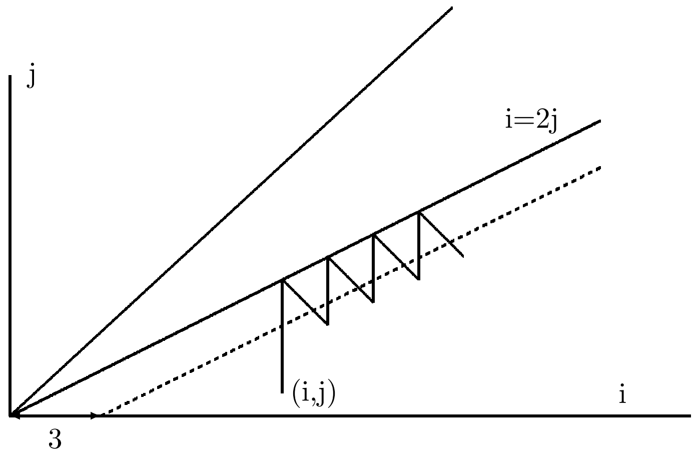
Lemma 3.4 *Let F be of any characteristic. Let α be as in proposition 2.1, and $\beta \in [0, \frac{\alpha}{h})$. Then there exists $C' > 0$, such that for any $C \in \mathbb{R}_+^*$, any $(E, \pi) \in \mathcal{E}_{G,C+\beta\ell}$, and any K-invariant vectors $\xi \in E$, $\eta \in E^*$ of norm 1, and $(i,j) \in \Lambda$ with $j \geq 2$, we have*

$$|c(i,j) - c(i+1, j-1)| \leq C' e^{2C+\beta i - (\frac{\alpha}{h} - \beta) j}.$$

Proof of proposition 3.1 assuming lemma 3.3 and 3.4: We adopt the zig-zag argument from [Laf08] to Sp_4. We put

$$S_\alpha = \{(i,j) \in \Lambda | 0 \leq i - 2j \leq \alpha\}.$$

First we move any $(i,j) \in \Lambda$ to the strip S_3. Then we show that we can move any $(i,j) \in S_3$ to the line $i = 2j$ using the moves inside S_4, and then we move (i,j) to infinity along this line as illustrated below.



Precisely, when $i \geq 2j \geq 0$, we have $(i, \lfloor i/2 \rfloor) \in S_2 \subset S_3$ and

$$\begin{aligned}
&|c(i,j) - c(i, \lfloor i/2 \rfloor)| \\
&\leq C' e^{2C - (\frac{\alpha}{h} - 2\beta) i + \frac{\alpha}{h} j} + \ldots + C' e^{2C - (\frac{\alpha}{h} - 2\beta) i + \frac{\alpha}{h} (\lfloor i/2 \rfloor - 1)} \\
&\leq C' e^{2C - (\frac{\alpha}{2h} - 2\beta) i}.
\end{aligned} \tag{3}$$

When $2j \geq i \geq 0$, we have $\left(i + \lceil \frac{2j-i}{3} \rceil, j - \lceil \frac{2j-i}{3} \rceil\right) \in S_3$, and

$$\begin{aligned}
&\left| c(i,j) - c(i + \lceil \frac{2j-i}{3} \rceil, j - \lceil \frac{2j-i}{3} \rceil) \right| \\
&\leq C' e^{2C - (\frac{\alpha}{h} - \beta) i + \beta j} + \ldots + C' e^{2C - (\frac{\alpha}{h} - \beta)(i + \lceil \frac{2j-i}{3} \rceil - 1) + \beta (j - \lceil \frac{2j-i}{3} \rceil + 1)} \\
&\leq C' e^{2C - (\frac{\alpha}{3h} - \beta)(i+j)}.
\end{aligned} \tag{4}$$

For any $(i,j) \in S_3$, if $i \in 2\mathbb{N}+k, k \in \{0,1\}$ then

$$\big|c(i,j) - c\big(i+k,(i+k)/2\big)\big| \leq C' e^{2C-(\frac{\alpha}{2h}-2\beta)i}. \tag{5}$$

In fact, by lemmas 3.3 and 3.4, when $(i,j) \in S_4$ we have

$$\max\Big(|c(i,j)-c(i,j+1)|, |c(i,j)-c(i+1,j-1)|\Big) \leq C' e^{2C-(\frac{\alpha}{2h}-2\beta)i}.$$

When $i \in 2\mathbb{N}$ and $(i,j) \in S_3$, we get inequality (5) by considering the move $(i,j) \mapsto (i,i/2)$. When $i \in 2\mathbb{N}+1$ and $(i,j) \in S_3$, there exists $k \in \{0,1\}$, suth that $(i+1,j+k-1) \in S_4$. Therefore, we obtain inequality (5) by considering the following moves inside S_4 : $(i,j) \mapsto (i,j+k) \mapsto (i+1,j+k-1) \mapsto (i+1,(i+1)/2)$.

Combining inequalities (3), (4) and (5) we obtain: when $i \geq 2j \geq 0$, and $i \in 2\mathbb{N}+k, k \in \{0,1\}$,

$$|c(i,j) - c(i+k,(i+k)/2)| \leq C' e^{2C-(\frac{\alpha}{2h}-2\beta)i}; \tag{6}$$

when $2j \geq i \geq j \geq 0$, there exists $k \in \{0,1,2\}$ such that

$$|c(i,j) - c\big(\lfloor\frac{2}{3}(i+j)\rfloor + k, \frac{1}{2}(\lfloor\frac{2}{3}(i+j)\rfloor + k)\big)| \leq C' e^{2C-(\frac{\alpha}{2h}-2\beta)i}. \tag{7}$$

Finally for any $j \geq 0$, we have

$$|c(2j,j) - c(2j+2,j+1)| \leq C' e^{2C-(\frac{\alpha}{2h}-2\beta)2j}.$$

Proposition 3.1 is then proved. □

It remains to prove lemmas 3.3 and 3.4. To prove these two lemmas, we use the following lemma in [Laf09] which is a variant of fast Fourier transform.

Lemma 3.5 *(lemma 4.4 in [Laf09]) Let $\chi : \mathbb{F} \to \mathbb{C}^*$ be a non trivial character. Let $h \in \mathbb{N}^*, \alpha \in \mathbb{R}_+^*, n \in \mathbb{N}^*$. Let E be a Banach space such that $\|T_{\mathcal{O}/\pi^h\mathcal{O}} \otimes 1_E\| \leq e^{-\alpha}$, and let $(\xi_{x,y})_{x,y\in\mathcal{O}/\pi^n\mathcal{O}}$ be a family of vectors of E. Then*

$$\mathop{\mathbb{E}}_{a,b\in\mathcal{O}/\pi^n\mathcal{O}} \Big\| \mathop{\mathbb{E}}_{x\in\mathcal{O}/\pi^n\mathcal{O},\varepsilon\in\mathbb{F}} \chi(\varepsilon)\xi_{x,ax+b+\pi^{n-1}\varepsilon} \Big\|^2 \leq q^{2h-2} e^{-2(\frac{n}{h}-1)\alpha} \mathop{\mathbb{E}}_{x,y\in\mathcal{O}/\pi^n\mathcal{O}} \|\xi_{x,y}\|^2.$$

Proof of lemma 3.3: Denote $m = \lfloor\frac{i+j}{2}\rfloor$, and $n_1 = 2m-2j-v_0$. Let $x,y,a,b \in \mathcal{O}/\pi^{n_1}\mathcal{O}$, and let $\sigma : \mathcal{O}/\pi^{n_1}\mathcal{O} \to \mathcal{O}$ be a section. Let $\beta(a,b)^{-1}, \alpha(x,y)$ be the elements in G defined as follows,

$$\beta(a,b)^{-1} = \begin{pmatrix} \pi^m & & & \\ & \pi^{i-m+j} & & \\ & & \pi^{-i+m-j} & \\ & & & \pi^{-m} \end{pmatrix} \cdot \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ \sigma(a) & 1 & 1 & \\ \sigma(a)^2 - 2\sigma(b) & \sigma(a) & 0 & 1 \end{pmatrix},$$

$$\alpha(x,y)=\begin{pmatrix}1&&&\\0&1&&\\\sigma(x)&0&1&\\\sigma(x)^2+2\sigma(y)&\sigma(x)&0&1\end{pmatrix}\cdot\begin{pmatrix}\pi^{-m+j}&&&\\&\pi^{-m+j}&&\\&&\pi^{m-j}&\\&&&\pi^{m-j}\end{pmatrix}.$$

Then

$$\beta(a,b)^{-1}\alpha(x,y)=\begin{pmatrix}\pi^m&&&\\&\pi^{i-m+j}&&\\&&\pi^{-i+m-j}&\\&&&\pi^{-m}\end{pmatrix}\times$$

$$\begin{pmatrix}1&&&\\0&1&&\\\sigma(a)+\sigma(x)&1&1&\\\sigma(a)^2-2\sigma(b)+\sigma(x)^2+2\sigma(y)&\sigma(a)+\sigma(x)&0&1\end{pmatrix}\cdot\begin{pmatrix}\pi^{-m+j}&&&\\&\pi^{-m+j}&&\\&&\pi^{m-j}&\\&&&\pi^{m-j}\end{pmatrix}.$$

Recall from the second section that for any $g\in KD(k,l)K$, q^k is the biggest norm of all matrix elements in g, and q^{k+l} is the biggest norm of all 2×2 minors of g. It is easy to see that

$$\|\Lambda^2\big(\beta(a,b)\big)\|=q^{i+j},\|\Lambda^2\big(\alpha(x,y)\big)\|=q^{2m-2j},$$

and

$$\|\beta(a,b)^{-1}\alpha(x,y)\|=q^i.$$

On the other hand, we calculate the minor of rows 3, 4 and columns 1, 2,

$$\det\Big(\begin{pmatrix}\pi^{-i+m-j}&\\&\pi^{-m}\end{pmatrix}\begin{pmatrix}\sigma(a)+\sigma(x)&1\\\sigma(a)^2-2\sigma(b)+\sigma(x)^2+2\sigma(y)&\sigma(a)+\sigma(x)\end{pmatrix}\times$$
$$\begin{pmatrix}\pi^{-m+j}&\\&\pi^{-m+j}\end{pmatrix}\Big)=-2\pi^{-i-2m+j}\big(\sigma(y)-\sigma(a)\sigma(x)-\sigma(b)\big).$$

