STRONG CONVERGENCE THEOREMS FOR INFINITE FAMILIES OF NONEXPANSIVE MAPPINGS IN GENERAL BANACH SPACES

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In 1979, Ishikawa proved a strong convergence theorem for finite families of nonexpansive mappings in general Banach spaces. Motivated by Ishikawa's result, we prove strong convergence theorems for infinite families of nonexpansive mappings.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. For an arbitrary set A, we also denote by $\sharp A$ the cardinal number of A.

Let C be a closed convex subset of a Banach space E. Let T be a nonexpansive mapping on C, that is,

$$\|Tx - Ty\| \leq \|x - y\| \tag{1.1}$$

for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. We know $F(T) \neq \varnothing$ in the case that E is uniformly convex and C is bounded; see Browder [2], Göhde [9], and Kirk [13]. Common fixed point theorems for families of nonexpansive mappings are proved in [2, 4, 5], and other references.

Many convergence theorems for nonexpansive mappings and families of nonexpansive mappings have been studied; see [1, 3, 6, 7, 10, 11, 12, 14, 15, 17, 18, 19, 20, 21] and others. For example, in 1979, Ishikawa proved the following theorem.

Theorem 1.1 [12]. *Let C be a compact convex subset of a Banach space E. Let $\{T_1, T_2, \ldots, T_k\}$ be a finite family of commuting nonexpansive mappings on C. Let $\{\beta_i\}_{i=1}^k$ be a finite sequence in $(0,1)$ and put $S_i x = \beta_i T_i x + (1-\beta_i)x$ for $x \in C$ and $i = 1, 2, \ldots, k$. Let $x_1 \in C$ and define a sequence $\{x_n\}$ in C by*

$$x_{n+1} = \left[\prod_{n_{k-1}=1}^{n}\left[S_k \prod_{n_{k-2}=1}^{n_{k-1}}\left[S_{k-1}\cdots\left[S_3 \prod_{n_1=1}^{n_2}\left[S_2 \prod_{n_0=1}^{n_1} S_1\right]\right]\cdots\right]\right]\right] x_1 \tag{1.2}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, \ldots, T_k\}$.

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The author thinks this theorem is one of the most interesting convergence theorems for families of nonexpansive mappings. In the case of $k = 4$, this iteration scheme is as follows:

$$\begin{aligned}
x_2 &= S_4S_3S_2S_1x_1,\\
x_3 &= S_4S_3S_2S_1S_1S_2S_1S_3S_2S_1x_2,\\
x_4 &= S_4S_3S_2S_1S_1S_1S_2S_1S_1S_2S_1S_3S_2S_1S_1S_2S_1S_3S_2S_1x_3,\\
x_5 &= S_4S_3S_2S_1S_1S_1S_1S_2S_1S_1S_1S_2S_1S_1S_2S_1S_3S_2S_1S_1\\
&\quad S_1S_2S_1S_1S_2S_1S_3S_2S_1S_1S_2S_1S_3S_2S_1x_4,\\
x_6 &= S_4S_3S_2S_1S_1S_1S_1S_1S_2S_1S_1S_1S_1S_2S_1S_1S_1S_2S_1S_1\\
&\quad S_2S_1S_3S_2S_1S_1S_1S_1S_2S_1S_1S_1S_2S_1S_1S_2S_1S_3S_2S_1\\
&\quad S_1S_1S_2S_1S_1S_2S_1S_3S_2S_1S_1S_2S_1S_3S_2S_1x_5,\\
x_7 &= S_4S_3S_2S_1S_1S_1S_1S_1S_1S_2S_1S_1S_1S_1S_1S_2S_1S_1S_1S_1\\
&\quad S_2S_1S_1S_1S_2S_1S_1S_2S_1S_3S_2S_1S_1S_1S_1S_1S_2S_1S_1S_1\\
&\quad S_1S_2S_1S_1S_1S_2S_1S_1S_2S_1S_3S_2S_1S_1S_1S_1S_2S_1S_1S_1\\
&\quad S_2S_1S_1S_2S_1S_3S_2S_1S_1S_1S_2S_1S_1S_2S_1S_3S_2S_1S_1S_2\\
&\quad S_1S_3S_2S_1x_6.
\end{aligned}\tag{1.3}$$

We remark that $S_iS_j = S_jS_i$ does not hold in general.

Very recently, in 2002, the following theorem was proved in [19].

Theorem 1.2 [19]. *Let C be a compact convex subset of a Banach space E and let S and T be nonexpansive mappings on C with $ST = TS$. Let $x_1 \in C$ and define a sequence $\{x_n\}$ in C by*

$$x_{n+1} = \frac{\alpha_n}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} S^iT^jx_n + (1-\alpha_n)x_n \tag{1.4}$$

for $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $0 < \liminf_n \alpha_n \le \limsup_n \alpha_n < 1$. Then $\{x_n\}$ converges strongly to a common fixed point z_0 of S and T.

This theorem is simpler than Theorem 1.1. However, this is not a convergence theorem for infinite families of nonexpansive mappings.

Under the assumption of the strict convexity of the Banach space, convergence theorems for infinite families of nonexpansive mappings were proved. In 1972, Linhart [15] proved the following; see also [20].

Theorem 1.3 [15]. *Let C be a compact convex subset of a strictly convex Banach space E. Let $\{T_n : n \in \mathbb{N}\}$ be an infinite family of commuting nonexpansive mappings on C. Let $\{\beta_n\}$ be a sequence in $(0,1)$. Put $S_ix = \beta_iT_ix + (1-\beta_i)x$ for $i \in \mathbb{N}$ and $x \in C$. Let f be a mapping on $\mathbb{N}$ satisfying $\sharp(f^{-1}(i)) = \infty$ for all $i \in \mathbb{N}$. Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and*

$$x_{n+1} = S_{f(n)} \circ S_{f(n-1)} \circ \cdots \circ S_{f(1)}x_1 \tag{1.5}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_n : n \in \mathbb{N}\}$.

The following mapping f on $\mathbb{N}$ satisfies the assumption in Theorem 1.3: if $n \in \mathbb{N}$ satisfies

$$\sum_{j=1}^{k-1} j < n \leq \sum_{j=1}^{k} j \tag{1.6}$$

for some $k \in \mathbb{N}$, then put

$$f(n) = n - \sum_{j=1}^{k-1} j. \tag{1.7}$$

That is,

$$\begin{array}{lllll} f(1) = 1, & & & & \\ f(2) = 1, & f(3) = 2, & & & \\ f(4) = 1, & f(5) = 2, & f(6) = 3, & & \\ f(7) = 1, & f(8) = 2, & f(9) = 3, & f(10) = 4, & \\ f(11) = 1, & f(12) = 2, & f(13) = 3, & f(14) = 4, & f(15) = 5, \\ f(16) = 1, & f(17) = 2, & \dots. & & \end{array} \tag{1.8}$$

It is a natural problem whether or not there exists an iteration to find a common fixed point for infinite families of commuting nonexpansive mappings without assuming the strict convexity of the Banach space. This problem has not been solved for twenty-five years. In this paper, we give such iteration. That is, our answer of this problem is positive.

2. Lemmas

In this section, we prove some lemmas. The following lemma is connected with Krasnosel'skiĭ and Mann's type sequences [14, 16]. This is a generalization of [19, Lemma 1]. See also [8, 20].

