Strong Coordination over a Three-Terminal Relay Network

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***Abstract*—We study the problem of strong coordination in a three-terminal relay network, in which agents communicate to ensure that their actions follow a joint behavior specified by a prescribed joint distribution of actions. The model unifies several coordination schemes, including line and broadcast coordination. We derive several inner bounds to the strong capacity region; in particular, we prove the achievability of a subset of coordination rate-tuples, which provides insight into the relative performance of line, broadcast, and relay coordination.**

I. Introduction

An important aspect in networked communication is the problem of coordinating the actions of nodes in a decentralized manner, while minimizing amount of communication. For the point-to-point case, such fundamental limits have been established for *empirical coordination* in [1], by which the histogram of induced actions at the agents is required to follow a prescribed target distribution, and *strong coordination*, by which the induced sequence of joint actions is required to be statistically indistinguishable from the target distribution. Both strong and empirical coordination have the potential to significantly lower the communication rate compared to a naive direct communication of the action sequence, which could be useful in applications such as distributed control or multi-agent based exploration and surveillance.

The limits of empirical coordination for small and large networks have been the subject of several works. For example, [2] considers a distributed multi-agent control problem, in which each agent generates actions based on its own observations of a source of randomness. The generation of dependent random variables in networks under a strong coordination constraint is considered in [3], [4] by investigating bidirectional transmissions in several rounds; however, these works only address the coordination of two nodes. An extension of the ideas to a two-way relay channel and strong coordination can be found in [5]. In [6], the synthesis of a discrete memoryless broadcast channel is considered, where a stochastic channel output is generated based on a compressed description of the channel input. The relation to coordination is given by the fact that the sequences generated at the channel output can be viewed as induced action sequences. Finally, [7], [8] address the characterization of the coordination capacity region for a line network consisting of three agents,[1] with and without secrecy constraints.

In this work, we address a general three-terminal setup and provide a unifying framework to derive inner and outer bounds to the strong capacity region for line, broadcast, and relay coordination. Outer bound are nevertheless omitted for brevity and will be reported in the full version of the paper.

II. Problem Statement and Main Results

We consider the setting illustrated in Fig. 1, in which three agents attempt to coordinate their actions. The actions taken by each agent $i \in \{1,2,3\}$ are described by a sequence of discrete actions $x_i^n \in \mathcal{X}_i^n$, and the behavior is captured by a memoryless joint probability distribution of the actions $q_{X_1X_2X_3}$. The network comprises four distinct links:

- a noiseless link between agent 1 and agent 2, over which agent 1 transmits messages $M_{12} \in \mathcal{M}_{12} \triangleq [\![1, 2^{nR_{12}}]\!]$;
- a noiseless link between agent 1 and agent 3, over which agent 1 transmits messages $M_{13} \in \mathcal{M}_{13} \triangleq [\![1, 2^{nR_{13}}]\!]$;
- a noiseless link between agent 2 and agent 3, over which agent 2 transmits messages $M_{23} \in \mathcal{M}_{23} \triangleq [\![1, 2^{nR_{23}}]\!]$;
- a noiseless link between agent 3 and agent 2, over which agent 3 transmits messages $M_{32} \in \mathcal{M}_{32} \triangleq [\![1, 2^{nR_{32}}]\!]$.

All agents have access to a common source of randomness, which produces uniform random numbers in $M_0 \in \mathcal{M}_0 \triangleq [\![1, 2^{nR_0}]\!]$. Agent 1 is considered as the *leader*, in that its

[1] The present paper corrects an unfortunate error in the achievability proof of [7], which, however, did not affect the main conclusions.

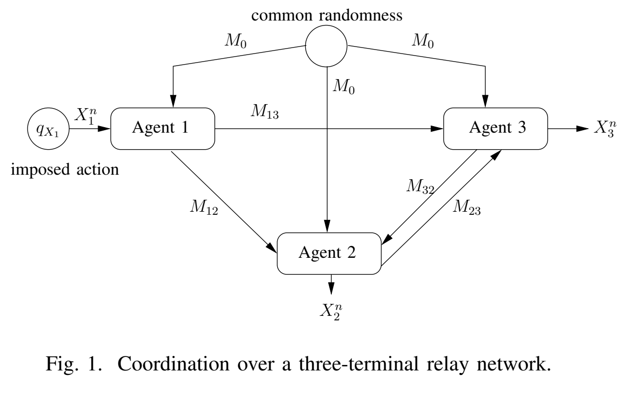

Fig. 1. Coordination over a three-terminal relay network.

