

STRONG CONSISTENCY OF LEAST SQUARES ESTIMATES IN NORMAL LINEAR REGRESSION¹

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In the usual linear regression model the sample regression coefficients converge with probability one to the population regression coefficients when the dependent variables are normally distributed and the inverse of the second-order moment matrix of the independent variables converges to the zero matrix.

In the usual linear model the least squares estimates of the regression coefficients are weakly consistent if the independent variables grow as the number of observations increases in such a way that the covariance matrix of the estimates converges to the matrix consisting of zeros. Here we prove that this condition implies strong consistency when the dependent variables are normally and independently distributed.

THEOREM 1. *Let y_1, y_2, \dots be a sequence of independently normally distributed random variables with variances σ^2 and expected values*

$$(1) \quad \mathcal{E}y_t = \boldsymbol{\beta}'\mathbf{x}_t, \quad t = 1, 2, \dots,$$

where $\boldsymbol{\beta}$ is a p -component vector of parameters and \mathbf{x}_t is a p -component nonstochastic vector. Let

$$(2) \quad \mathbf{A}_T = \sum_{t=1}^T \mathbf{x}_t \mathbf{x}_t',$$

and suppose \mathbf{A}_p is nonsingular. Define

$$(3) \quad \mathbf{b}_T = \mathbf{A}_T^{-1} \sum_{t=1}^T \mathbf{x}_t y_t, \quad T = p, p + 1, \dots$$

Then $\mathbf{b}_T \rightarrow \boldsymbol{\beta}$ as $T \rightarrow \infty$ with probability 1 if and only if

$$(4) \quad \mathbf{A}_T^{-1} \rightarrow \mathbf{0}.$$

PROOF. Let

$$(5) \quad u_t = y_t - \boldsymbol{\beta}'\mathbf{x}_t,$$

which has expected value 0 and variance σ^2 . Then

$$(6) \quad \mathbf{b}_T - \boldsymbol{\beta} = \mathbf{A}_T^{-1} \sum_{t=1}^T \mathbf{x}_t u_t.$$

We shall show (6) converges to $\mathbf{0}$ with probability 1 if (4) holds. Consider first the case of $p = 1$. Then the scalar $\sum_{t=1}^T x_t u_t$ is a martingale with $\sigma^2 A_T = \sum_{t=1}^T \mathcal{E}[(x_t u_t)^2 | u_{t-1}, u_{t-2}, \dots, u_1]$ diverging to infinity. Under these conditions

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(e.g., Neveu (1965), page 150) we have that $A_T^{-1}(\log A_T)^{-1-\epsilon} \sum_{t=1}^T x_t u_t$ converges to 0 for every $\epsilon > 0$ with probability 1. Then equation (6) converges to 0 with probability 1, since $A_T^{-1}(\log A_T)^{1+\epsilon} \rightarrow 0$ as $T \rightarrow \infty$ for $\epsilon = \frac{1}{2}$, say. Next, take $p > 1$ and consider the first component of $\mathbf{b}_T - \boldsymbol{\beta}$. Let

$$(7) \quad \boldsymbol{\beta} = \begin{pmatrix} \beta_1 \\ \boldsymbol{\beta}^{(2)} \end{pmatrix}, \quad \mathbf{x}_t = \begin{pmatrix} x_{1t} \\ \mathbf{x}_t^{(2)} \end{pmatrix},$$

$$(8) \quad \mathbf{b}_T = \begin{pmatrix} b_{1T} \\ \mathbf{b}_T^{(2)} \end{pmatrix}, \quad \mathbf{A}_T = \begin{pmatrix} a_{11T} & \mathbf{A}_{12T} \\ \mathbf{A}_{21T} & \mathbf{A}_{22T} \end{pmatrix}.$$

Then

$$(9) \quad b_{1T} - \beta_1 = \frac{Y_T}{S_T},$$

where

$$(10) \quad Y_T = \sum_{t=1}^T (x_{1t} - \mathbf{A}_{12T} \mathbf{A}_{22T}^{-1} \mathbf{x}_t^{(2)}) u_t,$$

$$(11) \quad S_T = \sum_{t=1}^T (x_{1t} - \mathbf{A}_{12T} \mathbf{A}_{22T}^{-1} \mathbf{x}_t^{(2)})^2.$$

