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Strong equilibria in games with the lexicographical improvement property

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Abstract We study a class of finite strategic games with the property that every deviation of a coalition of players that is profitable to each of its members strictly decreases the lexicographical order of a certain function defined on the set of strategy profiles. We call this property the *lexicographical improvement property* (*LIP*) and show that, in finite games, it is equivalent to the existence of a generalized strong potential function. We use this characterization to derive existence, efficiency and fairness properties of strong equilibria (SE). As our main result, we show that an important class of games that we call *bottleneck congestion games* has the LIP and thus the above mentioned properties. For infinite games, the LIP does neither imply the existence of a generalized strong potential nor the existence of SE. We therefore introduce the slightly more general concept of the *pairwise* LIP and prove that whenever the pairwise LIP is satisfied for a continuous function, then there exists a SE. As a consequence, we show that splittable bottleneck congestion games with continuous facility cost functions possess a SE.

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1 Introduction

The theory of non-cooperative games studies situations that involve rational and selfish agents who are motivated by optimizing their own utilities rather than reaching some social optimum. In his seminal work Nash (1950) showed that every finite non-cooperative game has an equilibrium in mixed strategies. It is well known that mixed or correlated strategies have no meaningful physical interpretation for many strategic games arising in practice; see also the discussion by Osborne and Rubinstein (1994, § 3.2) about critics on mixed Nash equilibria. For such games, one usually resorts to pure strategies, and pure Nash equilibria (PNE) are the solution concept of choice. A PNE is a strategy profile such that no player has an incentive to unilaterally change her pure strategy. While the PNE concept excludes the possibility that a single player can unilaterally improve her utility, it does not necessarily imply that a PNE is stable against coordinated deviations of a group of players if their joint deviation is profitable for each of its members. So when coordinated actions are possible, the Nash equilibrium concept is not sufficient to analyze stable states of a game. To cope with the issue of coordination, we adopt the solution concept of a *strong equilibrium* (SE for short) proposed by Aumann (1959). In a SE, no coalition (of any size) can deviate and strictly improve the utility of each of it members (while possibly lowering the utility of players outside the coalition). Clearly, every SE is a PNE, but not conversely. Thus, although SE may rarely exist, they constitute a very robust and appealing stability concept.

One of the most successful approaches to establish the existence of PNE in finite games is the potential function method introduced by Rosenthal (1973) and later formalized by Monderer and Shapley (1996). One defines a real-valued function P on the set of strategy profiles of the game and shows that every improving move of a single player strictly reduces the value of P. Since the set of strategy profiles of such a (finite) game is finite, every sequence of improving moves reaches a PNE. In particular, every local minimum[1] of P is a PNE. Holzman and Law-Yone (1997) generalized this concept to *generalized strong potential functions*. Here, it is required that every improving move of a coalition (that is profitable to each of its members) strictly reduces the value of P. Clearly, the global minimum of a generalized strong potential is an SE, while the local minima of P correspond to (potentially non-strong) PNE.

In a recent line of research (Fabrikant et al. 2004; Even-Dar et al. 2007; Andelman et al. 2009), lexicographical arguments have been used to prove the existence of SE. Here, it is argued that the strategy profile that minimizes the vector of the players' private costs with respect to the lexicographical order is a PNE or an SE. In this paper, we formalize and generalize this approach. We consider strategic games $G = (N, X, \pi)$, where N is the set of players, X the strategy space, and players experience non-negative private costs $\pi_i(x)$, $i \in N$, for a strategy profile x. We say

[1] Here, a local minimum is a strategy profile with the property that each other strategy profile that is reachable by a unilateral deviation has no smaller value of P.

that G has the *lexicographical improvement property* (*LIP*) if there exists a vector-valued function $\phi : X \to \mathbb{R}^q_+$, $q \in \mathbb{N}$, such that every improving move (profitable deviation of an arbitrary coalition) from $x \in X$ strictly reduces $\phi(x)$ with respect to the sorted lexicographical order. We say that G has the π-LIP if G satisfies the LIP with $\phi_i(x) = \pi_i(x)$, $i \in N$. Clearly, requiring $q = 1$ in the definition of the LIP reduces to the case of a generalized strong potential.

