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Research Article

Strong Convergence for the Split Common Fixed-Point Problem for Total Quasi-Asymptotically Nonexpansive Mappings in Hilbert Space

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In this paper, we study and modify the algorithm of Kraikaew and Saejung for the class of total quasi-asymptotically nonexpansive case so that the strong convergence is guaranteed for the solution of split common fixed-point problems in Hilbert space. Moreover, we justify our result through an example. The results presented in this paper not only extend the result of Kraikaew and Saejung but also extend, improve, and generalize some existing results in the literature.

1. Introduction

Let $\langle\cdot,\cdot\rangle$ be an inner product space, $\|\cdot\|$ the corresponding norm, E a Banach space, H_1, H_2 two Hilbert spaces, $A : H_1 \to H_2$ a bounded linear operator, and $A^* : H_2 \to H_1$ an adjoint of A. Let $\{C_i\}_{i=1}^{p}$ and $\{Q_j\}_{j=1}^{r}$ be a nonempty, closed, convex subsets of H_1 and H_2, respectively.

A Banach space E is said to satisfy *Opial's condition* (see [1]) if, for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ as $n \to \infty$ implies that

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|, \quad \forall y \in E, \quad y \neq x. \tag{1}$$

And also, a Banach space E is said to have *Kadec-Klee property* (see [1]), if, for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ as $n \to \infty$ implies that

$$x_n \longrightarrow x, \quad \text{as } n \longrightarrow \infty. \tag{2}$$

Remark 1. It is well known that each Hilbert space satisfied Opial and Kadec-Klee property.

The mapping $T : H \to H$ is said to be demiclosed at zero, if any sequence $\{x_n\}$ in H there holds the following implication:

$$x_n \rightharpoonup x, \quad Tx_n \longrightarrow 0, \quad \text{as } n \longrightarrow \infty \Longrightarrow Tx = 0. \tag{3}$$

T is said to be α-strongly quasi-nonexpansive if there exists $\alpha > 0$ with the property $\|Tx-z\|^2 \le \|x-z\|^2-\alpha\|x-Tx\|^2$, $\forall x \in H$ and $z \in \mathrm{Fix}(T)$; this is equivalent to

$$\langle x - z, Tx - x\rangle \le \frac{-1-\alpha}{2}\|x - Tx\|^2, \quad \forall x \in H, \quad z \in \mathrm{Fix}(T). \tag{4}$$

T is said to be quasi-nonexpansive, if $\mathrm{Fix}(T) \neq \emptyset$ such that $\|p - Tx\| \le \|p - x\|$, $\forall p \in \mathrm{Fix}(T)$ and $x \in H$, and $\{k_n\}$-quasi-asymptotically nonexpansive mapping, if $\mathrm{Fix}(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subseteq [1,\infty)$ with $k_n \to 1$ such that, for each $n \ge 1$, $\|p - T^n x\|^2 \le k_n\|p - x\|^2$, $\forall p \in \mathrm{Fix}(T)$ and $x \in H$, and it is said to be $(\{\nu_n\},\{\mu_n\},\xi)$-total quasi-asymptotically nonexpansive mapping if $\mathrm{Fix}(T) \neq \emptyset$; and there exist nonnegative real sequences $\{\nu_n\}$, $\{\mu_n\}$ in $[0,\infty)$

with $v_n \to 0$ and $\mu_n \to 0$ and a strictly continuous function $\xi : \mathfrak{R}^+ \to \mathfrak{R}^+$ with $\xi(0) = 0$ such that, for each $n \geq 1$,

$$\|p - T^n x\|^2 \leq \|p - x\|^2 + v_n \xi(\|p - x\|) + \mu_n, \quad \forall p \in \mathrm{Fix}(G), \quad x \in H. \tag{5}$$

Remark 2. It is known that, the class of quasi-nonexpansive mapping contained in the class of $\{k_n\}$-quasi-asymptotically nonexpansive mapping and the class of $\{k_n\}$-quasi-asymptotically nonexpansive mapping is contained in the class of $(\{v_n\}, \{\mu_n\}, \xi)$-total quasi-asymptotically nonexpansive mapping, see [2].

The mapping T is said to be uniformly L-Lipschitzian if $\exists$ a constant $L > 0$ such that, for each $n \geq 1$, $\|T^n x - T^n y\| \leq L\|x - y\|$, $\forall x, y \in H$, and it is said to be semicompact, if, for any bounded sequence $x_n \subset H$ with $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, there exists subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some point $x^* \in H$.

The convex feasibility problems (CFP) are finding a vector $x^* \in H_1$ satisfying

$$x^* \in \bigcap_{i=1}^{p} C_i. \tag{6}$$

The problem of solving (6) has been intensively studied by numerous authors due to its various application in several physical problems such as approximation theorem, image recovery, signal processing, control theory, biomedical engineering, communication and geophysics (see [3–5]) and reference therein.

