STRONG CONVERGENCE OF AN EXTENDED EXTRAGRADIENT METHOD FOR EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS

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ABSTRACT. In this paper, we introduced a new extended extragradient iteration algorithm for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of equilibrium problems for a monotone and Lipschitz-type continuous mapping. And we show that the iterative sequences generated by this algorithm converge strongly to the common element in a real Hilbert space.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H and f be a bifunction from $C \times C$ to $\mathbb{R}$. We consider the equilibrium problem: Find $x^* \in C$ such that

$$EP(f, C) \qquad\qquad f(x^*, y) \geq 0 \quad \forall y \in C.$$

The set of solutions of $EP(f, C)$ is denoted by $Sol(f, C)$. These problems appear frequently in many practical problems arising, for instance, physics, engineering, game theory, transportation, economics and network, and become an attractive field for many researchers both theory and applications (see [1, 2, 3, 4, 5, 18, 21]).

If $f(x, y) = \langle F(x), y - x \rangle$ for every $x, y \in C$, where F is a mapping from C to H, then the problem $EP(f, C)$ becomes the following variational inequality: Find $x^* \in C$ such that

$$VI(F, C) \qquad\qquad \langle F(x^*), y - x^* \rangle \geq 0 \quad \forall y \in C.$$

We denote $Sol(F, C)$ which is the set of solutions of $VI(F, C)$.

For solving $VI(F, C)$ in the Euclidean space $\mathbb{R}^n$ under the assumption that a subset $C \subseteq \mathbb{R}^n$ is nonempty closed convex, F is monotone, L-Lipschitz continuous and $Sol(F, C) \neq \emptyset$, Korpelevich in [9] introduced the following extragradient method:

$$\begin{cases} x^0 \in C, \\ y^k = Pr_C\big(x^k - \lambda F(x^k)\big), \\ x^{k+1} = Pr_C\big(x^k - \lambda F(y^k)\big), \end{cases}$$

for all $k \geq 0$, where $\lambda \in (0, \frac{1}{L})$ and Pr_C is denoted the projection on C. The author showed that the sequences $\{x^k\}$ and $\{y^k\}$ converge to the same point $\bar{x} \in Sol(F, C)$.

Takahashi and Toyoda in [17] introduced an extragradient method for finding a common element of $Sol(F, C)$ and the set of fixed points of a nonexpansive mapping T (shortly $Fix(T)$) under the assumption that a subset $C \subseteq H$ is closed convex and F is α-inverse strongly monotone:

$$\begin{cases} x^0 \in C, \\ x^{k+1} = \alpha_k x^k + (1 - \alpha_k) T Pr_C\big(x^k - \lambda_k F(x^k)\big), \end{cases}$$

for all $k \geq 0$, where $\{\alpha_k\}$ is a sequence in $(0, 1)$ and $\{\lambda_k\}$ is a sequence in $(0, 2\alpha)$. They proved that if $Fix(T) \cap Sol(F, C) \neq \emptyset$, then the sequence $\{x^k\}$ converges weakly to some $\bar{x} \in Sol(F, C) \cap Fix(T)$.

For obtaining a common element of $Sol(f, C)$ and the set of fixed points of a nonexpansive mapping T, Takahashi and Takahashi in [16] introduced an iterative scheme by the viscosity approximation method. Sequences $\{x^k\}$ and $\{y^k\}$ are defined by:

$$\begin{cases} x^0 \in H, \\ f(y^k, y) + \frac{1}{r_k}\langle y - y^k, y^k - x^k \rangle \geq 0 \quad \forall y \in C, \\ x^{k+1} = \alpha_k g(x^k) + (1 - \alpha_k) T(y^k) \quad \forall k \geq 0. \end{cases}$$

The authors showed that under certain conditions over $\{\alpha_k\}$ and $\{r_k\}$, sequences $\{x^k\}$ and $\{y^k\}$ converge strongly to $z = Pr_{Fix(T) \cap Sol(f, C)}\big(g(z)\big)$.

Recently, iterative algorithms for finding a common element of the set of solutions of equilibrium problems and the set of fixed points of a nonexpansive mapping in a real Hilbert space have further developed by some authors (see [6, 7, 8, 10, 12, 14, 15, 16, 18, 21, 22]). At each iteration k in all of these algorithms, it requires solving approximation auxiliary equilibrium problems.

In this paper, we introduce a new iterative algorithm for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of equilibrium problems for a monotone, Lipschitz-type continuous bifunction. At each iteration k, we only solve strongly convex problems on C. The iterative process is based on so-called extragradient method. We obtain a strong convergence theorem for three sequences generated by this process.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. We list some well known definitions.

Definition 2.1. Let C be a nonempty closed convex subset of H.