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actions x_1^n are drawn i.i.d. according to q_{X_1} ahead of time. A *coordination scheme*, which we define precisely next, is a means to exchange messages over the network so that agent 2 and agent 3 take actions x_2^n and x_3^n that appear as if drawn from the distribution $q_{X_2^n X_3^n | X_1^n = x_1^n}$.

Definition 1: A $(2^{nR_0}, 2^{nR_{12}}, 2^{nR_{13}}, 2^{nR_{23}}, 2^{nR_{32}})$ coordination scheme consists of:

- a stochastic encoder $f_{12} : \mathcal{M}_0 \times \mathcal{X}_1^n \to \mathcal{M}_{12}$;
- a stochastic encoder $f_{13} : \mathcal{M}_0 \times \mathcal{X}_1^n \to \mathcal{M}_{13}$;
- a stochastic encoder $f_{23} : \mathcal{M}_0 \times \mathcal{M}_{12} \to \mathcal{M}_{23}$;
- a stochastic encoder $f_{32} : \mathcal{M}_0 \times \mathcal{M}_{13} \to \mathcal{M}_{32}$;
- a stochastic actuator $g_2 : \mathcal{M}_0 \times \mathcal{M}_{12} \times \mathcal{M}_{32} \to \mathcal{X}_2^n$;
- a stochastic actuator $g_3 : \mathcal{M}_0 \times \mathcal{M}_{13} \times \mathcal{M}_{23} \to \mathcal{X}_3^n$;

The distribution of actions induced by a coordination scheme is denoted by $\tilde{p}_{X_1^n X_2^n X_3^n}$. We say that a coordination rate-tuple $(R_0, R_{12}, R_{13}, R_{23}, R_{32})$ is achievable if there exists a sequence of $(2^{nR_0}, 2^{nR_{12}}, 2^{nR_{13}}, 2^{nR_{23}}, 2^{nR_{32}})$ coordination schemes such that

$$\lim_{n\to\infty} \mathbb{V}\left(\tilde{p}_{X_1^n X_2^n X_3^n}, q_{X_1^n X_2^n X_3^n}\right) = 0,$$

where $\mathbb{V}$ is the variational distance. The *coordination capacity region* is the set of all achievable coordination rate-tuples. We also distinguish three coordination schemes that are special cases of the general scheme in Definition 1.

Definition 2: A *line coordination* scheme corresponds to the situation in which $R_{13} = R_{32} = 0$ or $R_{12} = R_{23} = 0$; in other words, agent 2 or agent 3 is a bottleneck for communication. This coordination scheme is the one considered in [7], [8].

Definition 3: A *broadcast coordination* scheme corresponds to the situation in which $R_{23} = R_{32} = 0$; in other words, agent 1 has to broadcast messages to all the other agents.
Note that broadcast coordination allows different messages to be sent to different agents, which differs from the setting in [6] in which the same message is broadcasted to all agents.

Definition 4: A *relay coordination* scheme corresponds to the situation in which $R_{23} = 0$ or $R_{32} = 0$; in other words, either agent 2 or agent 3 can serve as a relay to the other.
Our definition of relay coordination differs from the relay model introduced in [5], in which the relay only allows bidirectional communication.

The exact characterization of the entire set of coordination rate-tuples achieving a fixed behavior $q_{X_1X_2X_3}$ is beyond the scope of the present paper. Instead, we prove the achievability of a subset of coordination rate-tuples, which provides insight into the relative performance of the three special coordination schemes above. These regions are discussed in Section III, and include many constraints with auxiliary random variables. For now, we only highlight simple regions that are achievable when R_0 is so large that all constraints involving it become ineffective. In other words, we establish bounds on the rate of communication between agents, ignoring the potentially high amount of common randomness required.

Proposition 1: The rate-tuples in the region $\{R_{12} > \mathbb{I}(X_2X_3; X_1), R_{13} = 0, R_{23} > \mathbb{I}(X_3; X_1), R_{32} = 0\}$ are achievable with a line coordination scheme and enough common randomness. In addition, these rates are optimal among all line coordination schemes for which $R_{13} = R_{32} = 0$. Similarly, the rate-tuples in the region $\{R_{12} = 0, R_{13} > \mathbb{I}(X_2X_3; X_1), R_{23} = 0, R_{32} > \mathbb{I}(X_2; X_1)\}$ are achievable with a line coordination scheme and enough common randomness. In addition, these rates are optimal among all line coordination schemes for which $R_{12} = R_{23} = 0$.