In 2005, Censor et al. (see [6]) introduced and studied the problem of multiple set split feasibility problems (MSSFP) which is formulated as finding a vector $x^* \in H_1$ with the property

$$x^* \in \bigcap_{i=1}^{p} C_i, \quad Ax^* \in \bigcap_{j=1}^{r} Q_j. \tag{7}$$

If, in (7), we take $p = r = 1$, we get

$$x^* \in C, \quad Ax^* \in Q. \tag{8}$$

Equation (8) is known as the split feasibility problems (SFP) (see [7]), where C and Q are nonempty, closed, and convex subsets of H_1 and H_2, respectively. Since every closed convex subset of Hilbert space is the fixed-point set of its associating projection, then (6) and (7) become

$$x^* \in \bigcap_{i=1}^{p} \mathrm{Fix}(U_i), \tag{9}$$

$$x^* \in \bigcap_{i=1}^{p} \mathrm{Fix}(U_i), \quad Ax^* \in \bigcap_{j=1}^{r} \mathrm{Fix}(T_j). \tag{10}$$

Equations (9) and (10) are called the common fixed-point problems (CFPP) and split common fixed-point problems (SCFPP), respectively, where $U_i : H_1 \to H_1$ $(i = 1, 2, 3, \ldots, p)$ and $T_j : H_2 \to H_2$ $(j = 1, 2, 3, \ldots, r)$ are some nonlinear operators.

If we take $p = r = 1$, problem (10) is reduced to find a point $x^* \in H_1$ with property

$$x^* \in \mathrm{Fix}(U), \quad Ax^* \in \mathrm{Fix}(T). \tag{11}$$

Equation (11) is known as the two-set SCFPP.

In 2009, Censor and Segal [8] introduced the concept of SCFPP (10) in finite dimensional Hilbert space, who invented an algorithm for solving (11) which generate a sequence $\{x_n\}$ according to the following iterative procedure:

$$x_{n+1} := U(x_n + \gamma A^*(T - I)Ax_n), \quad \forall n \geq 0, \tag{12}$$

where the initial guess $x_0 \in H$ is choosing arbitrarily and $0 < \gamma < 2/\|A\|^2$.

In 2011, Moudafi [9] studied the convergence properties of relaxed algorithm for solving (10) for the class of quasi-nonexpansive operators T such that $(I - T)$ is demiclosed at zero and he obtained the weak convergence results. Note that, in finite dimensional Hilbert space, weak and strong convergence are equivalent. Differently, in infinite dimensional cases, they are not the same. Moudafi's results guarantee only weak convergence results. In 2013, Mohammed [10, 11] utilized the strongly quasi-nonexpansive operators and quasi-nonexpansive operators to solve Moudafi's algorithm and he obtained weak and strong convergence results, respectively.

In 2014, Kraikaew and Saejung [12] also modified Moudafi's algorithm [9] and they obtained the strong convergent results as shown below.

Theorem 3 (see [12]). *let $U : H_1 \to H_1$ be a strongly quasi-nonexpansive operator and let $T : H_2 \to H_2$ be a quasi-nonexpansive operator such that both $(I - U)$ and $(I - T)$ are demiclosed at zero. Let $A : H_1 \to H_2$ be a bounded linear operator with $L = \|AA^*\|$. Suppose that $\Gamma \neq \emptyset$. Let $\{x_n\} \subset H_1$ be a sequence generated by*

$$x_0 \in H, \quad x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) U(x_n + \gamma A^*(T - I)Ax_n), \tag{13}$$

where the parameters γ and $\{\alpha_n\}$ satisfy the following conditions:

(a) *$\gamma \in (0, 1/L)$;*

(b) *$\{\alpha_n\} \subset (0, 1), \lim_{n\to\infty}\alpha_n = 0$ and $\sum \alpha_n = \infty$. Then $x_n \to P_\Gamma x_0$.*

Motivated by these results, in this paper, we studied and modified the algorithm of Kraikaew and Saejung [12] for the class of total quasi-asymptotically nonexpansive mappings to solve the split common fixed-point problems (10) in the frame work of infinite dimensional Hilbert space. The results presented in this paper not only improve and extend some recent results of Kraikaew and Saejung [12], but also improve and extend some recent results of Censor and Segal [8], Moudafi [9], and Mohammed [10, 11] and many existing results.

Throughout this paper, we adopt the following notations.

(i) I is the identity operator.

(ii) $\operatorname{Fix}(T)$ is the fixed-point set of T; that is, $\operatorname{Fix}(T) = \{x \in H : Tx = x\}$.

(iii) "$\to$" and "$\rightharpoonup$" denote the strong and weak convergence, respectively.

(iv) $\omega_\omega(x_n)$ denote the set of the cluster point of $\{x_n\}$ in the weak topology, that is, $\{\exists\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup x\}$.

