Research Article

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Strong convergence of a self-adaptive inertial Tseng's extragradient method for pseudomonotone variational inequalities and fixed point problems

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Abstract: In this paper, we study the problem of finding a common solution of the pseudomonotone variational inequality problem and fixed point problem for demicontractive mappings. We introduce a new inertial iterative scheme that combines Tseng's extragradient method with the viscosity method together with the adaptive step size technique for finding a common solution of the investigated problem. We prove a strong convergence result for our proposed algorithm under mild conditions and without prior knowledge of the Lipschitz constant of the pseudomonotone operator in Hilbert spaces. Finally, we present some numerical experiments to show the efficiency of our method in comparison with some of the existing methods in the literature.

Keywords: Tseng's extragradient method, pseudomonotone, demicontractive, variational inequalities, fixed point, strong convergence, adaptive step size, inertial technique

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1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. In this paper, we consider the variational inequality problem (VIP) of finding a point $p \in C$ such that

$$\langle Ap, x - p \rangle \geq 0, \quad \forall x \in C, \tag{1}$$

where C is a nonempty closed convex subset of H, and $A : H \to H$ is a nonlinear operator. We denote by $VI(C, A)$ the solution set of the VIP (1).

Variational inequality theory, which was first introduced independently by Fichera [1] and Stampacchia [2], is a vital tool in mathematical analysis, and has a vast application across several fields of study, such as optimisation theory, engineering, physics, operator theory, economics, and many others (see [3–6] and references therein). Over the years, several iterative methods have been formulated and adopted in solving VIP (1) (see [7–11] and references therein). There are two common approaches to solving the VIP,

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namely, the regularised methods and the projection methods. These approaches usually require that the nonlinear operator A in VIP (1) has certain monotonicity. In this study, we adopt the projection method and consider the case in which the associated nonlinear operator is pseudomonotone (see definition below) – a larger class than monotone mappings.

Now, we review some nonlinear operators in nonlinear analysis.

Definition 1.1. A mapping $A : H \to H$ is said to be

(1) γ-*strongly monotone* on H if there exists a constant $\gamma > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \gamma\|x - y\|^2, \quad \forall x, y \in H. \tag{2}$$

(2) γ-*inverse strongly monotone* on H if there exists a constant $\gamma > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \gamma\|Ax - Ay\|^2, \quad \forall x, y \in H.$$

(3) *Monotone* on H, if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in H. \tag{3}$$

(4) γ-*strongly pseudomonotone* on H, if there exists a constant $\gamma > 0$ such that

$$\langle Ay, x - y\rangle \geq 0 \Rightarrow \langle Ax, x - y\rangle \geq \gamma\|x - y\|^2, \quad \forall x, y \in H. \tag{4}$$

(5) *Pseudomonotone* on H, if

$$\langle Ay, x - y\rangle \geq 0 \Rightarrow \langle Ax, x - y\rangle \geq 0, \quad \forall x, y \in H. \tag{5}$$

(6) *Lipschitz-continuous* on H, if there exists a constant $L > 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in H. \tag{6}$$

If $L \in [0, 1)$, then A is said to be a *contraction mapping*.

(7) *Sequentially weakly continuous* on H, if for each sequence $\{x_n\}$,

$$x_n \rightharpoonup x \quad \text{implies} \quad Tx_n \rightharpoonup Tx, \quad x \in H.$$

From the above definitions, we observe that (1) $\Rightarrow$ (3) $\Rightarrow$ (5) and (1) $\Rightarrow$ (4) $\Rightarrow$ (5). However, the converses are not generally true. Moreover, if A is γ- *strongly monotone* and L- *Lipschitz continuous*, then A is $\frac{\gamma}{L^2}$- *inverse strongly monotone* (see [12,13]).

The simplest known projection method for solving VIP is the gradient method (GM), which involves a single projection onto the feasible set C per iteration. However, the algorithm only converges weakly under some strict conditions that the operator is either strongly monotone or inverse strongly monotone, but fails to converge if A is monotone. The classical gradient projection algorithm proposed by Sibony [14] is given as follows:

$$x_{n+1} = P_C(x_n - \lambda Ax_n), \tag{7}$$

where A is strongly monotone and L-Lipschitz continuous, with step size $\lambda \in \left(0, \frac{2}{L}\right)$.

Korpelevich [15] and Antipin [16] proposed the extragradient method (EGM) for solving VIP (1), thereby relaxing the conditions placed in (7). The initial algorithm proposed by Korpelevich was employed in solving saddle point problems, but was later extended to VIPs in both Euclidean space and infinite dimensional Hilbert spaces. The EGM method is given as follows:

$$\begin{cases} x_0 \in C \\ y_n = P_C(x_n - \lambda Ax_n) \\ x_{n+1} = P_C(x_n - \lambda Ay_n), \end{cases} \tag{8}$$

where $\lambda \in \left(0, \frac{1}{L}\right)$, A is monotone and L-Lipschitz continuous, and P_C denotes the metric projection from H onto C. If the set $VI(C, A)$ is nonempty, then the algorithm only converges weakly to an element in $VI(C, A)$.

Over the years, EGM has been of interest to several researchers. Also, many results and variants have been developed from this method, using the assumptions of Lipschitz continuity, monotonicity, and pseudomonotonicity, see [17–20] and references therein.

Due to the extensive amount of time required in executing the EGM method, as a result of calculating two projections onto the closed convex set C in each iteration, Censor et al. [8] proposed the subgradient extragradient method (SEGM) in which they replaced the second projection onto C by a projection onto a half-space, thus, making computation easier and convergence rate faster. The SEGM is presented as follows:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n) \\ T_n = \{w \in H : \langle x_n - \lambda A x_n - y_n, w - y_n \rangle \leq 0\} \\ x_{n+1} = P_{T_n}(x_n - \lambda A y_n), \quad \forall n \geq 0, \end{cases} \tag{9}$$

where $\lambda \in \left(0, \frac{2}{L}\right)$. The authors only obtained a weak convergence result for the proposed method. However, they later introduced a hybrid SEGM in [7] and obtained a strong convergence result. Likewise, Tseng [21], in the bid to improve on the EGM, proposed Tseng's extragradient method (TEGM), which only requires one projection per iteration, as follows:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n) \\ x_{n+1} = y_n + \lambda(A x_n - A y_n), \quad \forall n \geq 0, \end{cases} \tag{10}$$

where A is monotone, L-Lipschitz continuous, and $\lambda \in \left(0, \frac{2}{L}\right)$. The TEGM (10) converges to a weak solution of the VIP with the assumption that $VI(C, A)$ is nonempty. The TEGM is also known as the forward-backward method. Recently, some authors have carried out some interesting works on the TEGM (see [22,23] and references therein).

In this work, we consider the *inertial algorithm*, which is a two-step iteration process and a technique for accelerating the speed of convergence of iterative schemes. The inertial extrapolation technique was derived by Polyak [24] from a dynamic system called the heavy ball with friction. Due to its efficiency, the inertial technique has become a centre of attraction and interest to many researchers in this field. Over the years, researchers have studied the inertial algorithm and applied it to solve different optimisation problems, see [25–28] and references therein.

Very recently, Tan and Qin [29] proposed the following Tseng's extragradient algorithm for solving pseudomonotone VIP:

$$\begin{cases} s_n = x_n + \delta_n(x_n - x_{n-1}) \\ y_n = P_C(s_n - \psi_n A s_n) \\ z_n = y_n - \psi_n(A y_n - A s_n) \\ x_{n+1} = \alpha_n f(z_n) + (1 - \alpha_n) z_n, \end{cases} \tag{11}$$

$$\delta_n = \begin{cases} \min\left\{\dfrac{\varepsilon_n}{\|x_n - x_{n-1}\|}, \delta\right\} & \text{if } x_n \neq x_{n-1} \\ \delta, & \text{otherwise.} \end{cases}$$

$$\psi_{n+1} = \begin{cases} \min\left\{\dfrac{\phi\|s_n - y_n\|}{\|A s_n - A y_n\|}, \psi_n\right\} & \text{if } A s_n - A y_n \neq 0 \\ \psi_n, & \text{otherwise,} \end{cases}$$

where f is a contraction and A is a pseudomonotone, Lipschitz continuous, and sequentially weakly continuous mapping. The authors proved a strong convergence result for the proposed method under mild conditions on the control parameters.

Another area of interest in this study is the fixed point theory. Let $U : H \to H$ be a nonlinear map. The fixed point problem (FPP) is to find a point $p \in H$ (called the *fixed point of* U) such that

$$Up = p. \tag{12}$$

In this work, we denote the set of fixed points of U by $F(U)$. Our interest in this study is to find a common element of the fixed point set, $F(U)$, and the solution set of the variational inequality, $VI(C, A)$. That is, the problem of finding a point $x^* \in H$ such that

$$x^* \in VI(C, A) \cap F(U). \tag{13}$$

Many algorithms have been proposed over the years and in recent times for solving the common solution problem (13) (see [30–40] and references therein). Common solution problem of this type has drawn the attention of researchers because of its potential application to mathematical models whose constraints can be expressed as FPP and VIP. This arises in areas like signal processing, image recovery, and network resource allocation. An instance of this is in network bandwidth allocation problem for two services in a heterogeneous wireless access networks in which the bandwidth of the services is mathematically related (see [37,41,42] and references therein).

Recently, Cai et al. [22] proposed the following inertial Tseng's extragradient algorithm for approximating the common solution of pseudomonotone VIP and FPP for nonexpansive mappings in real Hilbert spaces:

$$\begin{cases} x_0, x_1 \in H \\ w_n = x_n + \theta_n(x_n - x_{n-1}) \\ y_n = P_C(w_n - \psi A w_n) \\ z_n = y_n - \psi(Ay_n - Aw_n) \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)[\beta_n T z_n + (1 - \beta_n) z_n], \end{cases} \tag{14}$$

where f is a contraction, T is a nonexpansive mapping, A is pseudomonotone, L-Lipschitz and sequentially weakly continuous, and $\psi \in \left(0, \frac{1}{L}\right)$. They proved a strong convergence result for the proposed algorithm under some suitable conditions.