Proof: See Section III-A and Section III-B. The converse showing the optimality was already established in [7]. ∎

Proposition 2: The rate-tuples in the convex hull of the region $\{R_{12} > \mathbb{I}(X_2X_3; X_1), R_{13} > \mathbb{I}(X_3; X_1), R_{23} = R_{32} = 0\} \cup \{R_{12} > \mathbb{I}(X_2; X_1), R_{13} > \mathbb{I}(X_2X_3; X_1), R_{23} = R_{32} = 0\}$ are achievable with a broadcast coordination scheme and enough common randomness. Conversely, achievable rates with a broadcast coordination scheme must lie in the region defined by $\{R_{12} > \mathbb{I}(X_2; X_1), R_{13} \geqslant \mathbb{I}(X_3; X_1), R_{12}+R_{23} \geqslant \mathbb{I}(X_2X_3; X_1), R_{23} = R_{32} = 0\}$.

Proof: See Section III-A and Section III-C. The converse is omitted for brevity, but is similar to [7]. ∎

The case of relay coordination does not lend itself to such a simple analysis, and is only discussed in Section III-D.

III. Achievability Proof

A. Preliminaries

Following the approach first proposed in [9], we develop a generic achievable rate for the three-terminal coordination network by tying coordination to the notion of common information [10], [9]. We start by introducing discrete random variables U, V, W, X_1, X_2, X_3 with joint distribution

$$p(x_1, x_2, x_3, u, v, w) \triangleq W(x_1|uvw)W(x_2|uw) W(x_3|vw)p(u|w)p(v|w)p(w),$$

and with marginal distribution $p_{X_1X_2X_3} = q_{X_1X_2X_3}$. The existence of such variables can be proved by selecting trivial values of U, V, W, e.g. $U \triangleq X_2$ and $V \triangleq W \triangleq X_3$. The Bayesian network that captures the dependencies between all random variables is illustrated in Fig. 2. Intuitively, the role of W is to describe the common information of X_1, X_2, X_3 [10], [11]; the role of U is to describe the common information of X_1, X_2, which is not already described by W; the role of V is to describe the common information of X_1, X_3, which is not already described by W. For clarity and simplicity, we do

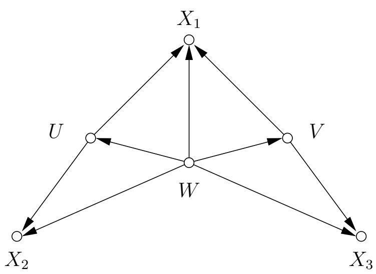

Fig. 2. Bayesian network formed by random variables U, V, W, X_1, X_2, X_3.

$$
\begin{aligned}
&\mathbb{E}(\mathbb{D}(\hat{p}_{Z^n}\|q_{Z^n}))\\
&= \frac{1}{M}\sum_{z^n}\sum_{ijk\ell}\mathbb{E}\left(W(z^n|U^n_{ijk},V^n_{ij\ell},W^n_{ij})\log\left(\frac{1}{M}\sum_{\tilde{i}\tilde{j}\tilde{k}\tilde{\ell}}\frac{W(z^n|U^n_{\tilde{i}\tilde{j}\tilde{k}},V^n_{\tilde{i}\tilde{j}\tilde{\ell}},W^n_{\tilde{i}\tilde{j}})}{q_{Z^n}(z^n)}\right)\right) &(1)\\
&= \frac{1}{M}\sum_{z^n}\sum_{ijk\ell}\mathbb{E}_{W^n_{ij}U^n_{ijk}V^n_{ij\ell}}\left(W(z^n|U^n_{ijk},V^n_{ij\ell},W^n_{ij})\mathbb{E}\left(\log\left(\frac{1}{M}\sum_{\tilde{i}\tilde{j}\tilde{k}\tilde{\ell}}\frac{W(z^n|U^n_{\tilde{i}\tilde{j}\tilde{k}},V^n_{\tilde{i}\tilde{j}\tilde{\ell}},W^n_{\tilde{i}\tilde{j}})}{q_{Z^n}(z^n)}\right)\middle| W^n_{ij}U^n_{ijk}V^n_{ij\ell}\right)\right) &(2)\\
&\leqslant \frac{1}{M}\sum_{z^n}\sum_{ijk\ell}\mathbb{E}_{W^n_{ij}U^n_{ijk}V^n_{ij\ell}}\left(W(z^n|U^n_{ijk},V^n_{ij\ell},W^n_{ij})\log\left(\frac{1}{M}\sum_{\tilde{i}\tilde{j}\tilde{k}\tilde{\ell}}\mathbb{E}\left(\frac{W(z^n|U^n_{\tilde{i}\tilde{j}\tilde{k}},V^n_{\tilde{i}\tilde{j}\tilde{\ell}},W^n_{\tilde{i}\tilde{j}})}{q_{Z^n}(z^n)}\middle| W^n_{ij}U^n_{ijk}V^n_{ij\ell}\right)\right)\right) &(3)
\end{aligned}
$$

not try to exploit the common information of X_2 and X_3 in this conference paper.

