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STRONG CONVERGENCE AND CONVERGENCE RATES OF APPROXIMATING SOLUTIONS FOR ALGEBRAIC RICCATI EQUATIONS IN HILBERT SPACES*

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Abstract

In this paper, we consider the linear quadratic optimal control problem on infinite time interval for linear time-invariant systems defined on Hilbert spaces. The optimal control is given by a feedback form in terms of solution Π to the associated algebraic Riccati equation (ARE). A Ritz type approximation is used to obtain a sequence Π^N of finite dimensional approximations of the solution to ARE. A sufficient condition that shows Π^N converges strongly to Π is obtained. Under this condition, we derive a formula which can be used to obtain a rate of convergence of Π^N to Π. We demonstrate and apply the results for the Galerkin approximation for parabolic systems and the averaging approximation for hereditary differential systems.

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1. Introduction

Assume Z, U and Y are Hilbert spaces. Consider the evolution equation on Z

$$(1.1) \qquad \dot{z}(t) = \mathcal{A}\, z(t) + \mathcal{B}\, u(t)\,, \qquad z(0) = z_0 \in Z$$

where $u(t)$ is a U-valued control function, $\mathcal{A}$ is the infinitesimal generator of strongly continuous semigroup $S(t)$ on Z, and $\mathcal{B} \in \mathfrak{L}(U,Z)$. The Y-valued observation function y is given by

$$(1.2) \qquad y(t) = \mathcal{C}\, z(t)\,, \qquad t \geq 0\,.$$

We assume that $\mathcal{C} \in \mathfrak{L}(Z,Y)$. We interpret the equation (1.1) in the mild sense; the solution of (1.1) is given by

$$(1.3) \qquad z(t) = S(t)z_0 + \int_0^t S(t-s)\mathcal{B}\, u(s)ds\,.$$

Consider the minimization problem; minimize the cost functional

$$(1.4) \qquad J(u,z_0) = \int_0^\infty (\|y(t)\|^2 + \|u(t)\|^2)dt$$

subject to (1.3). Then the following result is well-known [10],[11]:

Theorem 1.1 Assume $(\mathcal{A},\mathcal{B})$ is stabilizable and $(\mathcal{A},\mathcal{C})$ is detectable. Then there exists a unique nonnegative self-adjoint solution Π to the algebraic Riccati equation in Z:

$$(1.5) \qquad (\mathcal{A}^*\Pi + \Pi\mathcal{A} - \Pi\mathcal{B}\mathcal{B}^*\Pi + \mathcal{C}^*\mathcal{C})z = 0 \quad \text{for all } z \in \mathrm{dom}(\mathcal{A})\,,$$

and the optimal solution u^0 to (1.4) is given by

$$u^0(t) = -\mathcal{B}^*\Pi\, T(t)z_0$$

where $T(t)$ is the strongly continuous semigroup generated by $\mathcal{A} - \mathcal{B}\mathcal{B}^*\Pi$ and it is uniformly exponentially stable.

Here we have

Definition 1.2 (1) (A,B) is stabilizable if there exists an operator $K \in \mathfrak{L}(Z,U)$ such that $A-BK$ generates a uniformly exponentially stable semigroup on Z.

(2) (A,C) is detectable if there exists an operator $G \in \mathfrak{L}(Y,Z)$ such that $A - GC$ generates a uniformly exponentially stable semigroup.

The purpose here is to construct a finite dimensional approximation of the optimal feedback gain operator $B^*\Pi$. Let us consider a sequence of approximating problems (Z^N, A^N, B^N, C^N); let Z^N be a sequence of finite dimensional subspaces of Z and P^N be the orthogonal projection of Z onto Z^N. Assume $A^N : Z^N \to Z^N$, $B^N : U \to Z^N$ and $C^N : Z^N \to Y$ are continuous. Then consider the Nth approximating problem of (1.4)

$$\text{(1.6)} \qquad \text{minimize} \quad J^N(u,z_0) = \int_0^\infty (\|C^N z^N(t)\|^2 + \|u(t)\|^2)dt$$

subject to

$$\text{(1.7)} \qquad z^N(t) = S^N(t)P^N z_0 + \int_0^t S^N(t-s)B^N u(s)ds$$

where $S^N(t) = e^{A^N t}$, $t \geq 0$. Then the optimal control u^N of (1.6) is given by

$$u^N(t) = -B^{N^*}\Pi^N e^{(A^N - B^N B^{N^*}\Pi^N)t} P^N z_0 , \quad t \geq 0$$

where $\Pi^N : Z^N \to Z^N$ is self-adjoint and satisfies the Nth approximating algebraic Riccati equation in Z^N;

$$\text{(1.8)} \qquad A^{N^*}\Pi^N + \Pi^N A^N - \Pi^N B^N B^{N^*}\Pi^N + C^{N^*}C^N = 0 .$$

Here, $B^{N^*}\Pi^N$, $N \geq 1$ yields a sequence of finite dimensional approximations of the optimal feedback gain [3].