Since the norm of the minor of rows 3, 4 and columns 2, 4 is q^{i+j}, we have

$$\|\Lambda^2\big(\beta(a,b)^{-1}\alpha(x,y)\big)\|=\max(q^{i+2m-j-v},q^{i+j}),$$

where $v\in\{0,1,\ldots,2m-2j\}$ is the valuation of $2(y-ax-b)\in\mathcal{O}/\pi^{2m-2j}\mathcal{O}$. Let $y=ax+b+\pi^{n_1-1}\varepsilon$, where $\varepsilon\in\mathbb{F}$. When $\varepsilon=0$, we have $v=2m-2j$ and

$$\beta(a,b)^{-1}\alpha(x,y)\in KD(i,j)K.$$

When $\varepsilon\in\mathbb{F}^*$ we have $v=2(m-j)-1$, and then

$$\beta(a,b)^{-1}\alpha(x,y)\in KD(i,j+1)K.$$

Let $\chi : \mathbb{F} \to \mathbb{C}^*$ be a non trivial character. By Cauchy-Schwarz inequality and lemma 3.5 we have

$$\begin{aligned}
&|c(i,j) - c(i,j+1)| \\
&= q|\mathop{\mathbb{E}}_{a,b,x\in\mathcal{O}/\pi^{n_1}\mathcal{O},\varepsilon\in\mathbb{F}} \chi(\varepsilon)\langle {}^t\pi\big(\beta(a,b)\big)\eta, \pi\big(\alpha(x, ax+b+\pi^{n_1-1}\varepsilon)\big)\xi\rangle| \\
&\leq q\sqrt{\mathop{\mathbb{E}}_{a,b\in\mathcal{O}/\pi^{n_1}\mathcal{O}} \|{}^t\pi\big(\beta(a,b)\big)\eta\|^2} \times \\
&\sqrt{\mathop{\mathbb{E}}_{a,b\in\mathcal{O}/\pi^{n_1}\mathcal{O}} \|\mathop{\mathbb{E}}_{x\in\mathcal{O}/\pi^{n_1}\mathcal{O},\varepsilon\in\mathbb{F}} \chi(\varepsilon)\pi\big(\alpha(x, ax+b+\pi^{n_1-1}\varepsilon)\big)\xi\|^2} \\
&\leq q e^{C+\beta(i+j)} \cdot q^{h-1} \cdot e^{-(\frac{n_1}{h}-1)\alpha} \cdot e^{C+2\beta(m-j)}. \\
&\leq q^h \cdot e^{(\frac{v_0+2}{h}+1)\alpha} \cdot e^{2C-(\frac{\alpha}{h}-2\beta)i+\frac{\alpha}{h}j},
\end{aligned}$$

and the lemma follows immediately. □

Proof of lemma 3.4: Let $x, y, a, b \in \mathcal{O}/\pi^{j-1}\mathcal{O}$, and let $\sigma : \mathcal{O}/\pi^{j-1}\mathcal{O} \to \mathcal{O}$ be a section. Define

$$\beta(a,b)^{-1} = \begin{pmatrix} \pi^i & & & \\ & 1 & & \\ & & 1 & \\ & & & \pi^{-i} \end{pmatrix} \cdot \begin{pmatrix} 1 & & & \\ 1+\pi\sigma(a) & 1 & & \\ 0 & 0 & 1 & \\ -\pi\sigma(b) & 0 & -1-\pi\sigma(a) & 1 \end{pmatrix} \in G,$$

$$\alpha(x,y) = \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ \sigma(x) & 0 & 1 & \\ \pi\sigma(y)+\sigma(x) & \sigma(x) & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \pi^{-j} & & & \\ & 1 & & \\ & & 1 & \\ & & & \pi^j \end{pmatrix} \in G.$$

Then we have

$$\beta(a,b)^{-1}\alpha(x,y) =$$

$$\begin{pmatrix} \pi^i & & & \\ & 1 & & \\ & & 1 & \\ & & & \pi^{-i} \end{pmatrix} \cdot \begin{pmatrix} 1 & & & \\ 1+\pi\sigma(a) & 1 & & \\ \sigma(x) & 0 & 1 & \\ \pi(\sigma(y)-\sigma(a)\sigma(x)-\sigma(b)) & \sigma(x) & -1-\pi\sigma(a) & 1 \end{pmatrix} \times$$
$$\begin{pmatrix} \pi^{-j} & & & \\ & 1 & & \\ & & 1 & \\ & & & \pi^j \end{pmatrix}.$$

Firstly, we see that

$$\|\Lambda^2\big(\beta(a,b)\big)\| = q^i, \|\Lambda^2\big(\alpha(x,y)\big)\| = q^j,$$

and

$$\|\Lambda^2\big(\beta(a,b)^{-1}\alpha(x,y)\big)\| = q^{i+j},$$

which is the norm of the determinant of the submatrix of rows 2, 4 and columns 1, 3. Denote the valuation of $y - ax - b \in \mathcal{O}/\pi^{j-1}\mathcal{O}$ by $v \in \{0, 1, \dots, j-1\}$, and we have

$$\|\beta(a,b)^{-1}\alpha(x,y)\| = \max(q^i, q^{i+j-v-1}).$$

Let

$$y = ax + b + \pi^{j-2}\varepsilon, \varepsilon \in \mathbb{F}\cdot$$

When $\varepsilon = 0$, we see that $v = j - 1$ and

$$\beta(a,b)^{-1}\alpha(x, ax+b) \in KD(i,j)K.$$

When $\varepsilon \in \mathbb{F}^*$ we have $v = j - 2$, and therefore

$$\beta(a,b)^{-1}\alpha(x, ax + b + \pi^{j-2}\varepsilon) \in KD(i+1, j-1)K.$$

Let $\chi : \mathbb{F} \to \mathbb{C}^*$ be a non trivial character. By the same estimates as in the end of the proof of lemma 3.3 (n_1 replaced by $j-1$), we have

$$\begin{aligned}&|c(i,j) - c(i+1,j-1)|\\ \le& qe^{C+\beta j} \cdot q^{h-1} \cdot e^{-(\frac{j-1}{h}-1)\alpha} \cdot e^{C+\beta i}\\ =& q^h \cdot e^{(\frac{1}{h}+1)\alpha} \cdot e^{2C+\beta i-(\frac{\alpha}{h}-\beta)j}.\end{aligned}$$

□

As for proposition 3.2, we need two similar lemmas as follows for its proof.

Lemma 3.6 *Suppose char(F) $\neq 2$. Let α be as in proposition 2.1, $\beta \in [0, \frac{\alpha}{2h})$, and (V, τ) a non trivial irreducible unitary representation of K which factorizes through $Sp_4(\mathcal{O}/\pi^k\mathcal{O})$ for $k \ge 1$. There exists $C' > 0$, such that the following holds. Let $C \in \mathbb{R}_+^*$, (E, π) any element in $\mathcal{E}_{G,C+\beta\ell}$, and $\xi \in E$, $\eta \in V \otimes E^*$ any K-invariant vectors of norm 1. Then for any $(i,j) \in \Lambda$ with $i - j \ge 2k + v_0$, we have*

$$\|c(i,j) - c(i,j+1)\|_V \le C'e^{2C-(\frac{\alpha}{h}-2\beta)i+\frac{\alpha}{h}j}.$$

Lemma 3.7 *Let F be of any characteristic. Let α be as in proposition 2.1, $\beta \in [0, \frac{\alpha}{h})$, and (V, τ) a non trivial irreducible unitary representation of K which factorizes through $Sp_4(\mathcal{O}/\pi^k\mathcal{O})$ for $k \ge 1$. There exists $C' > 0$, such that the following holds. Let $C \in \mathbb{R}_+^*$, (E, π) any element in $\mathcal{E}_{G,C+\beta\ell}$, and $\xi \in E$, $\eta \in V \otimes E^*$ any K-invariant vectors of norm 1. Then for any $(i,j) \in \mathbb{Z}^2$ with $i + 1 \ge j \ge 2k + 2$, we have*

$$\|c(i,j) - c(i+1,j-1)\|_V \le C'e^{2C+\beta i-(\frac{\alpha}{h}-\beta)j}.$$

In particular,

$$\|c(j-1,j) - c(j,j-1)\|_V \le C'e^{2C-(\frac{\alpha}{h}-2\beta)j}.$$

Lemma 3.8 *Let h, α, n, E as in lemma 3.5. Let $k \in \{0, ..., \lfloor n/2 \rfloor\}$, $\varepsilon_0 \in \mathbb{F}^*$, and let $(\xi_{x,y})_{x\in\pi^k\mathcal{O}/\pi^n\mathcal{O}, y\in\pi^{2k}\mathcal{O}/\pi^n\mathcal{O}}$ be a family of vectors of E. Then there exists a constant C_2 depending only on q, such that*

$$\mathop{\mathbb{E}}_{a\in\pi^k\mathcal{O}/\pi^n\mathcal{O}, b\in\pi^{2k}\mathcal{O}/\pi^n\mathcal{O}} \Big\| \mathop{\mathbb{E}}_{x\in\pi^k\mathcal{O}/\pi^n\mathcal{O}} \xi_{x,ax+b+\pi^{n-1}\varepsilon_0} - \mathop{\mathbb{E}}_{x\in\pi^k\mathcal{O}/\pi^n\mathcal{O}} \xi_{x,ax+b} \Big\|^2$$

$$\leq C_2 q^{2h-2} e^{-2(\frac{n-2k}{h}-1)\alpha} \mathop{\mathbb{E}}_{x\in\pi^k\mathcal{O}/\pi^n\mathcal{O}, y\in\pi^{2k}\mathcal{O}/\pi^n\mathcal{O}} \|\xi_{x,y}\|^2.$$