Lemma 2.1. *Let* $\{z_n\}$ *and* $\{w_n\}$ *be sequences in a Banach space E and let* $\{\alpha_n\}$ *be a sequence in* $[0,1]$ *with* $\limsup_n \alpha_n < 1$. *Put*

$$d = \limsup_{n\to\infty} \|w_n - z_n\| \quad \text{or} \quad d = \liminf_{n\to\infty} \|w_n - z_n\|. \tag{2.1}$$

Suppose that $z_{n+1} = \alpha_n w_n + (1-\alpha_n) z_n$ *for all* $n \in \mathbb{N}$,

$$\limsup_{n\to\infty} \left(\|w_{n+1} - w_n\| - \|z_{n+1} - z_n\|\right) \leq 0, \tag{2.2}$$

and $d < \infty$. *Then*

$$\liminf_{n\to\infty} \left| \|w_{n+k} - z_n\| - (1 + \alpha_n + \alpha_{n+1} + \cdots + \alpha_{n+k-1}) d \right| = 0 \tag{2.3}$$

hold for all $k \in \mathbb{N}$.

Proof. Since

$$\begin{aligned} &\|w_{n+1}-z_{n+1}\|-\|w_n-z_n\| \\ &\quad\le \|w_{n+1}-w_n\|+\|w_n-z_{n+1}\|-\|w_n-z_n\| \\ &\quad= \|w_{n+1}-w_n\|-\|z_{n+1}-z_n\|, \end{aligned} \tag{2.4}$$

we have

$$\begin{aligned} &\limsup_{n\to\infty}\left(\|w_{n+j}-z_{n+j}\|-\|w_n-z_n\|\right) \\ &\quad= \limsup_{n\to\infty}\sum_{i=0}^{j-1}\left(\|w_{n+i+1}-z_{n+i+1}\|-\|w_{n+i}-z_{n+i}\|\right) \\ &\quad\le \limsup_{n\to\infty}\sum_{i=0}^{j-1}\left(\|w_{n+i+1}-w_{n+i}\|-\|z_{n+i+1}-z_{n+i}\|\right) \\ &\quad\le \sum_{i=0}^{j-1}\limsup_{n\to\infty}\left(\|w_{n+i+1}-w_{n+i}\|-\|z_{n+i+1}-z_{n+i}\|\right) \\ &\quad\le 0 \end{aligned} \tag{2.5}$$

for $j\in\mathbb{N}$. Put $a=(1-\limsup_n\alpha_n)/2$. We note that $0<a<1$. Fix $k,\ell\in\mathbb{N}$ and $\varepsilon>0$. Then there exists $m'\ge\ell$ such that $a\le 1-\alpha_n$, $\|w_{n+1}-w_n\|-\|z_{n+1}-z_n\|\le\varepsilon$, and $\|w_{n+j}-z_{n+j}\|-\|w_n-z_n\|\le\varepsilon/2$ for all $n\ge m'$ and $j=1,2,\dots,k$. In the case of $d=\limsup_n\|w_n-z_n\|$, we choose $m\ge m'$ satisfying

$$\|w_{m+k}-z_{m+k}\|\ge d-\frac{\varepsilon}{2} \tag{2.6}$$

and $\|w_n-z_n\|\le d+\varepsilon$ for all $n\ge m$. We note that

$$\|w_{m+j}-z_{m+j}\|\ge\|w_{m+k}-z_{m+k}\|-\frac{\varepsilon}{2}\ge d-\varepsilon \tag{2.7}$$

for $j=0,1,\dots,k-1$. In the case of $d=\liminf_n\|w_n-z_n\|$, we choose $m\ge m'$ satisfying

$$\|w_m-z_m\|\le d+\frac{\varepsilon}{2} \tag{2.8}$$

and $\|w_n-z_n\|\ge d-\varepsilon$ for all $n\ge m$. We note that

$$\|w_{m+j}-z_{m+j}\|\le\|w_m-z_m\|+\frac{\varepsilon}{2}\le d+\varepsilon \tag{2.9}$$

for $j=1,2,\dots,k$. In both cases, such m satisfies that $m\ge\ell$, $a\le 1-\alpha_n\le 1$, $\|w_{n+1}-w_n\|-\|z_{n+1}-z_n\|\le\varepsilon$ for all $n\ge m$, and

$$d-\varepsilon\le\|w_{m+j}-z_{m+j}\|\le d+\varepsilon \tag{2.10}$$

for $j=0,1,\dots,k$. We next show

$$\|w_{m+k}-z_{m+j}\|\ge\left(1+\alpha_{m+j}+\alpha_{m+j+1}+\cdots+\alpha_{m+k-1}\right)d-\frac{(k-j)(2k+1)}{a^{k-j}}\varepsilon \tag{2.11}$$

for $j = 0, 1, \dots, k-1$. Since

$$\begin{aligned} d - \varepsilon &\le \|w_{m+k} - z_{m+k}\| \\ &= \|w_{m+k} - \alpha_{m+k-1} w_{m+k-1} - (1 - \alpha_{m+k-1}) z_{m+k-1}\| \\ &\le \alpha_{m+k-1} \|w_{m+k} - w_{m+k-1}\| + (1 - \alpha_{m+k-1}) \|w_{m+k} - z_{m+k-1}\| \\ &\le \alpha_{m+k-1} \|z_{m+k} - z_{m+k-1}\| + \varepsilon + (1 - \alpha_{m+k-1}) \|w_{m+k} - z_{m+k-1}\| \\ &= \alpha_{m+k-1}^2 \|w_{m+k-1} - z_{m+k-1}\| + \varepsilon + (1 - \alpha_{m+k-1}) \|w_{m+k} - z_{m+k-1}\| \\ &\le \alpha_{m+k-1}^2 d + 2\varepsilon + (1 - \alpha_{m+k-1}) \|w_{m+k} - z_{m+k-1}\|, \end{aligned} \tag{2.12}$$

we obtain

$$\begin{aligned} \|w_{m+k} - z_{m+k-1}\| &\ge \frac{(1 - \alpha_{m+k-1}^2) d - 3\varepsilon}{1 - \alpha_{m+k-1}} \\ &\ge (1 + \alpha_{m+k-1}) d - \frac{2k+1}{a} \varepsilon. \end{aligned} \tag{2.13}$$

Hence (2.11) holds for $j = k-1$. We assume (2.11) holds for some $j \in \{1, 2, \dots, k-1\}$. Then since

$$\begin{aligned} &\left(1 + \sum_{i=j}^{k-1} \alpha_{m+i}\right) d - \frac{(k-j)(2k+1)}{a^{k-j}} \varepsilon \\ &\quad \le \|w_{m+k} - z_{m+j}\| \\ &\quad = \|w_{m+k} - \alpha_{m+j-1} w_{m+j-1} - (1 - \alpha_{m+j-1}) z_{m+j-1}\| \\ &\quad \le \alpha_{m+j-1} \|w_{m+k} - w_{m+j-1}\| + (1 - \alpha_{m+j-1}) \|w_{m+k} - z_{m+j-1}\| \\ &\quad \le \alpha_{m+j-1} \sum_{i=j-1}^{k-1} \|w_{m+i+1} - w_{m+i}\| + (1 - \alpha_{m+j-1}) \|w_{m+k} - z_{m+j-1}\| \\ &\quad \le \alpha_{m+j-1} \sum_{i=j-1}^{k-1} (\|z_{m+i+1} - z_{m+i}\| + \varepsilon) + (1 - \alpha_{m+j-1}) \|w_{m+k} - z_{m+j-1}\| \\ &\quad \le \alpha_{m+j-1} \sum_{i=j-1}^{k-1} \|z_{m+i+1} - z_{m+i}\| + k\varepsilon + (1 - \alpha_{m+j-1}) \|w_{m+k} - z_{m+j-1}\| \\ &\quad = \alpha_{m+j-1} \sum_{i=j-1}^{k-1} \alpha_{m+i} \|w_{m+i} - z_{m+i}\| + k\varepsilon + (1 - \alpha_{m+j-1}) \|w_{m+k} - z_{m+j-1}\| \\ &\quad \le \alpha_{m+j-1} \sum_{i=j-1}^{k-1} \alpha_{m+i} (d + \varepsilon) + k\varepsilon + (1 - \alpha_{m+j-1}) \|w_{m+k} - z_{m+j-1}\| \\ &\quad \le \alpha_{m+j-1} \sum_{i=j-1}^{k-1} \alpha_{m+i} d + 2k\varepsilon + (1 - \alpha_{m+j-1}) \|w_{m+k} - z_{m+j-1}\|, \end{aligned} \tag{2.14}$$

we obtain

$$
\begin{aligned}
\|w_{m+k}-z_{m+j-1}\| &\geq \frac{1+\sum_{i=j}^{k-1}\alpha_{m+i}-\alpha_{m+j-1}\sum_{i=j-1}^{k-1}\alpha_{m+i}}{1-\alpha_{m+j-1}}d-\frac{(k-j)(2k+1)/a^{k-j}+2k}{1-\alpha_{m+j-1}}\varepsilon \\
&\geq \left(1+\sum_{i=j-1}^{k-1}\alpha_{m+i}\right)d-\frac{(k-j+1)(2k+1)}{a^{k-j+1}}\varepsilon.
\end{aligned}
\tag{2.15}
$$