(v) Γ is the solution set of split common fixed-point problems (10); that is,

$$\Gamma = \left\{x^* \in C = \bigcap_{i=1}^{p} \operatorname{Fix}(U_i),\ Ax^* \in Q = \bigcap_{j=1}^{r} \operatorname{Fix}(T_j)\right\}. \tag{14}$$

2. Preliminaries

In the sequel, we will make use of the following lemmas in proving our main results.

Lemma 4 (see [2]). *Let $G : H \to H$ be a $(\{v_n\}, \{\mu_n\}, \xi)$-total quasi-asymptotically nonexpansive mapping. Then for each $q \in \operatorname{Fix}(G)$ and $x \in H$, the following inequalities are equivalent: for each $n \geq 1$*

$$\begin{aligned} \|q - G^n x\|^2 &\leq \|q - x\|^2 + v_n \xi(\|q - x\|) + \mu_n; \\ 2\langle x - G^n x, x - q\rangle &\geq \|x - G^n x\|^2 - v_n \xi(\|q - x\|) - \mu_n; \\ 2\langle x - G^n x, q - G^n\rangle &\leq \|x - G^n x\|^2 + v_n \xi(\|q - x\|) + \mu_n. \end{aligned} \tag{15}$$

Lemma 5 (see [2]). *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be a sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 + c_n) a_n + b_n. \tag{16}$$

If $\sum c_n < \infty$ and $\sum b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 6. *Let H be a Hilbert space, C a nonempty closed convex subset of H, and P_C a metric projection of H onto C satisfying $\langle x_n - x^*, x_n - P_C x_n\rangle \leq 0$, and then $\|P_C x_n - x_n\| \leq \|P_C x_n - x^*\|$, $\forall n \geq 1$.*

Proof. Let $x^* \in C$; then

$$\begin{aligned} \|x_n - P_C x_n\|^2 &= \|x_n - x^* + x^* - P_C x_n\|^2 \\ &= \|x_n - x^*\|^2 + \|x^* - P_C x_n\|^2 + 2\langle x_n - x^*, x^* - P_C x_n\rangle \\ &= \|x_n - x^*\|^2 + \|x^* - P_C x_n\|^2 + 2\langle x_n - x^*, x^* - x_n + x_n - P_C x_n\rangle \\ &= \|x_n - x^*\|^2 + \|x^* - P_C x_n\|^2 - 2\|x_n - x^*\|^2 + 2\langle x_n - x^*, x_n - P_C x_n\rangle \\ &= \|x^* - P_C x_n\|^2 - \|x_n - x^*\|^2 + 2\langle x_n - x^*, x_n - P_C x_n\rangle \\ &\leq \|x^* - P_C x_n\|^2, \\ \Longrightarrow \|x_n - P_C x_n\| &\leq \|x^* - P_C x_n\|. \end{aligned} \tag{17}$$

□

Lemma 7 (see [5]). *If a sequence $\{x_n\}$ is Fejer monotone with respect to nonempty closed convex subset C, then the following hold:*

(i) *$x_n \rightharpoonup x^* \in C$ if and only if $\omega_\omega \subset C$;*

(ii) *the sequence $\{P_C x_n\}$ converges strongly to some point in C;*

(iii) *if $x_n \rightharpoonup x^* \in C$, then $x^* = \lim_{n\to\infty} P_C x_n$.*

3. Main Results

Theorem 8. *Let H_1, H_2 be two Hilbert spaces and $G : H_1 \to H_1$, $T : H_2 \to H_2$ be $(\{v_{n_1}\}, \{\mu_{n_1}\}, \xi_1)$, $(\{v_{n_2}\}, \{\mu_{n_2}\}, \xi_2)$-total quasi-asymptotically nonexpansive mappings and uniformly L_1, L_2-Lipschitzian continuous mappings such that $(I - G)$ and $(I - T)$ are both demiclosed at zero. Let $A : H_1 \to H_2$ be a bounded linear operator and let $A^* : H_2 \to H_1$ be an adjoint of A with $L = \|AA^*\|$. Let M and M^* be positive constants such that $\xi(k) \leq \xi(M) + M^* k^2$, $\forall k \geq 0$. Assume that the solution set of SCFPP (14) is nonempty, and let P_Γ be a metric projection of H_1 onto Γ satisfying $\langle x_n - x^*, x_n - P_\Gamma x_n\rangle \leq 0$. Define the sequence $x_n \subset H_1$ by*

$$\begin{aligned} &x_0 \in H_1, \\ &u_n = (I + \gamma A^*(T^n - I) A) x_n, \\ &x_{n+1} = \alpha_n u_n + (1 - \alpha_n) G^n u_n, \quad \forall n \geq 1, \end{aligned} \tag{18}$$

where the parameter γ, L, $\{v_n\}$, $\{\mu_n\}$, $\{\xi_n\}$, and $\{\alpha_n\}$ satisfy the following conditions:

(a) *$0 < \alpha_n < 1$, $\gamma \in (0, 1/L)$, where $L = \max\{L_1, L_2\}$;*

(b) $\nu_n = \max\{\nu_{n_1}, \nu_{n_2}\}$, $\mu_n = \max\{\mu_{n_1}, \mu_{n_2}\}$, *and* $\xi = \max\{\xi_1, \xi_2\}$.