One of the major drawbacks of Algorithm (14) is the fact that the step size ψ of the algorithm depends on the Lipschitz constant of the cost operator. In many cases, this Lipschitz constant is unknown or even difficult to estimate. This makes it difficult to implement algorithms of this nature.

Very recently, Thong and Hieu [23] proposed an iterative scheme for finding a common element of the solution set of monotone variational inequality and set of fixed points of demicontractive mappings as follows:

$$\begin{cases} y_n = P_C(x_n - \psi_n A x_n) \\ z_n = y_n - \psi_n(Ay_n - Ax_n) \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)[\beta_n U z_n + (1 - \beta_n) z_n], \end{cases} \tag{15}$$

$$\psi_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\|x_n - y_n\|}{\|Ax_n - Ay_n\|}, \psi_n\right\} & \text{if } Ax_n - Ay_n \neq 0 \\ \psi_n, & \text{otherwise,} \end{cases}$$

where A is monotone and L-Lipschitz continuous, U is a demicontractive mapping such that $I - U$ is demiclosed at zero, and f is a contraction. The authors proved a strong convergence result under suitable conditions for the proposed method.

Motivated by the above results and the ongoing research activities in this direction, in this paper our aim is to introduce an effective iterative technique, which employs the efficient combination of the inertial technique, TEGM together with the viscosity method for finding a common solution of FPP of demicontractive mappings and pseudomonotone VIP with Lipschitz continuous and sequentially weakly continuous operator in Hilbert spaces. In line with this goal, we construct an algorithm with the following features:

(i) Our algorithm approximates the solution of a more general class of VIP and FPP.
(ii) The proposed method only requires one projection per iteration onto the feasible set, which guarantees the minimal cost of computation.

(iii) Moreover, our method is computationally efficient. It employs an efficient self-adaptive step size technique which makes the algorithm independent of the Lipschitz constant of the cost operator.
(iv) We employ the combination of the inertial technique together with the viscosity method, which are two of the efficient techniques for accelerating the rate of convergence of iterative schemes.
(v) We prove a strong convergence theorem for the proposed algorithm without following the conventional "**two-cases**" approach often employed by researchers (e.g. see [22,23,29,43–45]). This makes our results in this paper to be more concise and precise.

Furthermore, by several numerical experiments, we demonstrate the efficiency of our proposed method over many other existing methods in related literature.

The remainder of this paper is organised as follows. In Section 2, useful definitions and lemmas employed in the study are presented. In Section 3, we present the proposed algorithm and highlight some of its notable features. Section 4 presents the convergence analysis of the proposed method. In Section 5, we carry out some numerical experiments to illustrate the computational advantage of our method over some of the existing methods in the literature. Finally, in Section 6 we give a concluding remark.

2 Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. We denote the weak and strong convergence of sequence $\{x_n\}_{n=1}^{\infty}$ to x by $x_n \rightharpoonup x$, as $n \to \infty$ and $x_n \to x$, as $n \to \infty$.

The *metric projection* [46,47], $P_C : H \to C$ is defined, for each $x \in H$, as the unique element $P_C x \in C$ such that

$$\|x - P_C x\| = \inf\{\|x - z\| : z \in C\}.$$

It is a known fact that P_C is *nonexpansive*, i.e. $\|P_C x - P_C y\| \leq \|x - y\| \quad \forall x, y \in C$. Also, the mapping P_C is firmly nonexpansive, i.e.

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle,$$

for all $x, y \in H$. Some results on the metric projection map are given below.

Lemma 2.1. [48] *Let C be a nonempty closed convex subset of a real Hilbert space H. For any $x \in H$ and $z \in C$, Then,*

$$z = P_C x \Leftrightarrow \langle x - z, z - y\rangle \geq 0, \quad \textit{for all } y \in C.$$

Lemma 2.2. [48,49] *Let C be a nonempty, closed, and convex subset of a real Hilbert space H, $x \in H$. Then*:
(1) $||P_C x - P_C y||^2 \leq \langle x - y, P_C x - P_C y\rangle, \quad \forall y \in C.$
(2) $||x - P_C x||^2 + ||y - P_C x||^2 \leq ||x - y||^2, \quad \forall y \in C.$
(3) $||(I - P_C)x - (I - P_C)y||^2 \leq \langle x - y, (I - P_C)x - (I - P_C)y\rangle, \quad \forall y \in C.$

Definition 2.3. A mapping $T : H \to H$ is said to be
(1) *Nonexpansive* on H, if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H.$$

(2) *Quasi-nonexpansive* on H, if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\|, \quad \forall p \in F(T), x \in H.$$

(3) *λ-strictly pseudocontractive* on H with $0 \leq \lambda < 1$, if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \lambda\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in H.$$

(4) *β-demicontractive* with $0 \le \beta < 1$ if

$$\|Tx - p\|^2 \le \|x - p\|^2 + \beta\|(I - T)x\|^2, \quad \forall p \in F(T), x \in H,$$

or equivalently

$$\langle Tx - x, x - p\rangle \le \frac{\beta - 1}{2}\|x - Tx\|^2, \quad \forall p \in F(T), x \in H,$$

or equivalently

$$\langle Tx - p, x - p\rangle \le \|x - p\|^2 + \frac{\beta - 1}{2}\|x - Tx\|^2, \quad \forall p \in F(T), x \in H.$$

Remark 2.4. It is known that every *strictly pseudocontractive* mapping with a nonempty fixed point set is *demicontractive*. The class of demicontractive mappings includes all the other classes of mappings defined above (see [23]).

Next, we give some examples of the class of demicontractive mappings, as shown in [23,50].

Example 2.5.
(a) Let H be the real line and $C = [-1, 1]$. Define T on C by:

$$Tx = \begin{cases} \frac{2}{3}x \sin\frac{1}{x}, & x \ne 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Then T is demicontractive.
(b) Consider a mapping $T : [-2, 1] \to [-2, 1]$ defined such that,

$$Tx = -x^2 - x.$$

Then T is a demicontractive map that is neither quasi-nonexpansive nor strictly pseudocontractive.

We have the following lemmas which will be employed in our convergence analysis.

Lemma 2.6. [25] *For each $x, y \in H$, and $\delta \in \mathbb{R}$, we have the following results*:
(1) $||x + y||^2 \le ||x||^2 + 2\langle y, x + y\rangle$;
(2) $||x + y||^2 = ||x||^2 + 2\langle x, y\rangle + ||y||^2$;
(3) $||\delta x + (1 - \delta)y||^2 = \delta||x||^2 + (1 - \delta)||y||^2 - \delta(1 - \delta)||x - y||^2$.

Lemma 2.7. [51] *Let $\{a_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence in $(0, 1)$ with $\sum_{n=1}^{\infty}\alpha_n = \infty$, and $\{b_n\}$ be a sequence of real numbers. Assume that*

$$a_{n+1} \le (1 - \alpha_n)a_n + \alpha_n b_n, \quad \text{for all } n \ge 1,$$

if $\limsup_{k\to\infty} b_{n_k} \le 0$ *for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying* $\liminf_{k\to\infty}(a_{n_{k+1}} - a_{n_k}) \ge 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.8. [52] *Assume that $T : H \to H$ is a nonlinear operator with $F(T) \ne 0$. Then, $I - T$ is said to be demiclosed at zero if for any $\{x_n\}$ in H, the following implication holds*: $x_n \rightharpoonup x$ *and* $(I - T)x_n \to 0 \Rightarrow x \in F(T)$.

Lemma 2.9. [53] *Assume that D is a strongly positive bounded linear operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \le ||D||^{-1}$. Then $||I - \rho D|| \le 1 - \rho\bar{\gamma}$.*

Lemma 2.10. [54] *Let $U : H \to H$ be β-demicontractive with $F(U) \neq \varnothing$ and set $U_\lambda = (1-\lambda) + \lambda U, \lambda \in (0, 1-\beta)$. Then,*

(*i*) $F(U) = Fix(U_\lambda)$.

(*ii*) $\|U_\lambda x - z\|^2 \leq \|x - z\|^2 - \frac{1}{\lambda}(1-\beta-\lambda)\|(I-U_\lambda)x\|^2, \quad \forall x \in H, z \in F(U)$.

(*iii*) $F(U)$ *is a closed convex subset of* H.

Lemma 2.11. [55] *Consider the problem with C being a nonempty, closed, convex subset of a real Hilbert space H and $A : C \to H$ being pseudomonotone and continuous. Then p is a solution of VIP* (1) *if and only if*

$$\langle Ax, x - p\rangle \geq 0, \quad \forall x \in C.$$

3 Proposed algorithm

In this section, we propose an inertial viscosity-type Tseng's extragradient algorithm with self adaptive step size and highlight some of its important features. We establish the convergence of the algorithm under the following conditions:

Condition A

(A1) The feasible set C is closed, convex, and nonempty.

(A2) The solution set denoted by $\Omega = VI(C, A) \cap F(U)$ is nonempty.

(A3) The mapping A is pseudomonotone, L-Lipschitz continuous on H, and sequentially weakly continuous on C.

(A4) The mapping $U : H \to H$ is a τ-demicontractive map such that $I - U$ is demiclosed at zero.

(A5) $D : H \to H$ is a strongly positive bounded linear operator with coefficient $\bar{\gamma}$.

(A6) $f : H \to H$ is a contraction with coefficient $\rho \in (0, 1)$ such that $0 < \gamma < \frac{\bar{\gamma}}{\rho}$.

Condition B

(B1) $\{\alpha_n\} \subset (0, 1)$ such that $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$.