Proof: When $k = 0$, let f be the function on $\mathbb{F}$ defined by $f(\varepsilon_0) = q$, $f(0) = -q$, and zero elsewhere. The left hand side of the inequality is equal to

$$\mathop{\mathbb{E}}_{a,b\in\mathcal{O}/\pi^n\mathcal{O}} \| \mathop{\mathbb{E}}_{x\in\mathcal{O}/\pi^n\mathcal{O}, \varepsilon\in\mathbb{F}} f(\varepsilon)\xi_{x,ax+b+\pi^{n-1}\varepsilon}\|^2.$$

Write $f = \sum_{\chi\in\widehat{\mathbb{F}}, \chi\neq 1} f_\chi \chi$ with $f_\chi \in \mathbb{C}$, then by the triangular inequality and and lemma 3.5, the left hand side is equal to

$$\mathop{\mathbb{E}}_{a,b\in\mathcal{O}/\pi^n\mathcal{O}} \| \sum_{\chi\in\widehat{\mathbb{F}}, \chi\neq 1} f_\chi \mathop{\mathbb{E}}_{x\in\mathcal{O}/\pi^n\mathcal{O}, \varepsilon\in\mathbb{F}} \chi(\varepsilon)\xi_{x,ax+b+\pi^{n-1}\varepsilon}\|^2$$

$$\leq C_2 \max_{\chi\in\widehat{\mathbb{F}}, \chi\neq 1} \mathop{\mathbb{E}}_{a,b\in\mathcal{O}/\pi^n\mathcal{O}} \| \mathop{\mathbb{E}}_{x\in\mathcal{O}/\pi^n\mathcal{O}, \varepsilon\in\mathbb{F}} \chi(\varepsilon)\xi_{a,ax+b+\pi^{n-1}\varepsilon}\|^2$$

$$\leq C_2 q^{2h-2} e^{-2(\frac{n}{h}-1)\alpha} \mathop{\mathbb{E}}_{a,b\in\mathcal{O}/\pi^n\mathcal{O}} \|\xi_{x,y}\|^2,$$

where $C_2 = \big(\sum_{\chi\in\widehat{\mathbb{F}}, \chi\neq 1} |f_\chi|\big)^2$.

In general, let $s : \mathcal{O}/\pi^{n-2k}\mathcal{O} \to \mathcal{O}/\pi^{n-k}\mathcal{O}$ be a section, and for any $x_1, y_1 \in \mathcal{O}/\pi^{n-2k}\mathcal{O}$ let

$$\xi'_{x_1,y_1} = \mathop{\mathbb{E}}_{z\in\pi^{n-2k}\mathcal{O}/\pi^{n-k}\mathcal{O}} \xi_{\pi^k(s(x_1)+z), \pi^{2k}y_1}.$$

For any $a, x \in \pi^k\mathcal{O}/\pi^n\mathcal{O}$, the product $ax \in \mathcal{O}/\pi^n\mathcal{O}$ only depends on the images of a, x in $\pi^k\mathcal{O}/\pi^{n-k}\mathcal{O}$. So the left hand side of the inequality is equal to

$$\mathop{\mathbb{E}}_{a_1,b_1\in\mathcal{O}/\pi^{n-2k}\mathcal{O}} \| \mathop{\mathbb{E}}_{x_1\in\mathcal{O}/\pi^{n-2k}\mathcal{O}} \big(\xi'_{x_1,a_1x_1+b_1+\pi^{n-2k-1}\varepsilon_0} - \xi'_{x_1,a_1x_1+b_1}\big)\|^2.$$

By applying the lemma when $k = 0$ to $(\xi'_{x_1,y_1})_{x_1,y_1\in\mathcal{O}/\pi^{n-2k}\mathcal{O}}$ we get the inequality in the lemma with the same C_2. □

Proof of lemma 3.6:

Let $m, x, y, a, b, \varepsilon, \sigma, \alpha(x,y), \beta(a,b)$ be as in the proof of lemma 3.3. We recall also from the proof that

$$\|\Lambda^2\big(\beta(a,b)\big)\| = q^{i+j}, \|\Lambda^2\big(\alpha(x,y)\big)\| = q^{2m-2j}.$$

Let ε_0 be image of $\pi^{v_0}/2$ in $\mathbb{F}^*$, and let

$$\varepsilon_1 = 2\pi^{-2m+2j+1}(\sigma(y) - \sigma(a)\sigma(x) - \sigma(b)) \in \mathcal{O}.$$

Recall that $y = ax + b + \pi^{2m-2j-v_0-1}\varepsilon$, we have $\varepsilon_1 \bmod \pi\mathcal{O} = \varepsilon_0^{-1}\varepsilon$. Let k_1 be the element in K defined by

$$k_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -\pi^{i-2m+j}(\sigma(a)+\sigma(x)) & 1 \\ -1 & 0 & -\pi^{i-2m+3j+1}(\sigma(a)+\sigma(x)) & \pi^{2j+1} \\ -\pi^{i-2m+j}(\sigma(a)+\sigma(x)) & -1 & \pi^{2i-2m+2j} & 0 \end{pmatrix},$$

and let

$$g_1 = k_1\beta(a,b)^{-1}\alpha(x,y)$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -\pi^{i-2m+j}(\sigma(a)+\sigma(x)) & 1 \\ -1 & 0 & -\pi^{i-2m+3j+1}(\sigma(a)+\sigma(x)) & \pi^{2j+1} \\ -\pi^{i-2m+j}(\sigma(a)+\sigma(x)) & -1 & \pi^{2i-2m+2j} & 0 \end{pmatrix} \times$$

$$\begin{pmatrix} \pi^j & & & \\ 0 & \pi^{i-2m+2j} & & \\ \pi^{-i}(\sigma(a)+\sigma(x)) & \pi^{-i} & \pi^{-i+2m-2j} & \\ \pi^{-2m+j}\big(\sigma(a)+\sigma(x)\big)^2 + \pi^{-j-1}\varepsilon_1 & \pi^{-2m+j}(\sigma(a)+\sigma(x)) & 0 & \pi^{-j} \end{pmatrix}$$

$$= \begin{pmatrix} \pi^{-i}(\sigma(a)+\sigma(x)) & \pi^{-i} & \pi^{-i+2m-2j} & 0 \\ \pi^{-j-1}\varepsilon_1 & 0 & -\pi^{-j}(\sigma(a)+\sigma(x)) & \pi^{-j} \\ \pi^j(\varepsilon_1 - 1) & 0 & -\pi^{j+1}(\sigma(a)+\sigma(x)) & \pi^{j+1} \\ 0 & 0 & \pi^i & 0 \end{pmatrix}.$$

When $\varepsilon = 0$, we have

$$\begin{pmatrix} \pi^i & & & \\ & \pi^j & & \\ & & \pi^{-j} & \\ & & & \pi^{-i} \end{pmatrix} g_1 = \begin{pmatrix} \sigma(a)+\sigma(x) & 1 & \pi^{2m-2j} & 0 \\ \pi^{-1}\varepsilon_1 & 0 & -(\sigma(a)+\sigma(x)) & 1 \\ \varepsilon_1 - 1 & 0 & -\pi(\sigma(a)+\sigma(x)) & \pi \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

which is an element in K. When $\varepsilon = \varepsilon_0$, we have

$$\begin{pmatrix} \pi^i & & & \\ & \pi^{j+1} & & \\ & & \pi^{-j-1} & \\ & & & \pi^{-i} \end{pmatrix} g_1 = \begin{pmatrix} \sigma(a)+\sigma(x) & 1 & \pi^{2m-2j} & 0 \\ \varepsilon_1 & 0 & -\pi(\sigma(a)+\sigma(x)) & \pi \\ \pi^{-1}(\varepsilon_1 - 1) & 0 & -(\sigma(a)+\sigma(x)) & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

which is also in K. Denote $\xi_{x,y} = \pi(\alpha(x,y))\xi$, $\eta_{a,b} = \big(\mathrm{Id}_V \otimes {}^t\pi(\beta(a,b))\big)\eta$ and $n_1 = 2(m-j) - v_0$. Note that $c(k'gk'') = \tau(k')c(g)$ for any

$k', k'' \in K, g \in G$, we then have

$$\|c(i,j) - c(i,j+1)\|_V$$

$$= q\Big\|\underset{a,x\in\pi^k\mathcal{O}/\pi^{n_1}\mathcal{O}, b\in\pi^{2k}\mathcal{O}/\pi^{n_1}\mathcal{O}}{\mathbb{E}} \tau(k_1)\Big(\langle\eta_{a,b}, \xi_{x,ax+b+\pi^{n_1-1}\varepsilon_0}\rangle - \langle\eta_{a,b}, \xi_{x,ax+b}\rangle\Big)\Big\|_V. \tag{8}$$

When $i - j \geq k + v_0$ and $a, x \in \pi^k\mathcal{O}/\pi^{n_1}\mathcal{O}$, we have

$$k_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & \pi^{2j+1} \\ 0 & -1 & 0 & 0 \end{pmatrix} \bmod \pi^k\mathcal{O},$$

so (8) becomes

$$q\Big\|\underset{a,x\in\pi^k\mathcal{O}/\pi^{n_1}\mathcal{O}, b\in\pi^{2k}\mathcal{O}/\pi^{n_1}\mathcal{O}}{\mathbb{E}} \Big(\langle\eta_{a,b}, \xi_{x,ax+b+\pi^{n_1-1}\varepsilon_0}\rangle - \langle\eta_{a,b}, \xi_{x,ax+b}\rangle\Big)\Big\|_V.$$