Hence (2.11) holds for $j := j-1$. Therefore (2.11) holds for all $j = 0,1,\dots,k-1$. Specially, we have

$$
\|w_{m+k}-z_m\| \geq (1+\alpha_m+\alpha_{m+1}+\cdots+\alpha_{m+k-1})d-\frac{k(2k+1)}{a^k}\varepsilon. \tag{2.16}
$$

On the other hand, we have

$$
\begin{aligned}
\|w_{m+k}-z_m\| &\leq \|w_{m+k}-z_{m+k}\|+\sum_{i=0}^{k-1}\|z_{m+i+1}-z_{m+i}\| \\
&= \|w_{m+k}-z_{m+k}\|+\sum_{i=0}^{k-1}\alpha_{m+i}\|w_{m+i}-z_{m+i}\| \\
&\leq d+\varepsilon+\sum_{i=0}^{k-1}\alpha_{m+i}(d+\varepsilon) \\
&\leq d+\sum_{i=0}^{k-1}\alpha_{m+i}d+(k+1)\varepsilon.
\end{aligned}
\tag{2.17}
$$

From (2.16) and (2.17), we obtain

$$
\left|\|w_{m+k}-z_m\|-(1+\alpha_m+\alpha_{m+1}+\cdots+\alpha_{m+k-1})d\right| \leq \frac{k(2k+1)}{a^k}\varepsilon. \tag{2.18}
$$

Since $\ell \in \mathbb{N}$ and $\varepsilon > 0$ are arbitrary, we obtain the desired result. □

By using Lemma 2.1, we obtain the following useful lemma, which is a generalization of [19, Lemma 2] and [20, Lemma 6].

Lemma 2.2. *Let $\{z_n\}$ and $\{w_n\}$ be bounded sequences in a Banach space E and let $\{\alpha_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_n \alpha_n \leq \limsup_n \alpha_n < 1$. Suppose that $z_{n+1} = \alpha_n w_n + (1-\alpha_n)z_n$ for all $n \in \mathbb{N}$ and*

$$
\limsup_{n\to\infty}\left(\|w_{n+1}-w_n\|-\|z_{n+1}-z_n\|\right) \leq 0. \tag{2.19}
$$

Then $\lim_n \|w_n - z_n\| = 0$.

Proof. We put $a = \liminf_n \alpha_n > 0$, $M = 2\sup\{\|z_n\| + \|w_n\| : n \in \mathbb{N}\} < \infty$, and $d = \limsup_n \|w_n - z_n\| < \infty$. We assume $d > 0$. Then fix $k \in \mathbb{N}$ with $(1+ka)d > M$. By Lemma 2.1, we have

$$\liminf_{n\to\infty} \big| \|w_{n+k} - z_n\| - (1 + \alpha_n + \alpha_{n+1} + \cdots + \alpha_{n+k-1})d \big| = 0. \tag{2.20}$$

Thus, there exists a subsequence $\{n_i\}$ of a sequence $\{n\}$ in $\mathbb{N}$ such that

$$\lim_{i\to\infty} \big(\|w_{n_i+k} - z_{n_i}\| - (1 + \alpha_{n_i} + \alpha_{n_i+1} + \cdots + \alpha_{n_i+k-1})d \big) = 0, \tag{2.21}$$

the limit of $\{\|w_{n_i+k} - z_{n_i}\|\}$ exists, and the limits of $\{\alpha_{n_i+j}\}$ exist for all $j \in \{0,1,\dots,k-1\}$. Put $\beta_j = \lim_i \alpha_{n_i+j}$ for $j \in \{0,1,\cdots,k-1\}$. It is obvious that $\beta_j \geq a$ for all $j \in \{0,1,\dots, k-1\}$. We have

$$\begin{aligned} M &< (1+ka)d \\ &\leq (1 + \beta_0 + \beta_1 + \cdots + \beta_{k-1})d \\ &= \lim_{i\to\infty} (1 + \alpha_{n_i} + \alpha_{n_i+1} + \cdots + \alpha_{n_i+k-1})d \\ &= \lim_{i\to\infty} \|w_{n_i+k} - z_{n_i}\| \\ &\leq \limsup_{n\to\infty} \|w_{n+k} - z_n\| \\ &\leq M. \end{aligned} \tag{2.22}$$

This is a contradiction. Therefore $d = 0$. □

We prove the following lemmas, which are connected with real numbers.

Lemma 2.3. *Let $\{\alpha_n\}$ be a real sequence with $\lim_n(\alpha_{n+1} - \alpha_n) = 0$. Then every $t \in \mathbb{R}$ with $\liminf_n \alpha_n < t < \limsup_n \alpha_n$ is a cluster point of $\{\alpha_n\}$.*

Proof. We assume that there exists $t \in (\liminf_n \alpha_n, \limsup_n \alpha_n)$ such that t is not a cluster point of $\{\alpha_n\}$. Then there exist $\varepsilon > 0$ and $n_1 \in \mathbb{N}$ such that

$$\begin{gathered} \liminf_{n\to\infty} \alpha_n < t - \varepsilon < t < t + \varepsilon < \limsup_{n\to\infty} \alpha_n, \\ \alpha_n \in (-\infty, t-\varepsilon] \cup [t+\varepsilon, \infty), \end{gathered} \tag{2.23}$$

for all $n \geq n_1$. We choose $n_2 \geq n_1$ such that $|\alpha_{n+1} - \alpha_n| < \varepsilon$ for all $n \geq n_2$. Then there exist $n_3, n_4 \in \mathbb{N}$ such that $n_4 \geq n_3 \geq n_2$,

$$\alpha_{n_3} \in (-\infty, t-\varepsilon], \qquad \alpha_{n_4} \in [t+\varepsilon, \infty). \tag{2.24}$$

We put

$$n_5 = \max\{n : n < n_4,\ \alpha_n \le t - \varepsilon\} \ge n_3. \tag{2.25}$$

Then we have

$$\alpha_{n_5} \le t - \varepsilon < t + \varepsilon \le \alpha_{n_5+1} \tag{2.26}$$

and hence

$$\varepsilon \le 2\varepsilon \le \alpha_{n_5+1} - \alpha_{n_5} = \left|\alpha_{n_5+1} - \alpha_{n_5}\right| < \varepsilon. \tag{2.27}$$

This is a contradiction. Therefore we obtain the desired result. □

Lemma 2.4. *For $\alpha,\beta \in (0,1/2)$ and $n \in \mathbb{N}$,*

$$\begin{gathered} \left|\alpha^n - \beta^n\right| \le |\alpha - \beta|, \\ \sum_{k=1}^{\infty} \left|\alpha^k - \beta^k\right| \le 4|\alpha - \beta| \end{gathered} \tag{2.28}$$

hold.