(See, for example, Section 2.3 of Anderson (1971).) The variance of Y_T is $\sigma^2 S_T$. Define $\gamma_1^2 = S_p$, $v_1 = Y_p/\gamma_1$. For $T \geq p$ consider

$$(12) \quad Y_{T+1} = Y_T + (\mathbf{A}_{12T} \mathbf{A}_{22T}^{-1} - \mathbf{A}_{12, T+1} \mathbf{A}_{22, T+1}^{-1}) \sum_{t=1}^T \mathbf{x}_t^{(2)} u_t \\ + (x_{1, T+1} - \mathbf{A}_{12, T+1} \mathbf{A}_{22, T+1}^{-1} \mathbf{x}_{T+1}^{(2)}) u_{T+1}.$$

The second term on the right-hand side of (12) is uncorrelated with Y_R , $R = p, p + 1, \dots, T$, because

$$(13) \quad \sum_{t=1}^R (x_{1t} - \mathbf{A}_{12R} \mathbf{A}_{22R}^{-1} \mathbf{x}_t^{(2)}) \mathbf{x}_t^{(2)'} = \mathbf{0}.$$

The third term is uncorrelated with Y_R , $R = p, p + 1, \dots, T$, because u_{T+1} is uncorrelated with u_1, \dots, u_T . Thus the increment $Y_{T+1} - Y_T$ is uncorrelated with Y_p, Y_{p+1}, \dots, Y_T . Define

$$(14) \quad \gamma_t^2 = S_{t+p-1} - S_{t+p-2}, \quad t = 2, 3, \dots,$$

$$(15) \quad v_t = \frac{Y_{t+p-1} - Y_{t+p-2}}{\gamma_t}, \quad t = 2, 3, \dots$$

Then v_1, v_2, \dots constitute a sequence of independent random variables, each with distribution $N(0, \sigma^2)$. Since

$$(16) \quad \sum_{t=1}^{T-p+1} \gamma_t^2 = S_T = \sum_{t=1}^T (x_{1t} - \mathbf{A}_{12T} \mathbf{A}_{22T}^{-1} \mathbf{x}_t^{(2)})^2 = a_{11T} - \mathbf{A}_{12T} \mathbf{A}_{22T}^{-1} \mathbf{A}_{21T}$$

is the reciprocal of the upper left hand corner of \mathbf{A}_T^{-1} , it diverges to ∞ as $T \rightarrow \infty$. Then the argument for $p = 1$ above shows that

$$(17) \quad b_{1T} - \beta_1 = \frac{\sum_{t=1}^{T-p+1} \gamma_t v_t}{\sum_{t=1}^{T-p+1} \gamma_t^2}$$

converges to 0 with probability 1.

On the other hand $b_{1T} - \beta_1$ converges to 0 in probability only if its variance

σ^2/S_T converges to 0 since it is normally distributed. Because A_T^{-1} is positive definite, (4) holds if and only if every diagonal element of A_T^{-1} converges to 0. \square

For $p = 1$ the theorem holds under more general conditions.

THEOREM 2. *Let u_1, u_2, \dots and x_1, x_2, \dots be sequences of scalar random variables such that $\mathcal{E}(u_t | u_1, \dots, u_{t-1}, x_1, \dots, x_t) = 0$ and $\mathcal{E}(u_t^2 | u_1, \dots, u_{t-1}, x_1, \dots, x_t) = \sigma^2 < \infty$. Let $y_t = \beta x_t + u_t$, $t = 1, 2, \dots$. If x_1 is bounded away from 0 with probability 1 and if $\sum_{t=1}^T x_t^2 \rightarrow \infty$ with probability 1, then $\hat{\beta}_T = \sum_{t=1}^T x_t y_t / \sum_{t=1}^T x_t^2 \rightarrow \beta$ with probability 1.*

This theorem was implicit in Taylor (1974)

Alternative forms of (4) are (i) the maximum characteristic root of A_T^{-1} converges to 0, (ii) the minimum characteristic root of A_T diverges to ∞ and (iii) $\gamma' A_T \gamma \rightarrow \infty$ for every $\gamma \neq 0$.

Brown, Durbin and Evans (1975) have defined "recursive residuals," which are equivalent to v_1, v_2, \dots here, and used them in a different context.

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