(B2) The positive sequence $\{\varepsilon_n\}$ satisfies $\lim_{n\to\infty}\frac{\varepsilon_n}{\alpha_n} = 0$, $\{\beta_n\} \subset (a, 1-\tau)$ for some $a > 0$.

Now, the algorithm is presented as follows:

Algorithm 3.1. Inertial TEGM with self-adaptive stepsize

Step 0. Given $\delta > 0$, $\psi_1 > 0$, $\phi \in (0, 1)$. Select initial data $x_0, x_1 \in H$, and set $n = 1$.

Step 1. Given the $(n-1)$th and nth iterates, choose δ_n such that $0 \leq \delta_n \leq \hat{\delta}_n, \quad \forall n \in \mathbb{N}$ with $\hat{\delta}_n$ defined by

$$\hat{\delta}_n = \begin{cases} \min\left\{\dfrac{\varepsilon_n}{||x_n - x_{n-1}||}, \delta\right\}, & \text{if } x_n \neq x_{n-1}, \\ \delta, & \text{otherwise}. \end{cases} \tag{16}$$

Step 2. Compute

$$r_n = x_n + \delta_n(x_n - x_{n-1}).$$

Step 3. Compute

$$y_n = P_C(r_n - \psi_n A r_n).$$

If $y_n = r_n$, then set $z_n = r_n$ and go to Step 5. Else go to Step 4.

Step 4. Compute

$$z_n = y_n - \psi_n(Ay_n - Ar_n).$$

Step 5. Compute

$$x_{n+1} = \alpha_n \gamma f(r_n) + (I - \alpha_n D)[(1 - \beta_n)z_n + \beta_n U z_n].$$

Step 6. Compute

$$\psi_{n+1} = \begin{cases} \min\left\{\dfrac{\phi||r_n - y_n||}{||Ar_n - Ay_n||}, \psi_n\right\}, & \text{if } Ar_n - Ay_n \neq 0, \\ \psi_n, & \text{otherwise}. \end{cases} \tag{17}$$

Set $n := n + 1$ and return to Step 1.

Below are some of the interesting features of our proposed algorithm.

Remark 3.2.

(i) Observe that Algorithm 3.1 involves only one projection onto the feasible set C per iteration, which makes the algorithm computationally efficient.

(ii) The step size ψ_n in (17) is self-adaptive and supports easy and simple computations, which makes it possible to implement our algorithm without prior knowledge of the Lipschitz constant of the cost operator.

(iii) We also point out that in **Step 1** of the algorithm, the inertial technique employed can easily be implemented in numerical computation, since the value of $||x_n - x_{n-1}||$ is known prior to choosing δ_n.

Remark 3.3. It can easily be seen from (16) and condition (B1) that

$$\lim_{n\to\infty} \delta_n||x_n - x_{n-1}|| = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\delta_n}{\alpha_n}||x_n - x_{n-1}|| = 0.$$

4 Convergence analysis

First, we establish some lemmas which will be employed in the convergence analysis of our proposed algorithm.

Lemma 4.1. *The sequence* $\{\psi_n\}$ *generated by* (17) *is a nonincreasing sequence and* $\lim_{n\to\infty}\psi_n = \psi \geq \min\left\{\psi_1, \frac{\phi}{L}\right\}$.

Proof. It follows from (17) that $\psi_{n+1} \leq \psi_n, \ \forall n \in \mathbb{N}$. Hence, $\{\psi_n\}$ is nonincreasing. Also, since A is Lipschitz continuous, we have

$$\|Ar_n - Ay_n\| \leq L\|r_n - y_n\|,$$

which implies that

$$\frac{\|r_n - y_n\|}{\|Ar_n - Ay_n\|} \geq \frac{1}{L}.$$

Consequently, we obtain

$$\frac{\phi\|r_n - y_n\|}{\|Ar_n - Ay_n\|} \geq \frac{\phi}{L}, \quad \text{when } Ar_n - Ay_n \neq 0.$$

Combining this together with (17), we obtain

$$\psi_n \geq \min\left\{\psi_1, \frac{\phi}{L}\right\}.$$

Since $\{\psi_n\}$ is nonincreasing and bounded below, we can conclude that

$$\lim_{n\to\infty} \psi_n = \psi \geq \min\left\{\psi_1, \frac{\phi}{L}\right\}. \qquad \square$$

Lemma 4.2. *Let $\{r_n\}$ and $\{y_n\}$ be two sequences generated by Algorithm* 3.1, *and suppose that conditions* (A1)–(A3) *hold. If there exists a subsequence $\{r_{n_k}\}$ of $\{r_n\}$ convergent weakly to $z \in H$ and $\lim_{n\to\infty}\|r_{n_k} - y_{n_k}\| = 0$, then $z \in VI(C, A)$*

Proof. Using the property of the projection map and $y_n = P_C(r_n - \psi_n A r_n)$, we obtain

$$\langle r_{n_k} - \psi_{n_k} A r_{n_k} - y_{n_k}, x - y_{n_k}\rangle \leq 0 \quad \forall x \in C,$$

which implies that

$$\frac{1}{\psi_{n_k}}\langle r_{n_k} - y_{n_k}, x - y_{n_k}\rangle \leq \langle A r_{n_k}, x - y_{n_k}\rangle \quad \forall x \in C.$$

From this we obtain

$$\frac{1}{\psi_{n_k}}\langle r_{n_k} - y_{n_k}, x - y_{n_k}\rangle + \langle A r_{n_k}, y_{n_k} - r_{n_k}\rangle \leq \langle A r_{n_k}, x - r_{n_k}\rangle \quad \forall x \in C. \tag{18}$$

Since $\{r_{n_k}\}$ converges weakly to $z \in H$, we have that $\{r_{n_k}\}$ is bounded. Then, from the Lipschitz continuity of A and $\|r_{n_k} - y_{n_k}\| \to 0$, we obtain that $\{Ar_{n_k}\}$ and $\{y_{n_k}\}$ are also bounded. Since $\psi_{n_k} \geq \left\{\psi_1, \frac{\phi}{L}\right\}$, from (18) it follows that

$$\liminf_{k\to\infty} \langle A r_{n_k}, x - r_{n_k}\rangle \geq 0 \quad \forall x \in C. \tag{19}$$

Moreover, we have that

$$\langle A y_{n_k}, x - y_{n_k}\rangle = \langle A y_{n_k} - A r_{n_k}, x - r_{n_k}\rangle + \langle A r_{n_k}, x - r_{n_k}\rangle + \langle A y_{n_k}, r_{n_k} - y_{n_k}\rangle. \tag{20}$$

Since $\lim_{k\to\infty}\|r_{n_k} - y_{n_k}\| = 0$, then by the Lipschitz continuity of A we have $\lim_{k\to\infty}\|Ar_{n_k} - Ay_{n_k}\| = 0$. This together with (19) and (20) gives

$$\liminf_{k\to\infty} \langle A y_{n_k}, x - y_{n_k}\rangle \geq 0.$$

Now, choose a decreasing sequence $\{\theta_k\}$ of positive numbers such that $\theta_k \to 0$ as $k \to \infty$. For any k, we represent the smallest positive integer with N_k such that:

$$\langle A y_{n_j}, x - y_{n_j}\rangle + \theta_k \geq 0 \quad \forall j \geq N_k. \tag{21}$$

It is clear that the sequence $\{N_k\}$ is increasing since θ_k is decreasing. Furthermore, for any k, from $\{y_{N_k}\} \subset C$, we can assume $Ay_{N_k} \neq 0$ (otherwise, y_{N_k} is a solution) and set:

$$\upsilon_{N_k} = \frac{A y_{N_k}}{\|A y_{N_k}\|^2}.$$

Consequently, we have $\langle Ay_{N_k}, \upsilon_{N_k}\rangle = 1$, for each k. From (21), one can easily deduce that

$$\langle A y_{N_k}, x + \theta_k \upsilon_{N_k} - y_{N_k}\rangle \geq 0, \quad \forall k.$$

By the pseudomonotonicity of A, we have

$$\langle A\left(x + \theta_k \upsilon_{N_k}\right), x + \theta_k \upsilon_{N_k} - y_{N_k}\rangle \geq 0,$$

which implies that

$$\langle Ax, x - y_{N_k}\rangle \geq \langle Ax - A\left(x + \theta_k \upsilon_{N_k}\right), x + \theta_k \upsilon_{N_k} - y_{N_k}\rangle - \theta_k\langle Ax, \upsilon_{N_k}\rangle. \tag{22}$$

Next, we show that $\lim_{k\to\infty}\theta_k v_{N_k} = 0$. Indeed, since $r_{n_k} \rightharpoonup z$ and $\lim_{k\to\infty}\|r_{n_k} - y_{n_k}\| = 0$, we obtain $y_{N_k} \rightharpoonup z$, $k \to \infty$. Since $\{y_n\} \subset C$, we obtain $z \in C$. By the sequentially weakly continuity of A on C, we have $\{Ay_{n_k}\} \rightharpoonup Az$. We can assume that $Az \neq 0$ (otherwise, z is a solution). Since the norm mapping is sequentially weakly lower semicontinuous, we have

$$0 < \|Az\| \leq \liminf_{k\to\infty} \|Ay_{n_k}\|.$$

By the fact that $\{y_{N_k}\} \subset \{y_{n_k}\}$ and $\theta_k \to 0$ as $k \to \infty$, we obtain

$$0 \leq \limsup_{k\to\infty}\|\theta_k v_{N_k}\| = \limsup_{k\to\infty}\left(\frac{\theta_k}{\|Ay_{N_k}\|}\right) \leq \frac{\limsup_{k\to\infty}\theta_k}{\liminf_{k\to\infty}\|Ay_{n_k}\|} = 0,$$

and this implies that $\limsup_{k\to\infty}\theta_k v_{N_k} = 0$. Now, by the facts that A is Lipschitz continuous, sequences $\{y_{N_k}\}$, $\{v_{N_k}\}$ are bounded and $\lim_{k\to\infty}\theta_k v_{N_k} = 0$, we conclude from (22) that

$$\liminf_{k\to\infty} \langle Ax, x - y_{N_k}\rangle \geq 0.$$

Consequently, we have

$$\langle Ax, x - z\rangle = \lim_{k\to\infty} \langle Ax, x - y_{N_k}\rangle = \liminf_{k\to\infty} \langle Ax, x - y_{N_k}\rangle \geq 0, \quad \forall x \in C.$$