(I) The bifunction $f : C \times C \to \mathbb{R}$ is said to be

 (i) γ-strongly monotone on C if for each $x, y \in C$,
$$f(x,y) + f(y,x) \leq -\gamma \|x - y\|^2;$$

 (ii) monotone on C if for each $x, y \in C$,
$$f(x,y) + f(y,x) \leq 0;$$

 (iii) pseudomonotone on C if for each $x, y \in C$,
$$f(x,y) \geq 0 \Rightarrow f(y,x) \leq 0;$$

 (iv) Lipschitz-type continuous on C with constants $c_1 > 0$ and $c_2 > 0$, if for each $x, y \in C$,
$$f(x,y) + f(y,z) \geq f(x,z) - c_1 \|x - y\|^2 - c_2 \|y - z\|^2.$$

(II) The mapping $F : C \to H$ is said to be

 (i) monotone on C if for each $x, y \in C$,
$$\langle F(x) - F(y), x - y \rangle \geq 0;$$

 (ii) pseudomonotone on C if for each $x, y \in C$,
$$\langle F(y), x - y \rangle \geq 0 \Rightarrow \langle F(x), x - y \rangle \geq 0;$$

 (iii) L-Lipschitz continuous on C if for each $x, y \in C$,
$$\|F(x) - F(y)\| \leq L\|x - y\|.$$

If $L = 1$, then F is nonexpansive on C.

Now, we define the projection on C, denoted by $Pr_C(\cdot)$, i.e.,
$$Pr_C(x) = \operatorname{argmin}\{\|y - x\| : \ y \in C\} \quad \forall x \in H.$$

A space X is said to have Opial's condition ([13]) if for any sequence $\{x_n\}$ with $x_n \rightharpoonup \bar{x}$, the inequality
$$\liminf_{n \to \infty} \|x_n - \bar{x}\| < \liminf_{n \to \infty} \|x_n - y\|$$
holds for every $y \in H$ with $y \neq \bar{x}$,

Note that if F is L-Lipschitz on C, then for each $x, y \in C$, $f(x,y) = \langle F(x), y - x \rangle$ is Lipschitz-type continuous with constants $c_1 = c_2 = \frac{L}{2}$ on C. Indeed,
$$
\begin{aligned}
f(x,y) + f(y,z) - f(x,z) &= \langle F(x), y - x \rangle + \langle F(y), z - y \rangle + \langle F(x), z - x \rangle \\
&= -\langle F(y) - F(x), y - z \rangle \\
&\geq -\|F(x) - F(y)\| \|y - z\|
\end{aligned}
$$

$$\geq - L\|x - y\|\|y - z\|$$
$$\geq - \frac{L}{2}\|x - y\|^2 - \frac{L}{2}\|y - z\|^2$$
$$= - c_1\|x - y\|^2 - c_2\|y - z\|^2.$$

Thus f is Lipschitz-type continuous on C.

In this paper, for finding a point of the set $Sol(f, C) \cap Fix(T)$, we assume that the bifunction f satisfies the following conditions:

(i) f is monotone on C;

(ii) f is Lipschitz-type continuous on C;

(iii) for each $x \in C$, $y \mapsto f(x, y)$ is convex and subdifferentiable on C;

(iv) f is upper semicontinuous on C;

(v) $Sol(f, C) \cap Fix(T) \neq \emptyset$.

Now we are in a position to describe the extended extragradient algorithm for finding a common element of $Sol(f, C) \cap Fix(T)$.

Algorithm 2.2. *Choose $u \in H$, positive sequences $\{\lambda_n\}, \{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ satisfy the conditions*:

$$\begin{cases} \{\lambda_n\} \subset \left(0, \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}\right), \lim\limits_{n \to \infty} \lambda_n = \lambda \in (0, \frac{2\delta-1}{4c_2}], \\ \alpha_n + \beta_n + \gamma_n = 1, \lim\limits_{n \to \infty} \alpha_n = 0, \sum\limits_{n=0}^{\infty} \alpha_n = \infty, \lim\limits_{n \to \infty} \beta_n = \beta \in (0, 1). \end{cases}$$

Step 1. *Solve the strongly convex problems*:

$$\begin{cases} y^n := \operatorname{argmin}\{\frac{1}{2}\|y - x^n\|^2 + \lambda_n f(x^n, y) : \ y \in C\}, \\ t^n := \operatorname{argmin}\{\frac{1}{2}\|t - x^n\|^2 + \lambda_n f(y^n, t) : \ t \in C\}. \end{cases}$$

Step 2. *Set $x^{n+1} := \alpha_n u + \beta_n x^n + \gamma_n T(t^n)$.*

Increase k by 1 and go to Step 1.

In order to prove the main result in Section 3, we shall use the following lemmas in the sequel.

Lemma 2.3 (see [5])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H and $g : C \to \mathbb{R}$ be convex and subdifferentiable on C. Then x^* is a solution to the following convex problem*

$$\min\{g(x) : \ x \in C\}$$

if and only if

$$0 \in \partial g(x^*) + N_C(x^*),$$

where $\partial g(\cdot)$ denotes the subdifferential of g and $N_C(x^*)$ is the (outward) normal cone of C at $x^* \in C$.