Then, the sequence $\{x_n\}$ *defined by (18) converges strongly to* $x^* \in \Gamma$.

Proof. To show that $x_n \to x^*$ as $n \to \infty$, it suffices to show $x_n \rightharpoonup x^*$ and $\|x_n\| \to \|x^*\|$ as $n \to \infty$.

The proof is divided into five steps as follows.

Step 1. In this step, we show that, for each $x^* \in \Gamma$, the following limit exists:

$$\lim_{n\to\infty} \|x_n - x^*\| = \lim_{n\to\infty} \|u_n - x^*\|. \tag{19}$$

Let $x^* \in \Gamma$; this implies that $x^* \in C := \bigcap_{i=1}^{p} \mathrm{Fix}(U_i)$ and $Ax^* \in Q = \bigcap_{j=1}^{r} \mathrm{Fix}(T_j)$. From (18) and Lemma 4, we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad = \|\alpha_n u_n + (1-\alpha_n) G^n u_n - x^*\|^2 \\
&\quad = \alpha_n^2 \|u_n - G^n u_n\|^2 \\
&\qquad + 2\alpha_n \langle u_n - G^n u_n, G^n u_n - x^* \rangle + \|G^n u_n - x^*\|^2 \\
&\quad = \alpha_n^2 \|u_n - G^n u_n\|^2 \\
&\qquad + 2\alpha_n \langle u_n - x^* + x^* - G^n u_n, G^n u_n - x^* \rangle \\
&\qquad + \|G^n u_n - x^*\|^2 \\
&\quad = \alpha_n^2 \|u_n - G^n u_n\|^2 + 2\alpha_n \langle u_n - x^*, G^n u_n - x^* \rangle \\
&\qquad + (1 - 2\alpha_n) \|G^n u_n - x^*\|^2 \\
&\quad = \alpha_n^2 \|u_n - G^n u_n\|^2 + 2\alpha_n \langle u_n - x^*, G^n u_n - u_n + u_n - x^* \rangle \\
&\qquad + (1 - 2\alpha_n) \|G^n u_n - x^*\|^2 \\
&\quad \le -\alpha_n (1-\alpha_n) \|u_n - G^n u_n\|^2 + \nu_n \alpha_n \xi (\|u_n - x^*\|) \\
&\qquad + \mu_n \alpha_n + 2\alpha_n \|u_n - x^*\|^2 + (1 - 2\alpha_n) \\
&\qquad \cdot \left(\|u_n - x^*\|^2 + \nu_n \xi (\|u_n - x^*\|) + \mu_n \right) \\
&\quad = -\alpha_n (1-\alpha_n) \|u_n - G^n u_n\|^2 + (1-\alpha_n) \nu_n \xi \\
&\qquad \cdot (\|u_n - x^*\|) + \|u_n - x^*\|^2 + (1-\alpha_n) \mu_n \\
&\quad = -\alpha_n (1-\alpha_n) \|u_n - G^n u_n\|^2 + (1 + (1-\alpha_n) \nu_n M^*) \\
&\qquad \cdot \|u_n - x^*\|^2 + (1-\alpha_n) (\nu_n \xi (M) + \mu_n).
\end{aligned} \tag{20}$$

On the other hand,

$$\begin{aligned}
&\|u_n - x^*\|^2 \\
&\quad = \|x_n - x^* + \gamma A^* (T^n - I) A x_n\|^2 \\
&\quad = \|x_n - x^*\|^2 + \gamma^2 \|A^* (T^n - I) A x_n\|^2 \\
&\qquad + 2\gamma \langle x_n - x^*, A^* (T^n - I) A x_n \rangle,
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\gamma^2 \|A^* (T^n - I) A x_n\|^2 \\
&\quad = \gamma^2 \langle A^* (T^n - I) A x_n, A^* (T^n - I) A x_n \rangle \\
&\quad = \gamma^2 \langle A A^* (T^n - I) A x_n, (T^n - I) A x_n \rangle \\
&\quad \le \gamma^2 L \|(T^n - I) A x_n\|^2,
\end{aligned} \tag{22}$$

by Lemma 4, it follows that

$$\begin{aligned}
&2\gamma \langle x_n - x^*, A^* (T^n - I) A x_n \rangle \\
&\quad = 2\gamma \langle A x_n - A x^*, (T^n - I) A x_n \rangle \\
&\quad \le -\gamma \|(T^n - I) A x_n\|^2 + \gamma \nu_n M^* L \|x_n - x^*\|^2 \\
&\qquad + \gamma (\nu_n \xi (M) + \mu_n).
\end{aligned} \tag{23}$$

