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Research Article

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Strong consistency of regression function estimator with martingale difference errors

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Abstract: In this paper, we consider the regression model with fixed design: $Y_i = g(x_i) + \varepsilon_i, 1 \leq i \leq n$, where $\{x_i\}$ are the nonrandom design points, and $\{\varepsilon_i\}$ is a sequence of martingale, and g is an unknown function. Nonparametric estimator $g_n(x)$ of $g(x)$ will be introduced and its strong convergence properties are established.

Keywords: regression function, martingale difference, consistency

MSC 2020: 60F15, 62G05

1 Introduction

The estimation of a regression function $g(x) = E(y|x)$ is an important statistical problem. Usually, $g(x)$ has a specified functional form and parameter estimates are obtained according to certain desirable criteria. When the errors are normal random variables, we can test the appropriateness of the hypothesized model. However, one may wish to have an estimation technique applicable for an arbitrary $g(x)$. Priestley and Chao [1] considered the problem of estimating an unknown regression function $g(x)$ given observations at a fixed set of points. Their estimate is nonparametric in the sense that $g(x)$ is restricted only by certain smoothing requirements.

1.1 Priestley-Chao estimate

Let $Y_1, \ldots, Y_n$ be n observations at fixed $x_1, \ldots, x_n$ according to the model

$$Y_i = g(x_i) + \varepsilon_i, \quad 1 \leq i \leq n,$$

where $g(x)$ is an unknown function defined in $[0, 1]$ and the errors $\{\varepsilon_i\}$ are i.i.d. random variables with zero mean and finite variance σ^2. Without loss of generality we assume $0 \leq x_1 \leq \cdots \leq x_n \leq 1$. The Priestley-Chao estimate of $g(x)$ is

$$g_n(x) = \sum_{i=1}^{n} \frac{x_i - x_{i-1}}{h_n} K\left(\frac{x - x_i}{h_n}\right) Y_i, \tag{1.1}$$

where K is a weight function satisfying

$$K(u) \geq 0, \quad \text{for all } u; \quad \int_{-\infty}^{\infty} K(u)\mathrm{d}u = 1; \quad \int_{-\infty}^{\infty} K^2(u)\mathrm{d}u < \infty; \quad K(u) = K(-u) \tag{1.2}$$

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and $\{h_n\}$ is a sequence of positive real numbers with

$$h_n \to 0, \quad nh_n \to \infty, \quad \text{as } n \to \infty.$$

The estimate $g_n(x)$ can be viewed as a moving average of sample Ys whose weights are based on a class of kernels suggested by Rosenblatt [2] and Parzen [3].

Priestley and Chao [1] established the consistency of the estimate $g_n(x)$. Benedetti [4] studied the strong convergence and asymptotic normality for $g_n(x)$. Specially, the optimal choice of a weighting function was considered. Gasser and Müller [5] established a new kernel estimate which was superior to the one introduced by Priestley and Chao [1]. Their results were not restricted to positive kernels, but extended to classes of kernels satisfying certain moment conditions. Cheng [6] used linear combinations of sample quantile regression functions to estimate the unknown function g. Csörgő and Mielniczuk [7] considered the fixed-design regression model with long-range dependent normal errors and showed that the finite-dimensional distributions of the properly normalized Gasser and Mülller [5] and Priestley and Chao [1] estimators of the regression function converge to those of a white noise process. Burman [8] dealt with the convergence of spline regression estimators under mixing conditions. Robinson [9] studied central limit theorems for an estimator for the regression function of a fixed-design model when the residuals come from a linear process of martingale differences. Tran et al. [10] discussed the asymptotic normality of $g_n(x)$ assuming that the errors form a linear time series, more precisely, a weakly stationary linear process based on a martingale difference sequence. Yang and Wang [11] and Liang and Jing [12] established the strong consistency of regression function estimator for negative associated samples. Niu and Li [13] discussed the asymptotic normality of the weighted kernel estimators of $g(x)$ when the censoring variable is known or unknown. Zhang et al. [14] studied the strong convergence of the estimate $g_n(x)$ when the errors are the mixingale sequence. Yang [15] obtained the strong consistency of the Georgiev estimates of the regression function when the errors are martingale differences.

Motivated by the above works, in the present paper, we shall establish the strong consistency and uniform strong consistency of Priestley-Chao estimate of regression function based on the errors of martingale difference sequences and extend the results of Li [16] and Yin et al. [17]. In Section 2, we state the main results, and the proofs of these theorems are given in Section 3.

2 Main results

We consider the following regression model with fixed design

$$Y_i = g(x_i) + \varepsilon_i, \quad 1 \le i \le n, \tag{2.1}$$

where $0 \le x_1 \le \cdots \le x_n \le 1$ are the nonrandom design points, and $Y_1, \ldots, Y_n$ are the observed sample items. The sequence $\{\varepsilon_i, \mathcal{F}_i, \ i \ge 0\}$ is a martingale difference sequence with $\varepsilon_0 = 0$ and $g(x)$ is an unknown function. Denote $\delta_n = \max_{1\le i\le n}(x_i - x_{i-1})$, and we assume that the following regularity conditions are satisfied:

(a) Let $K(\cdot)$ be a weighted bounded function satisfying

$$\int_{-\infty}^{+\infty} K(u)\mathrm{d}u = 1, \qquad \int_{-\infty}^{+\infty} |K(u)|\mathrm{d}u < \infty.$$

(b) Let $g(\cdot), K(\cdot)$ satisfy Lipschitz conditions of orders α, β, respectively.

(c) Assume that the sequence $\delta_n \to 0$, as $n \to \infty$.

(d) Suppose that $\{h_n\}$ is a sequence of positive real numbers satisfying $h_n \to 0$ and

$$\frac{1}{h_n}\{(\delta_n/h_n)^\beta + \delta_n^\alpha\} \to 0,$$

as $n \to \infty$.