Thus, by Lemma 2.11, $z \in VI(C, A)$ as required. □

Lemma 4.3. *Let sequences $\{z_n\}$ and $\{y_n\}$ be two sequences generated by Algorithm* 3.1 *such that conditions* (A1)–(A3) *hold. Then, for all $p \in \Omega$ we have*

$$\|z_n - p\|^2 \leq \|r_n - p\|^2 - \left(1 - \phi^2\frac{\psi_n^2}{\psi_{n+1}^2}\right)\|r_n - y_n\|^2, \tag{23}$$

and

$$\|z_n - y_n\| \leq \phi\frac{\psi_n}{\psi_{n+1}}\|r_n - y_n\|. \tag{24}$$

Proof. By applying the definition of $\{\psi_n\}$, we have

$$\|Ar_n - Ay_n\| \leq \frac{\phi}{\psi_{n+1}}\|r_n - y_n\|, \quad \forall n \in \mathbb{N}. \tag{25}$$

Clearly, if $Ar_n = Ay_n$, then inequality (25) holds. Otherwise, from (17) we have

$$\psi_{n+1} = \min\left\{\frac{\psi\|r_n - y_n\|}{\|Ar_n - Ay_n\|}, \psi_n\right\} \leq \frac{\phi\|r_n - y_n\|}{\|Ar_n - Ay_n\|}.$$

It then follows that

$$\|Ar_n - Ay_n\| \leq \frac{\phi}{\psi_{n+1}}\|r_n - y_n\|.$$

Thus, the inequality (25) is valid both when $Ar_n = Ay_n$ and $Ar_n \neq Ay_n$. Now, from the definition of z_n and applying Lemma 2.6 we have

$$\begin{aligned}
\|z_n - p\|^2 &= \|y_n - \psi_n(Ay_n - Ar_n) - p\|^2 \\
&= \|y_n - p\|^2 + \psi_n^2\|Ay_n - Ar_n\|^2 - 2\psi_n\langle y_n - p, Ay_n - Ar_n\rangle \\
&= \|r_n - p\|^2 + \|y_n - r_n\|^2 + 2\langle y_n - r_n, r_n - p\rangle + \psi_n^2\|Ay_n - Ar_n\|^2 - 2\psi_n\langle y_n - p, Ay_n - Ar_n\rangle \\
&= \|r_n - p\|^2 + \|y_n - r_n\|^2 - 2\langle y_n - r_n, y_n - r_n\rangle + 2\langle y_n - r_n, y_n - p\rangle + \psi_n^2\|Ay_n - Ar_n\|^2 \\
&\quad - 2\psi_n\langle y_n - p, Ay_n - Ar_n\rangle \\
&= \|r_n - p\| - \|y_n - r_n\| + 2\langle y_n - r_n, y_n - p\rangle + \psi_n^2\|Ay_n - Ar_n\|^2 - 2\psi_n\langle yn - p, Ay_n - Ar_n\rangle.
\end{aligned} \tag{26}$$

Since $y_n = P_C(r_n - \psi_n A r_n)$, then by the projection property, we obtain

$$\langle y_n - r_n + \psi_n A r_n, y_n - p\rangle \le 0,$$

or equivalently,

$$\langle y_n - r_n, y_n - p\rangle \le -\psi_n \langle A r_n, y_n - p\rangle. \tag{27}$$

So, from (25), (26), and (27), we have

$$\begin{aligned}\|z_n - p\|^2 &\le \|r_n - p\|^2 - \|y_n - r_n\|^2 - 2\psi_n\langle A r_n, y_n - p\rangle + \phi^2\frac{\psi_n^2}{\psi_{n+1}^2}\|r_n - y_n\|^2 - 2\psi_n\langle y_n - p, A y_n - A r_n\rangle \\ &= \|r_n - p\|^2 - \left(1 - \phi^2\frac{\psi_n^2}{\psi_{n+1}^2}\right)\|r_n - y_n\|^2 - 2\psi_n\langle y_n - p, A y_n\rangle.\end{aligned} \tag{28}$$

Now, from $p \in VI(C, A)$, we have that

$$\langle Ap, y_n - p\rangle \ge 0, \quad y_n \in C.$$

Then, by the pseudomonotonicity of A, we obtain

$$\langle A y_n, y_n - p\rangle \ge 0. \tag{29}$$

Combining (28) and (29), we have that

$$\|z_n - p\|^2 \le \|r_n - p\|^2 - \left(1 - \phi^2\frac{\psi_n^2}{\psi_{n+1}^2}\right)\|r_n - y_n\|^2.$$

Moreover, from the definition of z_n and (25), we obtain

$$\|z_n - y_n\| \le \phi\frac{\psi_n}{\psi_{n+1}}\|r_n - y_n\|,$$

which completes the proof. □

Theorem 4.4. *Assume conditions* (*A*) *and* (*B*) *hold. Then, the sequence* $\{x_n\}$ *generated by Algorithm* 3.1 *converges strongly to an element* $p \in \Omega$, *where* $p = P_\Omega(I - D + \gamma f)(p)$ *is a solution of the variational inequality*

$$\langle (D - \gamma f)p, p - q\rangle \le 0, \quad \forall q \in \Omega.$$

Proof. We divide the proof of Theorem 4.4 as follows:

Claim 1. The sequence $\{x_n\}$ generated by Algorithm 3.1 is bounded.

First, we show that $P_\Omega(I - D + \gamma f)$ is a contraction of H. For all $x, y \in H$, we have

$$\begin{aligned}\|P_\Omega(I - D + \gamma f)(x) - P_\Omega(I - D + \gamma f)(y)\| &\le \|(I - D + \gamma f)(x) - (I - D + \gamma f)(y)\| \\ &\le \|(I - D)x - (I - D)y\| + \gamma\|fx - fy\| \\ &\le (1 - \bar{\gamma})\|x - y\| + \gamma\rho\|x - y\| \\ &= (1 - (\bar{\gamma} - \gamma\rho))\|x - y\|.\end{aligned}$$

It shows that $P_\Omega(I - D + \gamma f)$ is a contraction. Thus, by the Banach contraction principle there exists an element $p \in \Omega$ such that $p = P_\Omega(I - D + \gamma f)(p)$. Next, setting $g_n = (1 - \beta_n)z_n + \beta_n U z_n$ and applying (23) we have

$$\begin{aligned}\|g_n - p\|^2 &= \|(1 - \beta_n)z_n + \beta_n U z_n - p\|^2 \\ &= \|(1 - \beta_n)(z_n - p) + \beta_n(U z_n - p)\|^2 \\ &= (1 - \beta_n)^2\|z_n - p\|^2 + \beta_n^2\|U z_n - p\|^2 + 2(1 - \beta_n)\beta_n\langle U z_n - p, z_n - p\rangle\end{aligned} \tag{30}$$

$$
\begin{aligned}
&\leq (1-\beta_n)^2\|z_n-p\|^2+\beta_n^2[\|z_n-p\|^2+\tau\|z_n-Uz_n\|^2]+2(1-\beta_n)\beta_n\left[\|z_n-p\|^2-\frac{1-\tau}{2}\|z_n-Uz_n\|^2\right]\\
&=\|z_n-p\|^2+\beta_n(\beta_n\tau-(1-\beta_n)(1-\tau)\|z_n-Uz_n\|^2\\
&=\|z_n-p\|^2-\beta_n(1-\tau-\beta_n)\|z_n-Uz_n\|^2\\
&\leq \|r_n-p\|^2-\left(1-\phi^2\frac{\psi_n^2}{\psi_{n+1}^2}\right)\|r_n-y_n\|^2-\beta_n(1-\tau-\beta_n)\|Uz_n-z_n\|^2.
\end{aligned}
$$

By the condition on β_n, from this we obtain

$$
\|g_n-p\|^2\leq\|r_n-p\|^2-\left(1-\phi^2\frac{\psi_n^2}{\psi_{n+1}^2}\right)\|r_n-y_n\|^2. \tag{31}
$$

From Lemma 4.1, we have that

$$
\lim_{n\to\infty}\left(1-\phi^2\frac{\psi_n^2}{\psi_{n+1}^2}\right)=1-\phi^2>0.
$$

This implies that there exists $n_0\in\mathbb{N}$ such that $1-\phi^2\frac{\psi_n^2}{\psi_{n+1}^2}>0$ for all $n\geq n_0$. Hence, from (31) we obtain

$$
\|g_n-p\|^2\leq\|r_n-p\|^2\quad\forall n\geq n_0. \tag{32}
$$

Also, by definition of r_n and triangle inequality,

$$
\|r_n-p\|=\|x_n+\delta_n(x_n-x_{n-1}-p)\|\leq\|x_n-p\|+\delta_n\|x_n-x_{n-1}\|=\|x_n-p\|+\alpha_n\quad\frac{\delta_n}{\alpha_n}\|x_n-x_{n-1}\|. \tag{33}
$$

From Remark 3.3, we have $\frac{\delta_n}{\alpha_n}\|x_n-x_{n-1}\|\to 0$ as $n\to\infty$. Thus, there exists a constant $G_1>0$ that satisfies:

$$
\frac{\delta_n}{\alpha_n}\|x_n-x_{n-1}\|\leq G_1,\quad\forall n\geq 1. \tag{34}
$$

