

STRONG CONVERGENCE OF APPROXIMATING FIXED POINT SEQUENCES FOR NONEXPANSIVE MAPPINGS

HONG-KUN XU

Consider a nonexpansive self-mapping T of a bounded closed convex subset of a Banach space. Banach's contraction principle guarantees the existence of approximating fixed point sequences for T . However such sequences may not be strongly convergent, in general, even in a Hilbert space. It is shown in this paper that in a real smooth and uniformly convex Banach space, appropriately constructed approximating fixed point sequences can be strongly convergent.

1. INTRODUCTION

Let X be a real Banach space and C be a closed convex subset of X . Let $T : C \rightarrow C$ be a self-mapping of C . Recall that T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. We use $\text{Fix}(T)$ to denote the set of fixed points of T (that is, $\text{Fix}(T) = \{x \in C : Tx = x\}$). Throughout this article, we assume that $\text{Fix}(T)$ is nonempty.

Recall also that a sequence $\{x_n\}$ in C is said to be an approximating fixed point sequence for T if

$$\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0.$$

There are several ways to construct an approximating fixed point sequence for a nonexpansive mapping T . We mention two below.

Firstly we can use Banach's contraction principle to obtain a sequence $\{x_n\}$ in C such that

$$x_n = t_n x_0 + (1 - t_n)Tx_n, \quad n \geq 1$$

where the initial guess x_0 is taken arbitrarily in C and $\{t_n\}$ is a sequence in the interval $(0, 1)$ such that $t_n \rightarrow 0$ as $n \rightarrow \infty$. Due to the assumption that $\text{Fix}(T) \neq \emptyset$, this sequence $\{x_n\}$ is bounded (indeed $\|x_n - p\| \leq \|x_0 - p\|$ for all $p \in \text{Fix}(T)$). Hence

$$\|x_n - Tx_n\| = t_n \|x_0 - Tx_n\| \rightarrow 0$$

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and $\{x_n\}$ is an approximating fixed point sequence for T .

Secondly, we use Mann’s iteration process [8] to generate a sequence $\{x_n\}$ in C by the recursive formula

$$(1.1) \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n, \quad n \geq 0$$

where the initial guess $x_0 \in C$ is arbitrary, and the sequence $\{\alpha_n\}$ lies in the interval $(0, 1)$. This sequence $\{x_n\}$ is bounded since, for any $p \in \text{Fix}(T)$, we have

$$\|x_{n+1} - p\| \leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|Tx_n - p\| \leq \|x_n - p\|.$$

That is, $\{\|x_n - p\|\}$ is a nonincreasing sequence. Moreover, it is not hard to find that the sequence $\{\|x_n - Tx_n\|\}$ is also nonincreasing; hence $\lim_n \|x_n - Tx_n\|$ exists.

However, it is not known whether this sequence $\{x_n\}$ is always an approximating fixed point sequence of T . Only partial answers have been obtained. Indeed, if the space X is uniformly convex and if the control sequence $\{\alpha_n\}$ satisfies the condition $\sum_{n=0}^\infty \alpha_n(1 - \alpha_n) = \infty$, then Reich [12] showed that the sequence $\{x_n\}$ generated by Mann’s iteration process (1.1) is an approximating fixed point sequence of T . For the sake of completeness, we include a brief proof to this fact. Let δ_X be the modulus of convexity of X . Pick a $p \in \text{Fix}(T)$. Assuming $\|x_n - p\| > 0$ and noticing $\|Tx_n - p\| \leq \|x_n - p\|$, we deduce that

$$\|x_{n+1} - p\| \leq \|x_n - p\| \left[1 - 2\alpha_n(1 - \alpha_n)\delta_X \left(\frac{\|x_n - Tx_n\|}{\|x_n - p\|} \right) \right].$$

Hence

$$(1.2) \quad \sum_{n=0}^\infty \alpha_n(1 - \alpha_n)\|x_n - p\|\delta_X \left(\frac{\|x_n - Tx_n\|}{\|x_n - p\|} \right) \leq \|x_0 - p\| < \infty.$$

Put $r = \lim_n \|x_n - p\|$. If $r = 0$, we are done. So assume $r > 0$. If $\sum_{n=0}^\infty \alpha_n(1 - \alpha_n) = \infty$, we obtain from (1.2) that $\lim_n \delta_X(\|x_n - Tx_n\|/r) = 0$. This implies that $\lim_n \|x_n - Tx_n\| = 0$ and $\{x_n\}$ is an approximating fixed point sequence of T .