We now consider the channel resolvability problem with secrecy constraints illustrated in Fig. 3. Four uniformly distributed random messages $M_0 \in [\![1, 2^{nR_0}]\!]$, $M_w \in [\![1, 2^{nR_w}]\!]$, $M_u \in [\![1, 2^{nR_u}]\!]$, and $M_v \in [\![1, 2^{nR_v}]\!]$, are encoded into codewords $U^n \in \mathcal{U}^n$, $W^n \in \mathcal{W}^n$, and $V^n \in \mathcal{V}^n$ using the following deterministic encoding functions:

- $f_w^n : [\![1, 2^{nR_0}]\!] \times [\![1, 2^{nR_w}]\!] \to \mathcal{W}^n$;
- $f_u^n : [\![1, 2^{nR_0}]\!] \times [\![1, 2^{nR_w}]\!] \times [\![1, 2^{nR_u}]\!] \to \mathcal{U}^n$;
- $f_v^n : [\![1, 2^{nR_0}]\!] \times [\![1, 2^{nR_w}]\!] \times [\![1, 2^{nR_v}]\!] \to \mathcal{V}^n$.

The codewords U^n, V^n, and W^n, are then transmitted through the discrete memoryless channels (DMCs) with transition probabilities $W_{X_2|UW}$, $W_{X_1|UVW}$, $W_{X_3|VW}$, respectively. This procedure defines a joint distribution between input messages and channel outputs, which we denote by $\hat{p}_{X_1^nX_2^nX_3^nM_0M_wM_uM_v}$. The objective is to find rates (R_0, R_w, R_u, R_v) for which there exists a sequence of encoders f_u^n, f_v^n, f_w^n such that

$$
\begin{aligned}
&\lim_{n\to\infty}\mathbb{D}\big(\hat{p}_{X_1^nX_2^nX_3^n}\|q_{X_1^nX_2^nX_3^n}\big) = 0 &\text{(resolvability)}\\
&\lim_{n\to\infty}\mathbb{D}\big(\hat{p}_{X_1^nM_0}\|\hat{p}_{X_1^n}\hat{p}_{M_0}\big) = 0 &\text{(secrecy).}
\end{aligned}
$$

We now randomly generate a code as follows:

- we generate $2^{n(R_0+R_w)}$ sequences independently, which we label $w_{ij}^n \triangleq (w_{ij,1}, \cdots, w_{ij,n})$ for $i \in [\![1, 2^{nR_0}]\!]$ and $j \in [\![1, 2^{nR_w}]\!]$, according to $\prod_{m=1}^n p_W(w_{ij,m})$;

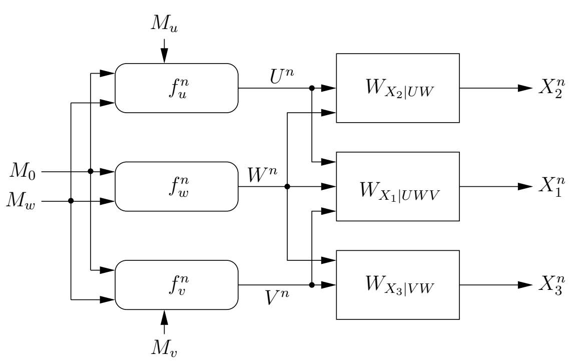

Fig. 3. Intermediate problem: channel resolvability with secrecy constraints.

- for each w_{ij}^n, we generate 2^{nR_u} sequences independently, which we label $u_{ijk}^n \triangleq (u_{ijk,1}, \cdots, u_{ijk,n})$ for $k \in [\![1, 2^{nR_u}]\!]$, according to $\prod_{m=1}^n p_{U|W}(u_{ijk,m}|w_{ij,m})$;
- for each w_{ij}^n, we generate 2^{nR_v} sequences independently, which we label $v_{ij\ell}^n \triangleq (v_{ij\ell,1}, \cdots, v_{ij\ell,n})$ for $\ell \in [\![1, 2^{nR_v}]\!]$, according to $\prod_{m=1}^n p_{V|W}(v_{ij\ell,m}|w_{ij,m})$.

The indices of the codewords implicitly define the mapping from messages to codewords.