So, from (32), (33), and (34) we obtain

$$
\|g_n-p\|\leq\|r_n-p\|\leq\|x_n-p\|+\alpha_nG_1,\quad\forall n\geq n_0. \tag{35}
$$

Now, by applying Lemma 2.6 and (35), $\forall\ n\geq n_0$ we have

$$
\begin{aligned}
\|x_{n+1}-p\|&=\|\alpha_n\gamma f(r_n)+(I-\alpha_nD)g_n-p\|\\
&=\|\alpha_n(\gamma f(r_n)-Dp)+(I-\alpha_nD)(g_n-p)\|\\
&\leq\alpha_n\|\gamma f(r_n)-Dp\|+(1-\alpha_n\bar{\gamma})\|g_n-p\|\\
&\leq\alpha_n\|\gamma f(r_n)-\gamma f(p)\|+\alpha_n\|\gamma f(p)-Dp\|+(1-\alpha_n\bar{\gamma})(\|x_n-p\|+\alpha_nG_1)\\
&\leq\alpha_n\gamma\rho\|r_n-p\|+\alpha_n\|\gamma f(p)-Dp\|+(1-\alpha_n\bar{\gamma})(\|x_n-p\|+\alpha_nG_1)\\
&\leq\alpha_n\gamma\rho(\|x_n-p\|+\alpha_nG_1)+\alpha_n\|\gamma f(p)-Dp\|+(1-\alpha_n\bar{\gamma})(\|x_n-p\|+\alpha_nG_1)\\
&=(1-\alpha_n(\bar{\gamma}-\gamma\rho))\|x_n-p\|+\alpha_n\|\gamma f(p)-Dp\|+(1-\alpha_n(\bar{\gamma}-\gamma\rho))\alpha_nG_1\\
&\leq(1-\alpha_n(\bar{\gamma}-\gamma\rho))\|x_n-p\|+\alpha_n(\bar{\gamma}-\gamma\rho)\left[\frac{\|\gamma f(p)-Dp\|}{\bar{\gamma}-\gamma\rho}+\frac{G_1}{\bar{\gamma}-\gamma\rho}\right]\\
&\leq\max\left\{\|x_n-p\|,\frac{\|\gamma f(p)-Dp\|}{\bar{\gamma}-\gamma\rho}+\frac{G_1}{\bar{\gamma}-\gamma\rho}\right\}\\
&\vdots\\
&\leq\max\left\{\|x_{n_0}-p\|,\frac{\|\gamma f(p)-Dp\|}{\bar{\gamma}-\gamma\rho}+\frac{G_1}{\bar{\gamma}-\gamma\rho}\right\}.
\end{aligned}
$$

Hence, the sequence $\{x_n\}$ is bounded, and so $\{r_n\}$, $\{y_n\}$, $\{z_n\}$ are also bounded.

Claim 2. The following inequality holds for all $p \in \Omega$ and $n \in \mathbb{N}$

$$\begin{aligned}||x_{n+1}-p||^2 \leq{}& \left(1-\frac{2\alpha_n(\bar{\gamma}-\gamma\rho)}{(1-\alpha_n\gamma\rho)}\right)||x_n-p||^2 + \frac{2\alpha_n(\bar{\gamma}-\gamma\rho)}{(1-\alpha_n\gamma\rho)}\left\{\frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma}-\gamma\rho)}G_3\right.\\ &\left.+ 3G_2\frac{((1-\alpha_n\bar{\gamma})^2+\alpha_n\gamma\rho)}{2(\bar{\gamma}-\gamma\rho)}\frac{\delta_n}{\alpha_n}||x_n-x_{n-1}|| + \frac{1}{(\bar{\gamma}-\gamma\rho)}\langle\gamma f(p)-Dp, x_{n+1}-p\rangle\right\}\\ &- \frac{(1-\alpha_n\bar{\gamma})^2}{(1-\alpha_n\gamma\rho)}\left\{\left(1-\phi^2\frac{\psi_n^2}{\psi_{n+1}^2}\right)\|r_n-y_n\|^2 + \beta_n(1-\tau-\beta_n)\|Uz_n-z_n\|^2\right\}.\end{aligned}$$

Using the Cauchy-Schwartz inequality and Lemma 2.6, we obtain

$$\begin{aligned}||r_n-p||^2 &= ||x_n+\delta_n(x_n-x_{n-1})-p||^2\\ &= ||x_n-p||^2 + \delta_n^2||x_n-x_{n-1}||^2 + 2\delta_n\langle x_n-p, x_n-x_{n-1}\rangle\\ &\leq ||x_n-p||^2 + \delta_n^2||x_n-x_{n-1}||^2 + 2\delta_n||x_n-x_{n-1}||||x_n-p||\\ &= ||x_n-p||^2 + \delta_n||x_n-x_{n-1}||(\delta_n||x_n-x_{n-1}|| + 2||x_n-p||)\\ &\leq ||x_n-p||^2 + 3G_2\delta_n||x_n-x_{n-1}||\\ &= ||x_n-p||^2 + 3G_2\alpha_n\frac{\delta_n}{\alpha_n}||x_n-x_{n-1}||,\end{aligned} \tag{36}$$

where $G_2 := \sup_{n\in\mathbb{N}}\{||x_n-p||, \theta_n||x_n-x_{n-1}||\} > 0$.

Now, by applying Lemma 2.6, (30), and (36) we have

$$\begin{aligned}||x_{n+1}-p||^2 &= ||\alpha_n\gamma f(r_n) + (I-\alpha_nD)g_n - p||^2\\ &= ||\alpha_n(\gamma f(r_n)-Dp) + (I-\alpha_nD)(g_n-p)||^2\\ &\leq (1-\alpha_n\bar{\gamma})^2||g_n-p||^2 + 2\alpha_n\langle\gamma f(r_n)-Dp, x_{n+1}-p\rangle\\ &\leq (1-\alpha_n\bar{\gamma})^2\left(\|r_n-p\|^2 - \left(1-\phi^2\frac{\psi_n^2}{\psi_{n+1}^2}\right)\|r_n-y_n\|^2 - \beta_n(1-\tau-\beta_n)\|Uz_n-z_n\|^2\right)\\ &\quad + 2\alpha_n\gamma\langle f(r_n)-f(p), x_{n+1}-p\rangle + 2\alpha_n\langle\gamma f(p)-Dp, x_{n+1}-p\rangle\\ &\leq (1-\alpha_n\bar{\gamma})^2\left(||r_n-p||^2 - \left(1-\phi^2\frac{\psi_n^2}{\psi_{n+1}^2}\right)\|r_n-y_n\|^2 - \beta_n(1-\tau-\beta_n)\|Uz_n-z_n\|^2\right)\\ &\quad + \alpha_n\gamma\rho(\|r_n-p\|^2 + \|x_{n+1}-p\|^2) + 2\alpha_n\langle\gamma f(p)-Dp, x_{n+1}-p\rangle\\ &\leq (1-\alpha_n\bar{\gamma})^2\left(||x_n-p||^2 + 3G_2\alpha_n\frac{\delta_n}{\alpha_n}||x_n-x_{n-1}|| - \left(1-\phi^2\frac{\psi_n^2}{\psi_{n+1}^2}\right)\|r_n-y_n\|^2\right.\\ &\quad \left.- \beta_n(1-\tau-\beta_n)\|Uz_n-z_n\|^2\right) + \alpha_n\gamma\rho\left(||x_n-p||^2 + 3G_2\alpha_n\frac{\delta_n}{\alpha_n}||x_n-x_{n-1}|| + \|x_{n+1}-p\|^2\right)\\ &\quad + 2\alpha_n\langle\gamma f(p)-Dp, x_{n+1}-p\rangle.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}||x_{n+1}-p||^2 &\leq \frac{(1-2\alpha_n\bar{\gamma}+(\alpha_n\bar{\gamma})^2+\alpha_n\gamma\rho)}{(1-\alpha_n\gamma\rho)}||x_n-p||^2 + 3G_2\frac{((1-\alpha_n\bar{\gamma})^2+\alpha_n\gamma\rho)}{(1-\alpha_n\gamma\rho)}\alpha_n\frac{\delta_n}{\alpha_n}||x_n-x_{n-1}||\\ &\quad + \frac{2\alpha_n}{(1-\alpha_n\gamma\rho)}\langle\gamma f(p)-Dp, x_{n+1}-p\rangle - \frac{(1-\alpha_n\bar{\gamma})^2}{(1-\alpha_n\gamma\rho)}\left\{\left(1-\phi^2\frac{\psi_n^2}{\psi_{n+1}^2}\right)\|r_n-y_n\|^2 + \beta_n(1-\tau-\beta_n)\|Uz_n-z_n\|^2\right\}\\ &= \frac{(1-2\alpha_n\bar{\gamma}+\alpha_n\gamma\rho)}{(1-\alpha_n\gamma\rho)}||x_n-p||^2 + \frac{(\alpha_n\bar{\gamma})^2}{(1-\alpha_n\gamma\rho)}||x_n-p||^2\\ &\quad + 3G_2\frac{((1-\alpha_n\bar{\gamma})^2+\alpha_n\gamma\rho)}{(1-\alpha_n\gamma\rho)}\alpha_n\frac{\delta_n}{\alpha_n}||x_n-x_{n-1}|| + \frac{2\alpha_n}{(1-\alpha_n\gamma\rho)}\langle\gamma f(p)-Dp, x_{n+1}-p\rangle\\ &\quad - \frac{(1-\alpha_n\bar{\gamma})^2}{(1-\alpha_n\gamma\rho)}\left\{\left(1-\phi^2\frac{\psi_n^2}{\psi_{n+1}^2}\right)\|r_n-y_n\|^2 + \beta_n(1-\tau-\beta_n)\|Uz_n-z_n\|^2\right\}\end{aligned}$$

$$\leq (1 - \frac{2\alpha_n(\bar{\gamma} - \gamma\rho)}{(1 - \alpha_n\gamma\rho)})||x_n - p||^2 + \frac{2\alpha_n(\bar{\gamma} - \gamma\rho)}{(1 - \alpha_n\gamma\rho)}\left\{\frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma} - \gamma\rho)}G_3 + 3G_2\frac{((1 - \alpha_n\bar{\gamma})^2 + \alpha_n\gamma\rho)}{2(\bar{\gamma} - \gamma\rho)}\frac{\delta_n}{\alpha_n}||x_n - x_{n-1}|| + \frac{1}{(\bar{\gamma} - \gamma\rho)}\langle\gamma f(p) - Dp, x_{n+1} - p\rangle\right\} - \frac{(1 - \alpha_n\bar{\gamma})^2}{(1 - \alpha_n\gamma\rho)}\left\{\left(1 - \phi^2\frac{\psi_n^2}{\psi_{n+1}^2}\right)\|r_n - y_n\|^2 + \beta_n(1 - \tau - \beta_n)\|Uz_n - z_n\|^2\right\},$$

where $G_3 := \sup\{||x_n - p||^2 : n \in \mathbb{N}\}$. This gives the required inequality.