By Cauchy-Schwarz inequality and lemma 3.8 (when $i - j \geq 2k + v_0$), it is less than

$$\leq q\sqrt{\underset{a\in\pi^k\mathcal{O}/\pi^{n_1}\mathcal{O}, b\in\pi^{2k}\mathcal{O}/\pi^{n_1}\mathcal{O}}{\mathbb{E}} \|\eta_{a,b}\|^2} \times$$

$$\sqrt{\underset{a\in\pi^k\mathcal{O}/\pi^{n_1}\mathcal{O}, b\in\pi^{2k}\mathcal{O}/\pi^{n_1}\mathcal{O}}{\mathbb{E}} \|\underset{x\in\pi^k\mathcal{O}/\pi^{n_1}\mathcal{O}}{\mathbb{E}} \xi_{x,ax+b+\pi^{n_1-1}\varepsilon_0} - \underset{x\in\pi^k\mathcal{O}/\pi^{n_1}\mathcal{O}}{\mathbb{E}} \xi_{x,ax+b}\|^2}$$

$$\leq qe^{C+\beta(i+j)} \cdot \sqrt{C_2} q^{h-1} \cdot e^{-(\frac{n_1-2k}{h}-1)\alpha} \cdot e^{C+2\beta(m-j)}$$

$$\leq \sqrt{C_2} q^h \cdot e^{(\frac{v_0+2+2k}{h}+1)\alpha} \cdot e^{2C-(\frac{\alpha}{h}-2\beta)i+\frac{\alpha}{h}j}.$$

□

Proof of lemma 3.7:

Let $x, y, a, b, \varepsilon, \sigma, \alpha(x,y), \beta(a,b)$ be as in the proof of lemma 3.4. From the proof we have

$$\|\Lambda^2\big(\beta(a,b)\big)\| = q^i, \|\Lambda^2\big(\alpha(x,y)\big)\| = q^j.$$

Denote $\varepsilon_1 = \pi^{-j+2}(\sigma(y) - \sigma(a)\sigma(x) - \sigma(b)) \in \mathcal{O}$, and we have $\varepsilon_1 \bmod \pi\mathcal{O} = \varepsilon$. Denote $a_1 = 1 + \pi\sigma(a) \in \mathcal{O}$. For any $i + 1 \geq j \geq 1$, let k_1 be the element in K defined by

$$k_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\pi^{i-j+1}a_1 \\ 0 & -\pi^{j-1}a_1^{-2}\varepsilon_1 - a_1^{-1}\sigma(x) & 1 & \pi^i a_1^{-1} \\ -1 & \pi^i a_1^{-1}(1-\varepsilon_1) - \pi^{i-j+1}\sigma(x) & \pi^{i-j+1}a_1 & \pi^{2i-j+1} \end{pmatrix}.$$

Denote

$$g_1 = k_1\beta(a,b)^{-1}\alpha(x,y)$$

$$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\pi^{i-j+1}a_1 \\ 0 & -\pi^{j-1}a_1^{-2}\varepsilon_1 - a_1^{-1}\sigma(x) & 1 & \pi^i a_1^{-1} \\ -1 & \pi^i a_1^{-1}(1-\varepsilon_1) - \pi^{i-j+1}\sigma(x) & \pi^{i-j+1}a_1 & \pi^{2i-j+1} \end{pmatrix} \times$$

$$\begin{pmatrix} \pi^{i-j} & & & \\ \pi^{-j}a_1 & 1 & & \\ \pi^{-j}\sigma(x) & 0 & 1 & \\ \pi^{-i-1}\varepsilon_1 & \pi^{-i}\sigma(x) & -\pi^{-i}a_1 & \pi^{-i+j} \end{pmatrix}$$

$$= \begin{pmatrix} \pi^{-i-1}\varepsilon_1 & \pi^{-i}\sigma(x) & -\pi^{-i}a_1 & \pi^{-i+j} \\ \pi^{-j}a_1(1-\varepsilon_1) & 1 - \pi^{-j+1}a_1\sigma(x) & \pi^{-j+1}a_1^2 & -\pi a_1 \\ 0 & -\pi^{j-1}a_1^{-2}\varepsilon_1 & 0 & \pi^j a_1^{-1} \\ 0 & \pi^i a_1^{-1}(1-\varepsilon_1) & 0 & \pi^{i+1} \end{pmatrix}$$

When $\varepsilon = 0$, we have

$$\begin{pmatrix} \pi^i & & & \\ & \pi^j & & \\ & & \pi^{-j} & \\ & & & \pi^{-i} \end{pmatrix} g_1$$

$$= \begin{pmatrix} \pi^{-1}\varepsilon_1 & \sigma(x) & -a_1 & \pi^j \\ a_1(1-\varepsilon_1) & \pi^j - \pi a_1\sigma(x) & \pi a_1^2 & -\pi^{j+1}a_1 \\ 0 & -\pi^{-1}a_1^{-2}\varepsilon_1 & 0 & a_1^{-1} \\ 0 & a_1^{-1}(1-\varepsilon_1) & 0 & \pi \end{pmatrix} \in K.$$

When $\varepsilon = 1$, we have

$$\begin{pmatrix} \pi^{i+1} & & & \\ & \pi^{j-1} & & \\ & & \pi^{-j+1} & \\ & & & \pi^{-i-1} \end{pmatrix} g_1$$

$$= \begin{pmatrix} \varepsilon_1 & \pi\sigma(x) & -\pi a_1 & \pi^{j+1} \\ \pi^{-1}a_1(1-\varepsilon_1) & \pi^{j-1} - a_1\sigma(x) & a_1^2 & -\pi^j a_1 \\ 0 & -a_1^{-2}\varepsilon_1 & 0 & \pi a_1^{-1} \\ 0 & \pi^{-1}a_1^{-1}(1-\varepsilon_1) & 0 & 1 \end{pmatrix} \in K$$

When $j \geq 2k+2$ and $a, x \in \pi^k\mathcal{O}/\pi^{j-1}\mathcal{O}$, we have

$$k_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\pi^{i-j+1} \\ 0 & 0 & 1 & 0 \\ -1 & 0 & \pi^{i-j+1} & 0 \end{pmatrix} \mod \pi^k\mathcal{O}.$$

By the same estimates (with n_1 replaced by $j-1$ and ε_0 by $1 \in \mathbb{F}^*$) as in the end of the proof of lemma 3.6 we have

$$\|c(i,j) - c(i+1,j-1)\|_V \leq q e^{C+\beta j} \cdot \sqrt{C_2} q^{h-1} \cdot e^{-(\frac{j-1-2k}{h}-1)\alpha} \cdot e^{C+\beta i}$$

$$= \sqrt{C_2} q^h \cdot e^{(\frac{1+2k}{h}+1)\alpha} \cdot e^{2C+\beta i-(\frac{\alpha}{h}-\beta)j}.$$

□

Let K_1 be the subgroup of K consisting of elements of the form

$$\begin{pmatrix} * & * & & \\ * & * & & \\ & & * & * \\ & & * & * \end{pmatrix}$$

, and K_2 consisting of elements of the form

$$\begin{pmatrix} 1 & & & \\ & * & * & \\ & * & * & \\ & & & 1 \end{pmatrix},$$

i.e.

$$K_1 = \{\begin{pmatrix} A & \\ & Q^t A^{-1} Q \end{pmatrix} | A \in GL_2(\mathcal{O})\},$$

where $Q = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, and

$$K_2 = \{\begin{pmatrix} 1 & & \\ & B & \\ & & 1 \end{pmatrix} | B \in SL_2(\mathcal{O})\}.$$

Lemma 3.9 *Let F be of any characteristic. Then $K = (K_1K_2)^{30}$.*

Proof: Denote B the lower triangular matrices in K, and W the Weyl group associated to $G = Sp_4(F)$. Denote

$$w_{21} = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}, w_{32} = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & -1 & 0 & \\ & & & 1 \end{pmatrix}.$$

The dihedral group W (of order 8) is generated by w_{21} and w_{32}, which are reflections w.r.t. the axes $x = y$ and $x = 0$, respectively. Since $w_{21} \in K_1$ and $w_{32} \in K_2$ we obtain $W \subset (K_1K_2)^4$.

Denote for any $a \in \mathcal{O}$,

$$\mu_{21}(a) = \begin{pmatrix} 1 & 0 & & \\ a & 1 & & \\ & & 1 & 0 \\ & & -a & 1 \end{pmatrix}, \mu_{32}(a) = \begin{pmatrix} 1 & & & \\ & 1 & 0 & \\ & a & 1 & \\ & & & 1 \end{pmatrix},$$

$$\mu_{31}(a) = \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ a & 0 & 1 & \\ 0 & a & 0 & 1 \end{pmatrix}, \mu_{41}(a) = \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ a & 0 & 0 & 1 \end{pmatrix}.$$

By calculations we have

$$\mu_{41}(a) = w_{21}\mu_{32}(a)w_{21} \in (K_1K_2)^3$$

and

$$\mu_{31}(a) = \mu_{21}(-a)\mu_{32}(1)\mu_{21}(a)\mu_{32}(-1)\mu_{41}(-a^2) \in (K_1K_2)^7.$$

Any element in B has the form

$$\begin{pmatrix} 1 & & & \\ a & 1 & & \\ c & b & 1 & \\ d & c-ab & -a & 1 \end{pmatrix} \cdot \begin{pmatrix} e & & & \\ & f & & \\ & & f^{-1} & \\ & & & e^{-1} \end{pmatrix}$$

where $a, b, c, d \in \mathcal{O}$ and $e, f \in \mathcal{O}^\times$, which is equal to

$$\mu_{21}(a)\mu_{32}(b)\mu_{31}(c)\mu_{41}(ac+d) \cdot \begin{pmatrix} e & & & \\ & f & & \\ & & f^{-1} & \\ & & & e^{-1} \end{pmatrix}.$$

So we have $B \subset (K_1K_2)^{13}$.