Throughout the following paper, C denotes a positive constant, which may take different values whenever it appears in different expression. $\mathrm{I}(A)$ denotes the indicator function of the event A. $\log x$ denotes $\ln \max\{x, e\}$, where ln is the natural logarithm. $\|\xi\|$ denotes the essential supremum of random variable ξ, namely $\|\xi\| = \inf\{c > 0 : P(|\xi| \le c) = 1\}$. The estimate $g_n(x)$ of the unknown function $g(x)$ is defined as (1.1).

Theorem 2.1. *Under conditions* (a)–(d), *assume that*
(i) *for some* $1 < r < 2$ *and* $b \ge 0$,

$$\sup_{i\ge 1} \mathbb{E}(|\varepsilon_i|^r|\mathcal{F}_{i-1}) \le b \quad a.s.,$$

(ii) $\delta_n/h_n = o\Big(n^{-\frac{1}{r}}(\log n)^{\frac{1}{r}-1}\Big)$,
then we have

$$\lim_{n\to\infty} g_n(x) = g(x) \quad a.s., \quad \forall x \in (0, 1). \tag{2.2}$$

Theorem 2.2. *Under conditions* (a)–(d), *assume that*
(i) *for some* $1 < r < 2$ *and* $b \ge 0$,

$$\sup_{i\ge 1} \mathbb{E}(|\varepsilon_i|^r|\mathcal{F}_{i-1}) \le b \quad a.s.,$$

(ii) $\delta_n/h_n = o\Big(n^{-\frac{1}{r}}(\log n)^{-1-\frac{1}{r}}\Big)$, *and there exists* $d > 0$ *satisfying*

$$n^d h_n \to \infty, \quad as\ n \to \infty,$$

then we have (2.2) *and*

$$\lim_{n\to\infty} \sup_{x\in[\tau,1-\tau]} g_n(x) = g(x) \quad a.s., \quad \forall \tau \in (0, 2^{-1}). \tag{2.3}$$

Remark 2.1. The assumptions of δ_n/h_n in Theorems 2.1 and 2.2 are weaker than those of Theorem 4 in [16], in which

$$\delta_n/h_n = O(n^{-\frac{1}{r}}(\log n)^{-(1+\rho)}), \quad \rho > 1.$$

Remark 2.2. The assumptions for the weighted function K in [4] are stronger than ones in the present paper. Besides the condition (a), Benedetti [4] assumed that the weighted function K satisfied

$$\int_{-\infty}^{\infty} K^2(u)\mathrm{d}u < \infty \quad \text{and} \quad K(u) = K(-u).$$

Suppose that the exponential moments of the errors $\{\varepsilon_i\}$ exist, then we have the following results.

Theorem 2.3. *Under conditions* (a)–(d), *assume that*
(i) *for some* $b \ge 0$,

$$\sup_{i\ge 1} \mathbb{E}(\exp(|\varepsilon_i|)) \le b,$$

(ii) $\delta_n/h_n = O\Big(\sqrt{\frac{\log n}{n^{1+\delta}}}\Big)$, *for some* $0 < \delta < 1$,
then we have (2.2).

Theorem 2.4. *Under conditions* (*a*)–(*d*) *and additional assumptions that,*
(*i*) *for some* $1 < r < 2$ *and* $b \geq 0$,

$$\mathbb{E}(\exp(|\varepsilon_i|^r)|\mathcal{F}_{i-1}) \leq b \quad a.s.,$$

(*ii*) $\delta_n/h_n = o\left(n^{-\frac{1}{r}}(\log n)^{\frac{1}{r}-1}\right)$, *and there exists* $d > 0$ *satisfying*

$$n^d h_n \to \infty, \quad as\ n \to \infty,$$

then we have (2.2) *and* (2.3).

3 Auxiliary results

In this section, we give some lemmas in order to prove our main results.

Lemma 3.1. [11, Theorem 1] *Under conditions* (*a*)–(*d*), *we have,*

$$\lim_{n\to\infty} \mathbb{E}(g_n(x)) = g(x), \quad \forall x \in (0,1), \tag{3.1}$$

$$\lim_{n\to\infty} \sup_{x\in[\tau,1-\tau]} |\mathbb{E}(g_n(x)) - g(x)| = 0, \quad \forall \tau \in (0, 2^{-1}). \tag{3.2}$$

Lemma 3.2. [11, Lemma 4] *Under conditions* (*a*), (*c*), *and* (*d*), *we have*

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{x_i - x_{i-1}}{h_n} \left| K\left(\frac{x - x_i}{h_n}\right) \right| = \int_{-\infty}^{+\infty} |K(u)| \mathrm{d}u, \quad \forall x \in (0,1), \tag{3.3}$$

$$\lim_{n\to\infty} \sup_{x\in[\tau,1-\tau]} \sum_{i=1}^{n} \frac{x_i - x_{i-1}}{h_n} \left| K\left(\frac{x - x_i}{h_n}\right) \right| = \int_{-\infty}^{+\infty} |K(u)| \mathrm{d}u, \quad \forall \tau \in (0, 2^{-1}). \tag{3.4}$$

Lemma 3.3. *If for some* $r > 1$, $\mathbb{E}|\xi|^r < \infty$, *then we have*

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{r}}(\log n)^{1-\frac{1}{r}}} \mathbb{E}|\xi| \mathrm{I}\left\{|\xi| > \left(\frac{n}{\log n}\right)^{\frac{1}{r}}\right\} < \infty. \tag{3.5}$$