The main focus of this paper is twofold. First, we show that arbitrary finite games with the π-LIP possess at least one SE with certain efficiency and fairness properties (a formal definition of these properties will be given in Sect. 3). Second, we identify an important and quite general class of games, the *bottleneck congestion games*, for which we can prove the π-LIP and, hence, prove that these games possess SE with the above mentioned properties.

Before we outline our results in more detail, let us give an informal definition of bottleneck congestion games. In a standard congestion game, there is a set of facilities, and the pure strategies of players are subsets of this set. Each facility f has a cost that is a function of its *load* that is usually defined as the number (or total weight) of players that select strategies containing f. The private cost of a player's strategy in a standard congestion game is the *sum* of the costs of the facilities in her strategy. In a bottleneck congestion game, the private cost of a player is equal to the cost of the *most expensive* facility that she uses (L_∞-norm of the vector of players' costs of the strategy). Bottleneck congestion games occur in many real-world applications, e.g., communication networks. Referring to Banner and Orda (2007), Cole et al. (2006), Keshav (1997) and Qiu et al. (2006), the throughput of a stream of packets in a communication network is usually determined by the available bandwidth or the capacity of the weakest links. This aspect is captured more realistically by bottleneck congestion games in which the individual cost of a player is the maximum (instead of the sum) of the delays in her strategy. Although these games constitute a more realistic model for network routing than classical congestion games, they have not received similar attention in the literature.

1.1 Our results

We first show that a finite game has the LIP if and only if it has a generalized strong potential. The proof is constructive, that is, given a game G having the LIP for a function ϕ, we explicitly construct a generalized strong potential P. We further investigate games having the π-LIP with respect to efficiency and fairness of SE. Our characterization implies that each game with the π-LIP possesses at least one SE that satisfies various efficiency and fairness properties, e.g., Pareto efficiency and min–max fairness. Moreover, we derive tight bounds on the strong prices of stability and anarchy.

One of our main results shows that bottleneck congestion games have the π-LIP and, thus, possess SE with the above mentioned properties. Moreover, our characterization of games having the LIP implies that bottleneck congestion games have the strong finite improvement property (FIP). Note that for congestion games with singleton strategies (where the concepts of standard congestion games and bottleneck congestion games coincide), Even-Dar et al. (2007) and Fabrikant et al. (2004) have already proved existence of PNE by arguing that the vector of facility costs decreases lexicographically for every improving move. Andelman et al. (2009) used the same argument to even establish existence of SE in this case. Our work generalizes these results to arbitrary strategy spaces and more general facility cost functions. In contrast to most congestion games considered so far, we require only that the facility cost functions satisfy three properties: "non-negativity", "independence of irrelevant choices", and "monotonicity". Roughly speaking, the second and third condition assume that the cost of a facility solely depends on the set of players using that facility and that the cost decreases if some players leave that facility, respectively. Thus, this framework extends classical *load-based* models in which the cost of a facility depends on the number or total weight of players using it. Our assumptions are weaker than in the load-based models and even allow that the cost of a facility may depend on the *set* of players using it.