We introduce the shorthand notation $Z^n = X_1^nX_2^nX_3^n$ and $M \triangleq 2^{n(R_0+R_w+R_u+R_w)}$ to simplify expressions, and we proceed to bound $\mathbb{E}(\mathbb{D}(\hat{p}_{Z^n}\|q_{Z^n}))$, where the expectation is over the random code generation. Recalling that

$$\hat{p}_{Z^n}(z^n) \triangleq \frac{1}{M}\sum_{ijk\ell} W_{Z^n|U^nV^nW^n}(z^n|u^n_{ijk}, v^n_{ij\ell}, w^n_{ij}),$$

we use the law of iterated expectation and Jensen's inequality to upper bound $\mathbb{E}(\mathbb{D}(\hat{p}_{Z^n}\|q_{Z^n}))$ as shown in Eq. (1) to (3) at the top of this page. We break down the sum in the term

$$\frac{1}{M}\sum_{\tilde{i}\tilde{j}\tilde{k}\tilde{\ell}}\mathbb{E}\left(\frac{W(z^n|U^n_{\tilde{i}\tilde{j}\tilde{k}},V^n_{\tilde{i}\tilde{j}\tilde{\ell}},W^n_{\tilde{i}\tilde{j}})}{q_{Z^n}(z^n)}\middle| W^n_{ij}U^n_{ijk}V^n_{ij\ell}\right)$$

in four distinct cases. If $(\tilde{i}, \tilde{j}, \tilde{k}, \tilde{\ell}) = (i, j, k, \ell)$, we obtain

$$\frac{1}{M}\frac{W(z^n|u^n_{ijk}, v^n_{ij\ell}, w^n_{ij})}{q_{Z^n}(z^n)}.$$

If $(\tilde{i}, \tilde{j}) = (i, j)$, $\tilde{k} \neq k$ and $\tilde{\ell} = \ell$, we obtain

$$
\begin{aligned}
&\frac{1}{M}\sum_{\tilde{k}\neq k}\mathbb{E}_{U_{ij\tilde{k}}}\left(\frac{W(z^n|U^n_{ij\tilde{k}}, v^n_{ij\ell}, w^n_{ij})}{q_{Z^n}(z^n)}\middle| W^n_{ij}\right)\\
&\qquad = \frac{1}{M}\sum_{\tilde{k}\neq k}\frac{p(z^n|v^n_{ij\ell}w^n_{ij})}{q_{Z^n}(z^n)} \leqslant \frac{2^{nR_u}}{M}\frac{p(z^n|v^n_{ij\ell}w^n_{ij})}{q_{Z^n}(z^n)},
\end{aligned}
$$

where the equality follows by Bayes' rule and the law of total probability. By symmetry, if $(\tilde{i}, \tilde{j}) = (i, j)$, $\tilde{\ell} \neq \ell$ and $\tilde{k} = k$, we obtain

$$
\begin{aligned}
&\frac{1}{M}\sum_{\tilde{\ell}\neq\ell}\mathbb{E}_{V_{ij\tilde{\ell}}}\left(\frac{W(z^n|u^n_{ijk}, V^n_{ij\tilde{\ell}}, w^n_{ij})}{q_{Z^n}(z^n)}\middle| W^n_{ij}\right)\\
&\qquad = \frac{1}{M}\sum_{\tilde{\ell}\neq\ell}\frac{p(z^n|u^n_{ijk}w^n_{ij})}{q_{Z^n}(z^n)} \leqslant \frac{2^{nR_v}}{M}\frac{p(z^n|u^n_{ijk}w^n_{ij})}{q_{Z^n}(z^n)}.
\end{aligned}
$$

$$\mathbb{E}(\mathbb{D}(\hat{p}_{Z^n}\|q_{Z^n})) \leqslant \sum_{z^n}\sum_{u^n}\sum_{v^n}\sum_{w^n} p(z^n,u^n,w^n,v^n) \log\left(\frac{1}{M}\frac{W(z^n|u^n,v^n,w^n)}{q_{Z^n}(z^n)} + \frac{2^{nR_u}}{M}\frac{p(z^n|v^nw^n)}{q_{Z^n}(z^n)} + \frac{2^{nR_v}}{M}\frac{p(z^n|u^nw^n)}{q_{Z^n}(z^n)} + \frac{2^{n(R_u+R_v)}}{M}\frac{p(z^n|w^n)}{q_{Z^n}(z^n)} + 1\right) \quad (4)$$