Claim 3. The sequence $\{\|x_n - p\|^2\}$ converges to zero.

Let $p = P_\Omega(I - D + \gamma f)(p)$. From **Claim 2**, we obtain

$$||x_{n+1} - p||^2 \leq \left(1 - \frac{2\alpha_n(\bar{\gamma} - \gamma\rho)}{(1 - \alpha_n\gamma\rho)}\right)||x_n - p||^2 + \frac{2\alpha_n(\bar{\gamma} - \gamma\rho)}{(1 - \alpha_n\gamma\rho)}\left\{\frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma} - \gamma\rho)}G_3 + 3G_2\frac{((1 - \alpha_n\bar{\gamma})^2 + \alpha_n\gamma\rho)}{2(\bar{\gamma} - \gamma\rho)}\frac{\delta_n}{\alpha_n}||x_n - x_{n-1}|| + \frac{1}{(\bar{\gamma} - \gamma\rho)}\langle\gamma f(p) - Dp, x_{n+1} - p\rangle\right\}. \tag{37}$$

To establish **Claim 3**, in view of Lemma 2.7, Remark 3.3, and the fact that $\lim_{n\to\infty}\alpha_n = 0$, it suffices to show that $\limsup_{k\to\infty}\langle\gamma f(p) - Dp, x_{n_k+1} - p\rangle \leq 0$ for every subsequence $\{\|x_{n_k} - p\|\}$ of $\{\|x_n - p\|\}$ satisfying

$$\liminf_{k\to\infty}\left(\|x_{n_k+1} - p\| - \|x_{n_k} - p\|\right) \geq 0.$$

Suppose that $\{\|x_{n_k} - p\|\}$ is a subsequence of $\{\|x_n - p\|\}$ such that

$$\liminf_{k\to\infty}\left(\|x_{n_k+1} - p\| - \|x_{n_k} - p\|\right) \geq 0. \tag{38}$$

Again, from **Claim 2** we obtain

$$\frac{(1 - \alpha_{n_k}\bar{\gamma})^2}{(1 - \alpha_{n_k}\gamma\rho)}\left(1 - \phi^2\frac{\psi_{n_k}^2}{\psi_{n_k+1}^2}\right)\|r_{n_k} - y_{n_k}\|^2 \leq \left(1 - \frac{2\alpha_{n_k}(\bar{\gamma} - \gamma\rho)}{(1 - \alpha_{n_k}\gamma\rho)}\right)||x_{n_k} - p||^2 - ||x_{n_k+1} - p||^2 + \frac{2\alpha_{n_k}(\bar{\gamma} - \gamma\rho)}{(1 - \alpha_{n_k}\gamma\rho)}\left\{\frac{\alpha_{n_k}\bar{\gamma}^2}{2(\bar{\gamma} - \gamma\rho)}G_3 + 3G_2\frac{((1 - \alpha_{n_k}\bar{\gamma})^2 + \alpha_{n_k}\gamma\rho)}{2(\bar{\gamma} - \gamma\rho)}\frac{\delta_{n_k}}{\alpha_{n_k}}||x_{n_k} - x_{n_k-1}|| + \frac{1}{(\bar{\gamma} - \gamma\rho)}\langle\gamma f(p) - Dp, x_{n_k+1} - p\rangle\right\}.$$

Applying (38) and the fact that $\lim_{k\to\infty}\alpha_{n_k} = 0$, we have

$$\frac{(1 - \alpha_{n_k}\bar{\gamma})^2}{(1 - \alpha_{n_k}\gamma\rho)}\left(1 - \phi^2\frac{\psi_{n_k}^2}{\psi_{n_k+1}^2}\right)\|r_{n_k} - y_{n_k}\|^2 \to 0, \quad k \to \infty.$$

By the conditions on the control parameters, we obtain

$$\|r_{n_k} - y_{n_k}\| \to 0, \quad k \to \infty. \tag{39}$$

Following similar argument, from **Claim 2** we have

$$\|Uz_{n_k} - z_{n_k}\| \to 0, \quad k \to \infty. \tag{40}$$

From (24) and (39), we obtain

$$\|z_{n_k} - y_{n_k}\| \to 0, \quad k \to \infty. \tag{41}$$

Combining (39) and (41), we have

$$\|r_{n_k} - z_{n_k}\| \leq \|r_{n_k} - y_{n_k}\| + \|y_{n_k} - z_{n_k}\| \to 0, \quad k \to \infty. \tag{42}$$

By Remark 3.3 and the definition of r_n, we obtain

$$\|x_{n_k} - r_{n_k}\| = \delta_{n_k}\|x_{n_k} - x_{n_k-1}\| \to 0, \quad k \to \infty. \tag{43}$$

From (39), (42), and (43), we obtain

$$\|x_{n_k} - y_{n_k}\| \to 0, \quad k \to \infty, \quad \|x_{n_k} - z_{n_k}\| \to 0, \quad k \to \infty. \tag{44}$$

Also, from (40) and (44), we obtain

$$\|x_{n_k} - Uz_{n_k}\| \to 0, \quad k \to \infty. \tag{45}$$

Using (44) and (45), we have

$$\|x_{n_k} - g_{n_k}\| \leq \left(1 - \beta_{n_k}\right)\|x_{n_k} - z_{n_k}\| + \beta_{n_k}\|x_{n_k} - Uz_{n_k}\| \to 0, \quad k \to \infty. \tag{46}$$

Combining this together with the fact that $\lim_{k\to\infty}\alpha_{n_k} = 0$, we obtain

$$\|x_{n_k+1} - x_{n_k}\| \leq \alpha_{n_k}\|\gamma f\left(r_{n_k}\right) - x_{n_k}\| + \left(1 - \alpha_{n_k}\bar{\gamma}\right)\|g_{n_k} - x_{n_k}\| \to 0, \quad k \to \infty. \tag{47}$$

To complete the proof, we need to show that $w_\omega(x_n) \subset \Omega$. Since $\{x_n\}$ is bounded, then $w_\omega(x_n)$ is nonempty. Let $x^* \in w_\omega(x_n)$ be an arbitrary element. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$ as $k \to \infty$. By Lemma 4.2 and (39), it follows that $x^* \in VI(C, A)$. Consequently, we have $w_\omega(x_n) \subset VI(C, A)$. From (44), we have that $z_{n_k} \rightharpoonup x^*$ as $k \to \infty$. Since $I - U$ is demiclosed at zero, then it follows from (40) that $x^* \in F(U)$. That is, $w_\omega(x_n) \subset F(U)$. Therefore, we have $w_\omega(x_n) \subset \Omega$.

Moreover, from (44) it follows that $w_\omega\{y_n\} = w_\omega\{x_n\} = w_\omega\{z_n\}$. By the boundedness of $\{x_{n_k}\}$, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup x^\dagger$ and

$$\lim_{j\to\infty}\langle \gamma f(p) - Dp, x_{n_{k_j}} - p\rangle = \limsup_{k\to\infty}\langle \gamma f(p) - Dp, x_{n_k} - p\rangle = \limsup_{k\to\infty}\langle \gamma f(p) - Dp, z_{n_k} - p\rangle. \tag{48}$$

Since $p = P_\Omega(I - D + \gamma f)(p)$, it follows from (48) that

$$\limsup_{k\to\infty}\langle \gamma f(p) - Dp, x_{n_k} - p\rangle = \lim_{j\to\infty}\langle \gamma f(p) - Dp, x_{n_{k_j}} - p\rangle = \langle \gamma f(p) - Dp, x^\dagger - p\rangle \leq 0. \tag{49}$$

Hence, from (47) and (49), we obtain

$$\begin{aligned}\limsup_{k\to\infty}\langle \gamma f(p) - Dp, x_{n_k+1} - p\rangle &= \limsup_{k\to\infty}\langle \gamma f(p) - Dp, x_{n_k+1} - x_{n_k}\rangle + \limsup_{k\to\infty}\langle \gamma f(p) - Dp, x_{n_k} - p\rangle \\ &= \langle \gamma f(p) - Dp, x^\dagger - p\rangle \leq 0.\end{aligned} \tag{50}$$

Applying Lemma 2.7 to (37), and using (50) together with the fact that $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| = 0$ and $\lim_{n\to\infty}\alpha_n = 0$, we deduce that $\lim_{n\to\infty}||x_n - p|| = 0$ as required. □

Taking $\gamma = 1$ and $D = I$ in Theorem 4.4, where I is the identity mapping, then we have the following corollary.