Proof. Note that $\{n/\log n\}$ is an increasing sequence, so it is easy to check that,

$$\mathbb{E}|\xi| \mathrm{I}\left\{|\xi| > \left(\frac{n}{\log n}\right)^{\frac{1}{r}}\right\} = \sum_{i=n}^{\infty} \mathbb{E}|\xi| \mathrm{I}\left\{\left(\frac{i}{\log i}\right)^{\frac{1}{r}} < |\xi| \leq \left(\frac{i+1}{\log(i+1)}\right)^{\frac{1}{r}}\right\}$$

and

$$\sum_{n=1}^{i} \frac{1}{n^{\frac{1}{r}}(\log n)^{1-\frac{1}{r}}} \leq C \int_{1}^{i} \frac{1}{x^{\frac{1}{r}}(\log x)^{1-\frac{1}{r}}} \mathrm{d}x \leq C\left(\frac{i}{\log i}\right)^{1-\frac{1}{r}}.$$

Thus, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{r}}(\log n)^{1-\frac{1}{r}}} \mathbb{E}|\xi| \mathrm{I}\left\{|\xi| > \left(\frac{n}{\log n}\right)^{\frac{1}{r}}\right\} &= \sum_{i=1}^{\infty} \mathbb{E}|\xi| \mathrm{I}\left\{\left(\frac{i}{\log i}\right)^{\frac{1}{r}} < |\xi| \leq \left(\frac{i+1}{\log(i+1)}\right)^{\frac{1}{r}}\right\} \sum_{n=1}^{i} \frac{1}{n^{\frac{1}{r}}(\log n)^{1-\frac{1}{r}}} \\ &\leq C \sum_{i=1}^{\infty} \mathbb{P}\left\{\left(\frac{i}{\log i}\right)^{\frac{1}{r}} < |\xi| \leq \left(\frac{i+1}{\log(i+1)}\right)^{\frac{1}{r}}\right\} \frac{i}{\log i} \leq C\mathbb{E}|\xi|^r < \infty.\end{aligned}$$

□

Lemma 3.4. [18, Theorem 1] *Let $\{\xi_n, n \geq 1\}$ be a sequence of nonnegative random variables defined on $(\Omega, \mathcal{F}, P)$ and $\{\mathcal{F}_n, n \geq 0\}$ be a sequence of sub-σ-algebras of $\mathcal{F}$ (to which $\{\xi_n, n \geq 1\}$ need to be adapted). If $\{\mathcal{F}_n, n \geq 0\}$ is nondecreasing, then*

$$\left\{\sum_{n=1}^{\infty} \xi_n < \infty\right\} \quad \textit{a.s. on} \quad \left\{\sum_{n=1}^{\infty} \mathbb{E}(\xi_n | \mathcal{F}_{n-1}) < \infty\right\}. \tag{3.6}$$

Lemma 3.5. *Let $\{\xi_i, \mathcal{F}_i, i \geq 1\}$ be a martingale difference sequence with $|\xi_i| \leq 1$ a.s., then for any $\varepsilon > 0$, we have*

$$\mathbb{P}\{|S_n| > \varepsilon\} \leq 2\exp\left(-\frac{\varepsilon^2}{2n}\right), \tag{3.7}$$

where $S_n = \sum_{i=1}^{n} \xi_i$.

Proof. For any $t > 0$, by Lemma 1 in [19] and Markov's inequality, we have

$$\mathbb{P}\{S_n > \varepsilon\} \leq \exp(-t\varepsilon)\mathbb{E}(\exp(tS_n)) \leq \exp(-t\varepsilon)\exp\left(\frac{t^2 n}{2}\right).$$

By taking $t = \frac{\varepsilon}{n}$, we get

$$\mathbb{P}\{S_n > \varepsilon\} \leq \exp\left(-\frac{\varepsilon^2}{2n}\right).$$

Similarly, we have

$$\mathbb{P}\{-S_n > \varepsilon\} \leq \exp\left(-\frac{\varepsilon^2}{2n}\right).$$

Thus, the proof is completed. □

4 Proof of main results

In this section, we give the proofs of main results. Let

$$a_{ni}(x) = \frac{x_i - x_{i-1}}{h_n} K\left(\frac{x - x_i}{h_n}\right),$$

then by the definition of $g_n(x)$ we get

$$|g_n(x) - g(x)| \leq \left|\sum_{i=1}^{n} a_{ni}(x)\varepsilon_i\right| + |\mathbb{E}g_n(x) - g(x)|.$$

From Lemma 3.1, the proofs of these theorems can be concluded by showing the following equations:

$$\lim_{n\to\infty} \left|\sum_{i=1}^{n} a_{ni}(x)\varepsilon_i\right| = 0 \quad \text{a.s.,} \quad \forall x \in (0, 1). \tag{4.1}$$

$$\lim_{n\to\infty} \sup_{x\in[\tau,1-\tau]} \left|\sum_{i=1}^{n} a_{ni}(x)\varepsilon_i\right| = 0 \quad \text{a.s.,} \quad \forall \tau \in (0, 2^{-1}). \tag{4.2}$$

Proof of Theorem 2.1. Define

$$\begin{aligned}\varepsilon_i(1) &:= \varepsilon_i \mathrm{I}\left\{|\varepsilon_i| \le \left(\frac{i}{\log i}\right)^{\frac{1}{r}}\right\}, \quad \varepsilon_i(2) := \varepsilon_i \mathrm{I}\left\{|\varepsilon_i| > \left(\frac{i}{\log i}\right)^{\frac{1}{r}}\right\},\\ \xi_i &:= \varepsilon_i(1) - \mathbb{E}(\varepsilon_i(1)|\mathcal{F}_{i-1}), \quad \eta_i := \varepsilon_i(2) - \mathbb{E}(\varepsilon_i(2)|\mathcal{F}_{i-1}),\\ S_n(x) &:= \sum_{i=1}^n a_{ni}(x)\varepsilon_i, \quad S_n'(x) := \sum_{i=1}^n a_{ni}(x)\xi_i, \quad S_n''(x) := \sum_{i=1}^n a_{ni}(x)\eta_i,\end{aligned} \tag{4.3}$$

then it follows that $\varepsilon_i = \xi_i + \eta_i$, $S_n(x) = S_n'(x) + S_n''(x)$, and $\{\xi_i, \mathcal{F}_i, i \ge 1\}$, $\{\eta_i, \mathcal{F}_i, i \ge 1\}$ are martingale difference sequences. Thus in order to prove (4.1), it is sufficient to show that $|S_n'(x)| \to 0$ a.s. and $|S_n''(x)| \to 0$ a.s. for any $x \in (0,1)$.