Lemma 2.4 (see [11])**.** *Assume that T is a nonexpansive self-mapping of a nonempty closed convex subset C of a real Hilbert space H. If $Fix(T) \neq \emptyset$, then $I - T$ is demiclosed; that is, whenever $\{x^n\}$ is a sequence in C weakly*

converging to some $\bar{x} \in C$ and the sequence $\{(I - T)(x^n)\}$ strongly converges to some $\bar{y}$, it follows that $(I - T)(\bar{x}) = \bar{y}$. Here I is the identity operator of H.

Lemma 2.5 (see [20]). *Let $\{x^n\}$ and $\{y^n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with*

$$0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1.$$

Suppose

$$x^{n+1} = (1 - \beta_n)y^n + \beta_n x^n \quad \forall n \geq 0 \quad and \quad \limsup_{n \to \infty}(\|y^{n+1} - y^n\| - \|x^{n+1} - x^n\|) \leq 0.$$

Then, $\lim_{n \to \infty} \|y^n - x^n\| = 0$.

Lemma 2.6 (see [19]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

$$\begin{cases} \sum_{n=1}^{\infty} \gamma_n = \infty, \\ \limsup_{n \to \infty} \frac{\delta_n}{\gamma_n} \leq 0 \quad or \quad \sum_{n=1}^{\infty} |\delta_n| < \infty. \end{cases}$$

Then $\lim_{n \to \infty} a_n = 0$.

3. Main results

In this section, we prove that the strong convergence of the sequences $\{x^n\}$, $\{y^n\}$ and $\{t^n\}$ defined by Algorithm 2.2 based on the extragradient method which solves the problem of finding a common element of two sets $Sol(f, C)$ and $Fix(T)$ for a monotone, Lipschitz-type continuous bifunction f in a real Hilbert space H.

Lemma 3.1. *Let $f(x, \cdot)$ be convex and subdifferentiable on C for all $x \in C$, and f be pseudomonotone on C. Then for $x^* \in Sol(f, C)$, we have*

$$\|t^n - x^*\|^2 \leq \|x^n - x^*\|^2 - (1 - 2\lambda_n c_2)\|t^n - y^n\|^2 - (1 - 2\lambda_n c_1)\|x^n - y^n\|^2 \quad \forall n \geq 0.$$

Proof. Since $f(x, \cdot)$ is convex on C for each $x \in C$ and Lemma 2.3, we obtain

$$t^n = \operatorname{argmin}\{\frac{1}{2}\|t - x^n\|^2 + \lambda_n f(y^n, t) : \ t \in C\}$$

if and only if

(3.1) $$0 \in \partial_2\{\lambda_n f(y^n, y) + \frac{1}{2}\|y - x^n\|^2\}(t^n) + N_C(t^n).$$

Since $f(y^n, \cdot)$ is subdifferentiable on C, by the well known Moreau-Rockafellar Theorem (see [5]), there exists $w \in \partial_2 f(y^n, t^n)$ such that

(3.2) $$f(y^n, t) - f(y^n, t^n) \geq \langle w, t - t^n \rangle \quad \forall t \in C.$$

For $t = x^* \in C$, this inequality becomes

$$(3.3) \qquad f(y^n, x^*) - f(y^n, t^n) \geq \langle w, x^* - t^n \rangle.$$

From (3.1), it follows that

$$0 = \lambda_n w + t^n - x^n + \bar{w},$$

where $w \in \partial_2 f(y^n, t^n)$ and $\bar{w} \in N_C(t^n)$. From the last inequality and the definition of the normal cone N_C, we have

$$(3.4) \qquad \langle t^n - x^n, t - t^n \rangle \geq \lambda_n \langle w, t^n - t \rangle \quad \forall t \in C.$$

Using $t = x^* \in C$, we obtain

$$(3.5) \qquad \langle t^n - x^n, x^* - t^n \rangle \geq \lambda_n \langle w, t^n - x^* \rangle.$$

It follows from (3.3) and (3.5) that

$$(3.6) \qquad \langle t^n - x^n, x^* - t^n \rangle \geq \lambda_n \big(f(y^n, t^n) - f(y^n, x^*) \big).$$

Since $x^* \in Sol(f, C)$, $f(x^*, y) \geq 0$ for all $y \in C$, and f is pseudomonotone on C, we have $f(y^n, x^*) \leq 0$. Then, (3.6) implies that

$$(3.7) \qquad \langle t^n - x^n, x^* - t^n \rangle \geq \lambda_n f(y^n, t^n).$$

Now applying Lipschitzian of f with $x = x^n, y = y^n$ and $z = t^n$, we get

$$(3.8) \qquad f(y^n, t^n) \geq f(x^n, t^n) - f(x^n, y^n) - c_1 \|y^n - x^n\|^2 - c_2 \|t^n - y^n\|^2.$$