In this paper we first obtain a condition on (Z^N, A^N, B^N, C^N) for which (1.8) admits a unique nonnegative solution Π^N, and $\Pi^N P^N$ converges strongly to Π in §2. Such a condition has been discussed in [2], [3] but the condition in this paper improves those in [2], [3], i.e., we introduce the uniform detectability condition (see, (H3) in §2, for the definition) which is additional to those considered in [2], and using this condition, we are able to show that there exists an integer N_0 such that for $N \geq N_0$

$$\|e^{(A^N - B^N B^{N*} \Pi^N)t} P^N\| \leq M\, e^{-\omega t}\ ,\ t \geq 0$$

for positive constants $M \geq 1$ and ω (independent of $N \geq N_0$). This assertion is a part of assumptions in [2, Theorem 2.2]. The uniform detectability condition is satisfied if C^*C is coercive, which is assumed in the discussions in [2,p. 693]. Thus, the uniform detectability condition can be regarded as a relaxation of the coercivity assumption mentioned above. Next, under the condition in §2 we derive a formula which provides a rate of convergence of Π^N to Π and apply the formula for specific examples.

2. Uniform Stability and Strong Convergence

We assume the following. Let $S^N(t) = e^{A^N t}$, $t \geqslant 0$

(H1) For each $z \in Z$, we have

(i) $S^N(t)P^N z \to S(t)z$, and

(ii) $S^N(t)^* P^N z \to S^*(t)z$,

where the convergences are uniform in t on bounded subsets of $[0,\infty)$.

(H2) (i) For each $u \in U$, $B^N u \to Bu$ and for each $z \in Z$ $B^{N^*}P^N z \to B^* z$.

(ii) For each $z \in Z$, $C^N P^N z \to Cz$ and for each $y \in Y$ $C^{N^*} y \to C^* y$.

(H3) (i) The family of the pairs (A^N, B^N) is uniformly stabilizable: i.e. there exists a sequence of operators $K^N \in \mathfrak{L}(Z^N, U)$ such that $\sup \|K^N\| < \infty$ and

$$\|e^{(A^N - B^N K^N)t} P^N\| \leqslant M_1 e^{-\omega_1 t}, \quad t \geqslant 0$$

for some positive constants $M_1 \geqslant 1$ and ω_1 .

(ii) The family of the pairs (A^N, C^N) is uniformly detectable; i.e. there exists a sequence of operators $G^N \in \mathfrak{L}(Y, Z^N)$ such that $\sup \|G^N\| < \infty$ and

$$\|e^{(A^N - G^N C^N)t} P^N\| \leqslant M_2 e^{-\omega_2 t}, \quad t \geqslant 0$$

for some positive constants $M_2 \geqslant 1$ and ω_2.

<u>Remark</u> **(1)** Suppose $B^N = P^N B$ and $C^N = CP^N$. Then (H2) holds since it follows from (H1) that $P^N z \to z$ for all $z \in Z$.

(2) The assumption (H3) is closely related to the preservation of exponential stability under approximation in [3,Conjecture 7.1] and it is shown in [2] that (H3) (i) ((POES) in [2]) is satisfied for parabolic systems using the

Galerkin approximation.

(3) A natural way to argue (H3) is to take $K^N = KP^N$ and $G^N = P^N G$ for some $K \in \mathfrak{L}(Z,U)$ and $G \in \mathfrak{L}(Y,Z)$ such that $A - BK$ and $A - GC$ generate uniformly exponentially stable semigroups on Z.

<u>Theorem 2.1</u> Suppose (H1)-(H3) are satisfied. Then for each N, (1.8) admits a unique nonnegative solution Π^N, $\sup\|\Pi^N\| < \infty$, and there exist positive constants $M_3 \geq 1$ and ω_3 (independent of N) such that

$$\|e^{(A^N - B^N B^{N*}\Pi^N)t}P^N\| \leq M_3 e^{-\omega_3 t}, \quad t \geq 0 .$$

<u>Proof:</u> The proof is based on the arguments in [11]. The existence and uniqueness of solutions to (1.8) follow from Theorem 1.1. Since $\langle \Pi^N P^N z,z\rangle = \min J^N(u,z)$, (H3) (i) implies that

$$\begin{aligned}\langle \Pi^N P^N z,z\rangle &\leq J^N(-K^N z^N(\cdot);z) \\ &= \int_0^\infty (\|C^N e^{(A^N-B^N K^N)t}P^N z\|^2 + \|K^N e^{(A^N-B^N K^N)t}P^N z\|^2)dt \\ &\leq \beta\|z\|^2 \quad \text{for some positive constant } \beta\end{aligned}$$