If $(\tilde{i},\tilde{j})=(i,j)$, $\tilde{\ell}\neq\ell$ and $\tilde{k}\neq k$, we obtain

$$\frac{1}{M}\sum_{\tilde{\ell}\neq\ell}\sum_{k\neq\tilde{k}}\mathbb{E}_{U_{ij\tilde{k}}V_{ij\tilde{\ell}}}\left(\frac{W(z^n|U^n_{ij\tilde{k}},V^n_{ij\tilde{\ell}},w^n_{ij})}{q_{Z^n}(z^n)}\middle| W^n_{ij}\right) = \frac{1}{M}\sum_{\tilde{\ell}\neq\ell}\sum_{\tilde{k}\neq k}\frac{p(z^n|w^n_{ij})}{q_{Z^n}(z^n)} \leqslant \frac{2^{n(R_u+R_v)}}{M}\frac{p(z^n|w^n_{ij})}{q_{Z^n}(z^n)}.$$

Finally, if $(\tilde{i},\tilde{j})\neq(i,j)$, and for any $\tilde{k}$ and $\tilde{\ell}$, we obtain

$$\frac{1}{M}\sum_{(\tilde{i},\tilde{j})\neq(i,j)}\sum_{\ell}\sum_{k}\mathbb{E}\left(\frac{W(z^n|U^n_{\tilde{i}\tilde{j}k},V^n_{\tilde{i}\tilde{j}\ell},W^n_{\tilde{i}\tilde{j}})}{q_{Z^n}(z^n)}\right) = \frac{1}{M}\sum_{(\tilde{i},\tilde{j})\neq(i,j)}\sum_{\ell}\sum_{k}\frac{q_{Z^n}(z^n)}{q_{Z^n}(z^n)} \leqslant 1.$$

Combining all these inequalities above, and since all codewords are generated according to the same distribution, we obtain the bound on $\mathbb{E}(\mathbb{D}(\hat{p}_{Z^n}\|q_{Z^n}))$ shown in Eq. (4). We then split the sum between the δ-typical sequences $(z^n,u^n,v^n,w^n)\in T^n_\delta(p_{ZUVW})$ and $(z^n,u^n,v^n,w^n)\notin T^n_\delta(p_{ZUVW})$. One can show that the sum over the sequences $(z^n,u^n,v^n,w^n)\notin T^n_\delta(p_{ZUVW})$ is upper bounded by

$$\mathbb{P}((Z^n,U^n,V^n,W^n)\notin T^n_\delta(p_{ZUVW}))\log\left(4\left(\frac{1}{\mu_Z}\right)^n+1\right),$$

with $\mu_z=\min_{z\in\text{supp}(q_Z)}q_Z(z)$. This term converges to 0 exponentially fast with n [12]. When summing over the sequences $(z^n,u^n,v^n,w^n)\in T^n_\delta(p_{ZUVW})$, one can upper bound the various terms using [12]. For instance, we have

$$\frac{1}{M}\frac{W(z^n|u^n,v^n,w^n)}{q_{Z^n}(z^n)} \leqslant \frac{1}{2^{n(R_0+R_w+R_u+R_v)}}\frac{2^{-n\mathbb{H}(Z|UVW)(1+\delta)}}{2^{-n\mathbb{H}(Z)(1-\delta)}} \leqslant \frac{2^{n\mathbb{I}(UVW;Z)+2\delta\mathbb{H}(Z)}}{2^{n(R_0+R_w+R_u+R_v)}},$$

which converges to 0 exponentially if

$$R_0+R_w+R_u+R_v > \mathbb{I}(UVW;Z)+2\delta\mathbb{H}(Z).$$

By analyzing all remaining terms, we find that $\mathbb{E}(\mathbb{D}(\hat{p}_{Z^n}\|q_{Z^n}))$ converges exponentially to 0 with n if

$$\begin{cases} R_0+R_w+R_u+R_v > \mathbb{I}(UVW;Z)+2\delta\mathbb{H}(Z)\,,\\ R_0+R_w+R_u > \mathbb{I}(UW;Z)+2\delta\mathbb{H}(Z)\,,\\ R_0+R_w+R_v > \mathbb{I}(VW;Z)+2\delta\mathbb{H}(Z)\,,\\ R_0+R_w > \mathbb{I}(W;Z)+2\delta\mathbb{H}(Z)\,. \end{cases} \quad (5)$$