By substituting (22) and (23) into (21), we obtained

$$\begin{aligned}
\|u_n - x^*\|^2 \le{} & (1 + \gamma \nu_n M^* L) \|x_n - x^*\|^2 - \gamma (1 - \gamma L) \\
& \cdot \|(T^n - I) A x_n\|^2 + \gamma (\nu_n \xi (M) + \mu_n).
\end{aligned} \tag{24}$$

Substituting (24) into (20) and then simplifying, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 \le{} & (1 + (1-\alpha_n) \nu_n M^*) \\
& \cdot \Big((1 + \gamma \nu_n M^* L) \|x_n - x^*\|^2 \\
& \quad - \gamma (1 - \gamma L) \|(T^n - I) A x_n\|^2 \\
& \quad + \gamma (\nu_n \xi (M) + \mu_n) \Big) \\
& - \alpha_n (1-\alpha_n) \|x_n - G^n u_n\|^2 \\
& + (1-\alpha_n) (\nu_n \xi (M) + \mu_n) \\
\le{} & (1 + (1-\alpha_n) \nu_n M^*) (1 + \gamma \nu_n M^* L) \\
& \cdot \|x_n - x^*\|^2 - \gamma (1 - \gamma L) \|(T^n - I) A x_n\|^2 \\
& - \alpha_n (1-\alpha_n) \|x_n - G^n u_n\|^2 \\
& + (1 + (1-\alpha_n) \nu_n M^*) \gamma (\nu_n \xi (M) + \mu_n) \\
& + (1-\alpha_n) (\nu_n \xi (M) + \mu_n),
\end{aligned} \tag{25}$$

and from (25), we deduce that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\\ &\le (1+\gamma v_n M^* L+(1-\alpha_n) v_n M^*(1+\gamma v_n M^* L))\\ &\quad\cdot\|x_n-x^*\|^2+(1+(1-\alpha_n) v_n M^*)\\ &\quad\cdot\gamma(v_n\xi(M)+\mu_n)+(1-\alpha_n)(v_n\xi(M)+\mu_n).\end{aligned}\quad (26)$$

Therefore, from (26), we have

$$\|x_{n+1}-x^*\|^2 \le (1+\beta_n)\|x_n-x^*\|^2+\eta_n, \quad (27)$$

where

$$\begin{aligned}\beta_n &= \gamma v_n M^* L+(1-\alpha_n) v_n M^*(1+\gamma v_n M^* L),\\ \eta_n &= (1+(1-\alpha_n) v_n M^*)\gamma(v_n\xi(M)+\mu_n)\\ &\quad+(1-\alpha_n)(v_n\xi(M)+\mu_n).\end{aligned}\quad (28)$$

Clearly, $\sum\beta_n<\infty$ and $\sum\eta_n<\infty$. Moreover, $\beta_n\to 0$ and $\eta_n\to 0$ as $n\to\infty$.

By Lemma 5, we conclude that $\lim_{n\to\infty}\|x_n-x^*\|$ exists.

We now prove that, for each $x^*\in\Gamma$, $\lim_{n\to\infty}\|u_n-x^*\|$ exists.

From (25), we deduce that

$$\begin{aligned}\gamma(1-\gamma L)\|(T^n-I)Ax_n\|^2 &\le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\ &\quad+\beta_n\|x_n-x^*\|^2+\eta_n,\\ \alpha_n(1-\alpha_n)\|u_n-G^n u_n\|^2 &\le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\ &\quad+\beta_n\|x_n-x^*\|^2+\eta_n.\end{aligned}\quad (29)$$

From (29), we deduce that

$$\begin{aligned}\lim_{n\to\infty}\|u_n-G^n u_n\|&=0,\\ \lim_{n\to\infty}\|Ax_n-T^n Ax_n\|&=0.\end{aligned}\quad (30)$$

From (24), (30), and the fact that $\lim_{n\to\infty}\|x_n-x^*\|$ exists, then $\lim_{n\to\infty}\|u_n-x^*\|$ exists. Moreover, from (20), we deduce that

$$\lim_{n\to\infty}\|u_n-x^*\|=\lim_{n\to\infty}\|x_n-x^*\|. \quad (31)$$

□

Step 2. In this step, we show that

$$\begin{aligned}\lim_{n\to\infty}\|x_{n+1}-x_n\|&=0,\\ \lim_{n\to\infty}\|u_{n+1}-u_n\|&=0.\end{aligned}\quad (32)$$

Proof. It follows from (18) that

$$\begin{aligned}\|x_{n+1}-x_n\| &\\ &=\|\alpha_n u_n+(1-\alpha_n)G^n u_n-x_n\|\\ &=\|(1-\alpha_n)(G^n u_n-u_n)+u_n-x_n\|\\ &=\|(1-\alpha_n)(G^n u_n-u_n)+A^*(T^n-I)Ax_n\|,\end{aligned}\quad (33)$$

and in view of (30), we obtain that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \quad (34)$$