Proof. We assume that $n \ge 2$ because the conclusion is obvious in the case of $n = 1$. Since

$$\alpha^n - \beta^n = (\alpha - \beta)\sum_{k=0}^{n-1} \alpha^{n-1-k}\beta^k, \tag{2.29}$$

we have

$$\begin{aligned} \left|\alpha^n - \beta^n\right| &= |\alpha - \beta| \sum_{k=0}^{n-1} \alpha^{n-1-k}\beta^k \\ &\le |\alpha - \beta| \sum_{k=0}^{n-1} \frac{1}{2^{n-1}} \\ &= |\alpha - \beta| \frac{n}{2^{n-1}} \\ &\le |\alpha - \beta|. \end{aligned} \tag{2.30}$$

We also have

$$\begin{aligned} \sum_{k=1}^{\infty} \left|\alpha^k - \beta^k\right| &= \left|\sum_{k=1}^{\infty} (\alpha^k - \beta^k)\right| \\ &= \left|\frac{\alpha}{1-\alpha} - \frac{\beta}{1-\beta}\right| \\ &= \left|\frac{\alpha - \beta}{(1-\alpha)(1-\beta)}\right| \\ &\le 4|\alpha - \beta|. \end{aligned} \tag{2.31}$$

This completes the proof. □

We know the following.

Lemma 2.5. *Let C be a subset of a Banach space E and let $\{V_n\}$ be a sequence of nonexpansive mappings on C with a common fixed point $w \in C$. Let $x_1 \in C$ and define a sequence $\{x_n\}$ in C by $x_{n+1} = V_n x_n$ for $n \in \mathbb{N}$. Then $\{\|x_n - w\|\}$ is a nonincreasing sequence in $\mathbb{R}$.*

Proof. We have $\|x_{n+1} - w\| = \|V_n x_n - V_n w\| \le \|x_n - w\|$ for all $n \in \mathbb{N}$. □

3. Three nonexpansive mappings

In this section, we prove a convergence theorem for three nonexpansive mappings. The purpose for this is that we give the idea of our results.

Lemma 3.1. *Let C be a closed convex subset of a Banach space E. Let T_1 and T_2 be nonexpansive mappings on C with $T_1 \circ T_2 = T_2 \circ T_1$. Let $\{t_n\}$ be a sequence in $(0,1)$ converging to 0 and let $\{z_n\}$ be a sequence in C such that $\{z_n\}$ converges strongly to some $w \in C$ and*

$$\lim_{n\to\infty} \frac{\|(1-t_n)T_1 z_n + t_n T_2 z_n - z_n\|}{t_n} = 0. \tag{3.1}$$

Then w is a common fixed point of T_1 and T_2.

Proof. It is obvious that

$$\sup_{m,n\in\mathbb{N}} \|T_1 z_m - T_1 z_n\| \le \sup_{m,n\in\mathbb{N}} \|z_m - z_n\|. \tag{3.2}$$

So $\{T_1 z_n\}$ is bounded because $\{z_n\}$ is bounded. Similarly, we have that $\{T_2 z_n\}$ is also bounded. Since

$$\lim_{n\to\infty} \|(1-t_n)T_1 z_n + t_n T_2 z_n - z_n\| = 0, \tag{3.3}$$

we have

$$\begin{aligned}
\|T_1 w - w\| &\le \limsup_{n\to\infty} \big(\|T_1 w - T_1 z_n\| + \|T_1 z_n - (1-t_n)T_1 z_n - t_n T_2 z_n\| \\
&\qquad + \|(1-t_n)T_1 z_n + t_n T_2 z_n - z_n\| + \|z_n - w\|\big) \\
&\le \limsup_{n\to\infty} \big(2\|w - z_n\| + t_n\|T_1 z_n - T_2 z_n\| + \|(1-t_n)T_1 z_n + t_n T_2 z_n - z_n\|\big) \\
&= 0
\end{aligned} \tag{3.4}$$

and hence w is a fixed point of T_1. We note that

$$T_1 \circ T_2 w = T_2 \circ T_1 w = T_2 w. \tag{3.5}$$

We assume that w is not a fixed point of T_2. Put

$$\varepsilon = \frac{||T_2 w - w||}{3} > 0. \tag{3.6}$$

Then there exists $m \in \mathbb{N}$ such that

$$||z_m - w|| < \varepsilon, \qquad \frac{||(1-t_m)T_1 z_m + t_m T_2 z_m - z_m||}{t_m} < \varepsilon. \tag{3.7}$$

Since

$$\begin{aligned} 3\varepsilon &= ||T_2 w - w|| \\ &\le ||T_2 w - z_m|| + ||z_m - w|| \\ &< ||T_2 w - z_m|| + \varepsilon, \end{aligned} \tag{3.8}$$

we have

$$2\varepsilon < ||T_2 w - z_m||. \tag{3.9}$$

So, we obtain

$$\begin{aligned} ||T_2 w - z_m|| &\le ||T_2 w - (1-t_m)T_1 z_m - t_m T_2 z_m|| \\ &\quad + ||(1-t_m)T_1 z_m + t_m T_2 z_m - z_m|| \\ &\le (1-t_m)||T_2 w - T_1 z_m|| + t_m ||T_2 w - T_2 z_m|| \\ &\quad + ||(1-t_m)T_1 z_m + t_m T_2 z_m - z_m|| \\ &= (1-t_m)||T_1 \circ T_2 w - T_1 z_m|| + t_m ||T_2 w - T_2 z_m|| \\ &\quad + ||(1-t_m)T_1 z_m + t_m T_2 z_m - z_m|| \\ &\le (1-t_m)||T_2 w - z_m|| + t_m ||w - z_m|| \\ &\quad + ||(1-t_m)T_1 z_m + t_m T_2 z_m - z_m|| \\ &< (1-t_m)||T_2 w - z_m|| + 2t_m \varepsilon \\ &< (1-t_m)||T_2 w - z_m|| + t_m ||T_2 w - z_m|| \\ &= ||T_2 w - z_m||. \end{aligned} \tag{3.10}$$

This is a contradiction. Hence, w is a common fixed point of T_1 and T_2. □

Lemma 3.2. *Let C be a closed convex subset of a Banach space E. Let T_1, T_2, and T_3 be commuting nonexpansive mappings on C. Let $\{t_n\}$ be a sequence in $(0,1/2)$ converging to 0 and let $\{z_n\}$ be a sequence in C such that $\{z_n\}$ converges strongly to some $w \in C$ and*

$$\lim_{n\to\infty} \frac{||(1-t_n-t_n^2)T_1 z_n + t_n T_2 z_n + t_n^2 T_3 z_n - z_n||}{t_n^2} = 0. \tag{3.11}$$

Then w is a common fixed point of T_1, T_2, and T_3.

Proof. We note that $\{T_1 z_n\}$, $\{T_2 z_n\}$ and $\{T_3 z_n\}$ are bounded sequences in C because $\{z_n\}$ is bounded. We have

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{||(1-t_n)T_1 z_n + t_n T_2 z_n - z_n||}{t_n} \\
&\quad \le \limsup_{n\to\infty} \frac{||(1-t_n-t_n^2)T_1 z_n + t_n T_2 z_n + t_n^2 T_3 z_n - z_n|| + t_n^2||T_1 z_n - T_3 z_n||}{t_n} \\
&\quad = \lim_{n\to\infty} \left(t_n \frac{||(1-t_n-t_n^2)T_1 z_n + t_n T_2 z_n + t_n^2 T_3 z_n - z_n||}{t_n^2} + t_n||T_1 z_n - T_3 z_n|| \right) \\
&\quad = 0.
\end{aligned} \tag{3.12}$$

So, by Lemma 3.1, we have that w is a common fixed point of T_1 and T_2. We note that

$$T_1 \circ T_3 w = T_3 \circ T_1 w = T_3 w, \qquad T_2 \circ T_3 w = T_3 \circ T_2 w = T_3 w. \tag{3.13}$$