Combining (3.7) and (3.8), we have

$$(3.9) \ \langle t^n - x^n, x^* - t^n \rangle \geq \lambda_n \big(f(x^n, t^n) - f(x^n, y^n) - c_1 \|y^n - x^n\|^2 - c_2 \|t^n - y^n\|^2 \big).$$

Similarly, since y^n is the unique solution to the strongly convex problem

$$\min \{ \frac{1}{2} \|y - x^n\|^2 + \lambda_n f(x^n, y) : \ y \in C \},$$

we have

$$(3.10) \qquad \lambda_n \big(f(x^n, y) - f(x^n, y^n) \big) \geq \langle y^n - x^n, y^n - y \rangle \quad \forall y \in C.$$

Substituting $y = t^n \in C$, we obtain

$$(3.11) \qquad \lambda_n \big(f(x^n, t^n) - f(x^n, y^n) \big) \geq \langle y^n - x^n, y^n - t^n \rangle.$$

From (3.9), (3.11) and

$$2 \langle t^n - x^n, x^* - t^n \rangle = \|x^n - x^*\|^2 - \|t^n - x^n\|^2 - \|t^n - x^*\|^2,$$

it implies that

$$\|x^n - x^*\|^2 - \|t^n - x^n\|^2 - \|t^n - x^*\|^2$$
$$\geq 2 \langle y^n - x^n, y^n - t^n \rangle - 2\lambda_n c_1 \|x^n - y^n\|^2 - 2\lambda_n c_2 \|t^n - y^n\|^2.$$

Hence, we have

$$\|t^n - x^*\|^2$$
$$\leq \|x^n - x^*\|^2 - \|t^n - x^n\|^2 - 2 \langle y^n - x^n, y^n - t^n \rangle + 2\lambda_n c_1 \|x^n - y^n\|^2$$

$$+ 2\lambda_n c_2 \|t^n - y^n\|^2$$
$$= \|x^n - x^*\|^2 - \|(t^n - y^n) + (y^n - x^n)\|^2 - 2\langle y^n - x^n, y^n - t^n\rangle$$
$$+ 2\lambda_n c_1 \|x^n - y^n\|^2 + 2\lambda_n c_2 \|t^n - y^n\|^2$$
$$\leq \|x^n - x^*\|^2 - \|t^n - y^n\|^2 - \|x^n - y^n\|^2 + 2\lambda_n c_1 \|x^n - y^n\|^2 + 2\lambda_n c_2 \|t^n - y^n\|^2$$
$$= \|x^n - x^*\|^2 - (1 - 2\lambda_n c_1)\|x^n - y^n\|^2 - (1 - 2\lambda_n c_2)\|y^n - t^n\|^2.$$

This completes the proof. $\qquad\square$

Lemma 3.2. *Suppose that assumptions* (i)-(v) *hold and T is nonexpansive on C, for each $x \in C$, $f(x, \cdot)$ is strongly convex with constant $\delta > 0$ on C. Then the sequences $\{x^n\}, \{y^n\}$ and $\{t^n\}$ generated by Algorithm 2.2 satisfy*

$$\|x^{n+1} - x^*\|^2 \leq \alpha_n \|u - x^*\|^2 + \|x^n - x^*\|^2 - (1 - 2\lambda_n c_1)\gamma_n \|x^n - y^n\|^2$$
(3.12)
$$- (1 - 2\lambda_n c_2)\gamma_n \|t^n - y^n\|^2.$$

Consequently,

$$\lim_{n\to\infty} \|x^{n+1} - x^n\| = \lim_{n\to\infty} \|t^n - y^n\| = 0,$$

provided $\lim_{n\to\infty} \|x^n - y^n\| = 0$.

Proof. For each n, it follows from (3.4) that

$$\langle t^n - x^n, t - t^n\rangle \geq \lambda_n \langle w, t^n - t\rangle \quad \forall w \in \partial_2 f(y^n, t^n), t \in C.$$

With $t = y^n \in C$, we have

$$\langle t^n - x^n, y^n - t^n\rangle \geq \lambda_n \langle w, t^n - y^n\rangle \quad \forall w \in \partial_2 f(y^n, t^n).$$

Combining $f(x, x) = 0$ for all $x \in C$, the last inequality and the definition of w,

$$f(y^n, t) - f(y^n, t^n) \geq \langle w, t - t^n\rangle \quad \forall t \in C,$$

we have

$$\langle t^n - x^n, y^n - t^n\rangle \geq -\lambda_n \langle w, y^n - t^n\rangle$$
$$\geq \lambda_n \big(f(y^n, t^n) - f(y^n, y^n)\big)$$
(3.13)
$$= \lambda_n f(y^n, t^n).$$

Substituting $y = t^n \in C$ into (3.10), we get

(3.14)
$$\langle y^n - x^n, t^n - y^n\rangle \geq \lambda_n \big(f(x^n, y^n) - f(x^n, t^n)\big).$$