We then study *infinite* games, that is, games with infinite strategy spaces that can be described by compact subsets of $\mathbb{R}^p$, $p \in \mathbb{N}$. We slightly generalize the LIP by introducing the notion of a pairwise vector-valued potential function $\phi: X \to \mathbb{R}^q_+ \times \mathbb{R}^q_+$, $q \in \mathbb{N}$. Informally, G has the *pairwise LIP* if every coalitional improving move from $x \in X$ strictly reduces a certain lexicographical order of $\phi(x)$ (see Sect. 5 for the formal definition). We prove that continuity of ϕ in the definition of the pairwise LIP is sufficient for the existence of SE. We then introduce *splittable bottleneck congestion games*. A splittable bottleneck congestion game arises from a bottleneck congestion game G by allowing players to fractionally distribute a certain *demand* over the pure strategies of G. We prove that these games have the pairwise LIP provided that the facility cost functions satisfy the three properties of "non-negativity", "independence of irrelevant choices", and "monotonicity". If the facility cost functions are also continuous, we obtain the pairwise LIP for a continuous function ϕ and thus obtain the existence of SE for splittable bottleneck congestion games. For *bounded* cost functions on the facilities (that may be discontinuous), we show that α-approximate SE exist for every $\alpha > 0$.

1.2 Further related work

The concept of a SE was introduced by Aumann (1959) and refined by Bernheim et al. (1987) to coalition-proof Nash equilibria (CPNE). These are states that are stable against those deviations that are themselves stable against further deviations by subsets of the original coalition. This implies that every SE is also a CPNE, but not conversely.

Congestion games were introduced by Rosenthal (1973) and have been further studied by Monderer and Shapley (1996). Holzman and Law-Yone (1997) explored the existence of SE in congestion games with monotone increasing cost functions. They showed that SE need not exist and gave a structural characterization of the strategy space for symmetric (and quasi-symmetric) congestion games that admit SE. Based on the previous work of Monderer and Shapley (1996), they also introduced the concept of a *generalized strong potential function*, i.e. a function on the set of strategy profiles that decreases for every profitable deviation of a coalition. Rozenfeld and Tennenholtz (2006) further explored the existence of (correlated) SE in congestion games with non-increasing cost functions.

Another generalization of congestion games has been proposed by Milchtaich (1996), who allows for player-specific facility cost functions (for subsequent work on weighted congestion games with player-specific facility cost functions see also Mavronicolas et al. 2007; Gairing et al. 2006; Ackermann et al. 2009). Milchtaich proves existence of PNE under restrictions on the strategy space (singleton strategies). As shown by Voorneveld et al. (1999), the model of Konishi et al. (1997a) is equivalent to that of Milchtaich, which is worth noting as Konishi et al. (1997a) established the existence of SE in these games.

Several authors studied the existence and efficiency of PNE and SE in specific classes of congestion games. For example, Even-Dar et al. (2007) showed that job scheduling games (on unrelated machines) have a PNE by arguing that the load-lexicographically minimal schedule is a PNE. Fabrikant et al. (2004) considered a scheduling model in which the processing time of a machine may depend on the set of jobs scheduled on that machine. For this model, they proved existence of PNE analogous to the proof of Even-Dar et al. Andelman et al. (2009) considered scheduling games on unrelated machines and proved that the load-lexicographically minimal schedule is even an SE. They also studied differences between PNE and SE and derived bounds on the (strong) price of anarchy and stability, respectively. Chien and Sinclair (2009) recently studied the strong price of anarchy of SE in general congestion games.

Bottleneck congestion games with network structure have been considered by Banner and Orda (2007). They studied existence of PNE in the unsplittable flow and in the splittable flow setting, respectively. They observed that standard techniques (such as Kakutani's fixed-point theorem) for proving existence of PNE do not apply to bottleneck routing games, as the players' private cost functions may be discontinuous. They proved existence of PNE by showing that bottleneck games are better reply secure, quasi-convex, and compact. Under these conditions, they recall Reny's existence theorem (1999) for better reply secure games with possibly discontinuous private cost functions. Banner and Orda, however, do not study SE. Note that our proof of the existence of SE is direct and constructive. Bottleneck routing with *non-atomic* players and elastic demands has been studied by Cole et al. (2006). Among other results, they derived bounds on the price of anarchy. For subsequent work on the price of anarchy in bottleneck routing games with atomic and non-atomic players, we refer to the paper by Mazalov et al. (2006).