By the Bruhat decomposition, we have $K = BWB = (K_1K_2)^{30}$.□

Lemma 3.10 *Let K be any compact group, $\{K_i\}_{1\leq i\leq n}$ a family of subgroups such that $K = (K_1K_2\dots K_n)^N$ for some $N \in \mathbb{N}^*$. Then for any finite dimensional unitary representation (V, τ) of K without invariant vector, and any $x \in V$, and $y_i \in V$ invariant by K_i for each $1 \leq i \leq n$, we have*

$$\|x\|_V \leq 2nN \max_{1\leq i\leq n} \{\|x - y_i\|_V\}.$$

Proof: Since $\int_K \|\tau(k)x - x\|_V^2 dk = 2\|x\|_V^2 \geq \|x\|_V^2$ we see that there exists a $k \in K$ such that $\|\tau(k)x - x\|_V \geq \|x\|_V$. Suppose that $k = (k_{11}\dots k_{n1})\dots(k_{1N}\dots k_{nN})$ with $k_{ij} \in K_i (1 \leq i \leq n, 1 \leq j \leq N)$. We then have

$$\|x\|_V \leq \|\tau(k)x - x\|_V \leq \sum_{1\leq i\leq n, 1\leq j\leq N} \|\tau(k_{ij})x - x\|_V$$

$$\leq 2 \sum_{1\leq i\leq n, 1\leq j\leq N} \|y_i - x\|_V \leq 2nN \max_{1\leq i\leq n} \{\|x - y_i\|_V\}$$

□

Proof of proposition 3.2: By lemmas 3.6 and 3.7, we obtain two similar inequalities as (6) and (7) in the proof of proposition 3.1 (using the same argument): when $i \geq 2j \geq 0$, and $i \in 2\mathbb{N}+k, k \in \{0,1\}$,

$$\|c(i,j) - c(i+k,(i+k)/2)\|_V \leq C' e^{2C-(\frac{\alpha}{2h}-2\beta)i};$$

when $2j \geq i \geq j \geq 0$, there exists $k \in \{0,1,2\}$ such that

$$\|c(i,j) - c\big(\lfloor \frac{2}{3}(i+j) \rfloor + k, \frac{1}{2}(\lfloor \frac{2}{3}(i+j) \rfloor + k)\big)\|_V \leq C' e^{2C-(\frac{\alpha}{2h}-2\beta)i}.$$

So it remains to prove

$$\|c(2j,j)\|_V \leq C' e^{2C-\frac{\alpha}{h}-2\beta)2j}.$$

First we see that

$$\max\Big(\|c(2j,j) - c(2j,0)\|_V, \|c(2j,j) - c(\lfloor 3j/2 \rfloor, \lfloor 3j/2 \rfloor)\|_V\Big) \leq C' e^{2C-(\frac{\alpha}{2h}-2\beta)2j}.$$

Moreover, $c(k'gk'') = \tau(k')c(g), \forall k', k'' \in K, g \in G$, and it follows that $c(\lfloor 3j/2 \rfloor, \lfloor 3j/2 \rfloor)$ is invariant by K_1, and that $c(2j,0)$ invariant by K_2. By applying lemma 3.10 to $K = (K_1K_2)^{30}$, we complete the proof of the proposition. □

4 Proof of theorem 2.3 when char$(F) = 2$

In this section we prove theorem 2.3 when $\mathrm{char}(F) = 2$. The proof for $\mathrm{char}(F) = 2$ is technically more difficult because it is only possible to prove a local estimate for the move $(0,2)$, and therefore we have two limits in the spherical propositions (proposition 4.2).

Throughout this section we assume F is of characteristic 2.

Lemma 4.1 *Let $\alpha > 0$ as in proposition 2.1, $\beta \in [0, \frac{\alpha}{4h})$. Let (V,τ) be an irreducible unitary representation of K which factorizes through $Sp_4(\mathcal{O}/\pi^k\mathcal{O})$ for $k \geq 0$. There exists $C' > 0$, such that the following holds. Let $C \in \mathbb{R}_+^*$, (E,π) any element in $\mathcal{E}_{G,C+\beta\ell}$, and $\xi \in E$, $\eta \in V \otimes E^*$ any K-invariant vectors of norm 1. Then for any $(i,j) \in \Lambda$ with $i - j \geq 4k+2$, we have*

$$\|c(i,j) - c(i,j+2)\|_V \leq C' e^{2C-(\frac{\alpha}{2h}-2\beta)i+\frac{\alpha}{2h}j}.$$

In particular when (V,τ) is the trivial representation of K (and $V = \mathbb{C}$), we have

$$|c(i,j) - c(i,j+2)| \leq C' e^{2C-(\frac{\alpha}{2h}-2\beta)i+\frac{\alpha}{2h}j},$$

for any $(i,j) \in \Lambda$ with $i - j \geq 2$.

Proof: Since $\mathrm{char}(F) = 2$, we have $-1 = 1$ in F. Let $m = \lfloor \frac{i+j}{2} \rfloor$, $x, y, a, b \in \mathcal{O}/\pi^{m-j-1}\mathcal{O}$ satisfying $y + ax + b \in \pi^{m-j-2}\mathcal{O}/\pi^{m-j-1}\mathcal{O}$, and put $\varepsilon \in \mathbb{F}$ with $y = ax+b+\pi^{m-j-2}\varepsilon$ as usual. Let $\sigma : \mathcal{O}/\pi^{m-j-1}\mathcal{O} \to \mathcal{O}$ be a section. Let

$$\beta(a,b)^{-1} = \begin{pmatrix} \pi^m & & & \\ 0 & \pi^{i-m+j} & & \\ \pi^{-i+m-j+1}\sigma(b) & \pi^{-i+m-j}(1+\pi\sigma(a))^2 & \pi^{-i+m-j} & \\ 0 & \pi^{-m+1}\sigma(b) & 0 & \pi^{-m} \end{pmatrix},$$

$$\alpha(x,y) = \begin{pmatrix} \pi^{-m+j} & & & \\ 0 & \pi^{-m+j} & & \\ \pi^{-m+j}(\sigma(x)+\pi\sigma(y)) & 0 & \pi^{m-j} & \\ \pi^{-m+j}\sigma(x)^2 & \pi^{-m+j}(\sigma(x)+\pi\sigma(y)) & 0 & \pi^{m-j} \end{pmatrix}.$$

Then

$$\beta(a,b)^{-1}\alpha(x,y) =$$

$$\begin{pmatrix} \pi^{j} & & & \\ 0 & \pi^{i-2m+2j} & & \\ \pi^{-i}(\pi\sigma(b)+\sigma(x)+\pi\sigma(y)) & \pi^{-i}(1+\pi\sigma(a))^2 & \pi^{-i+2m-2j} & \\ \pi^{-2m+j}\sigma(x)^2 & \pi^{-2m+j}(\pi\sigma(b)+\sigma(x)+\pi\sigma(y)) & 0 & \pi^{-j} \end{pmatrix}.$$

We see that

$$\|\Lambda^2\big(\beta(a,b)\big)\| = q^{i+j}, \|\Lambda^2\big(\alpha(x,y)\big)\| = q^{2m-2j}.$$

Denote

$$a_1 = (\pi\sigma(b)+\sigma(x)+\pi\sigma(y))(1+\pi\sigma(a))^{-2},$$

and

$$\varepsilon_1 = \pi^{-m+j+2}\big(\sigma(y)+\sigma(a)\sigma(x)+\sigma(b)\big) \in \mathcal{O}.$$

Let k_1 be the element in K defined by

$$k_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & \pi^{i-2m+j}a_1 & 1 \\ 1 & 0 & \pi^{i-2m+3j+2}a_1 & \pi^{2j+2} \\ \pi^{i-2m+j}a_1 & 1 & \pi^{2i-2m+2j}(1+\pi\sigma(a))^{-2} & 0 \end{pmatrix},$$

and let

$$g_1 = k_1\beta(a,b)^{-1}\alpha(x,y)$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & \pi^{i-2m+j}a_1 & 1 \\ 1 & 0 & \pi^{i-2m+3j+2}a_1 & \pi^{2j+2} \\ \pi^{i-2m+j}a_1 & 1 & \pi^{2i-2m+2j}(1+\pi\sigma(a))^{-2} & 0 \end{pmatrix} \times$$

$$\begin{pmatrix} \pi^{j} & & & \\ 0 & \pi^{i-2m+2j} & & \\ \pi^{-i}(\pi\sigma(b)+\sigma(x)+\pi\sigma(y)) & \pi^{-i}(1+\pi\sigma(a))^2 & \pi^{-i+2m-2j} & \\ \pi^{-2m+j}\sigma(x)^2 & \pi^{-2m+j}(\pi\sigma(b)+\sigma(x)+\pi\sigma(y)) & 0 & \pi^{-j} \end{pmatrix}$$

$$= \begin{pmatrix} \pi^{-i}a_1(1+\pi\sigma(a))^2 & \pi^{-i}(1+\pi\sigma(a))^2 & \pi^{-i+2m-2j} & 0 \\ \pi^{-j-2}\varepsilon_1^2(1+\pi\sigma(a))^{-2} & 0 & \pi^{-j}a_1 & \pi^{-j} \\ \pi^{j}\varepsilon_1^2(1+\pi\sigma(a))^{-2}+\pi^{j} & 0 & \pi^{j+2}a_1 & \pi^{j+2} \\ 0 & 0 & \pi^{i}(1+\pi\sigma(a))^{-2} & 0 \end{pmatrix}.$$