Similarly, it follows from (30) and (34) that

$$\begin{aligned}\|u_{n+1}-u_n\| &= \Big\|\left(I+\gamma A^*\left(T^{n+1}-I\right)A\right)x_{n+1}\\ &\qquad+\left(I+\gamma A^*\left(T^n-I\right)A\right)x_n\Big\|\\ &=\Big\|x_{n+1}-x_n\\ &\qquad+\gamma A^*\left(T^{n+1}-I\right)Ax_{n+1}-\gamma A^*\left(T^n-I\right)Ax_n\Big\|\\ &\longrightarrow 0 \quad \text{as } n\longrightarrow\infty\\ &\Longrightarrow \lim_{n\to\infty}\|u_{n+1}-u_n\|=0.\end{aligned}\quad (35)$$

□

Step 3. In this step, we show that

$$\|u_n-Gu_n\|\longrightarrow 0, \qquad \|Ax_n-Tx_n\|\longrightarrow 0, \quad \text{as } n\longrightarrow\infty. \quad (36)$$

Proof. From the fact that $\|u_n-G^n u_n\|\to 0$ and $\|u_{n+1}-u_n\|\to 0$ and G is uniformly L-Lipschitzian continuous, it follows that

$$\begin{aligned}\|u_n-Gu_n\| &\le \|u_n-G^n u_n\|+\|Gu_n-G^n u_n\|\\ &\le \|u_n-G^n u_n\|+L\|u_n-G^{n-1}u_n\|\\ &\le \|u_n-G^n u_n\|+L\|G^{n-1}u_n-G^{n-1}u_{n-1}\|\\ &\quad+L\|u_n-G^{n-1}u_{n-1}\|\\ &\le \|u_n-G^n u_n\|+L^2\|u_n-u_{n-1}\|\\ &\quad+L\|u_n-u_{n-1}+u_{n-1}-G^{n-1}u_{n-1}\|\\ &\le \|u_n-G^n u_n\|+L(L+1)\|u_n-u_{n-1}\|\\ &\quad+L\|u_{n-1}-G^{n-1}u_{n-1}\|\longrightarrow 0\\ &\Longrightarrow \|u_n-Gu_n\|\longrightarrow 0.\end{aligned}\quad (37)$$

Similarly, from the fact that $\|Ax_n-T^n Ax_n\|\to 0$, $\|x_{n+1}-x_n\|\to 0$, and T is uniformly L-Lipschitzian continuous, it follows that $\|Ax_n-TAx_n\|\to 0$. □

Step 4. In this step, we show that

$$x_n\rightharpoonup x^*, \quad u_n\rightharpoonup x^*, \quad \text{as } n\longrightarrow\infty. \quad (38)$$

Proof. Since $\{u_n\}$ is bounded, then there exists a subsequence $u_{n_i}\subset u_n$ such that

$$u_{n_i}\rightharpoonup x^*, \quad \text{as } i\longrightarrow\infty. \quad (39)$$

From (39) and (36), we have

$$\left\|u_{n_i} - Gu_{n_i}\right\| \longrightarrow 0, \quad \text{as } i \longrightarrow \infty. \tag{40}$$

From (39) and (40) and the fact that $(I - G)$ is demiclosed at zero, we get that $x^* \in \mathrm{Fix}(G)$.

Moreover, from (18), (39), and the fact $\|Ax_n - T^n Ax_n\| \to 0$, as $n \to \infty$, we have

$$x_{n_i} = \alpha_n u_{n_i} - \gamma A^* \left(T^{n_i} - I\right) Ax_{n_i} \longrightarrow x^*. \tag{41}$$

By the definition of A, we get

$$Ax_{n_i} \rightharpoonup Ax^*, \quad \text{as } i \longrightarrow \infty. \tag{42}$$

In view of (36), we get

$$\left\|Ax_{n_i} - TAx_{n_i}\right\| \longrightarrow 0, \quad \text{as } i \longrightarrow \infty. \tag{43}$$

From (42) and (43) and the fact that $(I - T)$ is demiclosed at zero, we have $Ax^* \in \mathrm{Fix}(T)$, and this implies that $x^* \in \Gamma$.

Now, we show that x^* is unique.

Suppose to the contrary that there exists another subsequence $u_{n_j} \subset u_n$ such that $u_{n_j} \rightharpoonup y^* \in \Gamma$ with $x^* \neq y^*$ by virtue of (19) and opial property of Hilbert space; we have

$$\begin{aligned}
&\liminf_{i\to\infty}\left\|u_{n_i} - x^*\right\| \\
&\quad < \liminf_{i\to\infty}\left\|u_{n_i} - y^*\right\| = \liminf_{n\to\infty}\left\|u_n - y^*\right\| \\
&\quad = \liminf_{j\to\infty}\left\|u_{n_j} - y^*\right\| < \liminf_{j\to\infty}\left\|u_{n_j} - x^*\right\| \\
&\quad = \liminf_{n\to\infty}\left\|u_n - x^*\right\| = \liminf_{i\to\infty}\left\|u_{n_i} - x^*\right\| \\
&\Longrightarrow \liminf_{i\to\infty}\left\|u_{n_i} - x^*\right\| < \liminf_{i\to\infty}\left\|u_{n_i} - x^*\right\|
\end{aligned} \tag{44}$$

which is contradiction. Therefore $u_n \rightharpoonup x^*$. By using (18) and (30), we have

$$x_n = u_n - \gamma A^* \left(T^n - I\right) Ax_n \rightharpoonup x^*, \quad \text{as } n \longrightarrow \infty. \tag{45}$$