Since Π^N is self-adjoint, nonnegative definite, this implies that $\|\Pi^N\| \leq \beta$. By the variation of constants formula

$$e^{(A^N-B^N B^{N*}\Pi^N)t} = T^N(t) + \int_0^t T^N(t-s)(G^N C^N - B^N B^{N*}\Pi^N)e^{(A^N-B^N B^{N*}\Pi^N)s}\, ds \tag{2.1}$$

where $T^N(t) = e^{(A^N-G^N C^N)t}$, $t \geq 0$. Here, from (1.8)

$$(A^N-B^N B^{N*}\Pi^N)^*\Pi^N + \Pi^N(A^N-B^N B^{N*}\Pi^N) + \Pi^N B^N B^{N*}\Pi^N + C^{N*}C^N = 0 ,$$

so that if $z^N(t) = e^{(A^N-B^N B^{N*}\Pi^N)t}P^N z$, $t \geq 0$, then

$$\frac{d}{dt}\langle z^N(t),\Pi^N z^N(t)\rangle + \|B^{N*}\Pi^N z^N(t)\|^2 + \|C^N z^N(t)\|^2 = 0$$

Thus, for all $t \geq 0$

$$(2.2)\qquad \langle \Pi^N z^N(t),z^N(t)\rangle + \int_0^t (\|B^{N*}\Pi^N z^N(t)\|^2 + \|C^N z^N(t)\|^2)dt$$
$$\leq \langle \Pi^N P^N z,z\rangle \leq \beta\|z\|^2$$

Now, from (2.1), we have for all $t \geq 0$

$$\int_0^t \|z^N(s)\|^2 ds \leq \frac{M_2^2}{\omega_2}\|z\|^2 + \frac{2M_2^2}{\omega_2^2}(\|G^N\|^2 + \|B^N\|^2)\int_0^t(\|B^{N*}\Pi^N z^N(s)\|^2$$
$$+ \|C^N z^N(s)\|^2)ds$$

where we have used the Young's inequality. From (2.2), we have

$$\int_0^\infty \|z^N(t)\|^2 dt \leq \frac{M_2^2}{\omega_2^2}(\omega_2 + 2\beta\ \sup(\|G^N\|^2 + \|B^N\|^2))\|z\|^2$$

for all $z \in Z$. Therefore, the theorem follows from the Datko's theorem [7].

(Q.E.D.)

The following is a consequence of [3, Theorem 6.9] and [2, Theorem 2.2].

__Corollary 2.2__ Suppose (A,B) is stabilizable and (A,C) is detectable and assume (H1) ~ (H3) are satisfied. Then the unique nonnegative solution Π^N to (1.8) converges strongly to Π, the unique solution to (1.5).

__Theorem 2.3__ Suppose that B is compact and $B^N = P^N B$ and that (H1)(i) and (H3)(i) are satisfied. Then (A,B) is stabilizable.

__Proof:__ Let us consider the case $C = I$ and $C^N = P^N$ with $Y = Z$. Then it is easy to show that (A,C) is detectable and (A^N,C^N), $N \geq 1$ are uniformly detectable since (H1)(i) implies that for some $M \geq 1$ and ω independent of N,

$\|S^N(t)P^N\| \leq Me^{\omega t}$, $t \geq 0$. It then follows from Theorem 1.1 and (H3)(i) that for each N, (1.8) with $C^N = P^N$ has a unique solution $\hat{\Pi}^N$. Using the same argument as in the proof of Theorem 2.1, we have $\|\hat{\Pi}^N\| \leq \hat{\beta}$ for some positive constant $\hat{\beta}$. Thus, by Theorem 6.5 in [3], there exists a subsequence of $\hat{\Pi}^N$ converges weakly to some nonnegative, self-adjoint operator $\hat{\Pi}$. We will show that $\hat{\Pi}$ satisfies (1.5) with $C = I$. Since $P^N\hat{\Pi}^N = \hat{\Pi}^N$, $B^{N*}\hat{\Pi}^N = B^*P^N\hat{\Pi}^N = B^*\hat{\Pi}^N$. Since B^* is compact, for each $z \in Z$, $B^{N_j*}\hat{\Pi}^{N_j}P^{N_j}z$ converges strongly to $B^*\hat{\Pi}z$. It now follows from [3, Theorem 6.7] that $\hat{\Pi}$ satisfies (1.5) But since (A,C) is detectable, by [10, Theorem 3.2], $A - BB^*\hat{\Pi}$ generates a uniformly exponentially stable semigroup on Z.

(Q.E.D.)

<u>Remark 2.4</u> Roughly speaking, Theorem 2.3 means that the uniform stabilizability implies the stabilizability of (A,B). The dual statement of Theorem 2.3 also holds: i.e., suppose C is compact, $C^N = CP^N$, then (H1)(ii) and (H3)(ii) imply that (A,C) is detectable. This statement can be proved by applying the exactly same arguments as in the proof of Theorem 2.3 to the dual Riccati equation

$$(A\Sigma + \Sigma A^* - \Sigma C^*C\Sigma + I)\, z = 0$$

for all $z \in \mathrm{dom}(A^*)$.

