TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 225, 1977

STRONG DIFFERENTIABILITY PROPERTIES OF BESSEL POTENTIALS

BY

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ABSTRACT. This paper is concerned with the "strong" L_p differentiability properties of Bessel potentials of order $\alpha > 0$ of L_p functions. Thus, for such a function f, we investigate the size (in the sense of an appropriate capacity) of the set of points x for which there is a polynomial $P_x(y)$ of degree $k \leq \alpha$ such that

$$\limsup_{\operatorname{diam}(S)\to 0} (\operatorname{diam} S)^{-k}\left\{|S|^{-1}\int |f(y) - P_x(y)|^p\,dy\right\}^{1/p} = 0$$

where, for example, S is allowed to run through the family of all oriented rectangles containing the origin.

1. Introduction. Let $\mathfrak{F}$ be a family of measurable sets in $\mathbf{R}^n$ which satisfy the following properties:

(1)
(i) $m(S) > 0$ for $S \in \mathfrak{F}$,
(ii) for every $\varepsilon > 0$, there is a set $S \in \mathfrak{F}$ such that $S \subset B(0,\varepsilon)$,
(iii) $\bigcap\{\bar{S}: S \in \mathfrak{F}\} = \{0\}$.

Here m denotes Lebesgue measure and $B(x,r)$ denotes the open n-ball centered at x with radius r. In this paper we shall investigate the "strong" L_p differentiability properties of a certain class of functions with respect to such a family $\mathfrak{F}$. Specifically, given a function f we are concerned with the existence of a polynomial $P_x(y)$ of degree k such that

$$\limsup_{\delta(S)\to 0} \delta(S)^{-k}\left\{m(S)^{-1}\int_{S+x} |f(y) - P_x(y)|^p\,dy\right\}^{1/p} = 0 \tag{2}$$

where it is understood that $S \in \mathfrak{F}$. The diameter of a set is denoted by $\delta(S)$ and $S + x = \{y + x: y \in S\}$. Whenever (2) is satisfied we shall say that $f \in t_p^k(x)$ with respect to the family $\mathfrak{F}$. This concept was introduced in [CZ] for the special case when $\mathfrak{F}$ is the family of balls centered at the origin.

Received by the editors July 14, 1975.
AMS (MOS) subject classifications (1970). Primary 26A54; Secondary 28A10.
[1] Research partially supported by a grant from the National Science Foundation.

We were led to the problem of strong L_p differentiability by certain questions that arose in the study of boundary regularity of solutions of parabolic differential equations [D]. In that study the family $\mathfrak{F}$ under consideration was the class of parabolic rectangles of the form $B(0, r) \times (0, r^2)$. This family is "irregular" in the sense that for every $\varepsilon > 0$, there is a set $S \in \mathfrak{F}$ with $m(s) \cdot m[B(0, \delta(S))]^{-1} < \varepsilon$. As in [D], we shall be primarily concerned with the size of the set where (2) fails to hold.

We will consider the question of strong L_p differentiability of functions in the space of Bessel potentials $\mathfrak{L}_p^\alpha = \{G_\alpha * g: g \in L_p(\mathbf{R}^n)\}$. Here G_α denotes the Bessel kernel of order $\alpha > 0$ and $*$ denotes the usual convolution operation. $\mathfrak{L}_p^\alpha$ is a Banach space under the norm $\|G_\alpha * f\|_{\mathfrak{L}_p^\alpha} = \|g\|_{L_p}$. When $p > 1$ and α is a positive integer, $\mathfrak{L}_p^\alpha$ is isometric to the Sobolev space W_p^α. For details see [ST].