From the condition (ii), there exists $0 < m_n \to 0$ satisfying

$$\left|\frac{\delta_n}{h_n}\right| = \frac{m_n}{n^{\frac{1}{r}}(\log n)^{1-\frac{1}{r}}}.$$

First, it is easy to see that

$$\mathbb{E}\left(\frac{\exp(\lambda S_n'(x))}{\prod_{i=1}^n \mathbb{E}(\exp(\lambda a_{ni}(x)\xi_i)|\mathcal{F}_{i-1})}\right) = \mathbb{E}\left(\frac{\exp(\lambda \sum_{i=1}^n a_{ni}(x)\xi_i)}{\prod_{i=1}^n \mathbb{E}(\exp(\lambda a_{ni}(x)\xi_i)|\mathcal{F}_{i-1})}\right) = 1,$$

then we have

$$\mathbb{E}(\exp(\lambda S_n'(x))) \le \left\|\prod_{i=1}^n \mathbb{E}(\exp(\lambda a_{ni}(x)\xi_i)|\mathcal{F}_{i-1})\right\|. \tag{4.4}$$

From the elementary inequality that,

$$1 + x \le e^x \le 1 + x + \frac{x^2}{2}e^{|x|}, \quad \forall x \in \mathbb{R},$$

and the definition of martingale difference, we have, for $1 \le i \le n$ and any $\lambda > 0$,

$$\begin{aligned}\mathbb{E}(\exp(\lambda a_{ni}(x)\xi_i)|\mathcal{F}_{i-1}) &\le \mathbb{E}\left(1 + \lambda a_{ni}(x)\xi_i + \frac{\lambda^2 a_{ni}^2(x)\xi_i^2}{2}\exp(\lambda|a_{ni}(x)\xi_i|)\,\middle|\,\mathcal{F}_{i-1}\right)\\ &\le 1 + \frac{\lambda^2 a_{ni}^2(x)}{2}\exp\left(\frac{C\lambda m_n}{\log n}\right)\mathbb{E}(\xi_i^2|\mathcal{F}_{i-1})\\ &\le 1 + \frac{C\lambda^2 a_{ni}^2(x)}{2}\exp\left(\frac{\lambda m_n}{\log n}\right)\mathbb{E}(\varepsilon_i^2(1)|\mathcal{F}_{i-1})\\ &= 1 + \frac{C\lambda^2 a_{ni}^2(x)}{2}\exp\left(\frac{\lambda m_n}{\log n}\right)\mathbb{E}\left(|\varepsilon_i|^r|\varepsilon_i|^{2-r}\mathrm{I}\left\{|\varepsilon_i|^{2-r} \le \left(\frac{i}{\log i}\right)^{\frac{2-r}{r}}\right\}\,\middle|\,\mathcal{F}_{i-1}\right)\\ &\le 1 + \frac{C\lambda^2 a_{ni}^2(x)}{2}\exp\left(\frac{\lambda m_n}{\log n}\right)\left(\frac{i}{\log i}\right)^{\frac{2-r}{r}}\\ &\le 1 + \frac{C\lambda^2 a_{ni}^2(x)}{2}\exp\left(\frac{\lambda m_n}{\log n}\right)\left(\frac{n}{\log n}\right)^{\frac{2-r}{r}}\\ &\le 1 + \frac{C\lambda^2 m_n^2}{2n\log n}\exp\left(\frac{\lambda m_n}{\log n}\right)\\ &\le \exp\left(\frac{C\lambda^2 m_n^2}{2n\log n}\exp\left(\frac{\lambda m_n}{\log n}\right)\right),\end{aligned} \tag{4.5}$$

in which the equality is valid as $1 < r < 2$. From (4.4) and (4.5), we get,

$$\mathbb{E}(\exp(\lambda S_n'(x))) \le \exp\left(\frac{C\lambda^2 m_n^2}{2\log n}\exp\left(\frac{\lambda m_n}{\log n}\right)\right).$$

By Markov's inequality, for any given $\varepsilon > 0$, $t > 0$, choosing $\lambda = \frac{2t\log n}{\varepsilon}$, we get,

$$\begin{aligned}\mathbb{P}\{S_n'(x) > \varepsilon\} &\le e^{-\lambda\varepsilon}\mathbb{E}(\exp(\lambda S_n'(x)))\\ &= \exp\left\{-2t\log n + \frac{Ct^2 m_n^2\exp\left(\frac{tm_n}{\varepsilon}\right)}{\varepsilon^2}\log n\right\}.\end{aligned} \tag{4.6}$$

Now letting $m_n \to 0$ in (4.6), we have, for all n large enough,

$$\mathbb{P}\{S_n'(x) > \varepsilon\} = o(n^{-t}), \quad \forall t > 0. \tag{4.7}$$

By choosing $t > 1$, we get,

$$\sum_{n=1}^{\infty}\mathbb{P}\{S_n'(x) > \varepsilon\} < \infty.$$

Similarly, we have

$$\sum_{n=1}^{\infty}\mathbb{P}\{-S_n'(x) > \varepsilon\} < \infty.$$

So by the Borel-Cantelli lemma, we get $|S_n'(x)| \to 0$ a.s.