Corollary 4.5. *Let H be a Hilbert space and suppose $U : H \to H$ is a τ-demicontractive map. Let $\{x_n\}$ be a sequence generated as follows*:

Algorithm 4.6.

Step 0. Given $\delta > 0$, $\phi \in (0, 1)$, select initial data $x_0, x_1 \in H$, $\lambda_0 > 0$, and set $n = 1$.

Step 1. Given the $(n-1)$th and nth iterates, choose δ_n such that $0 \leq \delta \leq \delta_n$, $\forall n \in \mathbb{N}$ with δ_n defined by:

$$\delta_n = \begin{cases} \min\left\{\dfrac{\varepsilon_n}{||x_n - x_{n-1}||}, \delta\right\}, & \text{if } x_n \neq x_{n-1}, \\ \delta, & \text{otherwise}. \end{cases} \tag{51}$$

Step 2. Compute

$$r_n = x_n + \delta_n(x_n - x_{n-1}). \tag{52}$$

Step 3. Compute the projection:

$$y_n = P_C(r_n - \psi_n A r_n), \tag{53}$$

If $y_n = r_n$, then set $y_n = r_n$ and go to Step 5. Else go to Step 4.

Step 4. Compute

$$z_n = y_n - \psi_n(Ay_n - Ar_n). \tag{54}$$

Step 5. Compute

$$\psi_{n+1} = \begin{cases} \min\left\{\dfrac{\phi||r_n - y_n||}{||Ar_n - Ay_n||}, \psi_n\right\}, & \text{if } Ar_n - Ay_n \neq 0, \\ \psi_n, & \text{otherwise}. \end{cases} \tag{55}$$

Step 6. Compute

$$x_{n+1} = \alpha_n f(r_n) + (1 - \alpha_n)[(1 - \beta_n)z_n + \beta_n U z_n]. \tag{56}$$

Set $n := n + 1$ and return to **Step 1**.

Assume that $\Omega = VI(C, A) \cap F(U) \neq 0$ and other assumptions in conditions A and B are satisfied. Then the sequence $\{x_n\}$ generated by Algorithm 4.6 converges strongly to a point $p \in \Omega$ where $p = P_\Omega \circ f(p)$ is a solution of the variational inequalities.

$$\langle (I - f)p, p - z\rangle \leq 0 \quad \textit{for all } z \in \Omega.$$

Remark 4.7. The result in Corollary 4.5 complements the result of Tan and Qin [29], Gang et al. [22] and Thong and Hieu [23] in the following ways:

(i) Our result in Corollary 4.5 extends the result of Tan and Qin [29] from pseudomonotone VIP to common solution problem of pseudomonotone variational inequality and FPPs of demicontractive maps.
(ii) Corollary 4.5 result extends the result of Cai et al. [22] from FPP of nonexpansive maps to FPP of demicontractive maps.
(iii) The result of Cai et al. [22] requires the knowledge of the Lipschitz constant of the cost operator while our result in Corollary 4.5 does not require any knowledge of the Lipschitz constant of the cost operator.
(iv) The result of Corollary 4.5 extends the result of Thong and Hieu [23] from monotone VIP to pseudomonotone VIP.
(v) Unlike the result of Thong and Hieu [23], our result in Corollary 4.5 employs inertial technique to speed up the rate of convergence of the algorithm.
(vi) As shown in our convergence analysis, we did not adopt the conventional "**two cases**" approach employed in several papers to prove strong convergence. Our procedure is more concise and easy to comprehend.

5 Numerical examples

In this section, we proceed to perform two numerical experiments to show the computational efficiency of our Algorithm 3.1 in comparison with some other algorithms in the literature. The graph of errors is plotted against the number of iterations in each case. All numerical computations were carried out using Matlab 2019(b). We use $\|x_{n+1} - x_n\| \leq 10^{-2}$ as the stopping criterion. The parameters are chosen as follows:

- Let $f(x) = \frac{1}{5}x$, then $\rho = \frac{1}{5}$ is the Lipschitz constant for f. Let $D(x) = \frac{x}{3}$ with constant $\bar{\gamma} = \frac{1}{3}$, then we take $\gamma = 1$, which satisfies $0 < \gamma < \frac{\bar{\gamma}}{\rho}$. Let $Ux = -\frac{3}{2}x$. Choose $\delta = 0.8$,$\psi_1 = 0.6$, $\phi = 0.7$, $\alpha_n = \frac{1}{n+3}$, $\varepsilon_n = \frac{1}{(n+3)^3}$, $\beta_n = \frac{3n+1}{5n+3}$ in our Algorithm 3.1.
- Take $Tx = \frac{x}{2}$, $\psi = \frac{0.8}{L}$, $\theta_n = \frac{1}{(n+3)^2}$ in Algorithm (14).
- Let $Gx = x - x_1$, $\gamma_n = \frac{1}{n+1}$, $\omega = 0.09$, $\rho_n = \frac{n}{2n+1}$ in Appendix 6.1.
- Take $T_n x = -\frac{2}{n \bmod 5}x$, $\lambda = m = \mu = \frac{1}{2}$, $\sigma_n = \frac{1}{n+3}$, $\tau_n = \frac{1}{3}$, $\gamma_n = \frac{1}{6}$, $\mu_n = \frac{1}{2}$, in Appendices 6.2 and 6.3.

Example 5.1. Consider the linear operator $A : \mathbb{R}^m \to \mathbb{R}^m$ ($m = 5, 10, 15, 20$) as follows: $A(x) = Fx + g$, where $g \in \mathbb{R}^m$ and $F = BB^T + M + E$, matrix $B \in \mathbb{R}^{m\times m}$, matrix $M \in \mathbb{R}^{m\times m}$, is skew symmetric, and matrix $E \in \mathbb{R}^{m\times m}$ is a diagonal matrix whose diagonal terms are nonnegative (which implies that F is positive symmetric definite). We choose the feasible set as $C = \{x \in \mathbb{R}^m : -2 \le x_i \le 5,\ i = 1, \ldots, m\}$. It can easily be verified that the mapping A is strongly pseudomonotone and Lipschitz continuous with $L = \|F\|$. In this example, both B and M entries are generated randomly in $[-2, 2]$, E is generated randomly in $[0, 2]$, and $g = 0$. The initial values $x_0 = x_1$ are generated randomly by rand$(m, 1)$.

The stopping criterion used for our computation is $\|x_{n+1} - x_n\| < 10^{-2}$. We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figure 1 and Table 1.

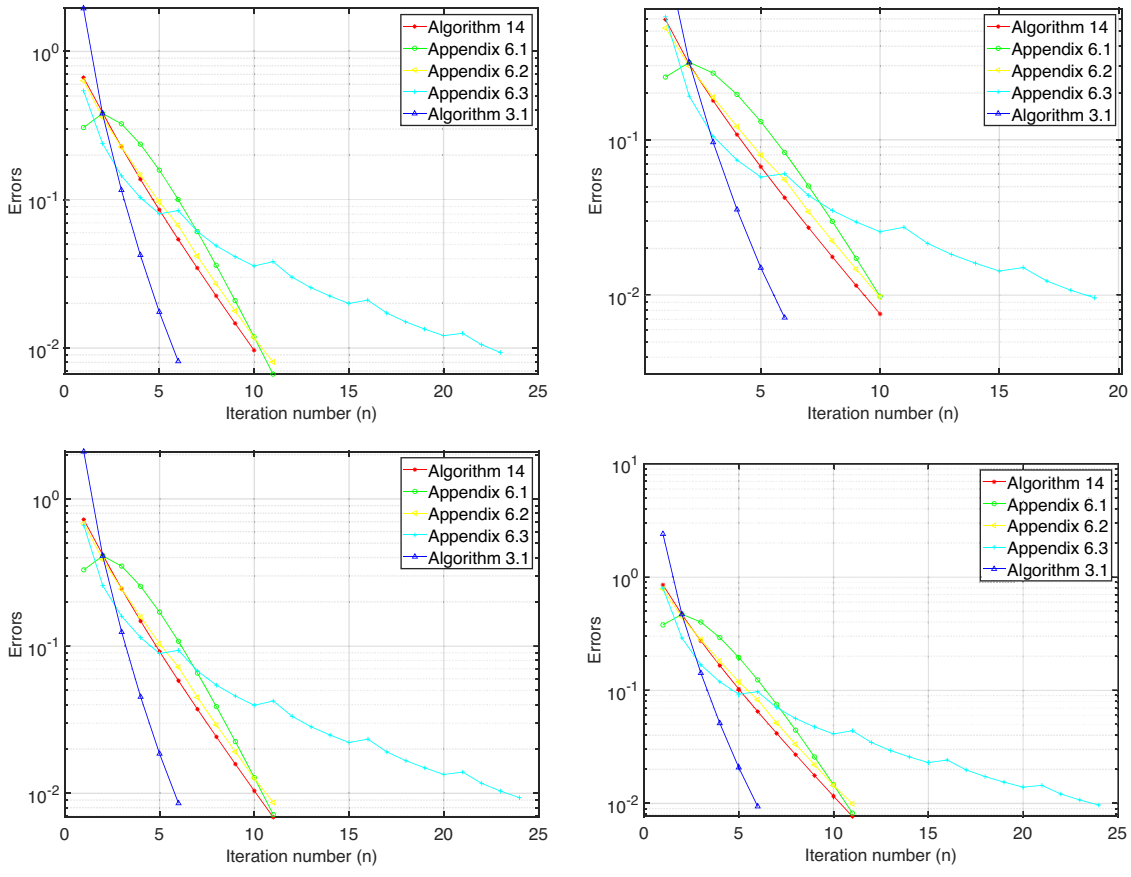

Figure 1: Top left: $m = 5$; top right: $m = 10$; bottom left: $m = 15$; bottom right: $m = 20$.