An approximating fixed point sequence is not necessarily always weakly convergent though it is true that in a Hilbert space every weak limit point of an approximating fixed point sequence is always a fixed point of T . This fact is called the demiclosedness principle for nonexpansive mappings which indeed holds in uniformly convex Banach spaces as stated in the next lemma.

LEMMA 1.1. (See [4].) *Let X be a uniformly convex Banach space, C a closed convex subset of C , and $T : C \rightarrow C$ a nonexpansive mapping with a fixed point. Then $I - T$ is demiclosed in the sense that if $\{x_n\}$ is a sequence in C and if $x_n \rightarrow x$ weakly and $(I - T)x_n \rightarrow y$ strongly for some x and y , then $(I - T)x = y$.*

In a summary, in the setting of real uniformly convex Banach spaces X , what is clear is that every weak limit point of an approximating fixed point sequence for T is a fixed point of T . However it remains unclear if the entire approximating fixed point sequence is weakly convergent. Reich [12] proves that if, in addition, X also has a Fréchet differentiable norm and if $\{x_n\}$ is an approximating fixed point sequence generated by Mann's iteration process (1.1), then $\{x_n\}$ is weakly convergent.

In general, an approximating fixed point sequence may fail to be strongly convergent even in the Hilbert space setting [3].

It is the purpose of this note to prove that an appropriately constructed approximating fixed point sequence can be strongly convergent in a smooth and uniformly convex Banach space. For more recent investigations on strong convergence for nonexpansive and maximal monotone mappings, see [5, 6, 7, 9, 10, 11, 13, 14, 15, 17] and the references therein.

2. PROJECTIONS IN UNIFORMLY CONVEX BANACH SPACES

Let X be a real uniformly convex Banach space X . Thus, for every $\varepsilon > 0$, $\delta_X(\varepsilon) > 0$, where δ_X is the modulus of convexity of X defined by

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \frac{1}{2} \|x + y\| : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon \right\}.$$

Let C be a nonempty closed convex subset of X . Like the Hilbert space case, we can define the nearest point projection P_C from X onto C by assigning to each $x \in X$ the only point $P_C x$ in C with the property

$$\|x - P_C x\| = \inf \{ \|x - y\| : y \in C \}.$$

This projection P_C , though continuous (indeed uniformly continuous on bounded sets), is however inconvenient to use because it is not nonexpansive anymore (hence $I - P_C$ lacks monotonicity), as contrast to the nonexpansivity of nearest point projections in a Hilbert space. Instead, another kind of projections has been introduced to replace the nearest point projections, which is however still denoted by the same notation P_C . That is, in the rest of the paper, by P_C we mean the projection from X onto C introduced as follows.

Let $J : X \rightarrow X^*$ be the duality map of X defined by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad x \in X.$$

Assume X is smooth so that J is single-valued on X and hence we can define a function φ on $X \times X$ by (see [1, 5])

$$(2.1) \quad \varphi(x, y) = \|x\|^2 - 2\langle x, J(y) \rangle + \|y\|^2, \quad x, y \in X.$$

It is easily seen that

$$(\|x\| - \|y\|)^2 \leq \varphi(x, y) \leq (\|x\| + \|y\|)^2, \quad x, y \in X.$$

Since for each fixed y , $\varphi(\cdot, y)$ is a continuous strictly convex function on X , there is a unique point $z \in C$ which solves the minimisation

$$(2.2) \quad \varphi(z, y) = \min\{\varphi(x, y) : x \in C\}.$$

This unique point z in C is called the (generalised) projection of y onto C . That is, we define the projection operator $P_C : X \rightarrow C$ by setting

$$(2.3) \quad P_C y = z,$$

where z is the only point in C satisfying (2.2). (Note that if X is a Hilbert space, $\varphi(x, y) = \|x - y\|^2$. Hence the projection P_C defined in (2.3) coincides with the nearest point projection onto C in the Hilbert space setting.)

The next proposition gathers some basic properties of P_C which will be used in the proof of the main result in the next section.