After publication of a preliminary version of this paper (Harks et al. 2009), there has been subsequent work on the computational complexity of SE and their worst-case inefficiency. Harks et al. (2010) settled the complexity of computing SE for the unit-demand model. Werth et al. (2011) studied bottleneck congestion games on networks with weighted demands and identified cases in which there are efficient algorithms computing SE; de Keijzer et al. (2010) investigated the worst-case inefficiency of SE in bottleneck congestion games with affine linear cost functions.

2 Preliminaries

We consider strategic games $G = (N, X, \pi)$, where $N = \{1, \dots, n\}$ is the non-empty and finite set of players, $X = \bigtimes_{i \in N} X_i$ is the non-empty strategy space, and

$\pi : X \to \mathbb{R}_+^n$ is the combined *private cost* function that assigns a private cost vector $\pi(x)$ to each strategy profile $x \in X$. These games are cost minimization games and we assume additionally that the private cost functions are non-negative. A strategic game is called *finite* if X is finite. We use standard game theory notation; for a coalition $S \subseteq N$ we denote by $-S$ its complement and by $X_S = \times_{i \in S} X_i$ we denote the set of strategy profiles of players in S.

Definition 1 (*Strong equilibrium* (*SE*)) A strategy profile x is a SE if there is no coalition $\emptyset \neq S \subseteq N$ that has an alternative strategy profile $y_S \in X_S$ such that $\pi_i(y_S, x_{-S}) - \pi_i(x) < 0$ for all $i \in S$.

A pair $\big(x, (y_S, x_{-S})\big) \in X \times X$ is called an *improving move* (or *profitable deviation*) of coalition S if $\pi_i(x_S, x_{-S}) - \pi_i(y_S, x_{-S}) > 0$ for all $i \in S$. We denote by $I(S)$ the set of improving moves of coalition $S \subseteq N$ in a strategic game $G = (N, X, \pi)$ and we set $I = \bigcup_{S \subseteq N} I(S)$. We call a sequence of strategy profiles $\gamma = (x^0, x^1, \ldots)$ an *improvement path* if every pair $(x^k, x^{k+1}) \in I$ for all $k = 0, 1, 2, \ldots$. One can interpret an improvement path as a path in the *improvement graph* $\mathcal{G}(G)$ derived from G, where every strategy profile $x \in X$ corresponds to a node in $\mathcal{G}(G)$ and two nodes x, x' are connected by a directed edge (x, x') if and only if $(x, x') \in I$. An important property of finite strategic games is the FIP. This property requires that each improvement path of unilateral improvements is finite. Equivalently, we say that G has the *strong FIP* (*SFIP*) if every improvement path is finite. Clearly, the SFIP implies the FIP, but not conversely. A necessary and sufficient condition for the SFIP is the existence of a generalized strong potential function, which we define below (see also Monderer and Shapley 1996; Holzman and Law-Yone 1997).

Definition 2 (*Generalized strong potential game*) A strategic game $G = (N, X, \pi)$ is called a *generalized strong potential game* if there is a function $P : X \to \mathbb{R}$ such that $P(x) - P(y) > 0$ for all $(x, y) \in I$. P is called a *generalized strong potential* of G.

In this paper, we define an equivalent property, the *LIP*. For this purpose, we will first define the sorted lexicographical order.

Definition 3 (*Sorted lexicographical order*) Let $a, b \in \mathbb{R}_+^q$ and denote by $\tilde{a}, \tilde{b} \in \mathbb{R}_+^q$ be the sorted vectors derived from a, b by permuting the entries in non-increasing order, that is, $\tilde{a}_1 \geq \cdots \geq \tilde{a}_q$ and $\tilde{b}_1 \geq \cdots \geq \tilde{b}_q$. Then, a is *strictly sorted lexicographically smaller* than b (written $a \prec b$) if there exists an index m such that $\tilde{a}_i = \tilde{b}_i$ for all $i < m$, and $\tilde{a}_m < \tilde{b}_m$. The vector a is *sorted lexicographically smaller* than b (written $a \preceq b$) if either $a \prec b$ or $\tilde{a} = \tilde{b}$.