□

Step 5. In this step, we show that

$$\left\|x_n\right\| \longrightarrow \left\|x^*\right\|, \quad \text{as } n \longrightarrow \infty. \tag{46}$$

To show (46), it suffices to show that $\|x_{n+1}\| \to \|x^*\|$ as $n \to \infty$.

Proof. From Lemmas 6, 7, (27), and the fact that $\beta_n \to 0$ and $\eta_n \to 0$, we have

$$\begin{aligned}
\left|\left\|x_{n+1}\right\| - \left\|x^*\right\|\right|^2 &\le \left\|x_{n+1} - x^*\right\|^2 \\
&\le \left(1 + \beta_n\right)\left\|x_n - x^*\right\|^2 + \eta_n \\
&= \left\|x_n - x^*\right\|^2 + \beta_n\left\|x_n - x^*\right\|^2 + \eta_n \\
&= \left\|x_n - P_\Gamma x_n + P_\Gamma x_n - x^*\right\|^2 \\
&\quad + \beta_n\left\|x_n - x^*\right\|^2 + \eta_n \\
&\le 4\left\|P_\Gamma x_n - x^*\right\|^2 + \beta_n\left\|x_n - x^*\right\|^2 + \eta_n \\
\Longrightarrow \left|\left\|x_{n+1}\right\| - \left\|x^*\right\|\right|^2 &\le 4\left\|P_\Gamma x_n - x^*\right\|^2 \\
&\quad + \beta_n\left\|x_n - x^*\right\|^2 + \eta_n \\
\Longrightarrow \lim_{n\to\infty}\left|\left\|x_{n+1}\right\| - \left\|x^*\right\|\right|^2 &\le 4\lim_{n\to\infty}\left\|P_\Gamma x_n - x^*\right\|^2 \\
&\quad + \lim_{n\to\infty}\beta_n\left\|x_n - x^*\right\|^2 \\
&\quad + \lim_{n\to\infty}\left(\eta_n\right) \\
\Longrightarrow \lim_{n\to\infty}\left|\left\|x_{n+1}\right\| - \left\|x^*\right\|\right|^2 &= 0.
\end{aligned} \tag{47}$$

From (38) and (46), we conclude that $x_n \to x^*$, as $n \to \infty$. □

Corollary 9. *Let H_1, H_2 A, and A^*, be as in Theorem 8. Let G and T be $\{k_{n_1}\}, (\{k_{n_2}\})$ quasi-asymptotically nonexpansive and uniformly L_1, L_2-Lipschitzians continuous mappings such that $(I - G)$ and $(I - T)$ are both demiclosed at zero. Let $L = \|AA^*\|$, and let M and M^* be constants such that $\xi(k) \le \xi(M) + M^* k^2$, $\forall k \ge 0$. Assume that the solution set of SCFPP (14) is nonempty, and let P_Γ be a metric projection of H_1 onto Γ satisfying $\langle x_n - x^*, x_n - P_\Gamma x_n\rangle \le 0$. Let the sequence $\{x_n\}$ be defined as in Theorem 8 where the parameters α_n, γ, $\{k_n\}$, and L satisfy the following conditions:*

(a) $\alpha_n \subset (0,1), \gamma \in (0, 1/L)$, *where* $L = \max\{L_1, L_2\}$;

(b) $k_n = \max\{k_{n_1}, k_{n_2}\}$.

Then the sequence $\{x_n\}$ defined as in Theorem 8 converges strongly to $x^* \in \Gamma$.

Proof. By Remark 2 G and T are $(\{\nu_n\}, \{\mu_n\}, \xi)$-total quasi-asymptotically nonexpansive mappings with $\{\nu_n\} = \{k_n - 1\}$, $\mu_n = 0$, and $\xi(k) = k^2$, $\forall k \ge 0$. Therefore, all the conditions in Theorem 8 are satisfied. The conclusions of this corollary follow directly from Theorem 8. □

Corollary 10. *Let H_1, H_2 A, and A^* be as in Theorem 8. Let G and T be two quasi-nonexpansive and uniformly L_1, L_2-Lipschitzian continuous mappings such that $(I - G)$ and $(I - T)$ are both demiclosed at zero. Let $L = \|AA^*\|$, and let M and M^* be positive constants such that $\xi(k) \le \xi(M) + M^* k^2$, $\forall k \ge 0$. Assume that the solution set of SCFPP (14) is nonempty, and let P_Γ be a metric projection of H onto Γ satisfying $\langle x_n - x^*, x_n - P_\Gamma x_n\rangle \le 0$. Let the sequence $\{x_n\}$ be defined as in Theorem 8 where the parameters $\{\alpha_n\}$, γ, and L, satisfy the following conditions:*

(a) $\{\alpha_n\} \subset (0,1)$, $\gamma \in (0, 1/L)$, *where* $L = \max\{L_1, L_2\}$.