PROPOSITION 2.1. *Assume that X is a smooth and uniformly convex Banach space and C is a nonempty closed convex subset of X .*

- (i) *Given sequences $\{x_n\}$ and $\{y_n\}$ in X . If one of them is bounded, then $\varphi(x_n, y_n) \rightarrow 0$ if and only if $\|x_n - y_n\| \rightarrow 0$.*
- (ii) *Given $y \in X$ and $z \in C$. Then $z = P_C y$ if and only if there holds the inequality:*

$$(2.4) \quad \langle v - z, J(z) - J(y) \rangle \geq 0 \quad \forall v \in C.$$

- (iii) *The following inequality holds:*

$$(2.5) \quad \varphi(x, P_C y) + \varphi(P_C y, y) \leq \varphi(x, y) \quad \forall x \in C, y \in X.$$

PROOF: (i) The necessity part is proved in [5] under the stronger condition that the space X be uniformly smooth. The uniform smoothness can be indeed weakened to smoothness. To see this, we notice that if $\varphi(x_n, y_n) \rightarrow 0$ and if one of the sequences $\{x_n\}$ and $\{y_n\}$ is bounded, then both $\{x_n\}$ and $\{y_n\}$ are bounded. Let $r > 0$ be such that the closed ball $B_r = \{u \in X : \|u\| \leq r\}$ contains all the points of $\{x_n\}$, $\{y_n\}$ and $\{x_n - y_n\}$. By Xu [16], we have a continuous strictly increasing function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ and satisfying the property:

$$\|u + v\|^2 \geq \|u\|^2 + 2\langle v, J(u) \rangle + g(\|v\|), \quad \forall u, v \in B_r.$$

In particular,

$$\begin{aligned}\|x_n\|^2 &= \|y_n + (x_n - y_n)\|^2 \\ &\geq \|y_n\|^2 + 2\langle x_n - y_n, J(y_n) \rangle + g(\|x_n - y_n\|) \\ &= -\|y_n\|^2 + 2\langle x_n, J(y_n) \rangle + g(\|x_n - y_n\|).\end{aligned}$$

It now follows from the definition of φ that

$$g(\|x_n - y_n\|) \leq \varphi(x_n, y_n) \rightarrow 0.$$

Therefore $\|x_n - y_n\| \rightarrow 0$.

To see the sufficiency part (true indeed in any smooth Banach space), we assume $\|x_n - y_n\| \rightarrow 0$ and thus both sequences $\{x_n\}$ and $\{y_n\}$ are bounded. That $\varphi(x_n, y_n) \rightarrow 0$ now follows from the following computations:

$$\begin{aligned}\varphi(x_n, y_n) &= \|x_n\|^2 - \|y_n\|^2 - 2\langle x_n - y_n, J(y_n) \rangle \\ &\leq \|x_n - y_n\|(\|x_n\| + 3\|y_n\|).\end{aligned}$$

(ii) Since for each fixed $y \in X$, $\varphi(\cdot, y)$ is convex, $z \in C$ is a minimiser of $\varphi(\cdot, y)$ over C if and only if there holds the optimality condition:

$$(2.6) \quad \langle \nabla \varphi(z, y), v - z \rangle \geq 0 \quad \forall v \in C$$

where $\nabla \varphi(z, y)$ is the gradient of $\varphi(\cdot, y)$ at z . Since it is easily computed that

$$\langle \nabla \varphi(z, y), v - z \rangle = 2\langle v - z, J(z) - J(y) \rangle$$

we obtain (2.4).

(iii) Using the definition of φ , we find that (2.5) is equivalent to the inequality:

$$\langle P_C y - x, J(P_C y) - J(y) \rangle \leq 0.$$

This is however the inequality (2.4) with v and z replaced by x and $P_C y$, respectively. \square

We shall use the notation:

1. \rightharpoonup for weak convergence and \rightarrow for strong convergence.
2. $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω -limit set of $\{x_n\}$.

LEMMA 2.2. *Let X be a real smooth and uniformly convex Banach space and K be a nonempty closed convex subset of X . Let $\{x_n\}$ be a bounded sequence in X and $u \in X$. Let $q = P_K u$. Assume that $\{x_n\}$ satisfies the conditions*

- (i) $\omega_w(x_n) \subset K$ and
- (ii) $\varphi(x_n, u) \leq \varphi(q, u)$ for all n .