Second, it is obvious that for every $k \ge 1$,

$$\max_{2^k \le n < 2^{k+1}}|S_n''(x)| \le \frac{C}{(2^{k+1})^{\frac{1}{r}}(\log 2^{k+1})^{1-\frac{1}{r}}}\sum_{i=1}^{2^{k+1}}|\eta_i|.$$

Hence, to show $|S_n''(x)| \to 0$ a.s., it suffices to prove that as $k \to \infty$,

$$\frac{1}{(2^k)^{\frac{1}{r}}(\log 2^k)^{1-\frac{1}{r}}}\sum_{i=1}^{2^k}|\eta_i| \to 0 \quad \text{a.s.} \tag{4.8}$$

From Markov's inequality, for any given $\varepsilon > 0$, we have

$$\begin{aligned}\sum_{k=1}^{\infty}\mathbb{P}\left\{\frac{1}{(2^k)^{\frac{1}{r}}(\log 2^k)^{1-\frac{1}{r}}}\sum_{i=1}^{2^k}|\eta_i| > \varepsilon\right\} &\le \frac{2}{\varepsilon}\sum_{k=1}^{\infty}\frac{1}{(2^k)^{\frac{1}{r}}(\log 2^k)^{1-\frac{1}{r}}}\sum_{i=1}^{2^k}\mathbb{E}|\varepsilon_i(2)|\\ &= \frac{2}{\varepsilon}\sum_{i=1}^{\infty}\mathbb{E}|\varepsilon_i(2)|\sum_{\{k:2^k\ge i\}}\frac{1}{(2^k)^{\frac{1}{r}}(\log 2^k)^{1-\frac{1}{r}}}\\ &\le C\sum_{i=1}^{\infty}\frac{1}{i^{\frac{1}{r}}(\log i)^{1-\frac{1}{r}}}\mathbb{E}|\varepsilon_i(2)|.\end{aligned} \tag{4.9}$$

From Lemma 3.3, we get

$$\sum_{i=1}^{\infty}\frac{1}{i^{\frac{1}{r}}(\log i)^{1-\frac{1}{r}}}\mathbb{E}|\varepsilon_i(2)| = \sum_{i=1}^{\infty}\frac{1}{i^{\frac{1}{r}}(\log i)^{1-\frac{1}{r}}}\mathbb{E}|\varepsilon_1|\mathrm{I}\left\{|\varepsilon_1| > \left(\frac{i}{\log i}\right)^{\frac{1}{r}}\right\} < \infty. \tag{4.10}$$

Thus by (4.9) and (4.10), we get (4.8). The proof of the theorem is completed. □

Proof of Theorem 2.2. For $1 \le i \le n$, denote

$$\varepsilon_i(1) := \varepsilon_i \mathrm{I}\left\{|\varepsilon_i| \le \left(\frac{n}{\log n}\right)^{\frac{1}{r}}\right\}, \quad \varepsilon_i(2) =: \varepsilon_i \mathrm{I}\left\{|\varepsilon_i| > \left(\frac{n}{\log n}\right)^{\frac{1}{r}}\right\},$$

and define ξ_i, η_i, $S_n(x)$, $S_n'(x)$, $S_n''(x)$ as (4.3).

In order to prove (4.1) and (4.2), it is sufficient to prove that for any $x \in (0,1)$ and any $\tau \in (0, 2^{-1})$,

$$|S_n'(x)| \to 0 \quad \text{a.s.}, \qquad \sup_{x\in[\tau,1-\tau]} |S_n'(x)| \to 0 \quad \text{a.s.}, \tag{4.11}$$

$$|S_n''(x)| \to 0 \quad \text{a.s.}, \qquad \sup_{x\in[\tau,1-\tau]} |S_n''(x)| \to 0 \quad \text{a.s.} \tag{4.12}$$

Since

$$\delta_n/h_n = o\left(\frac{1}{n^{\frac{1}{r}}(\log n)^{1+\frac{1}{r}}}\right) \quad \Rightarrow \quad \delta_n/h_n = o\left(\frac{1}{n^{\frac{1}{r}}(\log n)^{1-\frac{1}{r}}}\right),$$

then by using the similar proof to (4.7), we can get,

$$\mathbb{P}\{S_n'(x) > \varepsilon\} = o(n^{-t}), \quad \forall t > 0. \tag{4.13}$$

By choosing $t > 1$, we can obtain $|S_n'(x)| \to 0$ a.s.

Now let us choose $l(n)$ intervals $B_1, B_2, \dots, B_{l(n)}$ with radius

$$R_n = (h_n^{1+\beta}/n^{\frac{1}{r}}(\log n)^{1-\frac{1}{r}})^{\frac{1}{\beta}}$$

and centering at $t_1, t_2, \dots, t_{l(n)}$, respectively to cover $[0,1]$, where

$$l(n) < (h_n^{1+\beta}/n^{\frac{1}{r}}(\log n)^{1-\frac{1}{r}})^{-\frac{1}{\beta}}.$$

As $K(\cdot)$ satisfies Lipschitz conditions of order β, we have,

$$\begin{aligned}
\sup_{x\in B_k}|S_n'(x) - S_n'(t_k)| &= \sup_{x\in B_k}\left|\sum_{i=1}^{n} \frac{x_i - x_{i-1}}{h_n}\left(K\left(\frac{x - x_i}{h_n}\right) - K\left(\frac{t_k - x_i}{h_n}\right)\right)\xi_i\right| \\
&\le \sum_{i=1}^{n} \frac{x_i - x_{i-1}}{h_n} \sup_{x\in B_k}\left|\frac{x - t_k}{h_n}\right|^{\beta} |\xi_i| \\
&\le \frac{(2R_n)^{\beta}}{h_n^{\beta+1}} \frac{2n^{\frac{1}{r}}}{(\log n)^{\frac{1}{r}}} = \frac{C}{\log n} \to 0.
\end{aligned}$$