3. Convergence Rate

In this section, we assume that (H1) and (H3) are satisfied and let $B^N = P^N B$ and $C^N = CP^N$. Moreover, we assume

(H4) For each $z \in Z$, $\Pi z \in \mathrm{dom}(A^*)$ and B is compact.

From (1.5), we have for all $z \in \mathrm{dom}(A)$

$$2\langle \Pi z, Az\rangle - \langle B^*\Pi z, B^*\Pi z\rangle + \langle Cz, Cz\rangle = 0$$

Thus, (H4) and the density of $\mathrm{dom}(A)$ in Z imply that for all $z \in Z$

$$2\langle A^*\Pi z, z\rangle - \langle B^*\Pi z, B^*\Pi z\rangle + \langle Cz, Cz\rangle = 0 \ .$$

Define the self-adjoint operator $\hat{\Pi}^N = P^N \Pi P^N$. Then for all $x \in Z^N$

(3.1) $$2\langle A^{N^*}\hat{\Pi}^N x, x\rangle - \langle B^*\hat{\Pi}^N x, B^*\hat{\Pi}^N x\rangle + \langle C^N x, C^N x\rangle + \langle \Delta^N x, x\rangle = 0 \ ,$$

where $\Delta^N \in \mathfrak{L}(Z^N)$ is a self-adjoint operator defined by

(3.2) $$\langle \Delta^N x, x\rangle = 2\langle (A^* - A^{N^*}P^N)\Pi x, x\rangle + \langle B^*(\hat{\Pi}^N - \Pi^N)x, B^*(\hat{\Pi}^N + \Pi^N)x\rangle \quad \text{for all} \quad x \in Z^N \ .$$

From (1.8), for $x \in Z^N$

(3.3) $$2\langle A^{N^*}\Pi^N x, x\rangle - \langle B^{N^*}\Pi^N x, B^{N^*}\Pi^N x\rangle + \langle C^N x, C^N x\rangle = 0 \ .$$

Hence by subtracting (3.1) from (3.3)

$$2\langle (A^N - B^N B^{N^*}\Pi^N)x, (\Pi^N - \hat{\Pi}^N)x\rangle + \langle B^{N^*}(\Pi^N - \hat{\Pi}^N)x, B^{N^*}(\Pi^N - \hat{\Pi}^N)x\rangle - \langle \Delta^N x, x\rangle = 0$$

for all $x \in Z^N$. Or equivalently

(3.4) $$\Pi^N - \hat{\Pi}^N = \int_0^\infty e^{(A^N - B^N B^{N^*}\Pi^N)t}\left((\Pi^N - \hat{\Pi}^N)B^N B^{N^*}(\Pi^N - \hat{\Pi}^N) - \Delta^N\right) \times e^{(A^N - B^N B^{N^*}\Pi^N)t}\, dt \ .$$

Similarly, subtracting (3.3) from (3.1), we obtain

$$(3.5)\qquad \hat{\Pi}^N - \Pi^N = \int_0^\infty e^{(A^N-B^N B^{N*}\hat{\Pi}^N)t}((\Pi^N - \hat{\Pi}^N)B^N B^{N*}(\Pi^N - \hat{\Pi}^N) + \Delta^N) \times e^{(A^N-B^N B^{N*}\hat{\Pi}^N)t}dt\ .$$

Here, from Theorem 2.1, we have

$$\|e^{(A^N-B^N B^{N*}\Pi^N)t}\ P^N\| \leq M_3 e^{-\omega_3 t},\ t \geq 0$$

with $M_3 \geq 1$ and $\omega_3 > 0$. Since $\Pi^N P^N \to \Pi$, strongly by Corollary 2.2 and B is compact

$$\|B^{N*}(\Pi^N - \hat{\Pi}^N)\| = \|(\Pi^N - \Pi)P^N B\| \to 0 \quad \text{as} \quad N \to \infty\ .$$

Hence by the variation of constants formula and the Gronwall's lemma,

$$\|e^{(A^N-B^N B^{N*}\hat{\Pi}^N)t}\ P^N\| \leq M_3 e^{(-\omega_3+\|B\|\|B^{N*}(\Pi^N-\hat{\Pi}^N)\|)t}$$

It then follows that there exists an integer N_0 such that if $N \geq N_0$,

$$\|e^{(A^N-B^N B^{N*}\hat{\Pi}^N)t}\ P^N\| \leq M_3 e^{-\frac{\omega_3}{2}t},\ t \geq 0\ .$$

Now, from (3.4) for all $x \in Z^N$

$$(3.6)\qquad \langle(\Pi^N - \hat{\Pi}^N)x,x\rangle - \int_0^\infty \|B^N(\Pi^N - \hat{\Pi}^N)e^{(A^N-B^N B^{N*}\Pi^N)t}x\|^2\,dt = -\int_0^\infty \langle e^{(A^N-B^N B^{N*}\Pi^N)t}x,\ \Delta^N e^{(A^N-B^N B^{N*}\Pi^N)t}x\rangle dt$$

and from (3.5)

$$(3.7)\qquad \langle(\hat{\Pi}^N - \Pi^N)x,x\rangle - \int_0^\infty \|B^N(\Pi^N - \hat{\Pi}^N)e^{(A^N-B^N B^{N*}\hat{\Pi}^N)t}x\|^2 dt = \int_0^\infty \langle e^{(A^N-B^N B^{N*}\hat{\Pi}^N)t}x,\ \Delta^N e^{(A^N-B^N B^{N*}\hat{\Pi}^N)t}x\rangle dt\ .$$