Then $x_n \rightarrow q$.

PROOF: Since X is reflexive and $\{x_n\}$ is bounded, $\omega_w(x_n)$ is nonempty. Noticing the weak lower semi-continuity of $\varphi(\cdot, u)$, we derive from condition (ii) that

$$\varphi(v, u) \leq \varphi(q, u) \quad \forall v \in \omega_w(x_n).$$

However, since $\omega_w(x_n) \subset K$ and $q = P_K u$, we must have $v = q$ for all $v \in \omega_w(x_n)$. Thus $\omega_w(x_n) = \{q\}$ and $x_n \rightarrow q$.

To see $x_n \rightarrow q$, we observe that the inequality $\varphi(x_n, u) \leq \varphi(q, u)$ in condition (ii) is actually equivalent to the following one

$$\|x_n\|^2 \leq \|q\|^2 + 2\langle x_n - q, J(u) \rangle.$$

Since $x_n \rightarrow q$, it follows that

$$\limsup_n \|x_n\| \leq \|q\|.$$

This and the uniform convexity of X imply that $x_n \rightarrow q$. □

3. STRONG CONVERGENCE OF APPROXIMATING FIXED POINT SEQUENCES

Let C be a nonempty closed convex subset of a smooth and uniformly Banach space X and let $T : C \rightarrow C$ be a nonexpansive mapping with a fixed point. Starting an arbitrary initial guess x_0 , we can construct an approximating fixed point sequence of T as follows. Take a sequence $\{t_n\}$ in $(0,1)$ so that $t_n \rightarrow 0$ as $n \rightarrow \infty$. Once x_n has been constructed, we then construct two closed convex subsets C_n and Q_n such that

$$C_n = \overline{\text{co}}\{z \in C : \|z - Tz\| \leq t_n \|x_n - Tx_n\|\}$$

and

$$Q_n = \left\{v \in C : \langle x_n - v, J(x_0) - J(x_n) \rangle \geq 0\right\}.$$

Then we define the $(n + 1)$ th iterate x_{n+1} to be the projection of x_0 onto the intersection of C_n and Q_n :

$$(3.1) \quad x_{n+1} = P_{C_n \cap Q_n} x_0.$$

Before discussing the convergence of the sequence $\{x_n\}$, we first use induction to verify that $\text{Fix}(T) \subset C_n \cap Q_n$ and x_{n+1} is well-defined. As a matter of fact, it is trivial that $\text{Fix}(T) \subset C_n$ for all n . It is also trivial that $\text{Fix}(T) \subset Q_0 = C$ and thus $x_1 = P_{C_0 \cap Q_0} x_0$ is well-defined. Assume now $\text{Fix}(T) \subset Q_n$ and x_{n+1} is well-defined. We need to prove that $\text{Fix}(T) \subset Q_{n+1}$ and x_{n+2} is well-defined.

Since x_{n+1} is the projection of x_0 onto $C_n \cap Q_n$, by Proposition 2.1 (ii) we have

$$\langle x_{n+1} - z, J(x_0) - J(x_{n+1}) \rangle \geq 0 \quad \forall z \in C_n \cap Q_n.$$

As $\text{Fix}(T) \subset C_n \cap Q_n$, the last inequality holds, in particular, for all $z \in \text{Fix}(T)$. This together with the definition of Q_{n+1} implies that $\text{Fix}(T) \subset Q_{n+1}$. Now as the projection of x_0 onto the nonempty closed convex subset $C_{n+1} \cap Q_{n+1}$, x_{n+2} is well-defined.

We now state and prove the main result of this paper.