The LIP of a strategic game requires that there is a vector-valued function $\phi : X \to \mathbb{R}_+^q$ that is strictly decreasing with respect to the sorted lexicographical order on $\mathbb{R}_+^q$ for every improvement step.

Definition 4 (*Lexicographical improvement property, π-LIP*) A finite strategic game $G = (N, X, \pi)$ has the *LIP* if there exist $q \in \mathbb{N}$ and a function $\phi : X \to \mathbb{R}_+^q$ such that $\phi(x) \succ \phi(y)$ for all $(x, y) \in I$. G has the *π-LIP* if G has the LIP for $\phi = \pi$.

If a game G has the LIP for a function ϕ, we will call ϕ a *generalized strong vector-valued potential* of G. Clearly, the function ϕ is a generalized strong potential if $q = 1$. The next proposition states that the LIP is equivalent to the existence of a generalized strong potential, regardless of q.

Proposition 1 *Let $G = (N, X, \pi)$ be a finite strategic game. Then, the following statements are equivalent.*

1. *G has a generalized strong vector-valued potential $\phi : X \to \mathbb{R}^q_+$, $q \in \mathbb{N}$.*
2. *G has a generalized strong potential function $P : X \to \mathbb{R}_+$.*

Proof We only prove 1. $\Rightarrow$ 2. as the reverse direction is trivial. We will show that $P_M(x) = \sum_{i=1}^q \phi_i(x)^M$ is a generalized strong potential for M large enough. Let $S \subseteq N$ and $(x, (y_S, x_{-S})) \in I(S)$ be arbitrary. We will calculate $P_{M'}(x) - P_{M'}(y_S, x_{-S}) = \sum_{i=1}^q (\phi_i(x)^{M'} - \phi_i(y_S, x_{-S})^{M'})$ for some M'. To this end, let us denote by $\tilde{\phi}(x)$ and $\tilde{\phi}(y_S, x_{-S})$ the vectors that arise by sorting $\phi(x)$ and $\phi(y_S, x_{-S})$ in non-increasing order. As $\phi(y_S, x_{-S}) \prec \phi(x)$, there is an index $m \in \{1, \ldots, q\}$ such that $\tilde{\phi}_i(x) = \tilde{\phi}_i(y_S, x_{-S})$ for all $i < m$ and $\tilde{\phi}_m(x) < \tilde{\phi}_m(y_S, x_{-S})$. We then obtain

$$\begin{aligned} P_{M'}(x) - P_{M'}(y_S, x_{-S}) &= \sum_{i=1}^{q} \phi_i(x)^{M'} - \sum_{i=1}^{q} \phi_i(y_S, x_{-S})^{M'} \\ &= \tilde{\phi}_m(x)^{M'} - \tilde{\phi}_m(y_S, x_{-S})^{M'} + \sum_{i=m+1}^{q} \tilde{\phi}_i(x)^{M'} - \sum_{i=m+1}^{q} \tilde{\phi}_i(y_S, x_{-S})^{M'} \\ &\geq \tilde{\phi}_m(x)^{M'} - \tilde{\phi}_m(y_S, x_{-S})^{M'} - (q-m)\tilde{\phi}_m(y_S, x_{-S})^{M'} \\ &\geq \tilde{\phi}_m(x)^{M'} - q\tilde{\phi}_m(y_S, x_{-S})^{M'}. \end{aligned} \tag{1}$$

Standard calculus shows that the expression on the right hand side of (1) is positive if

$$M' > \log(q)/\big(\log(\tilde{\phi}_m(x)) - \log(\tilde{\phi}_m(y_S, x_{-S}))\big) > 0.$$

Clearly, M' depends on $(x, y) \in I$, but as the number of improvement steps is finite, we may chose $M = \max_{(x, y) \in I} M'(x, y)$ and obtain the claimed result. □

3 Efficiency and fairness of SE in games with the π-LIP

As the LIP implies the existence of SE, it is natural to investigate efficiency and fairness properties of these SE. We here consider strict Pareto efficiency, min–max fairness, strong price of anarchy, and strong price of stability.