These inequalities imply that for $x \in Z^N$

$$|\langle(\Pi^N - \hat{\Pi}^N)x,x\rangle| \leq \frac{2M_3^2}{\omega_3} \|\Delta^N\|\|x\|^2 ,$$

so that

$$\|\Pi^N - \hat{\Pi}^N\| \leq \frac{2M_3^2}{\omega_3} \|\Delta^N\| \tag{3.8}$$

where from (3.2)

$$\|\Delta^N\| \leq 2\|(A^* - A^{N^*}P^N)\Pi\| + 2\beta\|B\|\|(B^*-B^{N^*})\Pi\| \quad \text{for all} \quad N \geq N_0. \tag{3.9}$$

4. Examples

In this section we discuss the examples in which (H1)~(H4) are satisfied and then apply the formula (3.8) and (3.9) to obtain a convergence rate of Π^N to Π.

4.1 Parabolic Systems

Assume V and H are Hilbert spaces and $V \subset H$ with continuous dense injection i. Consider a bilinear form σ on V such that

(4.1) $$|\sigma(u,v)| \leq c\|u\|_V \|v\|_V \quad \text{for} \quad u,v \in V$$

(4.2) $$\sigma(u,u) \geq \omega\|u\|_V^2 - \rho\|u\|_H^2 \quad \text{for} \quad u \in V$$

where $\omega > 0$. It then follows from [9] that there exists an operator $A \in \mathfrak{L}(V,V^*)$ such that

(4.3) $$\sigma(u,v) = \langle -Au,v \rangle_{V^*,V} \quad \text{for} \quad u,v \in V ,$$

where $V \subset H = H^* \subset V^*$ and H being the pivoting space, and that A on H with

(4.4) $$\text{dom}(A) = \{x \in H : Ax \in H\} \quad \text{dense in} \quad V,$$

generates the analytic semigroup on H and V^*. For given $B \in \mathfrak{L}(U,H)$ and $C \in \mathfrak{L}(H,V)$ consider approximating problems (Z^N,A^N,B^N,C^N); i.e. let Z^N be a sequence of finite dimensional subspace of V and A^N: $Z^N \to Z^N$ is defined by

(4.5) $$\langle -A^N z,x \rangle = \sigma(z,x) \quad \text{for} \quad z,x \in Z^N .$$

Let P^N be the orthogonal projection of H onto Z^N and assume $B^N = P^N B$ and $C^N = CP^N$. We assume the approximation condition:

For each $z \in V$ there exists an element $z^N \in Z^N$

(4.6) such that $\|z - z^N\|_V \leq \epsilon(N)$ where $\epsilon(N) \to 0$ as $N \to \infty$.

It then follows from [2] that (H1) holds and if (A,B) is stabilizable and (A,C) is detectable, then (H3) holds. Thus from Corollary 2.2, Π^N converges strongly to Π. However one cannot apply the formula (3.8)-(3.9) as it is, since $\Pi Z \subset \mathrm{dom}(A^*)$ is the maximal regularity without assuming any regularity of C. This can be demonstrated by the following example. Consider the case when $H = L^2(0,1)$ and $V = H_0^1(0,1)$, and

$$\sigma(u,v) = \int_0^1 \frac{d}{dx} u(x) \frac{d}{dx} v(x)dx \quad \text{for} \quad u,v \in H_0^1 .$$

Let us consider the Liapunov equation on H

(4.7) $A\Sigma + \Sigma A + Q = 0$

where Q is self-adjoint operator on H. If for each $z \in Z$,

$$(\Sigma z)(x) = \int_0^1 \phi(x,y)z(y)dy \quad \text{and} \quad (Qz)(x) = \int_0^1 q(x,y)z(y)dy,$$

then ϕ satisfies $\Delta\phi + q = 0$ with Dirichlet boundary condition,

where $\Delta\phi = \frac{\partial^2}{\partial x^2}\phi + \frac{\partial^2}{\partial y^2}\phi$ for $\phi \in H^2([0,1] \times [0,1])$. In general (e.g., see [6],[8])

$$\int_0^1\int_0^1 \left(\left|\frac{\partial^2}{\partial x^2}\phi\right|^2 + \left|\frac{\partial^2}{\partial y^2}\phi\right|^2\right) dxdy \leq M \int_0^1\int_0^1 |q|^2\, dxdy .$$

This implies $\Sigma L^2 \subset \mathrm{dom}(A)$ is the maximal regularity.