We assume that w is not a fixed point of T_3. Put

$$\varepsilon = \frac{||T_3 w - w||}{3} > 0. \tag{3.14}$$

Then there exists $m \in \mathbb{N}$ such that

$$||z_m - w|| < \varepsilon, \qquad \frac{||(1-t_m-t_m^2)T_1 z_m + t_m T_2 z_m + t_m^2 T_3 z_m - z_m||}{t_m^2} < \varepsilon. \tag{3.15}$$

Since

$$\begin{aligned}
3\varepsilon &= ||T_3 w - w|| \\
&\le ||T_3 w - z_m|| + ||z_m - w|| \\
&< ||T_3 w - z_m|| + \varepsilon,
\end{aligned} \tag{3.16}$$

we have

$$2\varepsilon < ||T_3 w - z_m||. \tag{3.17}$$

So, we obtain

$$\begin{aligned}
||T_3w - z_m|| &\le ||T_3w - (1-t_m-t_m^2)T_1z_m - t_mT_2z_m - t_m^2T_3z_m|| \\
&\quad + ||(1-t_m-t_m^2)T_1z_m + t_mT_2z_m + t_m^2T_3z_m - z_m|| \\
&\le (1-t_m-t_m^2)||T_3w - T_1z_m|| \\
&\quad + t_m||T_3w - T_2z_m|| + t_m^2||T_3w - T_3z_m|| \\
&\quad + ||(1-t_m-t_m^2)T_1z_m + t_mT_2z_m + t_m^2T_3z_m - z_m|| \\
&= (1-t_m-t_m^2)||T_1\circ T_3w - T_1z_m|| \\
&\quad + t_m||T_2\circ T_3w - T_2z_m|| + t_m^2||T_3w - T_3z_m|| \\
&\quad + ||(1-t_m-t_m^2)T_1z_m + t_mT_2z_m + t_m^2T_3z_m - z_m|| \\
&\le (1-t_m-t_m^2)||T_3w - z_m|| \\
&\quad + t_m||T_3w - z_m|| + t_m^2||w - z_m|| \\
&\quad + ||(1-t_m-t_m^2)T_1z_m + t_mT_2z_m + t_m^2T_3z_m - z_m|| \\
&< (1-t_m^2)||T_3w - z_m|| + 2t_m^2\varepsilon \\
&< (1-t_m^2)||T_3w - z_m|| + t_m^2||T_3w - z_m|| \\
&= ||T_3w - z_m||.
\end{aligned} \tag{3.18}$$

This is a contradiction. Hence, w is a common fixed point of T_1, T_2, and T_3. □

Theorem 3.3. *Let C be a compact convex subset of a Banach space E. Let T_1, T_2, and T_3 be commuting nonexpansive mappings on C. Fix $\lambda \in (0,1)$. Let $\{\alpha_n\}$ be a sequence in $[0,1/2]$ satisfying*

$$\liminf_{n\to\infty} \alpha_n = 0, \qquad \limsup_{n\to\infty} \alpha_n > 0, \qquad \lim_{n\to\infty} (\alpha_{n+1} - \alpha_n) = 0. \tag{3.19}$$

Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and

$$x_{n+1} = \lambda(1-\alpha_n-\alpha_n^2)T_1x_n + \lambda\alpha_nT_2x_n + \lambda\alpha_n^2T_3x_n + (1-\lambda)x_n \tag{3.20}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a common fixed point of T_1, T_2, and T_3.

Proof. Put

$$\begin{aligned}
\alpha = \limsup_{n\to\infty} \alpha_n > 0, \qquad M = \sup_{x\in C} \|x\| < \infty, \\
y_n = (1-\alpha_n-\alpha_n^2)T_1x_n + \alpha_nT_2x_n + \alpha_n^2T_3x_n,
\end{aligned} \tag{3.21}$$

for $n \in \mathbb{N}$. We note that

$$x_{n+1} = \lambda y_n + (1-\lambda)x_n \tag{3.22}$$

for all $n \in \mathbb{N}$. Since

$$
\begin{aligned}
\|y_{n+1} - y_n\| = &\|(1-\alpha_{n+1}-\alpha_{n+1}^2)T_1x_{n+1} + \alpha_{n+1}T_2x_{n+1} + \alpha_{n+1}^2T_3x_{n+1} \\
&\quad - (1-\alpha_n-\alpha_n^2)T_1x_n - \alpha_nT_2x_n - \alpha_n^2T_3x_n\| \\
\le\ & (1-\alpha_{n+1}-\alpha_{n+1}^2)\|T_1x_{n+1} - T_1x_n\| \\
& + \alpha_{n+1}\|T_2x_{n+1} - T_2x_n\| + \alpha_{n+1}^2\|T_3x_{n+1} - T_3x_n\| \\
& + |\alpha_{n+1} + \alpha_{n+1}^2 - \alpha_n - \alpha_n^2|\|-T_1x_n\| \\
& + |\alpha_{n+1} - \alpha_n|\|T_2x_n\| + |\alpha_{n+1}^2 - \alpha_n^2|\|T_3x_n\| \\
\le\ & (1-\alpha_{n+1}-\alpha_{n+1}^2)\|x_{n+1}-x_n\| + \alpha_{n+1}\|x_{n+1}-x_n\| + \alpha_{n+1}^2\|x_{n+1}-x_n\| \\
& + |\alpha_{n+1} + \alpha_{n+1}^2 - \alpha_n - \alpha_n^2|M + |\alpha_{n+1}-\alpha_n|M + |\alpha_{n+1}^2 - \alpha_n^2|M \\
\le\ & \|x_{n+1}-x_n\| + 4|\alpha_{n+1}-\alpha_n|M
\end{aligned}
\tag{3.23}
$$

for $n \in \mathbb{N}$, we have

$$
\limsup_{n\to\infty}\left(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|\right) \le 0. \tag{3.24}
$$

So, by Lemma 2.2, we have $\lim_n \|x_n - y_n\| = 0$. Fix $t \in \mathbb{R}$ with $0 < t < \alpha$. Then by Lemma 2.3, there exists a subsequence $\{\alpha_{n_k}\}$ of $\{\alpha_n\}$ converging to t. Since C is compact, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ converging to some point $z_t \in C$. We have

$$
\begin{aligned}
&\|(1-t-t^2)T_1z_t + tT_2z_t + t^2T_3z_t - z_t\| \\
&\quad \le \|(1-t-t^2)T_1z_t + tT_2z_t + t^2T_3z_t - y_n\| + \|y_n - x_n\| + \|x_n - z_t\| \\
&\quad = \|(1-t-t^2)T_1z_t + tT_2z_t + t^2T_3z_t - (1-\alpha_n-\alpha_n^2)T_1x_n - \alpha_nT_2x_n - \alpha_n^2T_3x_n\| \\
&\qquad + \|y_n - x_n\| + \|x_n - z_t\| \\
&\quad \le (1-t-t^2)\|T_1z_t - T_1x_n\| + t\|T_2z_t - T_2x_n\| + t^2\|T_3z_t - T_3x_n\| \\
&\qquad + |t + t^2 - \alpha_n - \alpha_n^2|\|-T_1x_n\| + |t-\alpha_n|\|T_2x_n\| + |t^2 - \alpha_n^2|\|T_3x_n\| \\
&\qquad + \|y_n - x_n\| + \|x_n - z_t\| \\
&\quad \le (1-t-t^2)\|z_t - x_n\| + t\|z_t - x_n\| + t^2\|z_t - x_n\| \\
&\qquad + |t + t^2 - \alpha_n - \alpha_n^2|M + |t-\alpha_n|M + |t^2 - \alpha_n^2|M \\
&\qquad + \|y_n - x_n\| + \|x_n - z_t\| \\
&\quad \le 2\|z_t - x_n\| + 4|t-\alpha_n|M + \|y_n - x_n\|
\end{aligned}
\tag{3.25}
$$

for $n \in \mathbb{N}$, and hence

$$
\begin{aligned}
&\|(1-t-t^2)T_1z_t + tT_2z_t + t^2T_3z_t - z_t\| \\
&\quad \le \lim_{j\to\infty}\left(2\|z_t - x_{n_{k_j}}\| + 4|t - \alpha_{n_{k_j}}|M + \|y_{n_{k_j}} - x_{n_{k_j}}\|\right) \\
&\quad = 0.
\end{aligned}
\tag{3.26}
$$