When $\varepsilon = 0$, we have $|\varepsilon_1^2| \leq q^{-2}$ and

$$\begin{pmatrix} \pi^{i} & & & \\ & \pi^{j} & & \\ & & \pi^{-j} & \\ & & & \pi^{-i} \end{pmatrix} g_1$$

$$= \begin{pmatrix} a_1(1+\pi\sigma(a))^2 & (1+\pi\sigma(a))^2 & \pi^{2m-2j} & 0 \\ \pi^{-2}\varepsilon_1^2(1+\pi\sigma(a))^{-2} & 0 & a_1 & 1 \\ \varepsilon_1^2(1+\pi\sigma(a))^{-2}+1 & 0 & \pi^2 a_1 & \pi^2 \\ 0 & 0 & (1+\pi\sigma(a))^{-2} & 0 \end{pmatrix} \in K.$$

When $\varepsilon = 1$, we have

$$|\varepsilon_1^2(1+\pi\sigma(a))^{-2}+1| = \left|\left(\varepsilon_1(1+\pi\sigma(a))^{-1}+1\right)^2\right| \leq q^{-2},$$

and then

$$\begin{pmatrix} \pi^{i} & & & \\ & \pi^{j+2} & & \\ & & \pi^{-j-2} & \\ & & & \pi^{-i} \end{pmatrix} g_1$$

$$= \begin{pmatrix} a_1(1+\pi\sigma(a))^2 & (1+\pi\sigma(a))^2 & \pi^{2m-2j} & 0 \\ \varepsilon_1^2(1+\pi\sigma(a))^{-2} & 0 & \pi^2 a_1 & \pi^2 \\ \pi^{-2}\left(\varepsilon_1^2(1+\pi\sigma(a))^{-2}+1\right) & 0 & a_1 & 1 \\ 0 & 0 & (1+\pi\sigma(a))^{-2} & 0 \end{pmatrix} \in K.$$

When $i-j \geq 4k+2$, $a, x \in \pi^k\mathcal{O}/\pi^{m-j-1}\mathcal{O}$, and $b, y \in \pi^{2k}\mathcal{O}/\pi^{m-j-1}\mathcal{O}$, we have

$$k_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & \pi^{2j+2} \\ 0 & 1 & 0 & 0 \end{pmatrix} \mod \pi^k\mathcal{O}.$$

By replacing n_1 by $m-j-1$ at the end of the proof of lemma 3.6 we get

$$\begin{aligned} &\|c(i,j) - c(i,j+2)\|_V \\ &\leq qe^{C+\beta(i+j)} \cdot \sqrt{C_2}q^{h-1} \cdot e^{-(\frac{m-j-1-2k}{h}-1)\alpha} \cdot e^{C+2\beta(m-j)} \\ &\leq C'e^{2C-(\frac{\alpha}{2h}-2\beta)i+\frac{\alpha}{2h}j}. \end{aligned}$$

$\square$

Proposition 4.2 *Let $\alpha > 0$, $\beta \in [0, \frac{\alpha}{8h})$. There exists $C' > 0$, such that the following holds. Let $C \in \mathbb{R}_+^*$, (E, π) any element in $\mathcal{E}_{G,C+\beta\ell}$, and $\xi \in E$, $\eta \in E^*$ any K-invariant vectors of norm 1. There exist $c_0, c_1 \in \mathbb{C}$, such that*

$$|c(i,j) - c_l| \leq C' e^{2C - (\frac{\alpha}{4h} - 2\beta)i},$$

for any $(i,j) \in \Lambda$ with $i + j \in 2\mathbb{N} + l, l = 0, 1$.

Proof: We apply the same argument as in the proof of proposition 3.1, using lemma 3.4 (which is still true in characteristic 2) and lemma 4.1 (in the particular case when (V, τ) is the trivial representation of K). We will get two limits because the moves $(i,j) \mapsto (i+1, j-1)$ and $(i,j) \mapsto (i, j+2)$ generate a sublattice of $\mathbb{Z}^2$ of index 2.

First, we put $S_\alpha = \{(i,j) \in \Lambda | 0 \leq i - 2j \leq \alpha\}$. When $0 \leq 2j \leq i$, we have $\big(i, j + 2\lfloor \frac{i-2j}{4} \rfloor\big) \in S_4$, and by the particular case of lemma 4.1 when (V, τ) is the trivial representation of K, we get

$$|c(i,j) - c(i, j + 2\lfloor \frac{i-2j}{4} \rfloor)| \leq C' e^{2C - (\frac{\alpha}{4h} - 2\beta)i}.$$

When $0 \leq i \leq 2j$, we have $\big(i + \lceil \frac{2j-i}{3} \rceil, j - \lceil \frac{2j-i}{3} \rceil\big) \in S_3 \subset S_4$. By lemmas 3.4 we have

$$\Big|c(i,j) - c(i + \lceil \frac{2j-i}{3} \rceil, j - \lceil \frac{2j-i}{3} \rceil)\Big| \leq C' e^{2C - (\frac{\alpha}{h} - 3\beta)\frac{i+j}{3}}.$$

Moreover, when $(i,j) \in S_4$, there exists $k \in \{0,1,2\}$ such that

$$|c(i,j) - c\big(i + k, \frac{1}{2}(i+k)\big)| \leq C' e^{2C - (\frac{\alpha}{4h} - 2\beta)i}.$$

In fact, when $(i,j) \in S_8$, we first have

$$\max\Big(|c(i,j) - c(i, j+2)|, |c(i,j) - c(i+1, j-1)|\Big) \leq C' e^{2C - (\frac{\alpha}{4h} - 2\beta)i}.$$

It suffices to show the inequality when $i - 2j \in \{1,2,3,4\}$, by considering the following moves inside S_8. When $i - 2j = 1$, we obtain the inequality by considering $(2j+1, j) \mapsto (2j+2, j-1) \mapsto (2j+2, j+1)$. When $i - 2j = 2$, we consider $(2j+2, j) \mapsto (2j+4, j-2) \mapsto (2j+4, j+2)$. When $i - 2j = 3$ or 4, use the moves $(2j+3, j) \mapsto (2j+2, j+1)$ and $(2j+4, j) \mapsto (2j+4, j+2)$ respectively.

In sum, when $i \geq 2j \geq 0$, there exists $k \in \{0,1,2\}$, such that

$$|c(i,j) - c(i+k, \frac{1}{2}(i+k))| \leq C' e^{2C - (\frac{\alpha}{4h} - 2\beta)i}; \tag{9}$$

when $2j \geq i \geq j \geq 0$, there exists $k \in \{0,1,2,3\}$ such that

$$|c(i,j) - c\big(\lfloor\frac{2}{3}(i+j)\rfloor + k, \frac{1}{2}(\lfloor\frac{2}{3}(i+j)\rfloor + k)\big)| \\ \leq C' e^{2C - (\frac{\alpha}{4h} - 2\beta)i}. \qquad (10)$$

Finally the proposition follows from the inequality

$$|c(2j,j) - c(2j+4, j+2)| \leq C' e^{2C - (\frac{\alpha}{4h} - 2\beta)2j}.$$

□

Proposition 4.3 *Let $\alpha > 0$, $\beta \in [0, \frac{\alpha}{8h})$, and (V,τ) a non trivial irreducible unitary representation of K. There exists $C' > 0$, such that the following holds. Let $C \in \mathbb{R}_+^*$, (E,π) any element in $\mathcal{E}_{G,C+\beta\ell}$, and $\xi \in E$, $\eta \in V \otimes E^*$ any K-invariant vectors of norm 1. We have*

$$\|c(i,j)\|_V \leq C' e^{2C - (\frac{\alpha}{4h} - 2\beta)i}.$$

Proof: As (9) and (10) in the proof of the above proposition 4.2, by lemmas 3.7 and 4.1, we have the following inequalities. When $i \geq 2j \geq 0$, there exists $k \in \{0,1,2\}$, such that

$$\|c(i,j) - c(i+k, \frac{1}{2}(i+k))\|_V \leq C' e^{2C - (\frac{\alpha}{4h} - 2\beta)i}.$$

When $2j \geq i \geq j \geq 0$, there exists $k \in \{0,1,2,3\}$ such that

$$\|c(i,j) - c\big(\lfloor\frac{2}{3}(i+j)\rfloor + k, \frac{1}{2}(\lfloor\frac{2}{3}(i+j)\rfloor + k)\big)\|_V \leq C' e^{2C - (\frac{\alpha}{4h} - 2\beta)i}.$$

So it remains to prove that for any $j \in \mathbb{N}$ we have

$$\|c(2j,j)\|_V \leq C' e^{2C - (\frac{\alpha}{4h} - 2\beta)2j}. \qquad (11)$$

First when $j \in 2\mathbb{N}$, we know inequality (11) holds. In fact, by lemmas 3.7 and 4.1, when $j \in 2\mathbb{N}$ we have

$$\max\Big(\|c(2j,j) - c(2j,0)\|_V, \|c(2j,j) - c(3j/2, 3j/2)\|_V\Big) \leq C' e^{2C - (\frac{\alpha}{4h} - 2\beta)2j}.$$

Let K_1, K_2 be the subgroups of the group K as lemma 3.9. By lemmas 3.9 and 3.10 we get inequality (11).