Adding two inequalities (3.13) and (3.14), we obtain

$$\langle t^n - y^n, y^n - x^n - t^n + x^n\rangle \geq \lambda_n \big(f(x^n, y^n) + f(y^n, t^n) - f(x^n, t^n)\big).$$

Then, since f is Lipschitz-type continuous on C, we have

$$-\|t^n - y^n\|^2 \geq \lambda_n \big(-c_1 \|x^n - y^n\|^2 - c_2 \|y^n - t^n\|^2\big),$$

which follows that

(3.15)
$$(1 - \lambda_n c_2)\|t^n - y^n\|^2 \leq \lambda_n c_1 \|x^n - y^n\|^2.$$

So, we get

$$(3.16) \qquad \|t^n - y^n\|^2 \leq \frac{\lambda_n c_1}{1 - \lambda_n c_2} \|x^n - y^n\|^2.$$

For each $x^* \in Sol(f, C) \cap Fix(T)$, from Lemma 3.1, it implies that

$$
\begin{aligned}
\|x^{n+1} - x^*\|^2 &= \|\alpha_n u + \beta_n x^n + \gamma_n T(t^n) - x^*\|^2 \\
&= \|\alpha_n(u - x^*) + \beta_n(x^n - x^*) + \gamma_n(T(t^n) - x^*)\|^2 \\
&\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x^n - x^*\|^2 + \gamma_n \|T(t^n) - T(x^*)\|^2 \\
&\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x^n - x^*\|^2 + \gamma_n \|t^n - x^*\|^2 \\
&\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x^n - x^*\|^2 + \gamma_n \|x^n - x^*\|^2 \\
&= \alpha_n \|u - x^*\|^2 + (1 - \alpha_n) \|x^n - x^*\|^2 \\
&\leq \max\{\|u - x^*\|^2, \|u - x^*\|^2\}.
\end{aligned}
$$

(3.17)

Therefore $\{x^n\}$ is bounded and it follows from Lemma 3.1 that $\{x^n\}$, $\{t^n\}$, $\{y^n\}$ are bounded. Since $f(x, \cdot)$ is δ-strongly convex on C for all $x \in C$, we have

$$f(y^n, t^{n+1}) - f(y^n, t^n) \geq \langle w, t^{n+1} - t^n \rangle + \frac{\delta}{2} \|t^{n+1} - t^n\|^2,$$

where $w \in \partial_2 f(y^n, t^n)$. Substituting $t = t^{n+1}$ into (3.4), then we have

$$
\begin{aligned}
\langle t^n - x^n, t^{n+1} - t^n \rangle &\geq \lambda_n \langle w, t^n - t^{n+1} \rangle \\
&\geq \lambda_n \big(f(y^n, t^n) - f(y^n, t^{n+1})\big) + \frac{\lambda_n \delta}{2} \|t^{n+1} - t^n\|^2.
\end{aligned}
$$

(3.18)

Similarly, we also have

$$
\begin{aligned}
\langle t^{n+1} &- x^{n+1}, t^n - t^{n+1} \rangle \\
&\geq \lambda_{n+1} \big(f(y^{n+1}, t^{n+1}) - f(y^{n+1}, t^n)\big) + \frac{\lambda_{n+1} \delta}{2} \|t^{n+1} - t^n\|^2.
\end{aligned}
$$

(3.19)

Adding (3.18) and (3.19), we get

$$
\begin{aligned}
\langle t^{n+1} &- t^n, t^n - x^n - t^{n+1} + x^{n+1} \rangle \\
&\geq \lambda_n \big(f(y^n, t^n) - f(y^n, t^{n+1})\big) + \frac{\lambda_n \delta}{2} \|t^{n+1} - t^n\|^2 \\
&\quad + \lambda_{n+1} \big(f(y^{n+1}, t^{n+1}) - f(y^{n+1}, t^n)\big) + \frac{\lambda_{n+1} \delta}{2} \|t^{n+1} - t^n\|^2.
\end{aligned}
$$

From monotonicity and Lipschitz-type continuity of f and $\langle x, y \rangle \leq \frac{1}{2}(\|x\|^2 + \|y^2\|)$ for all $x, y \in H$, it implies that

$$
\begin{aligned}
\frac{1}{2}&(\|t^{n+1} - t^n\|^2 - \|x^{n+1} - x^n\|^2) \\
&\leq \|t^{n+1} - t^n\|^2 - \langle t^{n+1} - t^n, x^{n+1} - x^n \rangle \\
&= -\langle t^{n+1} - t^n, t^n - x^n - t^{n+1} + x^{n+1} \rangle
\end{aligned}
$$

$$\leq \lambda_n \big(f(y^n, t^{n+1}) - f(y^n, t^n) \big) - \frac{\lambda_n \delta}{2} \| t^{n+1} - t^n \|^2$$