Thus for all n large enough, we obtain from (4.13)

$$\begin{aligned}
\mathbb{P}\left\{\sup_{x\in[\tau,1-\tau]} |S_n'(x)| > \varepsilon\right\} &\le \sum_{k=1}^{l(n)}\left[\mathbb{P}\left\{\sup_{x\in B_k}|S_n'(x) - S_n'(t_k)| > \frac{\varepsilon}{2}\right\} + \mathbb{P}\left\{|S_n'(t_k)| > \frac{\varepsilon}{2}\right\}\right] \\
&= \sum_{k=1}^{l(n)} \mathbb{P}\left\{|S_n'(t_k)| > \frac{\varepsilon}{2}\right\} \le Cl(n)n^{-t} \\
&\le Ch_n^{-\frac{1+\beta}{\beta}} (\log n)^{(1-\frac{1}{r})\frac{1}{\beta}} n^{\frac{1}{r\beta}} n^{-t}.
\end{aligned} \tag{4.14}$$

By letting $t > 2 + \frac{\mathrm{d}(1+\beta)}{\beta} + \frac{1}{\beta}$ in (4.14), we get,

$$\mathbb{P}\left\{\sup_{x\in[\tau,1-\tau]} |S_n'(x)| > \varepsilon\right\} = o(n^{-2}).$$

From Borel-Cantelli lemma, we have $\sup_{x\in[\tau,1-\tau]}|S_n'(x)| \to 0$ a.s.

Next we show (4.12). By denoting $T_n(x) = \sum_{i=1}^{n} a_{ni}(x)\varepsilon_i(2)$, we have

$$S_n''(x) = T_n(x) - \sum_{i=1}^{n} a_{ni}(x)\mathbb{E}(\varepsilon_i(2)|\mathcal{F}_{i-1}).$$

It is not difficult to see

$$\begin{aligned}\sup_{x\in[\tau,1-\tau]}\left|\sum_{i=1}^{n} a_{ni}(x)\mathbb{E}(\varepsilon_i(2)|\mathcal{F}_{i-1})\right| &\le \sup_{x\in[\tau,1-\tau]}\sum_{i=1}^{n}|a_{ni}(x)|\mathbb{E}(|\varepsilon_i(2)||\mathcal{F}_{i-1})\\ &= \sup_{x\in[\tau,1-\tau]}\sum_{i=1}^{n}|a_{ni}(x)|\mathbb{E}\left(|\varepsilon_i|^r|\varepsilon_i|^{1-r}\mathrm{I}\left\{|\varepsilon_i|^{1-r} \le \left(\frac{n}{\log n}\right)^{\frac{1-r}{r}}\right\}\middle|\mathcal{F}_{i-1}\right)\\ &\le C\sup_{x\in[\tau,1-\tau]}\sum_{i=1}^{n}|a_{ni}(x)|\left(\frac{n}{\log n}\right)^{\frac{1}{r}-1}\\ &\le C\left(\frac{n}{\log n}\right)^{\frac{1}{r}-1} \to 0,\end{aligned} \tag{4.15}$$

in which the equality is valid as $r > 1$ and the last inequality is from Lemma 3.2. Furthermore, we have

$$\begin{aligned}\sup_{x\in[\tau,1-\tau]}|T_n(x)| &\le C\frac{\delta_n}{h_n}\sum_{i=1}^{n}|\varepsilon_i|\mathrm{I}\left\{|\varepsilon_i| > \left(\frac{n}{\log n}\right)^{\frac{1}{r}}\right\}\\ &\le \frac{C}{n^{\frac{1}{r}}(\log n)^{1+\frac{1}{r}}}\sum_{i=1}^{n}|\varepsilon_i|\mathrm{I}\left\{|\varepsilon_i| > \left(\frac{i}{\log i}\right)^{\frac{1}{r}}\right\}.\end{aligned} \tag{4.16}$$

Let

$$\zeta_i = \frac{1}{i^{\frac{1}{r}}(\log i)^{1+\frac{1}{r}}}\mathbb{E}\left(|\varepsilon_i|\mathrm{I}\left\{|\varepsilon_i| > \left(\frac{i}{\log i}\right)^{\frac{1}{r}}\right\}\middle|\mathcal{F}_{i-1}\right),$$

then it is easy to check (see the second inequality in (4.15))

$$\zeta_i \le \frac{C}{i(\log i)^{\frac{2}{r}}} \quad \text{a.s.},$$

which implies $\sum_{i=1}^{\infty}\zeta_i < \infty$ a.s. as $1 < r < 2$. From Lemma 3.4, we get

$$\sum_{i=1}^{\infty}\frac{1}{i^{\frac{1}{r}}(\log i)^{1+\frac{1}{r}}}|\varepsilon_i|\mathrm{I}\left\{|\varepsilon_i| > \left(\frac{i}{\log i}\right)^{\frac{1}{r}}\right\} < \infty, \quad \text{a.s.},$$

then from Kronecker lemma and (4.16), we have,

$$\sup_{x\in[\tau,1-\tau]}|T_n(x)| \to 0 \quad \text{a.s.} \tag{4.17}$$

Combining (4.15) and (4.17), it is obvious that

$$\sup_{x\in[\tau,1-\tau]}|S_n''(x)| \to 0 \quad \text{a.s.}, \quad \forall\tau \in (0, 2^{-1}).$$