If $\mathfrak{F} = \mathfrak{B}$, the family of balls centered at the origin, then the classical theorem of Lebesgue can be rephrased to say that $f \in t_1^0(x)$ for almost every x, whenever f is locally integrable on $\mathbf{R}^n$. For this same family $\mathfrak{B}$, higher order differentiability was first considered by Calderón and Zygmund in [CZ], where they prove that if $f \in \mathfrak{L}_p^k$ for integer k, then $f \in t_p^k(x)$ for almost every $x \in \mathbf{R}^n$. This result has recently been improved by considering $f \in \mathfrak{L}_p^\alpha$ for arbitrary α and by using an appropriate capacity (or Hausdorff measure) to measure the size of the exceptional sets (cf. [FZ], [BZ], [M2]). All of these results continue to hold if $\mathfrak{B}$ is replaced by any family $\mathfrak{F}$ that is "regular".

One of the first results to deal with an "irregular" family of sets is due to Zygmund [Z1]. In that paper he considers $\mathfrak{F} = \mathfrak{R}$, the family of all oriented rectangles which contain the origin. He proves that $f \in t_1^0(x)$ with respect to $\mathfrak{R}$ for almost every x provided that $f \in L_p(\mathbf{R}^n)$ for some $p > 1$. This assumption can be weakened slightly by requiring only that

$$\int_{\mathbf{R}^n} |f|(\log^+|f|)^{n-1}$$

be finite. See [JMZ] or [Z2] for a complete discussion. However, some integrability requirement is necessary, for Saks [S] has provided an example of a function $f \in L_1$ such that $f \notin t_1^0(x)$ for each $x \in \mathbf{R}^n$.

A final result worth mentioning is due to Riviere [R]. He considers a Vitali family $\mathcal{V} = \{U_\alpha\}_{\alpha>0}$ of bounded open sets with the following properties:

1. $\alpha < \beta \Rightarrow U_\alpha \subset U_\beta$ (nestedness),
2. $\bigcap_{\alpha>0} \overline{U}_\alpha = \{0\}$,
3. $m(U_\alpha - U_\alpha) \leqslant A\, m(U_\alpha)$ for some constant A, independent of α,
4. $m(U_\alpha)$ is left continuous as a function of α.

Here $U_\alpha - U_\alpha = \{x - y: x, y \in U_\alpha\}$. He then proves that for any locally integrable function f, $f \in t_1^0(x)$ with respect to $\mathcal{V}$ for almost every $x \in \mathbf{R}^n$.

In this paper we shall first generalize the results of Zygmund and Riviere to the function classes $\mathfrak{L}_p^\alpha$, and prove that the exceptional set is null with respect to an appropriate capacity. We then consider the question of higher order differentiability. Theorem 2 will show, for example, that the results of **[BZ]** and **[M2]** will hold for the family $\mathfrak{R}$ as well as $\mathfrak{B}$, provided we assume that $p > 1$.

2. **First order differentiability.** The Bessel potential space $\mathfrak{L}_p^\alpha$ was defined above. The Bessel capacity associated with this function space is defined as follows. For any set $E \subset \mathbf{R}^n$, define

$$B_{\alpha,p}(E) = \operatorname{Inf}\{\|g\|_{L_p}^p : g \in L_p^+, G_\alpha * g \geqslant 1 \text{ on } E\}.$$

This set function is an outer measure on $\mathbf{R}^n$ and moreover $B_{\alpha,p}(E) = \operatorname{Inf}\{B_{\alpha,p}(G): G \supset E$ and G is open$\}$. If $\alpha = 0$ we set $\mathfrak{L}_p^0 = L_p$ and $B_{0,p} = m$. If $p > 1$ and $\alpha p < n$, then $B_{\alpha,p}$ is related to Hausdorff measure of dimension $n - \alpha p$ by the following relations:

1. $H^{n-\alpha p}(E) = 0 \Rightarrow B_{\alpha,p}(E) = 0$,
2. $B_{\alpha,p}(E) = 0 \Rightarrow H^{n-\alpha p+\varepsilon}(E) = 0$ for any $\varepsilon > 0$.

See Meyers **[M1]** for further details.