THEOREM 3.1. *Let X be a real smooth and uniformly convex Banach space, C a nonempty closed convex subset of X , and $T : C \rightarrow C$ a nonexpansive mapping such that $\text{Fix}(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by the process (3.1). Then $\{x_n\}$ is an approximating fixed point sequence for T and strongly convergent to a fixed point of T .*

PROOF: First we observe that $\{x_n\}$ is bounded. As a matter of fact, by the definition of Q_n , we have $x_n = P_{Q_n}x_0$. Hence by Proposition 2.1 (iii)

$$(3.2) \quad \varphi(y, x_n) + \varphi(x_n, x_0) \leq \varphi(y, x_0) \quad \forall y \in Q_n.$$

Since $\text{Fix}(T) \subset Q_n$, we get

$$(3.3) \quad \varphi(x_n, x_0) \leq \varphi(p, x_0) \quad \forall p \in \text{Fix}(T).$$

This implies the boundedness of $\{x_n\}$. Because x_{n+1} belongs to Q_n , we can substitute it for y in (3.2) to get

$$(3.4) \quad \varphi(x_{n+1}, x_n) \leq \varphi(x_{n+1}, x_0) - \varphi(x_n, x_0).$$

This implies that the real sequence $\{\varphi(x_n, x_0)\}$ is increasing (and also bounded) and thus $\lim_n \varphi(x_n, x_0)$ exists. Back to (3.4), we conclude that $\varphi(x_{n+1}, x_n) \rightarrow 0$ which implies $\|x_{n+1} - x_n\| \rightarrow 0$ by virtue of Proposition 2.1 (i).

We now claim that $\{x_n\}$ is an approximating fixed point sequence of T . Let \tilde{C} be a bounded closed convex subset of C which contains all the points x_n and Tx_n for all n and let $\eta = \text{diam}(\tilde{C})$. Since $x_{n+1} \in C_n$ and by definition of C_n , we have

$$\left\| x_{n+1} - \sum_{i=1}^l \lambda_i z_i \right\| < t_n$$

where $\lambda_i > 0$ satisfying $\sum_{i=1}^l \lambda_i = 1$ and each $z_i \in C$ satisfies

$$\|z_i - Tz_i\| \leq t_n \|x_n - Tx_n\| \leq \eta t_n.$$

By Bruck [2], there exists a continuous strictly increasing function γ (depending only on η) with $\gamma(0) = 0$ and such that

$$\gamma \left(\left\| T \left(\sum_{i=1}^m \mu_i v_i \right) - \sum_{i=1}^m \mu_i T v_i \right\| \right) \leq \max(\|v_i - v_j\| - \|T v_i - T v_j\| : 1 \leq i, j \leq m)$$

for all integers $m > 1$, all points $\{v_i\}$ in \tilde{C} , and all nonnegative numbers $\{\mu_i\}$ such that $\sum_{i=1}^m \mu_i = 1$. It follows that

$$\begin{aligned} \|x_{n+1} - Tx_{n+1}\| &\leq \left\| x_{n+1} - \sum_{i=1}^l \lambda_i z_i \right\| + \left\| \sum_{i=1}^l \lambda_i (z_i - Tz_i) \right\| \\ &\quad + \left\| \sum_{i=1}^l \lambda_i Tz_i - T \left(\sum_{i=1}^l \lambda_i z_i \right) \right\| + \left\| T \left(\sum_{i=1}^l \lambda_i z_i \right) - Tx_{n+1} \right\| \\ &\leq (2 + \eta)t_n + \gamma^{-1} \left(\max(\|z_i - z_j\| - \|Tz_i - Tz_j\| : 1 \leq i, j \leq l) \right) \\ &\leq (2 + \eta)t_n + \gamma^{-1} \left(\max(\|z_i - Tz_i\| + \|z_j - Tz_j\| : 1 \leq i, j \leq l) \right) \\ &\leq (2 + \eta)t_n + \gamma^{-1}(2\eta t_n) \rightarrow 0. \end{aligned}$$

Therefore, $\{x_n\}$ is an approximating fixed point sequence.

Finally let us prove that $\{x_n\}$ is strongly convergent to a fixed point of T . By the demiclosedness principle (Lemma 1.1), we have $\omega_w(x_n) \subset \text{Fix}(T)$. Let $q = P_{\text{Fix}(T)}x_0$. By (3.3) we see that $\varphi(x_n, x_0) \leq \varphi(q, x_0)$ for all n . Therefore, applying Lemma 2.2 to the nonempty closed convex subset $K := \text{Fix}(T)$, we conclude that $x_n \rightarrow q$. \square

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School of Mathematical Sciences
University of KwaZulu-Natal
Westville Campus
Private Bag X54001
Durban 4000
South Africa
e-mail: xuhk@ukzn.ac.za