Therefore we have

$$(1-t-t^2)T_1z_t + tT_2z_t + t^2T_3z_t = z_t \tag{3.27}$$

for all $t \in \mathbb{R}$ with $0 < t < \alpha$. Since C is compact, there exists a real sequence $\{t_n\}$ in $(0,\alpha)$ such that $\lim_n t_n = 0$, and $\{z_{t_n}\}$ converges strongly to some point $w \in C$. By Lemma 3.2, we obtain that such w is a common fixed point of T_1, T_2, and T_3. We note that w is a cluster point of $\{x_n\}$ because so are z_{t_n} for all $n \in \mathbb{N}$. Hence, $\liminf_n \|x_n - w\| = 0$. We also have that $\{\|x_n - w\|\}$ is nonincreasing by Lemma 2.5. Thus, $\lim_n \|x_n - w\| = 0$. This completes the proof. □

We give an example concerning $\{\alpha_n\}$.

Example 3.4. Define a sequence $\{\beta_n\}$ in $[-1/2, 1/2]$ by

$$\beta_n = \begin{cases} \dfrac{1}{2k} & \text{if } 2\displaystyle\sum_{j=1}^{k-1} j < n \le 2\sum_{j=1}^{k-1} j + k \text{ for some } k \in \mathbb{N}, \\ -\dfrac{1}{2k} & \text{if } 2\displaystyle\sum_{j=1}^{k-1} j + k < n \le 2\sum_{j=1}^{k} j \text{ for some } k \in \mathbb{N}. \end{cases} \tag{3.28}$$

Define a sequence $\{\alpha_n\}$ in $[0, 1/2]$ by

$$\alpha_n = \sum_{k=1}^{n} \beta_k \tag{3.29}$$

for $n \in \mathbb{N}$. Then $\{\alpha_n\}$ satisfies the assumption of Theorem 3.3.

Remark 3.5. The sequence $\{\alpha_n\}$ is as follows:

$$\begin{array}{lllll} \alpha_1 = 1/2, & \alpha_2 = 0, & \alpha_3 = 1/4, & \alpha_4 = 2/4, & \alpha_5 = 1/4, \\ \alpha_6 = 0, & \alpha_7 = 1/6, & \alpha_8 = 2/6, & \alpha_9 = 3/6, & \alpha_{10} = 2/6, \\ \alpha_{11} = 1/6, & \alpha_{12} = 0, & \alpha_{13} = 1/8, & \alpha_{14} = 2/8, & \alpha_{15} = 3/8, \\ \alpha_{16} = 4/8, & \alpha_{17} = 3/8, & \alpha_{18} = 2/8, & \alpha_{19} = 1/8, & \alpha_{20} = 0, \\ \alpha_{21} = 1/10, & \alpha_{22} = 2/10, & \alpha_{23} = 3/10, & \alpha_{24} = 4/10, & \alpha_{25} = 5/10, \\ \alpha_{26} = 4/10, & \alpha_{27} = 3/10, & \alpha_{28} = 2/10, & \dots. & \end{array} \tag{3.30}$$

4. Main results

In this section, we prove our main results.

Lemma 4.1. *Let C be a closed convex subset of a Banach space E. Let $\ell \in \mathbb{N}$ with $\ell \ge 2$ and let $T_1, T_2, \dots, T_\ell$ be commuting nonexpansive mappings on C. Let $\{t_n\}$ be a sequence in $(0, 1/2)$ converging to 0 and let $\{z_n\}$ be a sequence in C such that $\{z_n\}$ converges strongly to some*

$w \in C$ and

$$\lim_{n\to\infty} \frac{\left\|\left(1-\sum_{k=1}^{\ell-1} t_n^k\right) T_1 z_n + \sum_{k=2}^{\ell} t_n^{k-1} T_k z_n - z_n\right\|}{t_n^{\ell-1}} = 0. \tag{4.1}$$

Then w is a common fixed point of $T_1, T_2, \dots, T_\ell$.

Proof. We will prove this lemma by induction. We have already proved the conclusion in the case of $\ell = 2$, 3. Fix $\ell \in \mathbb{N}$ with $\ell \geq 4$. We assume that the conclusion holds for every integer less than ℓ and greater than 1. We note that $\{T_1 z_n\}, \{T_2 z_n\}, \dots, \{T_\ell z_n\}$ are bounded sequences in C because $\{z_n\}$ is bounded. We have

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{\left\|\left(1-\sum_{k=1}^{\ell-2} t_n^k\right) T_1 z_n + \sum_{k=2}^{\ell-1} t_n^{k-1} T_k z_n - z_n\right\|}{t_n^{\ell-2}} \\
&\quad\leq \limsup_{n\to\infty} \frac{\left\|\left(1-\sum_{k=1}^{\ell-1} t_n^k\right) T_1 z_n + \sum_{k=2}^{\ell} t_n^{k-1} T_k z_n - z_n\right\| + t_n^{\ell-1}\left\|T_1 z_n - T_\ell z_n\right\|}{t_n^{\ell-2}} \\
&\quad= \lim_{n\to\infty} \left(t_n \frac{\left\|\left(1-\sum_{k=1}^{\ell-1} t_n^k\right) T_1 z_n + \sum_{k=2}^{\ell} t_n^{k-1} T_k z_n - z_n\right\|}{t_n^{\ell-1}} + t_n \left\|T_1 z_n - T_\ell z_n\right\| \right) \\
&\quad= 0.
\end{aligned} \tag{4.2}$$

So, by the assumption of induction, we have that w is a common fixed point of $T_1, T_2, \dots, T_{\ell-1}$. We note that

$$T_k \circ T_\ell w = T_\ell \circ T_k w = T_\ell w \tag{4.3}$$

for all $k \in \mathbb{N}$ with $1 \leq k < \ell$. We assume that w is not a fixed point of T_ℓ. Put

$$\varepsilon = \frac{\left\|T_\ell w - w\right\|}{3} > 0. \tag{4.4}$$

Then there exists $m \in \mathbb{N}$ such that

$$\left\|z_m - w\right\| < \varepsilon, \qquad \frac{\left\|\left(1-\sum_{k=1}^{\ell-1} t_m^k\right) T_1 z_m + \sum_{k=2}^{\ell} t_m^{k-1} T_k z_m - z_m\right\|}{t_m^{\ell-1}} < \varepsilon. \tag{4.5}$$