For any family $\mathfrak{F}$ defined as in §1, we introduce the maximal operator

$$M_{\mathfrak{F}} f(x) = \operatorname{Sup}\left\{\frac{1}{M(S)}\int_{S+x} f(y)\,dy : S \in \mathfrak{F}\right\}$$

defined for every $f \in L_1(\mathbf{R}^n)$. Often $\mathfrak{F}$ may be chosen to contain only bounded sets, in which case f need be only locally integrable. To prove that functions from a Lebesgue class $L_p(\mathbf{R}^n)$ are in $t_p^0(x)$ almost everywhere with respect to $\mathfrak{F}$, it suffices to prove a weak type (1, 1) estimate on the maximal operator, i.e.

$$m\{M_{\mathfrak{F}} f(x) > t\} \leqslant C\|f\|_{L_1}/t \tag{3}$$

for some constant C independent of $f \in L_1(\mathbf{R}^n)$. Given (3), we choose functions $f_n \in C^\infty(\mathbf{R}^n)$ such that $\|f - f_n\|_{L_p} \to 0$ and observe that

$$\frac{1}{m(S)}\int_{S+x} |f(y) - f(x)|^p\,dy \leqslant \frac{C_p}{m(S)}\int_{S+x} |f(y) - f_n(y)|^p\,dy$$
$$+ \frac{C_p}{m(S)}\int_{S+x} |f_n(y) - f_n(x)|^p\,dy + C_p|f_n(x) - f(x)|^p.$$

The first term on the right is dominated by $C_p M_{\mathfrak{F}}(|f - f_n|^p)$, and since f_n is continuous we have

$$\overline{\lim_{\delta(S)\to 0}}\, \frac{1}{m(S)}\int_{S+x} |f(y) - f(x)|^p\,dy \leqslant C_p M_{\mathfrak{F}}(|f - f_n|^p) + C_p|f_n(x) - f(x)|^p.$$

In view of (3) and the fact that $f_n \to f$ in measure, we can choose n sufficiently large so that the right side is less than ε except on a set of measure less then ε, hence the left side is zero almost everywhere.

For functions in $\mathfrak{L}_p^\alpha$ we will prove that in fact the same conclusion holds $B_{\alpha,p}$ almost everywhere, by replacing estimate (3) by a capacitary weak type inequality. This can be done provided the maximal operator is a bounded operator on L_p (see Adams [A]). The following result was first proved in [D] for the case $\mathfrak{F} = \mathfrak{R}$.

THEOREM 1. *Let $\mathfrak{F}$ be a family of measurable sets satisfying conditions* (1), *and suppose $M_{\mathfrak{F}}$ is of strong type (p, p). That is,*

$$\|M_{\mathfrak{F}}f\|_{L_p} \leqslant C\|f\|_{L_p} \tag{4}$$

for some constant C independent of $f \in L_p(\mathbf{R}^n)$, $1 \leqslant p < \infty$. If $f \in \mathfrak{L}_p^\alpha$, $\alpha \geqslant 0$, then $f \in t_1^0(x)$ with respect to $\mathfrak{F}$ for $B_{\alpha,p}$ almost every $x \in \mathbf{R}^n$.

PROOF. The result for $\mathfrak{L}_p^0 = L_p$ is precisely that detailed above, so let $\alpha > 0$ and let $f = G_\alpha * g \in \mathfrak{L}_p^\alpha$. We first make use of the strong type estimate (4) to prove the following capacitary weak type estimate

$$B_{\alpha,p}\{M_{\mathfrak{F}}f(x) > t\} \leqslant (C/t^p)\|f\|_{\mathfrak{L}_p^\alpha}^p. \tag{5}$$