Hence we will modify the arguments in Section 3 to improve the formula (3.8)-(3.9) for this example. First we note that in (3.2) for $x \in Z^N$

$$\begin{aligned} |\langle (A^* - A^{N*}P^N)\Pi x, x\rangle| &= |\sigma(x,(\hat{\Pi}^N - \Pi)x)| \\ &\leq c\|x\|_V\|(\Pi - \hat{\Pi}^N)x\|_V \quad \text{by} \quad (4.1) . \end{aligned}$$

Thus from (3.2)

$$(4.8) \qquad \gamma = \sup_{\|x\|_V \neq 0} \frac{|\langle \Delta^N x,x\rangle|}{\|x\|_V^2} \leq 2\,c \sup_{x\neq 0} \frac{\|(\Pi-\hat{\Pi}^N)x\|_V}{\|x\|_V}$$

$$+ \; 2\alpha\beta\|B\| \sup_{x\neq 0} \frac{\|B^*(\Pi-\hat{\Pi}^N)x\|_V}{\|x\|_V},$$

where $\|x\|_H \leq \alpha\|x\|_V$.

<u>Lemma 4.1</u> There exists a positive constant M such that

$$\int_0^\infty \|e^{(A^N-B^NB^{N^*}\Pi^N)t}\;P^Nx\|_V^2\;dt \leq M\|x\|_H^2\;,\;\text{and}$$

$$\int_0^\infty \|e^{(A^N-B^NB^{N^*}\hat{\Pi}^N)t}\;P^N)\|_V^2\;dt \leq M\|x\|_H^2\;.$$

<u>Proof:</u> Let $\xi^N(t) = e^{(A^N-B^NB^N{}^*\!\hat{\Pi}^N)t}\;P^Nx$, $t \geq 0$. Then $\xi^N(t)$ satisfies

$$\frac{d}{dt}\,\xi^N(t) = (A^N-B^NB^{N^*}\Pi^N)\xi^N(t)\;,\;t \geq 0\;,$$

so that from (4.5)

$$\frac{1}{2}\,\frac{d}{dt}\,\|\xi^N(t)\|_H^2 + \sigma(\xi^N,\xi^N) = -\;\langle B^{N^*}\xi^N(t),B^{N^*}\Pi^N\xi^N(t)\rangle_H$$

and from (4.2)

$$\frac{1}{2}\,\frac{d}{dt}\,\|\xi^N(t)\|_H^2 + \omega\|\xi^N(t)\|_V^2 \leq (\rho + 2\beta\|B\|^2)\;\|\xi^N(t)\|_H^2\;.$$

The integration of this inequality with respect to t yields

$$\frac{1}{2}\,\|\xi^N(t)\|_H^2 + \omega\int_0^t \|\xi^N(t)\|_V^2 \leq \frac{1}{2}\,\|\xi^N(0)\|^2 + (\rho + 2\beta\|B\|^2)\int_0^t \|\xi^N(s)\|_H^2 ds.$$

Now the lemma follows from Theorem 2.1.

Q.E.D.

It then follows from Lemma 4.1 and (4.8) that

$$\int_0^\infty \langle \Delta^N e^{(A^N - B^N B^{N*} \Pi^N)t} x , e^{(A^N - B^N B^{N*} \Pi^N)t} x \rangle$$

$$\leq \gamma \int_0^\infty \| e^{(A^N - B^N B^{N*} \Pi^N)t} P^N x \|_V^2 \, dt \leq M\gamma \|x\|_H^2 .$$

Similary for $e^{(A^N - B^N B^{N*} \hat{\Pi}^N)t}$, $t \geq 0$. Therefore we obtain, using (3.6) and (3.7),

(4.9) $\quad \|\Pi^N - \hat{\Pi}^N\| \leq M\gamma$.

where γ is given by (4.8).

Consider the (1-dimensional) parabolic control system [2];

$$\frac{\partial}{\partial t} z(t,x) = \frac{\partial}{\partial x} (p(x) \frac{\partial}{\partial x} z) + q(x) \frac{\partial}{\partial x} z + r(x)z$$

$$+ \sum_{i=1}^{m} b_i(x) u_i(t) \quad \text{in } (0,1)$$

with boundary condition $z(t,0) = z(t,1) = 0$, where $p \in C^1(0,1)$, being bounded below by a positive constant ω, $\frac{d}{dx} q$, $r \in L^\infty(0,1)$, and $b_i \in L^2(0,1)$, $i = 1,\ldots,m$. In this case, $H = L^2(0,1)$ and $V = H_0^1(0,1)$, and the bilinear form σ is given by

$$\sigma(u,v) = \int_0^1 [p(x) \frac{d}{dx}u \frac{d}{dx}v - (q(x) \frac{d}{dx}u + r(x)u)v]dx .$$