Moreover, it follows that

$$|S_n''(x)| \to 0 \quad \text{a.s.}, \quad \forall x \in (0, 1).$$

From the above discussions, we can obtain the desired results. ☐

Proof of Theorem 2.3. Fixed $a > 0$, denote

$$\varepsilon_i(1) := \varepsilon_i \mathrm{I}\{|\varepsilon_i| \le an\}, \quad \varepsilon_i(2) := \varepsilon_i \mathrm{I}\{|\varepsilon_i| > an\}$$

and define ξ_i, η_i, $S_n(x)$, $S_n'(x)$, $S_n''(x)$ as (4.3), then it is easy to see that $\varepsilon_i = \xi_i + \eta_i$, $S_n(x) = S_n'(x) + S_n''(x)$, and $\{\xi_i, \mathcal{F}_i, i \ge 1\}$, $\{\eta_i, \mathcal{F}_i, i \ge 1\}$ are martingale difference sequences. For any given $\varepsilon > 0$ and $t \in (0,1)$, it is obvious that

$$\mathbb{P}\{|S_n(x)| > \varepsilon\} \le \mathbb{P}\{|S_n'(x)| > t\varepsilon\} + \mathbb{P}\{|S_n''(x)| > (1-t)\varepsilon\}. \tag{4.18}$$

By noting that

$$|a_{ni}\xi_i| \le 2anC\sqrt{\frac{\log n}{n^{1+\delta}}} = 2aC\sqrt{n^{1-\delta}\log n}, \quad 1 \le i \le n,$$

and combining with Lemma 3.5, we have

$$\mathbb{P}\{|S_n'(x)| > t\varepsilon\} = \mathbb{P}\left\{\frac{|S_n'(x)|}{2aC\sqrt{n^{1-\delta}\log n}} > \frac{t\varepsilon}{2aC\sqrt{n^{1-\delta}\log n}}\right\} \le 2\exp\left(-\frac{t^2\varepsilon^2}{8a^2C^2n^{2-\delta}\log n}\right). \tag{4.19}$$

Letting $F_i(x) = \mathbb{P}\{|\varepsilon_i| > x\}$, then from $\mathbb{E}(\exp(|\varepsilon_i|)) \le b$, we have

$$F_i(x) \le b\exp(-x) \quad \text{for all } x \ge 0.$$

Thus, we get

$$\begin{aligned}
\mathbb{E}(\eta_i^2) &= \mathbb{E}(\varepsilon_i^2 \mathrm{I}_{\{|\varepsilon_i| > an\}}) - \mathbb{E}(\mathbb{E}(\varepsilon_i \mathrm{I}_{\{|\varepsilon_i| > an\}}|\mathcal{F}_{i-1}))^2 \\
&\le \mathbb{E}(\varepsilon_i^2 \mathrm{I}_{\{|\varepsilon_i| > an\}}) \\
&= -\int_{an}^{\infty} x^2 \mathrm{d}F_i(x) \\
&= -\lim_{M\to\infty}\left(M^2F_i(M) - a^2n^2F_i(an) - \int_{an}^{M} 2xF_i(x)\mathrm{d}x\right) \\
&\le ba^2n^2\exp(-an) + 2b\int_{an}^{\infty} x\exp(-x)\mathrm{d}x \\
&= b(a^2n^2 + 2an + 2)\exp(-an),
\end{aligned}$$

which can yield

$$\begin{aligned}
\mathbb{P}\{|S_n''(x)| > (1-t)\varepsilon\} &\le \frac{1}{(1-t)^2\varepsilon^2}\mathbb{E}(S_n''(x))^2 \\
&= \frac{1}{(1-t)^2\varepsilon^2}\sum_{i=1}^{n} a_{ni}^2 \mathbb{E}(\eta_i^2) \\
&\le \frac{1}{(1-t)^2\varepsilon^2}\frac{C\log n}{n^\delta} b(a^2n^2 + 2an + 2)\exp(-an).
\end{aligned} \tag{4.20}$$

Here the last equality holds by the fact that $\{\eta_i, \mathcal{F}_i, i \ge 1\}$ is a martingale difference sequence. Furthermore, by comparing (4.19) with (4.20), we can choose the constant a satisfying $an = \frac{t^2\varepsilon^2}{8a^2C^2n^{2-\delta}\log n}$, namely,

$$a = \frac{(t\varepsilon)^{\frac{2}{3}}}{2C^{\frac{2}{3}}n^{\frac{3-\delta}{3}}(\log n)^{\frac{1}{3}}}. \tag{4.21}$$

From (4.19), (4.20), and (4.21), we obtain

$$\mathbb{P}\{|S_n(x)| > \varepsilon\} \le H(t,n)\exp\left(-\frac{(t\varepsilon)^{\frac{2}{3}}}{2C^{\frac{2}{3}}n^{-\frac{\delta}{3}}(\log n)^{\frac{1}{3}}}\right), \tag{4.22}$$

where

$$H(t,n) = 2 + \frac{1}{(1-t)^2\varepsilon^2}\left(C_1(t\varepsilon)^{\frac{4}{3}}\left(\frac{\log n}{n^\delta}\right)^{\frac{1}{3}} + C_2(t\varepsilon)^{\frac{2}{3}}\left(\frac{\log n}{n^\delta}\right)^{\frac{2}{3}} + C_3\frac{\log n}{n^\delta}\right)$$

and C_1, C_2, C_3 are constants.

Finally, we choose $t = \frac{8C(\log n)^3}{\varepsilon n^{\frac{\delta}{2}}}$. Then for all n large enough, we have $0 < t < 1$ and $H(t,n) \le H$, where H is a constant. Then we get

$$\mathbb{P}\{|S_n(x)| > \varepsilon\} \le Hn^{-2},$$

which implies

$$\sum_{n=1}^{\infty}\mathbb{P}\{|S_n(x)| > \varepsilon\} < \infty.$$

From Borel-Cantelli lemma, (4.1) follows. □

Proof of Theorem 2.4. For $1 \le i \le n$, denote

$$\varepsilon_i(1) = \varepsilon_i\mathrm{I}\left\{|\varepsilon_i| \le \left(\frac{n}{\log n}\right)^{\frac{1}{r}}\right\}, \quad \varepsilon_i(2) = \varepsilon_i\mathrm{I}\left\{|\varepsilon_i| > \left(\frac{n}{\log n}\right)^{\frac{1}{r}}\right\},$$

and define ξ_i, η_i, $S_n(x)$, $S_n'(x)$, $S_n''(x)$ as (4.3).