By remarking that

$$\mathbb{D}\big(\hat{p}_{X_1^nM_0}\|\hat{p}_{X_1^n}\hat{p}_{M_0}\big) = \frac{1}{2^{nR_0}}\sum_{m_0}\mathbb{D}\big(\hat{p}_{X_1^n|M_0=m_0}\|\hat{p}_{X_1^n}\big) \leqslant \frac{1}{2^{nR_0}}\sum_{m_0}\mathbb{D}\big(\hat{p}_{X_1^n|M_0=m_0}\|q_{X_1^n}\big),$$

the analysis of $\mathbb{D}\big(\hat{p}_{X_1^nM_0}\|\hat{p}_{X_1^n}\hat{p}_{M_0}\big)$ is performed in a similar manner.[2] Replacing Z^n by X_1^n and dropping the summation over $i\in[\![1,2^{nR_0}]\!]$ in the previous analysis, we obtain the following sufficient condition to ensure that $\mathbb{D}\big(\hat{p}_{X_1^nM_0}\|\hat{p}_{X_1^n}\hat{p}_{M_0}\big)$ converges to 0 exponentially fast with n:

$$\begin{cases} R_w+R_u+R_v > \mathbb{I}(UVW;X_1)+2\delta\mathbb{H}(X_1)\,,\\ R_w+R_u > \mathbb{I}(UW;X_1)+2\delta\mathbb{H}(X_1)\,,\\ R_w+R_v > \mathbb{I}(VW;X_1)+2\delta\mathbb{H}(X_1)\,,\\ R_w > \mathbb{I}(W;X_1)+2\delta\mathbb{H}(X_1)\,. \end{cases} \quad (6)$$

By Markov's inequality, we conclude that if the rate-tuple (R_0,R_w,R_u,R_v) satisfies the constraints in Eq. (5) and Eq. (6), there exists a sequence of codes satisfying the resolvability and secrecy constraints. By Pinsker's inequality, this also implies that

$$\lim_{n\to\infty}\mathbb{V}\big(\hat{p}_{X_1^nX_2^nX_3^n},q_{X_1^nX_2^nX_3^n}\big)=0 \quad \text{(resolvability)} \quad (7)$$
$$\lim_{n\to\infty}\mathbb{V}\big(\hat{p}_{X_1^nM_0},\hat{p}_{X_1^n}\hat{p}_{M_0}\big)=0 \quad \text{(secrecy).} \quad (8)$$

B. Achievable line coordination rates

We construct a line coordination scheme from the code for the intermediate problem in Section III-A as follows.

- Upon observing m_0 and x_1^n, agent 1 generates m_w, m_v, m_u according to $\hat{p}(m_w,m_v,m_u|x_1^n,m_0)$; agent 1 transmits m_w, m_v, m_u to agent 2, which defines f_{12}^n;
- Upon receiving m_0, m_w, m_v, m_u, agent 2 generates x_2^n according to $W(x_2^n|u^n_{m_0m_wm_u}w^n_{m_0m_w})$, which defines g_2^n; agent 2 forwards m_w, m_v to agent 3, which defines f_{23}^n.
- Upon receiving m_0, m_w, m_v, agent 3 generates x_3^n according to $W(x_3^n|v^n_{m_0m_wm_v}w^n_{m_0m_w})$, which defines g_3^n.

This scheme induces a joint probability distribution $\tilde{p}_{X_1^nX_2^nX_3^nM_0M_wM_uM_v}$. Using the triangle inequality,

$$\mathbb{V}\big(\tilde{p}_{X_1^nX_2^nX_3^n},q_{X_1^nX_2^nX_3^n}\big) \leqslant \mathbb{V}\big(\tilde{p}_{X_1^nX_2^nX_3^n},\hat{p}_{X_1^nX_2^nX_3^n}\big) + \mathbb{V}\big(\hat{p}_{X_1^nX_2^nX_3^n},q_{X_1^nX_2^nX_3^n}\big). \quad (9)$$

[2] This is where the unfortunate error in the analysis of [7] appears, as the absence of randomization over M_w in the encoding of W^n prevents one obtain the desired independence of M_0 and X_1^n.

The distribution $\hat{p}$ only differs from $\tilde{p}$ because M_0 may not be independent of X_1^n; fortunately, Eq. (8) guarantees near independence so that the first term in the right-hand side of Eq. (9) vanishes. The second term vanishes because of Eq. (7). Hence, we have constructed a line coordination schemes that operates with rate R_0, $R_{12} = R_w + R_u + R_v$, and $R_{23} = R_w + R_v$. Since R_v only appears through the sum-rate $R_w + R_v$, we set $V = W$ and $R_v = 0$ in Eq. (5) and Eq. (6), and we conclude that the rate-tuples $(R_0, R_{12}, R_{13} = 0, R_{23}, R_{32} = 0)$ are achievable with a line coordination scheme if

$$\begin{cases} R_0 + R_{12} > \mathbb{I}(UW; X_1X_2X_3) \\ R_0 + R_{23} > \mathbb{I}(W; X_1X_2X_3) \\ R_{12} > \mathbb{I}(UW; X_1) \\ R_{23} > \mathbb{I}(W; X_1). \end{cases}$$