$B: \mathbb{R}^m \to L^2(0,1)$ is defined by

$$(Bu)(x) = \sum_{i=1}^{m} b_i(x)u_i \quad \text{for} \quad u \in \mathbb{R}^m ,$$

and $\text{dom}(A) = \text{dom}(A^*) = H^2(0,1) \cap H_0^1(0,1)$. Let us consider the following finite dimensional subspace Z^N of V:

$$Z^N = \{z \in H : z(x) = \sum_{i=1}^{N-1} \alpha_i B_i^N(x) , \alpha_i \in \mathbb{R}\}$$

where $B_i^N(\cdot)$, $i = 1,\ldots,N\text{-}1$ are the linear B-spline elements on the interval [0,1]; i.e.,

$$B_i^N = \begin{cases} -N(x-\frac{i+1}{N}), & x \in \left[\frac{i}{N}, \frac{i+1}{N}\right] \\ N(x-\frac{i-1}{N}), & x \in \left[\frac{i-1}{N}, \frac{i}{N}\right] \\ 0, & \text{elsewhere} \end{cases}.$$

Then the approximation condition (4.6) is satisfied [8]. Suppose (A,B) is stabilizable and (A,C) is detectable. Then (1.5) has the unique solution Π and using a similar arguments to those given above to show the regularity of solutions to Liapanov equation (4.7), one can show that for $x \in H$, $\Pi x \in \mathrm{dom}(A^*)$. Since A^* is closed in H and $\mathrm{dom}(A^*) \subset V$, by the closed graph theorem, there exists a positive constant k_1, such that $\|\Pi z\|_{H^2(0,1)} \leq k_1\|z\|_{L^2(0,1)}$. Hence the fundamental error estimate (e.g., [8]) gives

$$\|\Pi z - \hat{\Pi}^N z\|_{L^2} \leq k_2 \left(\frac{1}{N}\right)^2 \|z\|_{L^2}$$

$$\|\Pi z - \hat{\Pi}^N z\|_{H_0^1} \leq k_3 \left(\frac{1}{N}\right) \|z\|_{L^2}$$

for some positive constants k_2,k_3. Now it follows from (4.8) and (4.9) that

$$\|\Pi^N - \hat{\Pi}^N\| \leq k\left(\frac{1}{N}\right) \quad \text{for some constant } k.$$

4.2 <u>Hereditary Differential Systems</u>

Consider the hereditary differential system in $\mathbb{R}^N$;

$$(4.10) \quad \begin{cases} \dot{x}(t) = A_0x(t) + A_1x(t-r) + \int_{-r}^{0} A(\theta)x(t+\theta)d\theta + Bu(t) \\ x(0) = \eta \quad \text{and} \quad x(\theta) = \phi(\theta), \ -r \leq \theta < 0 \\ y(t) = C\,x(t) \end{cases}$$

and the optimal control problem; for given initial data $z = (\eta,\phi) \in \mathbb{R}^n \times L^2(-r,0;\mathbb{R}^n)$, minimize the cost functional

$$J(u,z) = \int_0^\infty (|y(t)|^2 + |u(t)|^2)dt \; . \tag{4.11}$$

Here, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ and the element of $A(\cdot)$ is square integrable. It is shown [1] that (4.10) and (4.11) are equivalently formulated as the problem (1.1) ~ (1.4) on the product space $Z = \mathbb{R}^n \times L^2(-r,0;\mathbb{R}^n)$; i.e., $z(t) = (x(t),x(t+\cdot)) \in Z$ is the mild solution of (1.1) with

$$\mathrm{dom}(\mathcal{A}) = \{(\eta,\phi) \in Z : \phi \in H^1(-r,0) \text{ and } \phi(0) = \eta\} \; ,$$

for $(\phi(0),\phi) \in \mathrm{dom}(\mathcal{A})$

$$\mathcal{A}(\phi(0),\phi) = (A_0\phi(0) + A_1\phi(-r) + \int_{-r}^0 A(\theta)\phi(\theta)d\theta,\ \dot{\phi}) \; .$$

The input operator $\mathcal{B} : \mathbb{R}^m \to Z$ and the output operator $\mathcal{C} : Z \to \mathbb{R}^p$ are given by

$$\mathcal{B}u = (Bu,0) \in Z \quad \text{and} \quad \mathcal{C}(\eta,\phi) = C\eta \; .$$

Let us consider the averaging approximation [1] of (4.10); let

$$Z^N = \{z \in Z : z = (a_0, \sum_{k=1}^N a_k \chi_{[-\frac{i}{N}r, -\frac{i-1}{N}r)}),\ a_k \in R^n,\ 0 \le k \le N\} \subset Z \; ,$$

and $\mathcal{A}^N$ has the matrix representation $(Q^N)^{-1}H^N$ on $\mathbb{R}^{n(N+1)}$ when Z^N is identified with $\mathbb{R}^{n(N+1)}$ by its coordinate vector $\mathrm{col}(a_0^T,\ldots,a_N^T)$, where the block diagonal matrix Q^N and the block Hessenberg matrix H^N are given by