It remains to show inequality (11) when $j \in 2\mathbb{N}+1$. We first have

$$\max\Big(\|c(2j,j) - c(2j+1,0)\|_V, \|c(2j,j) - c\big(2j - \lfloor\frac{j}{2}\rfloor, j + \lfloor\frac{j}{2}\rfloor\big)\|_V\Big) \\ \leq C' e^{2C - (\frac{\alpha}{4h} - 2\beta)2j}.$$

Note that lemma 3.7 is still valid for $i = j - 1$, i.e.

$$\|c\big(2j-\lfloor\frac{j}{2}\rfloor, j+\lfloor\frac{j}{2}\rfloor\big)-c\big(2j-\lfloor\frac{j}{2}\rfloor-1, j+\lfloor\frac{j}{2}\rfloor+1\big)\|_V \leq C'e^{2C-(\frac{3\alpha}{h}-3\beta)j}.$$

Then we have

$$\|c(2j,j) - c\big(2j - \lfloor\frac{j}{2}\rfloor - 1, j + \lfloor\frac{j}{2}\rfloor + 1\big)\|_V \leq C'e^{2C-(\frac{\alpha}{2h}-2\beta)2j}.$$

Denote B_1, B_2 the image in K_1 of $\begin{pmatrix}\mathcal{O}^\times & \pi\mathcal{O}\\ \mathcal{O} & \mathcal{O}^\times\end{pmatrix}$ and $\begin{pmatrix}\mathcal{O}^\times & \mathcal{O}\\ \pi\mathcal{O} & \mathcal{O}^\times\end{pmatrix}$ respectively, under the group isomorphism $GL_2(\mathcal{O}) \to K_1$. We see that $K_1 = (B_1B_2)^2$. Moreover $c(k'gk'') = \tau(k')c(g)$ for any $k', k'' \in K, g \in G$, it follows that $c(2j+1, 0), c\big(2j - \lfloor\frac{j}{2}\rfloor, j + \lfloor\frac{j}{2}\rfloor\big), c\big(2j - \lfloor\frac{j}{2}\rfloor - 1, j + \lfloor\frac{j}{2}\rfloor + 1\big)$ are invariant by K_2, B_1, B_2 repectively. By applying lemma 3.10 to $K = (B_1B_2K_2)^{60}$, we obtain inequality (11) for $j \in 2\mathbb{N}+1$. $\square$

Proof of theorem 2.3 when char$(F) = 2$: For simplicity we say that an element $g \in G$ is even (resp. odd) when $g \in KD(i,j)K, i \geq j \geq 0$ and $i+j$ is even (resp. odd). By proposition 4.2, we see that when g is even (resp. odd) and tends to infinity, the limit of $e_Ke_ge_K$ exists in $\mathcal{C}^{\mathcal{E}}_{C+\beta\ell}(G)$, which we denote by T_0 (resp. T_1). Let $\mathrm{p} = \frac{1}{2}(T_0 + T_1)$.

We have for any $g \in G$, $e_Ke_g\mathrm{p} = \mathrm{p}$, and thus $\mathrm{p}^2 = \mathrm{p}$. In fact, it suffices to show that for any $g \in G$ there exist $\alpha(g), \beta(g) > 0$ with $\alpha(g) + \beta(g) = 1$, such that

$$e_Ke_gT_0 = \alpha(g)T_0 + \beta(g)T_1, \tag{12}$$

and

$$e_Ke_gT_1 = \beta(g)T_0 + \alpha(g)T_1. \tag{13}$$

Let $\alpha(g)$ (resp. $\beta(g)$) be the volume of the set of elements $(k_1, k_2, k_3, k_4) \in K$ (where k_i are vectors in F^4 with norms ≤ 1) such that

$$\|gk_1 \wedge gk_2\|_{\wedge(F^4)} \in q^{2\mathbb{Z}}(\text{resp. } q^{2\mathbb{Z}+1}).$$

We see that for any $k = (k_1, k_2, k_3, k_4) \in K$, when $i + j \in 2\mathbb{N}$ (resp. $2\mathbb{N}+1$) with $(i,j) \in \Lambda$ and when $\|gk_1 \wedge gk_2\| \geq q^{-2j}$, $gkD(i,j)$ is even exactly when $\|gk_1 \wedge gk_2\| \in q^{2\mathbb{Z}}$ (resp. $q^{2\mathbb{Z}+1}$). Hence we have

$$\lim_{i+j\in 2\mathbb{N}(\text{ resp. } 2\mathbb{N}+1), j\to\infty} \mathrm{vol}\{k \in K, gkD(i,j) \text{ is even}\} = \alpha(g)(\text{ resp. } \beta(g)),$$

and also

$$\lim_{i+j\in 2\mathbb{N}(\text{ resp. } 2\mathbb{N}+1), j\to\infty} \mathrm{vol}\{k \in K, gkD(i,j) \text{ is odd}\} = \beta(g)(\text{ resp. } \alpha(g)).$$

And thus equalities (12) and (13) follow.

By proposition 4.3, for any non trivial irreducible representation V of K we have $e_K^V e_g T_0 = e_K^V e_g T_1 = 0$. By the same argument as in the proof of theorem when $\mathrm{char}(F) \neq 2$ in section 2, we have

$$e_g \mathrm{p} = e_K e_g \mathrm{p} = \mathrm{p}.$$

We complete the proof by taking

$$\mathrm{p}_n = \frac{1}{2}\big(e_K e_{D(2\lfloor \frac{n}{2} \rfloor, 0)} e_K + e_K e_{D(2\lfloor \frac{n}{2} \rfloor - 1, 0)} e_K\big),$$

and $t = \frac{\alpha}{4h} - 2\beta$. □

5 Extension to simple algebraic groups of higher split rank

Let F be a non archimedean local field. This section is dedicated to the proof the the following theorem, which is theorem 1.3 in the introduction.

Theorem 5.1 *Let G be a connected almost F-simple algebraic group with F-split rank ≥ 2. Then $G(F)$ has strong Banach property (T).*

We begin the proof with some lemmas. The following lemma is proposition 8.2 in [Bor].

Lemma 5.2 *Let k be a field and H an abelian k-group. Let $\pi : H \to \mathrm{GL}_n$ be a k-rational representation. Then $\pi(H)$ is conjugate over k to some subgroup of the group of diagonal elements in GL_n.*

The following lemma is a consequence of theorem 7.2 in [BT], which is also proposition I.1.6.2 in [Mar].

Lemma 5.3 *Let k be any field and G a connected almost k-simple group with k-split rank ≥ 2. Then there exists a k-rational group homomorphism with finite kernel from SL_3 or Sp_4 to G.*

The following lemma is a direct consequence of propositions I.1.3.3 (ii) and I.1.5.4 (iii), and theorem I.2.3.1 (a) in [Mar].

Lemma 5.4 *Let G be a simply connected and almost F-simple group. Let S be a maximal F-split torus of G, $\Phi(G,S)$ the root system with some ordering and ϑ a proper subset of simple roots. Then there exist two unipotent F-subgroups $V_\vartheta, V_\vartheta^-$ of G, and two S-equivariant F-isomorphisms $\mathrm{Lie}V_\vartheta \to V_\vartheta, \mathrm{Lie}V_\vartheta^- \to V_\vartheta^-$, such that*

- *(i)* $\mathrm{Lie}V_\vartheta$ *(resp.* $\mathrm{Lie}V_\vartheta^-$ *) is the direct sum of eigenspaces of positive (resp. negative) roots which are not integral linear combinations of* ϑ, *and*
- *(ii)* $V_\vartheta(F) \cup V_\vartheta^-(F)$ *generates* $G(F)$.

The next two lemmas reduce the proof to the simply connected covering of our algebraic group.

Lemma 5.5 *(proposition I.1.4.11 in [Mar]) Let* k *be a field, and let* G *be connected semisimple* k*-group. Then there exists a simply connected* k*-group* $\tilde{G}$ *and a* k*-isogeny (i.e. surjective* k*-group homomorphism with finite kernel) from* $\tilde{G}$ *to* G.

Lemma 5.6 *Let* G_1 *be a locally compact group and* G_2 *its quotient by a finite normal subgroup. Then* G_1 *has strong Banach property (T) if and only if* G_2 *has strong Banach property (T).*

Proof: Let H be the kernel of $G_1 \to G_2$. Suppose G_1 has strong Banach property (T), and let $\mathrm{p}_n \in C_c(G_1)$ be real and self-adjoint elements (otherwise take $\mathrm{p}_n + \bar{\mathrm{p}}_n + \mathrm{p}_n^* + \bar{\mathrm{p}}_n^*$) that tends to the idempotent element in $\mathcal{C}^{\mathcal{E}}_{C+s\ell}(G_1)$. Then $\big(\mathop{\mathbb{E}}_{h\in H} h\big)\mathrm{p}_n$ tends to a real and self-adjoint (since H is normal) idempotent element p' in $\mathcal{C}^{\mathcal{E}}_{C+s\ell}(G_2)$ such that $e_g\mathrm{p}' = \mathrm{p}'$ for any $g \in G_2$.

On the other direction, if G_2 has strong Banach property (T), let $\mathrm{p}_n \in C_c(G_2)$ tend to the idempotent element in $\mathcal{C}^{\mathcal{E}}_{C+s\ell}(G_2)$, and denote its lifting to $C_c(G_1)$ by $\tilde{\mathrm{p}}_n$ (i.e. $\tilde{\mathrm{p}}_n(gh) = \mathrm{p}_n(g)$ for any $g \in G_1, h \in H$). For any $(E,\pi) \in \mathcal{E}_{G_1,C+s\ell}$, we have $\pi(\tilde{\mathrm{p}}_n)\xi = \pi(\tilde{\mathrm{p}}_n)(\mathop{\mathbb{E}}_{h\in H}\pi(h)\xi)$, and thus

$$\|\pi(\tilde{\mathrm{p}}_n) - \pi(\tilde{\mathrm{p}}_m)\|_{\mathcal{L}(E)} \le \max_{h\in H} \|\pi(h)\| \|\pi(\tilde{\mathrm{p}}_n) - \pi(\tilde{\mathrm{p}}_m)\|_{\mathcal{L}(E^H)},$$

where E^H denotes the space of H-invariant vectors. We conclude that $\tilde{\mathrm{p}}_n$ tends to a real and self-adjoint idempotent element p in $\mathcal{C}^{\mathcal{E}}_{C+s\ell}(G_1)$ such that $e_g\mathrm{p} = \mathrm{p}$ for any $g \in G_1$. □

Proof of theorem 5.1: In view of lemmas 5.5 and 5.6, we can assume G is simply connected (in order to apply lemma 5.4 as indicated below). By lemma 5.3 there exist a subgroup R of $G(F)$ and a surjective group homomorphism I from $SL_3(F)$ or $Sp_4(F)$ to R with finite kernel. Let $\rho : F^* \to SL_3(F)$ (resp. $Sp_4(F)$) be the group homomorphism defined by

$$x \mapsto \begin{pmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x^{-1} \end{pmatrix} \text{ (resp. } \begin{pmatrix} x & & & \\ & 1 & & \\ & & 1 & \\ & & & x^{-1} \end{pmatrix})$$

for any $x \in F$, and let $a = I \circ \rho(\pi)$, where π is a uniformizer of F. By lemma 5.2, the set of eigenvalues of $Ad(a)$ is a subset of $\pi^{\mathbb{Z}}$ which contains $\{1\}$ as a proper subset. Let S be a maximal F-split torus of G whose F points contains a. We can choose an ordering of $\Phi(S, G)$ such that $|\chi(a)| \leq 1$ for any simple root χ. Let ϑ be the proper subset of simple roots χ such that $|\chi(a)| = 1$, and let $V_\vartheta, V_\vartheta^-$ be as in lemma 5.4.