In particular, if we assume that R_0 is large enough so that the first two constraints are always satisfied, and if we substitute $U \triangleq X_2$ and $W \triangleq X_3$, which trivially satisfy the constraints in Section III, we obtain

$$R_{12} > \mathbb{I}(X_2X_3; X_1) \quad R_{23} > \mathbb{I}(X_3; X_1).$$

By swapping the roles of agent 2 and agent 3, U and V, and messages m_u and m_v, we conclude that the rate-tuples $(R_0, R_{12} = 0, R_{13}, R_{23} = 0, R_{32})$ are achievable with a line coordination scheme if

$$\begin{cases} R_0 + R_{13} > \mathbb{I}(VW; X_1X_2X_3) \\ R_0 + R_{32} > \mathbb{I}(W; X_1X_2X_3) \\ R_{13} > \mathbb{I}(VW; X_1) \\ R_{32} > \mathbb{I}(W; X_1). \end{cases}$$

Assuming R_0 is always large enough, and if we substitute $V \triangleq X_3$ and $W \triangleq X_2$, we obtain

$$R_{13} > \mathbb{I}(X_2X_3; X_1) \quad R_{32} > \mathbb{I}(X_2; X_1).$$

C. Achievable broadcast coordination rates

We now construct a broadcast coordination scheme from the code for the intermediate problem in Section III-A as follows.

- Upon observing m_0 and x_1^n, agent 1 generates m_w, m_v, m_u according to $\hat{p}(m_w, m_v, m_u | x_1^n, m_0)$; agent 1 transmits m_w, m_u to agent 2, which defines f_{12}^n; agent 1 transmits m_w, m_v to agent 3, which defines f_{13}^n;
- Upon receiving m_0, m_w, m_u, agent 2 generates x_2^n according to $W(x_2^n | u^n_{m_0 m_w m_u} w^n_{m_0 m_w})$, which defines g_2^n;
- Upon receiving m_0, m_w, m_v, agent 3 generates x_3^n according to $W(x_3^n | v^n_{m_0 m_w m_v} w^n_{m_0 m_w})$, which defines g_3^n.

As in Section III-B, this defines a coordination scheme that operates with rate R_0, $R_{12} = R_w + R_u$, $R_{13} = R_w + R_u$. We conclude that the rate-tuples $(R_0, R_{12}, R_{13}, R_{23} = 0, R_{32} = 0)$ are achievable with a broadcast coordination scheme if R_0, R_W, R_u, R_v satisfy the constraints in Eq. (5) and Eq. (6); unfortunately, in general, these constraints do not simplify to a simple region in terms of R_0, R_{12}, R_{23} only.

If we assume that R_0 is always large enough, and if we substitute $V \triangleq X_3$ and $U \triangleq W \triangleq X_2$ in the constraints, we obtain the following set of achievable (R_{12}, R_{13}):

$$R_{12} > \mathbb{I}(X_2; X_1) \quad R_{13} > \mathbb{I}(X_2X_3; X_1).$$

Similarly, by setting $V \triangleq W \triangleq X_3$ and $U \triangleq X_2$ in the constraints, we obtain

$$R_{12} > \mathbb{I}(X_2X_3; X_1) \quad R_{13} > \mathbb{I}(X_3; X_1).$$

D. Achievable relay coordination rates

Finally, we highlight an example of relay coordination scheme constructed from the code for the intermediate problem in Section III-A as follows.

- Upon observing m_0 and x_1^n, agent 1 generates m_w, m_v, m_u according to $\hat{p}(m_w, m_v, m_u | x_1^n, m_0)$; agent 1 transmits m_w, m_u to agent 2, which defines f_{12}^n; agent 1 transmits m_v to agent 3, which defines f_{13}^n;
- Upon receiving m_0, m_w, m_u, agent 2 generates x_2^n according to $W(x_2^n | u^n_{m_0 m_w m_u} w^n_{m_0 m_w})$, which defines g_2^n; agent 2 transmits m_w to agent 3, which defines f_{23}^n;
- Upon receiving m_0, m_w, m_v, agent 3 generates x_3^n according to $W(x_3^n | v^n_{m_0 m_w m_v} w^n_{m_0 m_w})$, which defines g_3^n.

This defines a coordination scheme that operates with rate R_0, $R_{12} = R_w + R_u$, $R_{13} = R_v$, $R_{23} = R_w$. As in Section III-C, the rate constraints in Eq. (5) and Eq. (6) are not amenable to simplifications that would express the achievable region in terms of R_0, R_{12}, R_{13}, R_{23} only.

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