Since

$$\begin{aligned}
3\varepsilon &= \left\|T_\ell w - w\right\| \\
&\leq \left\|T_\ell w - z_m\right\| + \left\|z_m - w\right\| \\
&< \left\|T_\ell w - z_m\right\| + \varepsilon,
\end{aligned} \tag{4.6}$$

we have

$$2\varepsilon < \|T_\ell w - z_m\|. \tag{4.7}$$

So, we obtain

$$\begin{aligned}
\|T_\ell w - z_m\| &\le \left\| T_\ell w - \left(1 - \sum_{k=1}^{\ell-1} t_m^k\right) T_1 z_m - \sum_{k=2}^{\ell} t_m^{k-1} T_k z_m \right\| \\
&\quad + \left\| \left(1 - \sum_{k=1}^{\ell-1} t_m^k\right) T_1 z_m + \sum_{k=2}^{\ell} t_m^{k-1} T_k z_m - z_m \right\| \\
&< \left(1 - \sum_{k=1}^{\ell-1} t_m^k\right) \|T_\ell w - T_1 z_m\| + \sum_{k=2}^{\ell} t_m^{k-1} \|T_\ell w - T_k z_m\| + t_m^{\ell-1}\varepsilon \\
&= \left(1 - \sum_{k=1}^{\ell-1} t_m^k\right) \|T_1 \circ T_\ell w - T_1 z_m\| \\
&\quad + \sum_{k=2}^{\ell-1} t_m^{k-1} \|T_k \circ T_\ell w - T_k z_m\| + t_m^{\ell-1} \|T_\ell w - T_\ell z_m\| + t_m^{\ell-1}\varepsilon \\
&\le \left(1 - \sum_{k=1}^{\ell-1} t_m^k\right) \|T_\ell w - z_m\| + \sum_{k=2}^{\ell-1} t_m^{k-1} \|T_\ell w - z_m\| + t_m^{\ell-1} \|w - z_m\| + t_m^{\ell-1}\varepsilon \\
&< (1 - t_m^{\ell-1}) \|T_\ell w - z_m\| + 2 t_m^{\ell-1}\varepsilon \\
&< (1 - t_m^{\ell-1}) \|T_\ell w - z_m\| + t_m^{\ell-1} \|T_\ell w - z_m\| \\
&= \|T_\ell w - z_m\|.
\end{aligned} \tag{4.8}$$

This is a contradiction. Hence, w is a common fixed point of $T_1, T_2, \ldots, T_\ell$. By induction, we obtain the desired result. □

Lemma 4.2. *Let C be a bounded closed convex subset of a Banach space E. Let $\{T_n : n \in \mathbb{N}\}$ be an infinite family of commuting nonexpansive mappings on C. Let $\{t_n\}$ be a sequence in $(0, 1/2)$ converging to 0 and let $\{z_n\}$ be a sequence in C such that $\{z_n\}$ converges strongly to some $w \in C$ and*

$$\left(1 - \sum_{k=1}^{\infty} t_n^k\right) T_1 z_n + \sum_{k=2}^{\infty} t_n^{k-1} T_k z_n = z_n \tag{4.9}$$

for all $n \in \mathbb{N}$. Then w is a common fixed point of $\{T_n : n \in \mathbb{N}\}$.

Proof. Fix $\ell \in \mathbb{N}$ with $\ell \geq 2$. We put $M = 2\sup\{\|x\| : x \in C\} < \infty$. We have

$$\begin{aligned}
\limsup_{n\to\infty} & \frac{\left\|\left(1-\sum_{k=1}^{\ell-1} t_n^k\right) T_1 z_n + \sum_{k=2}^{\ell} t_n^{k-1} T_k z_n - z_n\right\|}{t_n^{\ell-1}} \\
&\leq \limsup_{n\to\infty} \left(\frac{\left\|\left(1-\sum_{k=1}^{\infty} t_n^k\right) T_1 z_n + \sum_{k=2}^{\infty} t_n^{k-1} T_k z_n - z_n\right\|}{t_n^{\ell-1}} + \frac{\sum_{k=\ell+1}^{\infty} t_n^{k-1} \left\|T_1 z_n - T_k z_n\right\|}{t_n^{\ell-1}} \right) \\
&= \limsup_{n\to\infty} \frac{\sum_{k=\ell+1}^{\infty} t_n^{k-1} \left\|T_1 z_n - T_k z_n\right\|}{t_n^{\ell-1}} \\
&\leq \limsup_{n\to\infty} \sum_{k=\ell+1}^{\infty} t_n^{k-\ell} M \\
&= \lim_{n\to\infty} \frac{t_n}{1-t_n} M \\
&= 0.
\end{aligned} \tag{4.10}$$

So, by Lemma 4.1, we have that w is a common fixed point of $T_1, T_2, \ldots, T_\ell$. Since $\ell \in \mathbb{N}$ is arbitrary, we obtain that w is a common fixed point of $\{T_n : n \in \mathbb{N}\}$. This completes the proof. □

THEOREM 4.3. *Let C be a compact convex subset of a Banach space E. Let $\{T_n : n \in \mathbb{N}\}$ be an infinite family of commuting nonexpansive mappings on C. Fix $\lambda \in (0,1)$. Let $\{\alpha_n\}$ be a sequence in $[0,1/2]$ satisfying*

$$\liminf_{n\to\infty} \alpha_n = 0, \qquad \limsup_{n\to\infty} \alpha_n > 0, \qquad \lim_{n\to\infty} (\alpha_{n+1} - \alpha_n) = 0. \tag{4.11}$$

Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and

$$x_{n+1} = \lambda\left(1 - \sum_{k=1}^{n-1} \alpha_n^k\right) T_1 x_n + \lambda\left(\sum_{k=2}^{n} \alpha_n^{k-1} T_k x_n\right) + (1-\lambda) x_n \tag{4.12}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_n : n \in \mathbb{N}\}$.

Remark 4.4. We know that $\bigcap_{n=1}^{\infty} F(T_n) \neq \varnothing$ by DeMarr's result in [5]. We define $\sum_{k=1}^{0} \alpha_1^k = 0$ and $\sum_{k=2}^{1} \alpha_1^{k-1} T_k x_1 = 0$.

Proof. Put

$$\begin{gathered}
\alpha = \limsup_{n\to\infty} \alpha_n > 0, \qquad M = \sup_{x\in C} \|x\| < \infty, \\
y_n = \left(1 - \sum_{k=1}^{n-1} \alpha_n^k\right) T_1 x_n + \sum_{k=2}^{n} \alpha_n^{k-1} T_k x_n
\end{gathered} \tag{4.13}$$

for $n \in \mathbb{N}$. We note that

$$x_{n+1} = \lambda y_n + (1-\lambda) x_n \tag{4.14}$$

for all $n \in \mathbb{N}$. Since

$$\begin{aligned}
&||y_{n+1} - y_n|| \\
&= \left|\left|\left(1 - \sum_{k=1}^{n} \alpha_{n+1}^k\right) T_1 x_{n+1} + \sum_{k=2}^{n+1} \alpha_{n+1}^{k-1} T_k x_{n+1} - \left(1 - \sum_{k=1}^{n-1} \alpha_n^k\right) T_1 x_n - \sum_{k=2}^{n} \alpha_n^{k-1} T_k x_n\right|\right| \\
&\leq \left(1 - \sum_{k=1}^{n} \alpha_{n+1}^k\right) ||T_1 x_{n+1} - T_1 x_n|| + \sum_{k=2}^{n+1} \alpha_{n+1}^{k-1} ||T_k x_{n+1} - T_k x_n|| \\
&\quad + \left|\sum_{k=1}^{n} \alpha_{n+1}^k - \sum_{k=1}^{n} \alpha_n^k\right| ||-T_1 x_n|| + \sum_{k=2}^{n+1} |\alpha_{n+1}^{k-1} - \alpha_n^{k-1}| \, ||T_k x_n|| \\
&\quad + |\alpha_n^n| \, ||-T_1 x_n|| + |\alpha_n^n| \, ||T_{n+1} x_n|| \\
&\leq \left(1 - \sum_{k=1}^{n} \alpha_{n+1}^k\right) ||x_{n+1} - x_n|| + \sum_{k=2}^{n+1} \alpha_{n+1}^{k-1} ||x_{n+1} - x_n|| \\
&\quad + 2M \sum_{k=1}^{n} |\alpha_{n+1}^k - \alpha_n^k| + 2M |\alpha_n^n| \\
&\leq ||x_{n+1} - x_n|| + 8M |\alpha_{n+1} - \alpha_n| + 2M \frac{1}{2^n}
\end{aligned} \tag{4.15}$$

for $n \in \mathbb{N}$, we have

$$\limsup_{n\to\infty} \left(||y_{n+1} - y_n|| - ||x_{n+1} - x_n||\right) \leq 0. \tag{4.16}$$