This follows by observing that $M_{\mathfrak{F}}f \leqslant G_\alpha * M_{\mathfrak{F}}g$ and, hence,

$$\begin{aligned} B_{\alpha,p}\{M_{\mathfrak{F}}f > t\} &\leqslant B_{\alpha,p}\{G_\alpha * M_{\mathfrak{F}}g > t\} = B_{\alpha,p}\{G_\alpha * (M_{\mathfrak{F}}g/t) > 1\} \\ &\leqslant t^{-p}\|M_{\mathfrak{F}}g\|_{L_p}^p \leqslant Ct^{-p}\|g\|_{L_p}^p = Ct^{-p}\|f\|_{L_p^\alpha}^p \end{aligned}$$

using the definition of capacity and estimate (4). Secondly, we will show that any function $f \in \mathfrak{L}_p^\alpha$ can be approximated by continuous functions $f_\varepsilon \in \mathfrak{L}_p^\alpha$ such that

$$\|f_\varepsilon - f\|_{\mathfrak{L}_p^\alpha} \to 0 \quad \text{as } \varepsilon \to 0 \tag{6}$$

and also that

$$B_{\alpha,p}\{x: |f_\varepsilon - f|(x) > \delta\} \to 0 \quad \text{as } \varepsilon \to 0 \text{ for every } \delta > 0. \tag{7}$$

To this end we use a standard mollifier argument. Let $\phi \in C_c^\infty(\mathbf{R}^n)$ be positive with $\int_{\mathbf{R}^n} \phi = 1$. Set $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(x/\varepsilon)$ and define

$$f_\varepsilon = \phi_\varepsilon * f = G_\alpha * (\phi_\varepsilon * g).$$

Then

$$\|f_\varepsilon - f\|_{\mathfrak{L}_p^\alpha} = \|G_\alpha * (G_\varepsilon - G)\|_{\mathfrak{L}_p^\alpha} = \|g_\varepsilon - g\|_{L_p},$$

and since $g_\varepsilon \to g$ in L_p, (6) is proved. Also notice that

$$B_{\alpha,p}\{x\colon |f_\varepsilon(x) - f(x)| > \delta\} \leqslant B_{\alpha,p}\{x\colon G_\alpha * |g_\varepsilon - g|(x) > \delta\}$$
$$\leqslant \delta^{-p}\|g_\varepsilon - g\|^p_{L_p^\alpha} \to 0 \quad \text{as } \varepsilon \to 0$$

so that (7) holds as well. Now fix $\eta > 0$ and choose ε so small that $\|f_\varepsilon - f\|^p_{\mathfrak{L}_p^\alpha} \leqslant \eta^{p+1}/C$, where C is the constant of (4), such that

$$B_{\alpha,p}\{x\colon |f_\varepsilon - f|(x) > \eta\} < \eta.$$

Then

$$\frac{1}{m(S)}\int_{S+x} |f(y) - f(x)|\,dy \leqslant \frac{1}{m(S)}\int_{S+x} |f(y) - f_\varepsilon(y)|\,dy$$
$$+\frac{1}{m(S)}\int_{S+x} |f_\varepsilon(y) - f_\varepsilon(x)|\,dy + |f_\varepsilon(x) - f(x)|.$$

Note that the last term is $<\eta$ except on a set of $B_{\alpha,p}$ capacity $<\eta$. The first term is bounded by $M_{\mathfrak{F}}(|f - f_\varepsilon|)$, which, according to (5) is less than η except on a set of $B_{\alpha,p}$ capacity less than $(C/\eta^p)\|f - f_\varepsilon\|^p_{\mathfrak{L}_p^\alpha} < \eta$. The middle term becomes negligible as $\delta(S) \to 0$ since f_ε is continuous. Altogether, we conclude

$$\overline{\lim_{\delta(S)\to 0}}\ \frac{1}{m(S)}\int_{S+x} |f(y) - f(x)|\,dy < 2\eta$$

except on a set of $B_{\alpha,p}$ capacity less than 2η. Since η is arbitrary, this proves the theorem. Notice that we have specified the constant polynomial P_x to be $f(x)$, which is defined, of course, $B_{\alpha,p}$ almost everywhere. Q.E.D.