Table 1: Numerical results for Example 5.1

		Algorithm 14	Appendix 6.1	Appendix 6.2	Appendix 6.3	Algorithm 3.1
$m = 5$	No. of Iter.	10	11	11	22	6
	CPU time (s)	1.7148	0.9880	0.8846	1.9379	0.4792
$m = 10$	No. of Iter.	11	11	11	22	6
	CPU time (s)	1.4375	1.1026	0.9923	1.9712	0.5256
$m = 15$	No. of Iter.	11	11	11	24	6
	CPU time (s)	1.4107	0.9337	1.0554	1.9888	0.6284
$m = 20$	No. of Iter.	11	11	12	25	6
	CPU time (s)	1.2953	0.8184	0.9771	1.6142	0.4390

Example 5.2. We consider the next example in the infinite dimensional Hilbert space $H = L^2([0, 1])$ with inner product

$$\langle x, y\rangle := \int_0^1 x(t)y(t)\mathrm{d}t \quad \text{for all } x, y \in H, \tag{57}$$

and induced norm

$$||x|| := \left(\int_0^1 |x(t)|^2\mathrm{d}t\right)^{\frac{1}{2}} \quad \text{for all } x \in H. \tag{58}$$

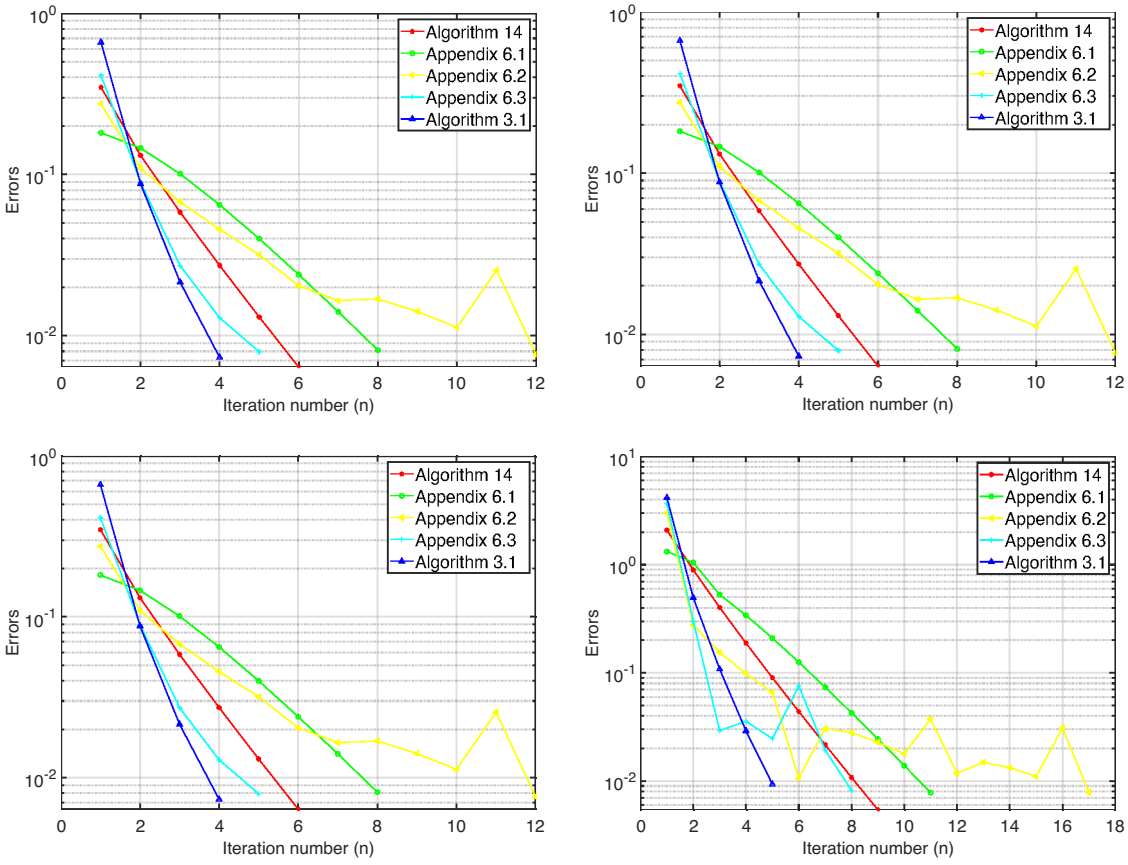

Figure 2: Top left: Case I; top right: Case II; bottom left: Case III; bottom right: Case IV.

Now, define $A : H \to H$ by $A(x)(t) = \max\{0, x(t)\}$, for all $t \in [0, 1]$, $x \in H$. It is easy to see that A is pseudomonotone and 1-Lipschitz continuous on H. It can easily be verified that all the conditions of Theorem 4.4 are satisfied.

We choose four different initial values as follows:

Case I: $x_0 = \frac{2t^3+1}{3}$ and $x_1 = 3t^5 + t^2 + 1$;

Case II: $x_0 = \exp(-t)$ and $x_1 = \cos 2t$;

Case III: $x_0 = t^3 + t + 5$ and $x_1 = \exp(-2t)$;

Case IV: $x_0 = 2t^5 + t^2 + 3$ and $x_1 = 2t^3 - t^2 + 3$.

The stopping criterion used for our computation is $\|x_{n+1} - x_n\| < 10^{-2}$. We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figure 2 and Table 2.

Table 2: Numerical results for Example 5.2

	Algorithm 14	Appendix 6.1	Appendix 6.2	Appendix 6.3	Algorithm 3.1
No. of Iter.	6	8	12	5	4
No. of Iter.	6	8	12	5	4
No. of Iter.	6	8	12	5	4
No. of Iter.	9	11	17	8	5

6 Conclusion

We studied the pseudomonotone VIP with a fixed point constraint. We introduced a new inertial TEGM with an adaptive step size for approximating a solution of the pseudomonotone VIP, which is also a fixed point of demicontractive mappings. We proved strong convergence results for the proposed algorithm without the knowledge of the Lipschitz constant of the cost operator. Finally, we presented several numerical experiments to demonstrate the efficiency of our proposed method in comparison with some of the existing methods in the literature.

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Appendix

Appendix 6.1. (Algorithm 3.3 in [56])

Take $x_1 \in H$, $\psi_1 > 0$, $\omega \in \left(0, \frac{1-\beta}{2}\right]$ and $\phi \in (0, 1)$. Choose the sequences $\{\alpha_n\}$ and $\{\gamma_n\}$ satisfying the assumptions made on the control parameters

Step 1. Compute

$$y_n = P_C(x_n - \psi_n A(x_n)).$$

Step 2. Compute

$$z_n = P_{H_n}(x_n - \psi_n A(y_n)),$$

where

$$H_n \coloneqq \{x \in H : \langle x_n - \psi_n A(x_n) - y_n, x - y_n\rangle \leq 0\},$$

and

$$\psi_{n+1} \coloneqq \begin{cases} \min\left\{\frac{\phi \,||\, x_n - y_n \,||}{||\, A(x_n) - A(y_n) \,||}, \psi_n\right\}, & \text{if } A(x_n) - A(y_n) \neq 0, \\ \psi_n, & \text{otherwise}. \end{cases}$$

Step 3. Compute

$$t_n \coloneqq (1 - \rho_n)x_n + \rho_n z_n.$$

Step 4. Compute

$$v_n = t_n - \gamma_n G(t_n).$$

Step 5. Compute

$$x_{n+1} = [(1 - \omega)I + \omega U]v_n.$$

Let $n = n + 1$ and return to Step 1.

Appendix 6.2. (Algorithm 1 in [57])

Initial step: Given $x_0, x_1 \in H$ arbitrarily. Let $\gamma > 0$, $m \in (0, 1)\mu \in (0, 1)$

Iteration steps: Compute x_{n+1} below:

Step 1. Put $v_n = x_n - \sigma_n(x_{n-1} - x_n)$ and calculate $u_n = P_C(v_n - l_n A v_n)$, where l_n is picked to be the largest $l \in \{\lambda, \lambda^m, \lambda^{m^2}, \ldots\}$ s.t

$$l\|Av_n - Au_n\| \leq \mu\|v_n - u_n\|.$$

Step 2. Calculate

$$z_n = (1 - \alpha_n)P_{C_n}(v_n - l_n A(u_n)) + \alpha_n f(x_n),$$

where

$$C_n \coloneqq \{v \in H : \langle v_n - l_n A v_n - u_n, u_n - v\rangle \geq 0\}.$$

Step 3. Compute

$$x_{n+1} = \gamma_n P_{C_n}(v_n - l_n A u_n) + \mu_n T_n z_n + \tau_n x_n.$$

Update $n = n + 1$ and return to Step 1.

Appendix 6.3. (Algorithm 2 in [57])

Initial step: Given $x_0, x_1 \in H$ arbitrarily. Let $\gamma > 0$, $m \in (0,1) \mu \in (0,1)$
Iteration steps: Compute x_{n+1} below:

Step 1. Put $v_n = x_n - \sigma_n(x_{n-1} - x_n)$ and calculate

$u_n = P_c(v_n - l_n A v_n)$, where l_n is picked to

be the largest $l \in \{\lambda, \lambda^m, \lambda^{m^2}, \ldots\}$ s.t

$$l\|Av_n - Au_n\| \leq \mu\|v_n - u_n\|$$

Step 2. Calculate

$$z_n = (1 - \alpha_n)P_{C_n}(v_n - l_n A(u_n)) + \alpha_n f(x_n),$$

where

$$C_n := \{v \in H : \langle v_n - l_n A v_n - u_n, u_n - v\rangle \geq 0\}$$

Step 3. Compute

$$x_{n+1} = \gamma_n P_{C_n}(v_n - l_n A u_n) + \delta_n T_n z_n + \beta_n v_n$$

Update $n = n + 1$ and return to Step 1.