So, by Lemma 2.2, we have $\lim_n \|x_n - y_n\| = 0$. Fix $t \in \mathbb{R}$ with $0 < t < \alpha$. Then by Lemma 2.3, there exists a subsequence $\{\alpha_{n_k}\}$ of $\{\alpha_n\}$ converging to t. Since C is compact, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ converging strongly to some point $z_t \in C$. We have

$$\begin{aligned}
&\left|\left|\left(1 - \sum_{\ell=1}^{\infty} t^\ell\right) T_1 z_t + \sum_{\ell=2}^{\infty} t^{\ell-1} T_\ell z_t - z_t\right|\right| \\
&\quad \leq \left|\left|\left(1 - \sum_{\ell=1}^{\infty} t^\ell\right) T_1 z_t + \sum_{\ell=2}^{\infty} t^{\ell-1} T_\ell z_t - y_n\right|\right| + ||y_n - x_n|| + ||x_n - z_t|| \\
&\quad = \left|\left|\left(1 - \sum_{\ell=1}^{\infty} t^\ell\right) T_1 z_t + \sum_{\ell=2}^{\infty} t^{\ell-1} T_\ell z_t - \left(1 - \sum_{\ell=1}^{n-1} \alpha_n^\ell\right) T_1 x_n - \sum_{\ell=2}^{n} \alpha_n^{\ell-1} T_\ell x_n\right|\right| \\
&\qquad + ||y_n - x_n|| + ||x_n - z_t|| \\
&\quad \leq \left(1 - \sum_{\ell=1}^{\infty} t^\ell\right) ||T_1 z_t - T_1 x_n|| + \sum_{\ell=2}^{\infty} t^{\ell-1} ||T_\ell z_t - T_\ell x_n|| \\
&\qquad + \left|\sum_{\ell=1}^{n-1} t^\ell - \sum_{\ell=1}^{n-1} \alpha_n^\ell\right| ||-T_1 x_n|| + \sum_{\ell=2}^{n} |t^{\ell-1} - \alpha_n^{\ell-1}| \, ||T_\ell x_n|| \\
&\qquad + \sum_{\ell=n}^{\infty} t^\ell ||-T_1 x_n|| + \sum_{\ell=n+1}^{\infty} t^{\ell-1} ||T_\ell x_n|| + ||y_n - x_n|| + ||x_n - z_t||
\end{aligned}$$

$$
\begin{aligned}
&\le \left(1-\sum_{\ell=1}^{\infty} t^{\ell}\right)||z_t - x_n|| + \sum_{\ell=2}^{\infty} t^{\ell-1}||z_t - x_n|| + \left|\sum_{\ell=1}^{n-1} t^{\ell} - \sum_{\ell=1}^{n-1} \alpha_n^{\ell}\right| M \\
&\quad + \sum_{\ell=2}^{n} |\, t^{\ell-1} - \alpha_n^{\ell-1} \,| M + \sum_{\ell=n}^{\infty} t^{\ell} M + \sum_{\ell=n+1}^{\infty} t^{\ell-1} M + ||y_n - x_n|| + ||x_n - z_t|| \\
&\le 2||z_t - x_n|| + 8\,|\, t - \alpha_n \,| M + \frac{2t^n}{1-t} M + ||y_n - x_n||
\end{aligned}
\tag{4.17}
$$

for $n \in \mathbb{N}$, and hence

$$
\begin{aligned}
&\left\|\left(1-\sum_{\ell=1}^{\infty} t^{\ell}\right) T_1 z_t + \sum_{\ell=2}^{\infty} t^{\ell-1} T_{\ell} z_t - z_t\right\| \\
&\quad \le \limsup_{j\to\infty}\left(2||z_t - x_{n_{k_j}}|| + 8\,|\, t - \alpha_{n_{k_j}} \,| M + \frac{2t^{n_{k_j}}}{1-t} M + ||y_{n_{k_j}} - x_{n_{k_j}}||\right) = 0.
\end{aligned}
\tag{4.18}
$$

Therefore we have

$$
\left(1-\sum_{\ell=1}^{\infty} t^{\ell}\right) T_1 z_t + \sum_{\ell=2}^{\infty} t^{\ell-1} T_{\ell} z_t = z_t \tag{4.19}
$$

for all $t \in \mathbb{R}$ with $0 < t < \alpha$. Since C is compact, there exists a real sequence $\{t_n\}$ in $(0,\alpha)$ such that $\lim_n t_n = 0$ and $\{z_{t_n}\}$ converges strongly to some point $w \in C$. By Lemma 4.2, we obtain that such w is a common fixed point of $\{T_n : n \in \mathbb{N}\}$. We note that w is a cluster point of $\{x_n\}$ because so are z_{t_n} for all $n \in \mathbb{N}$. Hence, $\liminf_n \|x_n - w\| = 0$. We also have that $\{\|x_n - w\|\}$ is nonincreasing by Lemma 2.5. Thus, $\lim_n \|x_n - w\| = 0$. This completes the proof. □

Similarly, we can prove the following.

Theorem 4.5. *Let C be a compact convex subset of a Banach space E. Let $\{T_n : n \in \mathbb{N}\}$ be an infinite family of commuting nonexpansive mappings on C. Fix $\lambda \in (0,1)$. Let $\{\alpha_n\}$ be a sequence in $[0,1/2]$ satisfying*

$$
\liminf_{n\to\infty} \alpha_n = 0, \qquad \limsup_{n\to\infty} \alpha_n > 0, \qquad \lim_{n\to\infty} \left(\alpha_{n+1} - \alpha_n\right) = 0. \tag{4.20}
$$

Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and

$$
x_{n+1} = \lambda\left(1-\sum_{k=1}^{\infty} \alpha_n^{k}\right) T_1 x_n + \lambda\left(\sum_{k=2}^{\infty} \alpha_n^{k-1} T_k x_n\right) + (1-\lambda) x_n \tag{4.21}
$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_n : n \in \mathbb{N}\}$.

As direct consequences, we obtain the following.

Theorem 4.6. *Let C be a compact convex subset of a Banach space E. Let S and T be nonexpansive mappings on C with $ST = TS$. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ satisfying*

$$\liminf_{n\to\infty} \alpha_n = 0, \qquad \limsup_{n\to\infty} \alpha_n > 0, \qquad \lim_{n\to\infty} \left(\alpha_{n+1} - \alpha_n\right) = 0. \tag{4.22}$$

Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and

$$x_{n+1} = \frac{1-\alpha_n}{2} S x_n + \frac{\alpha_n}{2} T x_n + \frac{1}{2} x_n \tag{4.23}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a common fixed point of S and T.

Remark 4.7. This theorem is simpler than Theorem 1.2.

Theorem 4.8. *Let C be a compact convex subset of a Banach space E. Let $\ell \in \mathbb{N}$ with $\ell \geq 2$ and let $\{T_1, T_2, \dots, T_\ell\}$ be a finite family of commuting nonexpansive mappings on C. Let $\{\alpha_n\}$ be a sequence in $[0,1/2]$ satisfying*

$$\liminf_{n\to\infty} \alpha_n = 0, \qquad \limsup_{n\to\infty} \alpha_n > 0, \qquad \lim_{n\to\infty} \left(\alpha_{n+1} - \alpha_n\right) = 0. \tag{4.24}$$

Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and

$$x_{n+1} = \frac{1}{2}\left(1 - \sum_{k=1}^{\ell-1} \alpha_n^k\right) T_1 x_n + \frac{1}{2}\left(\sum_{k=2}^{\ell} \alpha_n^{k-1} T_k x_n\right) + \frac{1}{2} x_n \tag{4.25}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, \dots, T_\ell\}$.

Remark 4.9. This theorem is simpler than Theorem 1.1.

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