Corollary 1. *If $f \in \mathfrak{L}_p^\alpha$ with $\alpha > 0$ and $1 < p < \infty$, then $f \in t_1^0(x)$ with respect to $\mathfrak{F}$ for $B_{\alpha,p}$ almost every x if*

1. *$\mathfrak{F} = \mathfrak{B}$, the family of all balls centered at the origin,*
2. *$\mathfrak{F} = \mathfrak{R}$, the family of all rectangles containing the origin,*
3. *$\mathfrak{F} = \mathfrak{V}$, the family defined in* §1.

Proof. In view of Theorem 1, we need only verify hypothesis (4) for each family. For $\mathfrak{B}$, this result is well known and appears, for example, in Stein **[ST]**. For $\mathfrak{R}$, estimate (4) is proved in **[JMZ]**.

The proof of (4) for the family $\mathfrak{V}$ follows from the weak type (1, 1) estimate

$$m\{x\colon M_{\mathfrak{V}}f(x) > t\} \leqslant (C/t)\|f\|_{L_1} \tag{8}$$

for $f \in L_1(\mathbf{R}^n)$. This extimate is proved in **[R]**. To prove (4) from this, we use the familiar argument which appears, for example, on p. 7 of **[ST]**. Set

$$f_s(x) = \begin{cases} f(x) & \text{if } |f(x)| \geqslant s/2, \\ 0 & \text{otherwise}. \end{cases}$$

Then $M_{\mathcal{V}}f(x) \leqslant M_{\mathcal{V}}f_s(x) + s/2$ and so

$$m\{x\colon M_{\mathcal{V}}f(x) > s\} \leqslant m\{x\colon M_{\mathcal{V}}f_s(x) > s/2\} \leqslant (2C/s)\|f_s\|_{L_1}$$

by (8) since $f_s \in L_1(\mathbf{R}^n)$. If we set $\lambda(s) = m\{x\colon M_{\mathcal{V}}f(x) > s\}$, then

$$\begin{aligned}\int_{\mathbf{R}^n} |M_{\mathcal{V}}f|^p\,dx &= p\int_0^\infty s^{p-1}\lambda(s)\,ds \leqslant p\int_0^\infty 2Cs^{p-2}\|f_s\|_{L_1}\,ds \\ &= 2Cp\int_0^\infty s^{p-2}\int_{|f|>s/2}|f(x)|\,dx\,ds = 2Cp\int_{\mathbf{R}^n}|f(x)|\int_0^{2|f(x)|}s^{p-2}\,ds\,dx \\ &= \frac{2Cp}{p-1}\int_{\mathbf{R}^n}|f(x)|2|f(x)|^{p-1}\,dx = \frac{4Cp}{p-1}\|f\|_{L_p}^p\end{aligned}$$

which verifies (4) and completes the proof of the corollary.

3. **Higher order differentiability.** In this section we shall discuss the concept of higher order differentiability with respect to irregular families. Theorem 2 will imply, for example, that $\mathfrak{L}_p^\alpha$ functions are in $t_1^k(x)$ with respect to $\mathfrak{R}$ for $B_{\alpha-k,p}$ almost every $x \in \mathbf{R}^n$. Unfortunately, the additional assumption we shall make in Theorem 2 does not apply to arbitrary families $\mathcal{V}$, and the question of higher order differentiability with respect to such a family remains open.

THEOREM 2. *Let $\mathfrak{F}$ be a family of measurable sets satisfying conditions* (1) *and suppose in addition that $\mathfrak{F}$ is closed under dilations. That is, if* $0 < t < 1$, $t \cdot S = \{ty\colon y \in S\}$ *is in $\mathfrak{F}$ whenever S is. Suppose $M_{\mathfrak{F}}$ satisfies condition* (4).

If $f \in \mathfrak{L}_p^\alpha$, $\alpha \geqslant 1$, and if $k \leqslant \alpha$ is an integer, then $f \in t_1^k(x)$ with respect to $\mathfrak{F}$ for $B_{\alpha-k,p}$ almost every $x \in \mathbf{R}^n$.