$$+ \lambda_{n+1} \big(f(y^{n+1}, t^n) - f(y^{n+1}, t^{n+1}) \big) - \frac{\lambda_{n+1} \delta}{2} \| t^{n+1} - t^n \|^2$$

$$\leq \lambda_n \big(f(t^n, t^{n+1}) + c_1 \| y^n - t^n \|^2 + (c_2 - \frac{\delta}{2}) \| t^{n+1} - t^n \|^2 \big)$$

$$+ \lambda_{n+1} \big(f(t^{n+1}, t^n) + c_1 \| y^{n+1} - t^{n+1} \|^2 + (c_2 - \frac{\delta}{2}) \| t^{n+1} - t^n \|^2 \big)$$

$$\leq (\lambda_n - \lambda_{n+1}) f(t^n, t^{n+1}) + \big((\lambda_n + \lambda_{n+1}) c_2 - \delta \big) \| t^{n+1} - t^n \|^2 + c_1 \lambda_n \| y^n - t^n \|^2$$

$$+ c_1 \lambda_{n+1} \| y^{n+1} - t^{n+1} \|^2.$$

Then we have

$$m_n \| t^{n+1} - t^n \|^2 \leq \| x^{n+1} - x^n \|^2 + 2(\lambda_n - \lambda_{n+1}) f(t^n, t^{n+1}) + 2 c_1 \lambda_n \| y^n - t^n \|^2$$

$$(3.20) \qquad\qquad + 2 c_1 \lambda_{n+1} \| y^{n+1} - t^{n+1} \|^2,$$

where $m_n = 1 + 2\delta - 2(\lambda_n + \lambda_{n+1}) c_2$. It follows from $\lambda \leq \frac{2\delta - 1}{4 c_2}$ that there exists n_0 such that $m_n > 0$ for all $n \geq n_0$ and we have

$$\frac{\gamma_{n+1}^2}{(1 - \beta_{n+1})^2} \| t^{n+1} - t^n \|^2 - \frac{1}{2} \| x^{n+1} - x^n \|^2$$

$$\leq M_n \| x^{n+1} - x^n \|^2 + \frac{2 \gamma_{n+1}^2 (\lambda_n - \lambda_{n+1})}{m_n (1 - \beta_{n+1})^2} f(t^n, t^{n+1})$$

$$(3.21) \qquad + \frac{2 c_1 \lambda_n \gamma_{n+1}^2}{(1 - \beta_{n+1})^2} \| y^n - t^n \|^2 + \frac{2 c_1 \lambda_{n+1} \gamma_{n+1}^2}{(1 - \beta_{n+1})^2} \| y^{n+1} - t^{n+1} \|^2,$$

where

$$M_n = \frac{\gamma_{n+1}^2}{m_n (1 - \beta_{n+1})^2} - \frac{1}{2}.$$

From (iv), (3.21), Lemma 3.1, (3.17), (3.16), $\lim_{n \to \infty} \| x^n - y^n \| = 0$ and

$$\lim_{n \to \infty} M_n = \frac{1 - 2\delta + 4\lambda c_2}{2(1 + 2\delta - 4\lambda c_2)} \leq 0,$$

we have

$$(3.22) \qquad \lim_{n \to \infty} \Big(\frac{\gamma_{n+1}^2}{(1 - \beta_{n+1})^2} \| t^{n+1} - t^n \|^2 - \frac{1}{2} \| x^{n+1} - x^n \|^2 \Big) \leq 0.$$

Set $x^{n+1} = (1 - \beta_n) z^n + \beta_n x^n$. Then, we obtain

$$z^{n+1} - z^n = \frac{\alpha_{n+1} u + \gamma_{n+1} T(t^{n+1})}{1 - \beta_{n+1}} - \frac{\alpha_n u + \gamma_n T(t^n)}{1 - \beta_n}$$

$$= \Big(\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \Big) u + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \big(T(t^{n+1}) - T(t^n) \big)$$

$$(3.23) \qquad + \Big(\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n} \Big) T(t^n).$$

Hence, we have

$$\frac{1}{2}(\|z^{n+1} - z^n\|^2 - \|x^{n+1} - x^n\|^2)$$

$$\leq |\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}|^2 \|u\|^2 + \left(\frac{\gamma_{n+1}}{1 - \beta_{n+1}}\right)^2 \|t^{n+1} - t^n\|^2$$

$$+ |\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n}|^2 \|T(t^n)\|^2 - \frac{1}{2}\|x^{n+1} - x^n\|^2,$$

Combining this, (3.23), and boundedness of the sequences $\{x^n\}, \{y^n\}, \{t^n\}$ and $\{T(t^n)\}$, we obtain