$$Q^N = \begin{bmatrix} I & & & \\ & \frac{r}{N}I & & \\ & & \ddots & \\ & & & \frac{r}{N}I \end{bmatrix} \quad \text{and} \quad H^N = \begin{bmatrix} A_0^N & A_1^N & \cdots & A_N^N \\ I & -I & & \\ & \ddots & \ddots & \\ & & I & -I \end{bmatrix}$$

with $A_0^N = A_0$, $A_i^N = \int_{-\frac{i}{N}r}^{-\frac{i-1}{N}r} A(\theta)d\theta$ and $A_N^N = A_1 + \int_{-r}^{-\frac{N-1}{N}r} A(\theta)d\theta$. Note that $P^N B = B$ and $CP^N = C$. Set $B^N = B$ and $C^N = C$. Then $(A^N)^*$ has the matrix representation $(Q^N)^{-1}H^{N^T}$ on $\mathbb{R}^{n(N+1)}$. (Hl)(i) is proved in [1] and (Hl)(ii) is proved in [3]. Using the arguments in [5], [7] one can show that (H3) is satisfied (i.e., (i) is straightforward but (ii) is not so). Thus, the formula (3.8)-(3.9) yields

$$\|\Pi^N - \hat{\Pi}^N\| \leq 2\|(A^* - A^{N^*}P^N)\Pi\|$$

By the regularity result in [4], if $A(\cdot) \in H^1(-r,0;R^{n\times n})$, then

$$A^*\Pi + C^*C \in \mathrm{dom}(A^*)$$

where $\mathrm{dom}(A^*) = \{(y,\psi) \in Z : \psi \in H^1 \text{ and } \psi(-r) = A_1^T y\}$ and $A^*(y,\psi) = (\psi(0) + A_0^T y, -\dot{\psi}(\theta) + A^T(\cdot)y) \in Z$ [3]. Since $C^*C(\eta,\phi) = (C^TC\eta,0) \in Z$ for $(\eta,\phi) \in Z$, this implies that if $\Pi z = (y,\psi)$, then $\dot{\psi} \in H^1$ so that $\psi \in H^2$, and since A^* is closed, $\|\psi\|_{H^2} \leq M\|z\|_Z$ for some constant M. It then follows from the arguments and error estimate in [1], [3] that

$$\|(A^*-A^{N^*}P^N)\Pi(y,\psi)\| \leq \frac{\tilde{M}}{\sqrt{N}}(|y| + \|\psi\|_{H^2})$$

Hence we obtain $\|\Pi^N - \hat{\Pi}^N\| = 0(\frac{1}{\sqrt{N}})$.

References

[1] H.T. Banks and J.A. Burns. Hereditary Control Problems: Numerical Method Based on Averaging Approximation, *SIAM J. Control and Optimization*, 16 (1978), 169-208.

[2] H.T. Banks and K. Kunisch. The Linear Regulator Problem for Parabolic Systems, *SIAM J. Control and Optimization*, 22 (1984), 684-698.

[3] J.S. Gibson. Linear-quadratic Optimal Control of Hereditary Differential Systems: Infinite Dimensional Riccati Equations and Numerical Approximations, *SIAM J. Control and Optimization*, 21 (1983), 95-139.

[4] K. Ito. On the Regularity of Solutions of An Operator Riccati Equation Arising in Linear Quadratic Optimal Control Problems for Hereditary Differential Systems, *LCDS Report*, Brown University, May 1986.

[5] I. Lasiecka and A. Manitius. Differentiability and Convergence Rates of Approximating Semigroups for Retarded Functional Differential Equations, *Preprint*, May 1984.

[6] A. Pazy. Semigroups of Linear Operators and Applications to Partial Differential Equations, *Springer-Verlag*, 44, 1983.

[7] D. Salamon. Structure and Stability of Finite Dimensional Approximations for Functional Differential Equations, *SIAM J. control and Optimization*, 23 (1985), 928-951.

[8] M.H. Schultz. Spline Analysis, Prentice-Hall, Englewood Cliffs, NJ, 1973.

[9] H. Tanabe. Equations of Evolution, Pitman, London, 1979.

[10] R.B. Vinter. Filter Stability for Stochastic Evolution Equations, *SIAM J. Control and Optimization*, 15 (1977), 465-485.

[11] J. Zabczyk. Remark on the Albebraic Riccati Equation in Hilbert Spaces, *Applied Math. Optim.*, 2 (1976), 251-258.

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16. Abstract

In this paper, we consider the linear quadratic optimal control problem on infinite time interval for linear time-invariant systems defined on Hilbert spaces. The optimal control is given by a feedback form in terms of solution π to the associated algebraic Riccati equation (ARE). A Ritz type approximation is used to obtain a sequence π^N of finite dimensional approximations of the solution to ARE. A sufficient condition that shows π^N converges strongly to π is obtained. Under this condition, we derive a formula which can be used to obtain a rate of convergence of π^N to π. We demonstrate and apply the results for the Galerkin approximation for parabolic systems and the averaging approximation for hereditary differential systems.

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