In order to prove the theorem, we shall make use of the following result, which is an improvement of a similar result proved in **[GZ]**.

LEMMA 1. *Let $f \in \mathfrak{L}_p^\alpha$ with $1 \leqslant \alpha < \infty$. Then, for $B_{\alpha,p}$ almost every $x \in \mathbf{R}^n$, f is absolutely continuous on H^{n-1} almost every ray emanating from x. Moreover, on such a ray,*

$$f(x+z) - f(x) = \int_0^1 \nabla f(x+tz)\cdot z\,dt.$$

PROOF OF LEMMA 1. Let $f = g_\alpha * g \in \mathfrak{L}_p^\alpha$ and let $f_i = G_\alpha * g_i$ be C^∞ mollifiers of f. Then $\|f_i - f\|_{\mathfrak{L}_p^\alpha} \to 0$ as $i \to \infty$, and there is a subsequence $\{f_j\}$ that converges for $B_{\alpha,p}$ almost every x. That is, for some set E with $B_{\alpha,p}(E) = 0$,

$$(9) \qquad f_j(x) \to f(x) \quad \text{for every } x \in \mathbf{R}^n \setminus E.$$

For this subsequence, $\nabla f_j \to \nabla f$ in $\mathfrak{L}_p^{\alpha-1}$ (see [ST, Lemma 3, p. 136]), and therefore

$$\nabla f_j - \nabla f = G_{\alpha-1} * h_j$$

where $h_j \to 0$ in $L_p(\mathbf{R}^n)$. But then

$$G_1 * |\nabla f_j - \nabla f|(x) = G_1 * |G_{\alpha-1} * h_j|(x) \leqslant G_\alpha * |h_j|(x).$$

Since $h_j \to 0$ in $L^p(\mathbf{R}^n)$, $G_\alpha * |h_j|(x) \to 0$ for $B_{\alpha,p}$ almost every x, hence so does $G_1 * |\nabla f_j - \nabla f|(x)$ by the above inequality. For any such point x we have

$$\begin{aligned} G_1 * |\nabla f_j - \nabla f|(x) &= \int_{\mathbf{R}^n} G_1(z)|\nabla f_j(x-z) - \nabla f(x-z)|\,dz \\ &\geqslant \int_{B(0,R)} G_1(z)|\nabla f_j(x-z) - \nabla f(x-z)|\,dz \\ &\cdot \int_{\partial B(0,1)} \int_0^R \frac{G_1(ry)}{|ry|^{n-1}} |\nabla f_j(x-ry) - \nabla f(x-ry)|\,dr\,dH^{n-1}(y) \end{aligned}$$

where the last step is integration in polar coordinates. Using the estimate $G_1(z) \geqslant C|z|^{n-1}$ for $|z| < R$ (see [ST, p. 132]) we conclude from the above that

$$\int_{\partial B(0,1)} \int_0^R |\nabla f_j(x-ry) - \nabla f(x-ry)|\,dr\,dH^{n-1}(y) \to 0$$

for $B_{\alpha,p}$ almost every x. For such a point x, we may choose a further subsequence f_k such that

$$(10) \qquad \int_0^R |\nabla f_k(x-ry) - \nabla f(x-ry)|\,dr \to 0$$

for H^{n-1} almost every $y \ni \partial B(0,1)$. Moreover we may assume that $x - ry \notin E$ for any such point y, $0 < r < R$ (see [GZ]). If $z = x - ry$ is any such point, since $f_k \in C^\infty(\mathbf{R}^n)$ we have

$$f_k(x+z) - f_k(x) = \int_0^1 \nabla f_k(x+tz) \cdot z\,dt.$$

Letting $k \to \infty$, in view of (9) and (10) we may replace f_k by f in the above. Finally, choose a sequence $R_j \to \infty$ to prove the lemma. Q.E.D.