By the similar proof of Theorem 2.2 and noting

$$\mathbb{E}(|\varepsilon_i|^r|\mathcal{F}_{i-1}) \le \mathbb{E}(\exp(|\varepsilon_i|^r)|\mathcal{F}_{i-1}) \le b \quad \text{a.s.},$$

then in order to obtain Theorem 2.4, it is sufficient to prove that for any $\tau \in (0, 2^{-1})$,

$$\sup_{x\in[\tau,1-\tau]}|S_n''(x)| \to 0 \quad \text{a.s.}$$

Let $T_n(x) = \sum_{i=1}^n a_{ni}(x)\varepsilon_i(2)$, then $S_n''(x) = T_n(x) - \sum_{i=1}^n a_{ni}(x)\mathbb{E}(\varepsilon_i(2)|\mathcal{F}_{i-1})$. It is easy to check that

$$\sup_{x\in[\tau,1-\tau]}\left|\sum_{i=1}^n a_{ni}(x)\mathbb{E}(\varepsilon_i(2)|\mathcal{F}_{i-1})\right| \le C\left(\frac{n}{\log n}\right)^{\frac{1}{r}-1} \to 0 \quad (n\to\infty) \tag{4.23}$$

and

$$\sup_{x\in[\tau,1-\tau]}|T_n(x)| \le C\frac{\delta_n}{h_n}\sum_{i=1}^n|\varepsilon_i|\mathrm{I}\left\{|\varepsilon_i| > \left(\frac{n}{\log n}\right)^{\frac{1}{r}}\right\}. \tag{4.24}$$

Thus for any $\varepsilon > 0$, $\lambda > 0$, we have

$$\begin{aligned}\mathbb{P}\left\{\sup_{x\in[\tau,1-\tau]}|T_n(x)| > \varepsilon\right\} &\le \mathbb{P}\left\{C\frac{\delta_n}{h_n}\sum_{i=1}^n|\varepsilon_i|\mathrm{I}\left\{|\varepsilon_i| > \left(\frac{n}{\log n}\right)^{\frac{1}{r}}\right\} > \varepsilon\right\}\\ &\le e^{-\lambda\varepsilon}\mathbb{E}\left(\exp\left(\lambda C\frac{\delta_n}{h_n}\sum_{i=1}^n|\varepsilon_i|\mathrm{I}\left\{|\varepsilon_i| > \left(\frac{n}{\log n}\right)^{\frac{1}{r}}\right\}\right)\right).\end{aligned} \tag{4.25}$$

Now by taking $\lambda = \varepsilon^{-1}\log(n^2b^n)$, we get

$$\mathbb{P}\left\{\sup_{x\in[\tau,1-\tau]}|T_n(x)| > \varepsilon\right\} \le n^{-2}b^{-n}\mathbb{E}\left(\exp\left(\varepsilon^{-1}\log(n^2b^n)C\frac{\delta_n}{h_n}\sum_{i=1}^n|\varepsilon_i|\mathrm{I}\left\{|\varepsilon_i| > \left(\frac{n}{\log n}\right)^{\frac{1}{r}}\right\}\right)\right). \tag{4.26}$$

Furthermore, it is easy to see that for every $1 \le i \le n$,

$$|\varepsilon_i|\mathrm{I}\left\{|\varepsilon_i| > \left(\frac{n}{\log n}\right)^{\frac{1}{r}}\right\} = |\varepsilon_i|^r|\varepsilon_i|^{1-r}\mathrm{I}\left\{|\varepsilon_i|^{1-r} \le \left(\frac{n}{\log n}\right)^{\frac{1-r}{r}}\right\} \le |\varepsilon_i|^r\left(\frac{n}{\log n}\right)^{\frac{1-r}{r}} \quad \text{a.s.},$$

which yields that for any $1 \le i \le n$,

$$\mathbb{E}\left(\exp\left(C\varepsilon^{-1}\log(n^2b^n)\frac{\delta_n}{h_n}\left(\frac{n}{\log n}\right)^{\frac{1-r}{r}}|\varepsilon_i|^r\right)\,\middle|\,\mathcal{F}_{i-1}\right) = \mathbb{E}(\exp(c_n|\varepsilon_i|^r)|\mathcal{F}_{i-1}), \qquad \text{a.s.},$$

where $c_n = o(1)$. Therefore, for all n large enough, we get

$$\mathbb{E}\left(\exp\left(C\varepsilon^{-1}\log(n^2b^n)\frac{\delta_n}{h_n}\left(\frac{n}{\log n}\right)^{\frac{1-r}{r}}\sum_{i=1}^n|\varepsilon_i|^r\right)\right) \le b^n.$$

From (4.23) and (4.26), we have, for all n large enough,

$$\mathbb{P}\left\{\sup_{x\in[\tau,1-\tau]}|S_n''(x)| > \varepsilon\right\} \le \mathbb{P}\left\{\sup_{x\in[\tau,1-\tau]}|T_n(x)| > \frac{\varepsilon}{2}\right\} + \mathbb{P}\left\{\sup_{x\in[\tau,1-\tau]}\left|\sum_{i=1}^n a_{ni}(x)\mathbb{E}(\varepsilon_i(2)|\mathcal{F}_{i-1})\right| > \frac{\varepsilon}{2}\right\} \le Cn^{-2}.$$

From Borel-Cantelli lemma, we can obtain the desired results. □

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