Then, the sequence $\{x_n\}$ defined as in (18) converges strongly to $x^* \in \Gamma$.

Proof. By Remark 2 G and T are $(\{1\})$-quasi-asymptotically nonexpansive mappings. Therefore, all the conditions in Corollary 9 are satisfied. The conclusions of this corollary follow directly from Corollary 9. □

Now we give an example of our theorem.

Example 11. Let B be a unit ball in a real Hilbert space l_2, and let $T : B \to B$ be a mapping define by

$$T : (x_1, x_2, x_3, \dots) \longrightarrow \left(0, x_1^2, a_2 x_2, a_3 x_3, \dots\right), \quad (x_1, x_2, x_3, \dots) \in B, \tag{48}$$

where $\{a_i\}$ is a sequence in $(0, 1)$ such that $\prod_{i=2}^{\infty}(a_i) = 1/2$.

It is proved in Goebel and Kirk [13] that

(a) $\|Tx - Ty\| \le 2\|x - y\|$,

(b) $\|T^n x - T^n y\| \le 2\prod_{i=2}^{n}(a_i)\|x - y\|\ \forall x, y \in B$ and $n \ge 2$.

Let $k_1^{1/2} = 2$ such that $k_n^{1/2} = 2\prod_{i=2}^{n}(a_i)$, for $n \ge 2$; then

$$\lim_{n\to\infty} k_n = \lim_{n\to\infty}\left(2\prod_{i=2}^{n} a_i\right) = 1. \tag{49}$$

Let $v_n = k_n - 1$, $\forall n \ge 1$, let $\xi(t) = t^2$, $\forall t \ge 0$, let and $\{\mu_n\}$ be a nonnegative real sequence such that $\mu_n \to \infty$ as $n \to \infty$. From (a), (b) and $\forall x, y \in B$ and $n \ge 1$, we have

$$\left\|T^n x - T^n y\right\|^2 \le \left\|x - y\right\|^2 + v_n\left\|x - y\right\|^2 + \mu_n. \tag{50}$$

Again, since $0 \in B$ and $0 \in \mathrm{Fix}(T)$, this implies that $\mathrm{Fix}(T) \ne \emptyset$. From the above equation, we have

$$\left\|p - T^n y\right\|^2 \le \left\|p - y\right\|^2 + v_n \xi\left(\|p - y\|\right) + \mu_n. \tag{51}$$

This show that T is total quasi-asymptotically nonexpansive mapping.

Example 12. Let $H_1 = H_2 = l_2$ be a real Hilbert spaces, C, Q two unit balls in l_2, and $T : C \to C$, $G : Q \to Q$ two mappings defined by

$$\begin{aligned} T &: (x_1, x_2, x_3, \dots) \longrightarrow \left(0, x_1^2, a_2 x_2, a_3 x_3, \dots\right), \quad (x_1, x_2, x_3, \dots) \in C, \\ G &: (y_1, y_2, y_3, \dots) \longrightarrow \left(0, y_1^2, b_2 y_2, b_3 y_3, \dots\right), \quad (y_1, y_2, y_3, \dots) \in Q, \end{aligned} \tag{52}$$

such that $(I - T)$ and $(I - G)$ are both demiclosed at zero, where $\{a_i\}$ and $\{b_i\}$ are sequences in $(0, 1)$ such that $\prod_{i=2}^{\infty}(a_i) = \prod_{i=2}^{\infty}(b_i) = 1/2$. Let A, A^*, M, $M^*\xi(k)$, and P_Γ be as in Theorem 8. And assume that conditions (a)–(b) in Theorem 8 are satisfied. Then, the sequence $\{x_n\}$ defined in Theorem 8 converges strongly to $x^* \in \Gamma$.

Proof. By Example 11, it follows that G and T are both $(\{v_n\}, \{\mu_n\}, \xi)$-total quasi-asymptotically nonexpansive mappings; moreover from Example 11 (b) we have that G and T are both uniformly L_1, L_2-Lipschitzian with $L_1 = 2\prod_{i=2}^{n}(a_i)$ and $L_2 = 2\prod_{i=2}^{n}(b_i)$; also by our hypothesis $(I - G)$ and $(I - T)$ are both demiclosed at zero. Therefore, all the conditions in Theorem 8 are satisfied. Hence, the conclusions of this corollary follow directly from Theorem 8. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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