PROOF OF THEOREM 2. Let $f \in \mathfrak{L}_p^\alpha$, $\alpha \geqslant 1$, and let $k \leqslant \alpha$ be an integer. Let $P_x(y)$ be the Taylor polynomial of degree k, i.e.

$$P_x(y) = \sum_{0 \leqslant |\beta| \leqslant k} \frac{D^\beta f(x)}{\beta!}(y-x)^\beta.$$

Here $\beta = (\beta_1, \ldots, \beta_n)$ is a multi-index,

$$|\beta| = \sum_{j=1}^n \beta_j, \quad D^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}, \quad \beta! = \beta_1! \beta_2! \cdots \beta_n!,$$

and $z^\beta = z_1^{\beta_1} z_2^{\beta_2} \cdots z_n^{\beta_n}$. Note that if $|\beta| \leqslant k$, then $D^\beta f \in \mathfrak{L}_p^{\alpha-k}$, and since such functions are defined up to sets of $B_{\alpha-k,p}$ capacity zero, $P_x(y)$ is defined for $B_{\alpha-k,p}$ almost all $x \in \mathbf{R}^n$.

Let $R_x(y) = f(y) - P_x(y)$ and notice that if $|\gamma| = k$ then

$$D^\gamma R_x(y) = D^\gamma f(y) - D^\gamma f(x).$$

Since $D^\gamma f \in \mathfrak{L}_p^{\alpha-k}$, we know from Theorem 1 that

$$\overline{\operatorname{Lim}}_{\delta(S)\to 0; S \in \mathfrak{F}} \frac{1}{m(S)} \int_{S+x} |D^\gamma R_x(y)|\, dy = 0 \tag{11}$$

at $B_{\alpha-k,p}$ almost every x, whenever $|\gamma| = k$. Now let ν be a multi-index such that $|\nu| = k - 1$. Since $D^\nu f \in \mathfrak{L}_p^{\alpha-k}$, we know from Lemma 1 that for $B_{\alpha-k,p}$ almost every point x, $D^\nu f$ is absolutely continuous on H^{n-1} almost every ray emanating from x. Hence on such a ray, $D^\nu R_x(y)$ is also absolutely continuous and

$$|D^\nu R_x(x+z) - D^\nu R_x(x)| \leqslant \int_0^1 |\nabla(D^\nu R_x)(x+tz) \cdot z|\, dt.$$

Since this estimate holds for almost every $z \in \mathbf{R}^n$, we may integrate both sides with respect to z over a set $S \in \mathfrak{F}$, yielding

$$\begin{aligned}\int_S |D^\nu R_x(x+z) - D^\nu R_x(x)|\, dz &\leqslant \int_S \int_0^1 |\nabla(D^\nu R_x)(x+tz) \cdot z|\, dt\, dz \\ &\leqslant \int_0^1 \int_S \delta(S) |\nabla(D^\nu R_x)(x+tz)|\, dz\, dt \\ &= \int_0^1 \frac{\delta(S)}{t^n} \int_{t\cdot S} |\nabla(D^\nu R_x)(x+z)|\, dz\, dt.\end{aligned}$$

The second inequality above is obtained by interchanging the order of integration and using the Schwartz inequality, and the final step is just the change of variables $z \mapsto tz$. We now divide both sides by $m(s)\delta(s)$ and use the transformation $x + z \mapsto z$, noting that $t^n m(S) = m(t \cdot S)$.

$$\frac{1}{\delta(S)} \frac{1}{m(S)} \int_{S+x} |D^\nu R_x(z) - D^\nu R_x(0)|\, dz$$
$$\leqslant \int_0^1 \frac{1}{m(t \cdot S)} \int_{t \cdot S+x} |\nabla(D^\nu R_x)(z)|\, dz\, dt.$$