(3.24) $$\limsup_{n \to \infty}(\|z^{n+1} - z^n\| - \|x^{n+1} - x^n\|) \leq 0.$$

Hence by Lemma 2.5, we obtain $\lim_{n \to \infty} \|z^n - x^n\| = 0$. Consequently,

$$\lim_{n \to \infty} \|x^{n+1} - x^n\| = \lim_{n \to \infty} (1 - \beta_n)\|z^n - x^n\| = 0.$$

It follows from (3.20) that $\lim_{n \to \infty} \|t^{n+1} - t^n\| = 0$. By (3.17) and Lemma 3.1, we have

$$\|x^{n+1} - x^*\|^2$$

$$\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x^n - x^*\|^2 + \gamma_n \|x^n - x^*\|^2 - (1 - 2\lambda_n c_2)\gamma_n \|t^n - y^n\|^2$$

$$- (1 - 2\lambda_n c_1)\gamma_n \|x^n - y^n\|^2$$

$$\leq \alpha_n \|u - x^*\|^2 + \|x^n - x^*\|^2 - (1 - 2\lambda_n c_2)\gamma_n \|t^n - y^n\|^2$$

$$- (1 - 2\lambda_n c_1)\gamma_n \|x^n - y^n\|^2.$$

This implies (3.12) and

(3.25) $$(1 - 2\lambda_n c_2)\gamma_n \|t^n - y^n\|^2$$

$$\leq \alpha_n \|u - x^*\|^2 + \|x^n - x^*\|^2 - \|x^{n+1} - x^*\|^2$$

$$= \alpha_n \|u - x^*\|^2 + (\|x^n - x^*\| - \|x^{n+1} - x^*\|)(\|x^n - x^*\| + \|x^{n+1} - x^*\|)$$

$$\leq \alpha_n \|u - x^*\|^2 + (\|x^n - x^{n+1}\|)(\|x^n - x^*\| + \|x^{n+1} - x^*\|).$$

From $\lim_{n \to \infty} \alpha_n = 0$ and (3.25), it follows

$$\lim_{n \to \infty} \|t^n - y^n\| = 0.$$ $\square$

Theorem 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Suppose that assumptions* (i)-(v) *hold, for each $x \in C$, $f(x, \cdot)$ is strongly convex with constant $\delta > 0$ on C and T is nonexpansive on C. Then the sequences $\{x^n\}, \{y^n\}$ and $\{t^n\}$ generated by Algorithm 2.2 converge strongly to the same point $\bar{x}$ provided $\lim_{n \to \infty} \|x^n - y^n\| = 0$, where*

$$\bar{x} = Pr_{Sol(f,C) \cap Fix(T)}(u).$$

Proof. It follows from Lemma 3.1 that
$$\|t^n - x^*\| \leq \|x^n - x^*\| \quad \forall n \geq 0,$$
and hence, we have
$$
\begin{aligned}
\|T(x^n) - x^n\| \leq & \|T(x^n) - T(t^n)\| + \|T(t^n) - x^{n+1}\| + \|x^{n+1} - x^n\| \\
\leq & \|x^n - t^n\| + \|T(t^n) - x^{n+1}\| + \|x^{n+1} - x^n\| \\
\leq & \|x^n - t^n\| + \alpha_n\|T(t^n) - u\| + \beta_n\|T(t^n) - x^n\| + \|x^{n+1} - x^n\| \\
\leq & \|x^n - t^n\| + \alpha_n\|T(t^n) - u\| + \beta_n\|T(t^n) - T(x^n)\| \\
& + \beta_n\|T(x^n) - x^n\| + \|x^{n+1} - x^n\| \\
\leq & \|x^n - t^n\| + \alpha_n\|T(t^n) - u\| + \beta_n\|t^n - x^n\| + \beta_n\|T(x^n) - x^n\| \\
& + \|x^{n+1} - x^n\|.
\end{aligned}
$$
Consequently, from Lemma 3.2 and $\lim_{n\to\infty} \beta_n \in (0,1)$, it follows that
$$
(3.26) \qquad \lim_{n\to\infty} \|T(x^n) - x^n\| = 0.
$$
Then, we also have
$$
\begin{aligned}
\|T(t^n) - t^n\| \leq & \|T(t^n) - T(x^n)\| + \|T(x^n) - x^n\| + \|x^n - t^n\| \\
\leq & \|t^n - x^n\| + \|T(x^n) - x^n\| + \|x^n - t^n\| \\
(3.27) \qquad & \to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$
Since $\{x^n\}$ is bounded, there exists a subsequence $\{x^{n_j}\}$ of $\{x^n\}$ so that
$$
(3.28) \qquad \limsup_{n\to\infty}\langle u - x^*, x^n - x^*\rangle = \lim_{j\to\infty}\langle u - x^*, x^{n_j} - x^*\rangle,
$$
where $x^* := Pr_{Sol(f,C)\cap Fix(T)}(u)$. Without loss of generality, we may further assume that $\{x^{n_j}\}$ converges weakly to $\bar{x} \in H$. Hence, (3.28) reduces to
$$
(3.29) \qquad \limsup_{n\to\infty}\langle u - x^*, x^n - x^*\rangle = \langle u - x^*, \bar{x} - x^*\rangle.
$$
From Lemma 2.4, (3.26) and $x^{n_j} \rightharpoonup \bar{x}$ as $j \to \infty$, it follows
$$
(3.30) \qquad T(\bar{x}) = \bar{x}.
$$
In fact, assume that $\bar{x} \notin Fix(T)$. From Opial's condition in [13], we have
$$
\begin{aligned}
\liminf_{j\to\infty}\|t^{n_j} - \bar{x}\| < & \liminf_{j\to\infty}\|t^{n_j} - T(\bar{x})\| \\
\leq & \liminf_{j\to\infty}(\|t^{n_j} - T(t^{n_j})\| + \|T(t^{n_j}) - T(\bar{x})\|) \\
= & \liminf_{j\to\infty}\|T(t^{n_j}) - T(\bar{x})\| \\
\leq & \liminf_{j\to\infty}\|t^{n_j} - \bar{x}\|.
\end{aligned}
$$
This is a contradiction. Thus, $\bar{x} = T(\bar{x})$.