For simplicity denote $G(F)$ and $V_\vartheta(F), V_\vartheta^-(F)$ by G and $V_\vartheta, V_\vartheta^-$ from now on. Let $\|\cdot\|$ be the norm on Lie$G$ defined w.r.t. some F-basis. Let ℓ' be the length function on G defined by

$$\ell'(g) = \log \|Ad(g)\|_{\mathrm{End}(\mathrm{Lie}G)}.$$

Note that for any length function $\tilde{\ell}$ on $SL_3(F)$ or $Sp_4(F)$, there exist $\kappa \in \mathbb{R}_+^*$ such that $\tilde{\ell} \leq \kappa\ell$, where ℓ is the length function on $SL_3(F)$ or $Sp_4(F)$ defined in section 2. In fact for $SL_3(F)$, let K be the compact generating set

$$\{SL_3(\mathcal{O}), SL_3(\mathcal{O})\pi \begin{pmatrix} \pi^{-3} & & \\ & 1 & \\ & & 1 \end{pmatrix} SL_3(\mathcal{O}), SL_3(\mathcal{O})\pi^2 \begin{pmatrix} \pi^{-3} & & \\ & \pi^{-3} & \\ & & 1 \end{pmatrix} SL_3(\mathcal{O})\}.$$

Then we have $\ell(g) \geq \min\{n : g \in K^n\}$ (note that it holds for diagonal elements and K is a $SL_3(\mathcal{O})$ bi-invariant set). Therefore, we have

$$\left(\max_{g \in K} \tilde{\ell}(g)\right) \ell(g) \geq \tilde{\ell}(g).$$

It can be shown for $Sp_4(F)$ using the same argument by replacing the compact generating set K by

$$\{Sp_4(\mathcal{O}), Sp_4(\mathcal{O}) \begin{pmatrix} \pi^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & \pi \end{pmatrix} Sp_4(\mathcal{O}), Sp_4(\mathcal{O}) \begin{pmatrix} \pi^{-1} & & & \\ & \pi^{-1} & & \\ & & \pi & \\ & & & \pi \end{pmatrix} Sp_4(\mathcal{O})\}.$$

Let $\mathcal{E}$ be a class of Banach spaces of type > 1 stable under duality and complex conjugation. Let $s, t, C, C' \in \mathbb{R}_+^*, \mathrm{p} \in \mathcal{C}^{\mathcal{E}}_{s\kappa\ell+C}(R), \mathrm{p}_m \in C_c(R)$ verify the conditions (i) and (ii) of theorem 2.2 if R is isogenous to $SL_3(F)$, or of theorem 2.3 if R is isogenous to $Sp_4(F)$, where $\kappa \in \mathbb{R}_+^*$ such that $\ell'|_R \leq \kappa\ell$ (in view of lemma 5.6). Let U be an open compact subgroup of G and $f = \frac{e_U}{vol(e_U)}$. Then to establish that G has strong Banach property (T) it suffices to show that if s is small enough the series $\mathrm{p}_m f \in C_c(G)$ converges in $\mathcal{C}^{\mathcal{E}}_{s\ell'+C}(G)$ to a self adjoint idempotent p' such that for any $(E, \pi) \in \mathcal{E}_{G,s\ell'+C}$, the image of $\pi(\mathrm{p}')$ consists of all G-invariant vectors of E. First it is clear that the series $\mathrm{p}_m f$ is a Cauchy series in $\mathcal{C}^{\mathcal{E}}_{s\ell'+C}(G)$ and we note p' its limit (in fact p is a multiplier of $\mathcal{C}^{\mathcal{E}}_{s\ell'+C}(G)$ and $\mathrm{p}' = \mathrm{p}f$). Let $(E,\pi) \in \mathcal{E}_{G,s\ell'+C}$. It is obvious that $\pi(\mathrm{p}')$ acts by identity over any G-invariant vector. It remains to show that for any $x \in E$, $\pi(\mathrm{p}')x$ is G-invariant (in fact it follows that $\mathrm{p}' = f^*\mathrm{p}' = f^*\mathrm{p}f$, so p' is self-adjoint). In view of statement (ii) of lemma 5.4, it suffices to show that $\pi(\mathrm{p}')x$ is V_ϑ-invariant and V_ϑ^--invariant.

We first show that $\pi(\mathrm{p}')x$ is V_ϑ-invariant. Let $E : \mathrm{Lie}V_\vartheta \to V_\vartheta$ be as in lemma 5.4. We know that $\pi(\mathrm{p}')x$ is fixed by R, then in particular by a. It suffices to show that for any $Y \in \mathrm{Lie}V_\vartheta$,

$$\pi(E(Y))\pi(\mathrm{p}')x - \pi(\mathrm{p}')x = \pi(E(Y))\pi(a^{-n})\pi(\mathrm{p}')x - \pi(a^{-n})\pi(\mathrm{p}')x$$

$$= \pi(a^{-n})\big(\pi(a^nE(Y)a^{-n})-1\big)\pi(\mathrm{p}')x = \pi(a^{-n})\big(\pi(E(Ad(a^n)Y))-1\big)\pi(\mathrm{p}')x$$

tends to 0 when $n \in \mathbb{N}$ tends to infinity.

Let $Y = \sum_{\lambda\in\Lambda} Y_\lambda$ be the decomposition of Y under the adjoint action of a in $\mathrm{Lie}V_\vartheta$, where $\Lambda \subset F$ denotes the set of eigenvalues of the action. Due to the way ϑ is chosen, the eigenvalues of $Ad(a)|_{\mathrm{Lie}V_\vartheta}$ are all of the form $\pi^{\mathbb{N}^*}$. Since U is an open subgroup of G, there exists $r > 0$ such that when $Y' \in V$ and $\|Y'\| \leq r$, we have $E(Y') \in U$. For any $n \in \mathbb{N}$, we put

$$m = \lfloor n\kappa^{-1}\log\min_{\lambda\in\Lambda}|\lambda|^{-1} + \kappa^{-1}\log(r/\max_{\lambda\in\Lambda}\|Y_\lambda\|)\rfloor.$$

When n is big enough such that $m > 0$, we have

$$\big(\pi(E(Ad(a^n)Y)) - 1\big)\pi(\mathrm{p}_m f)x$$

$$= \int_R \mathrm{p}_m(g)\pi(g)\big(\pi(E(Ad(g^{-1}a^n)Y)) - 1\big)\pi(f)xdg.$$

When $\ell(g) \leq m$, we have

$$\|Ad(g^{-1}a^n)Y\| \leq e^{\ell'(g)}\max_{\lambda\in\Lambda}|\lambda|^n\|Y_\lambda\|_{\mathrm{Lie}V_\vartheta} \leq r,$$

and hence

$$\big(\pi(E(Ad(g^{-1}a^n)Y)) - 1\big)\pi(f)x = 0.$$

Therefore we have

$$\pi(a^{-n})\big(\pi(E(Ad(a^n)Y)) - 1\big)\pi(\mathrm{p}_m f)x = 0$$

when n is big enough.

On the other hand for any $n \in \mathbb{N}$, we always have

$$Ad(a^n)Y = \sum_{\lambda\in\Lambda}\lambda^nY_\lambda \in \bigoplus_{\lambda\in\Lambda}\mathcal{O}Y_\lambda.$$

Hence

$$\begin{aligned}&\|\pi(a^{-n})\big(\pi(E(Ad(a^n)Y))-1\big)\pi(\mathrm{p}'-\mathrm{p}_m f)x\|_E\\&\quad\leq e^{C+s\ell'(a)n}(1+C'')\|\pi(\mathrm{p}'-\mathrm{p}_m f)x\|_E,\end{aligned}$$

where

$$C''=\sup_{t_\lambda\in\mathcal{O}}\|\pi(E(\max_{\lambda\in\Lambda}t_\lambda Y_\lambda))\|_{\mathcal{L}(E)}<\infty$$

depends only on Y. But

$$\|\pi(\mathrm{p}'-\mathrm{p}_m f)x\|_E\leq C'e^{2C-tm}\|\pi(f)x\|_E$$

by statement (ii) of theorem 2.2 if R is isogenous to $SL_3(F)$, or of theorem 2.3 if R is isogenous to $Sp_4(F)$ (we recall that C' and t are the constants of theorem 2.2 and theorem 2.3). In total, when n is big enough

$$\|\pi(a^{-n})\big(\pi(E(Ad(a^n)Y))-1\big)\pi((\mathrm{p}'-\mathrm{p}_m f))x\|_E$$

$$\leq e^{C+s\ell'(a)n}(1+C'')C'e^{2C-tm}\|\pi(f)x\|_E,$$

and if

$$s<\frac{t}{\kappa\ell'(a)}\log\min_{\lambda\in\Lambda}|\lambda|^{-1},$$

it tends to 0 when n tends to infinity.

We prove $\pi(\mathrm{p}')x$ is V_ϑ^--invariant by exactly the same argument (with a replaced by a^{-1} and the ordering of $\Phi(S,G)$ by its inverse, i.e. the ordering such that $|\chi(a^{-1})|\leq 1$ for any simple root χ). □

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