Since all derivatives of R_x of order k satisfy (11) and since for $0 < t < 1$, $t \cdot S \in \mathfrak{F}$ with $\delta(t \cdot S) \leqslant \delta(S)$, we conclude the existence of a number δ such that the right side of the above is less than ε whenever $\delta(S) < \delta$. Since $D^\nu R_x(0) = 0$, this implies

$$\text{(12)} \qquad \overline{\operatorname{Lim}}_{\delta(S)\to 0; S \in \mathfrak{F}} \frac{1}{\delta(S)} \frac{1}{m(S)} \int_{S+x} |D^\nu R_x(y)|\, dy = 0$$

for $B_{\alpha-k,p}$ almost every x, whenever $|\nu| = k - 1$. This estimate replaces (11) for derivatives of order k. Now let η be a multi-index with $|\eta| = k - 2$. Arguing just as above we conclude that

$$\text{(13)} \qquad \overline{\operatorname{Lim}}_{\delta(S)\to 0; S \in \mathfrak{F}} \frac{1}{\delta(S)^2} \frac{1}{m(S)} \int_{S+x} |D^\eta R_x(y)|\, dy = 0$$

and inductively that

$$\overline{\operatorname{Lim}}_{\delta(S)\to 0; S \in \mathfrak{F}} \frac{1}{\delta(S)^k} \frac{1}{m(S)} \int_{S+x} |D^0 R_x(y)|\, dy = 0$$

which proves the theorem. Q.E.D.

COROLLARY 2. *If $f \in \mathfrak{L}_p^\alpha$ with $\alpha \geqslant 1$, $1 < p < \infty$, and if $k \leqslant \alpha$ is an integer, then $f \in t_1^k(x)$ with respect to $\mathfrak{R}$ for $B_{\alpha-k,p}$ almost every $x \in \mathbf{R}^n$.*

The exponent 1 in Theorems 1 and 2 can be improved if one is willing to accept a larger exceptional set. For example, if $1 < q < p$ and $k \leqslant \alpha - 1$, then one can show that $f \in t_q^k(x)$ for $B_{1,s}$ almost every x, where $s < p$ satisfies the equation $sp/(p - s) = np/(n - p)(q - 1)$. This follows from the observation that if $|\gamma| = k$, $|D^\gamma f|^q \in W^{1,s}(\mathbf{R}^n) = \mathfrak{L}_s^1$. Applying Theorem 1 to $|D^\gamma f|^q$ we obtain estimate (11) with exponent q for $B_{1,s}$ almost every x. Making the appropriate changes in the rest of the argument now proves the claim.

The question as to whether the exponent 1 can be improved in the general case without altering the capacity seems to be open. Saks' example [S], seems to indicate that one cannot expect f to lie in $t_p^k(x)$ even almost everywhere.

Finally, in the definition of strong L_p differentiability (see (2)), one may be tempted to replace $\delta(S)^k$ by $m(S)^{k/n}$. We conclude by exhibiting an example which shows that $\delta(S)^k$ cannot be replaced by $m(S)$ raised to any power.

Let $f(x,y) = x^2$ and consider the family $\mathfrak{F} = \{R_\varepsilon\}_{\varepsilon>0}$ of rectangles of the form $R_\varepsilon = \{(x,y): -\varepsilon < x < \varepsilon, -\varepsilon^3 < y < \varepsilon^3\}$. After mollification outside a larger set, we may assume that $f \in \mathfrak{L}_p^\alpha(\mathbf{R}^2)$ for all $\alpha \geqslant 0, p \geqslant 1$. The Taylor polynomial of degree one at the origin is simply $P_0(y) \equiv 0$, and it is easily verified that

$$\frac{1}{m(R_\varepsilon)^{1/2}} \frac{1}{m(R_\varepsilon)} \int_{R_\varepsilon} |f(x,y)| \, dx \, dy = \frac{1}{(4\varepsilon^4)^{1/2}} \frac{1}{4\varepsilon^4} \frac{4\varepsilon^6}{3} = \frac{1}{6} \nrightarrow 0.$$

Similar behavior is exhibited at points other than the origin. Moreover, by modifying the rectangles R_ε, one can show it is impossible to replace $\delta(S)^k$ by $m(S)$ to *any* power.

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