From Lemma 3.2 and $x^{n_j} \rightharpoonup \bar{x}$ as $j \to \infty$, it follows
$$
y^{n_j} \rightharpoonup \bar{x}, t^{n_j} \rightharpoonup \bar{x} \quad \text{as } j \to \infty.
$$

Then, from (3.10), $\lim_{n\to\infty} \lambda_n = \lambda \in (0,1)$ and assumptions of f, it follows

$$\lambda_{n_j}\big(f(x^{n_j}, y) - f(x^{n_j}, y^{n_j})\big) \geq \langle y^{n_j} - x^{n_j}, y^{n_j} - y\rangle \quad \forall y \in C,$$

and when $j \to \infty$, we have $f(\bar{x}, y) \geq 0$ for all $y \in C$. It means that $\bar{x} \in Sol(f, C)$. Combining this and (3.30), we have

$$\bar{x} \in Sol(f, C) \cap Fix(T).$$

Then, it is easy to see that

$$\langle u - x^*, \bar{x} - x^*\rangle \leq 0.$$

Thus, combining with (3.29), we have

(3.31)
$$\limsup_{n\to\infty} \langle u - x^*, x^n - x^*\rangle \leq 0.$$

Now, with $x^* = Pr_{Sol(f,C)\cap Fix(T)}(u)$, from $\|t^n - x^*\| \leq \|x^n - x^*\|$, Lemma 3.1 and

$$\langle x, y\rangle \leq \frac{1}{2}(\|x\|^2 + \|y\|^2) \quad \forall x, y \in H,$$

it implies

$$\|x^{n+1} - x^*\|^2$$
$$= \langle \alpha_n u + \beta_n x^n + \gamma_n T(t^n) - x^*, x^{n+1} - x^*\rangle$$
$$= \alpha_n\langle u - x^*, x^{n+1} - x^*\rangle + \beta_n\langle x^n - x^*, x^{n+1} - x^*\rangle$$
$$\quad + \gamma_n\langle T(t^n) - x^*, x^{n+1} - x^*\rangle$$
$$\leq \alpha_n\langle u - x^*, x^{n+1} - x^*\rangle + \frac{\beta_n}{2}(\|x^n - x^*\|^2 + \|x^{n+1} - x^*\|^2)$$
$$\quad + \frac{\gamma_n}{2}(\|T(t^n) - x^*\|^2 + \|x^{n+1} - x^*\|^2)$$
$$\leq \alpha_n\langle u - x^*, x^{n+1} - x^*\rangle + \frac{\beta_n}{2}(\|x^n - x^*\|^2 + \|x^{n+1} - x^*\|^2)$$
$$\quad + \frac{\gamma_n}{2}(\|t^n - x^*\|^2 + \|x^{n+1} - x^*\|^2)$$
$$\leq \alpha_n\langle u - x^*, x^{n+1} - x^*\rangle + \frac{1}{2}(1 - \alpha_n)(\|x^n - x^*\|^2 + \|x^{n+1} - x^*\|^2),$$
$$\leq \alpha_n\langle u - x^*, x^{n+1} - x^*\rangle + (1 - \alpha_n)\|x^n - x^*\|^2.$$

This implies that

$$\|x^{n+1} - x^*\|^2 \leq (1 - \alpha_n)\|x^n - x^*\|^2 + \alpha_n\beta_n,$$

where $\beta_n := \langle x^{n+1} - x^*, u - x^*\rangle$. Then, from an application of Lemma 2.6 and (3.31), it yields that $\lim_{n\to\infty} \|x^n - x^*\| = 0$. From Lemma 3.1, it follows $\lim_{n\to\infty} \|y^n - x^*\| = 0$ and $\lim_{n\to\infty} \|t^n - x^*\| = 0$. $\qquad\square$

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