3.1 Pareto efficiency

Pareto efficiency is one of the fundamental concepts studied in the economics literature, see Mas-Colell et al. (1995). For a strategic game $G = (N, X, \pi)$, a strategy profile x is called *weakly Pareto efficient* if there is no $y \in X$ such that $\pi_i(y) < \pi_i(x)$ for all $i \in N$. A strategy profile x is *strictly Pareto efficient* if there is no $y \in X$ such that $\pi_i(y) \leq \pi_i(x)$ for all $i \in N$, where at least one inequality is strict. So strictly Pareto efficient strategy profiles are those for which every improvement of a coalition of players is at the expense of at least one player not in the coalition. Pareto efficiency has also been studied in the context of standard congestion games (with sum-objective). Holzman and Law-Yone (1997) give sufficient conditions on the strategy spaces of congestion games that guarantee the existence of an SE which is strictly Pareto efficient, and Chien and Sinclair (2009) quantify the social welfare achieved in weakly Pareto efficient PNE.

Clearly, every SE is weakly Pareto optimal as it is resilient against a profitable deviation of the whole player set N. In games with the π-LIP this result can be strengthened in the sense that there always is an SE, that is even *strictly* Pareto efficient.

Theorem 1 *Let G be a finite strategic game having the π-LIP. Then there exists an SE that is strictly Pareto optimal.*

Proof The sorted lexicographical minimum x of π is an SE. To see that it also strictly Pareto efficient, assume by contradiction that there is $y \in X$ and a player i such that $\pi_i(y) < \pi_i(x)$ and $\pi_j(y) \leq \pi_j(x)$ for all $j \in N \setminus \{i\}$. Then, $y \prec x$, contradicting the minimality of x. □

3.2 Min–max-fairness

Min–max fairness is a central topic in resource allocation in communication networks, see Srikant (2003) for an overview and pointers to the large body of research in this area. While strict Pareto efficiency requires that there is no alternative profile that improves the cost for a single player without strictly deteriorating the costs of the other players, the notion of min–max-fairness is stronger. A profile x is called *min–max fair* if for any other strategy profile y with $\pi_i(y) < \pi_i(x)$ for some $i \in N$, there exists either $j \in N \setminus \{i\}$ such that $\pi_j(x) \geq \pi_i(x)$ and $\pi_j(y) > \pi_j(x)$, or there exists $j \in N \setminus \{i\}$ such that $\pi_j(x) < \pi_i(x)$ and $\pi_j(y) \geq \pi_i(x)$. Note that in contrast to Pareto efficiency, an improvement that increases the cost of a player with smaller original cost is allowed (up to the threshold $\pi_i(x)$). It is easy to see that every min–max-fair strategy profile is a strictly Pareto efficient state, but not conversely.

Theorem 2 *Let G be a finite strategic game having the π-LIP. Then, there exists an SE that is min–max fair.*

Proof We show that the strategy profile x minimizing π with respect to the sorted lexicographical order $\preceq$ is min–max fair. Assume by contradiction that there is another strategy profile y such that $\pi_i(y) < \pi_i(x)$ for some $i \in N$ and the following two statements hold:

1. $\pi_j(y) \leq \pi_j(x)$ for all $j \in N \setminus \{i\}$ with $\pi_j(x) \geq \pi_i(x)$.
2. $\pi_j(y) < \pi_i(x)$ for all $j \in N \setminus \{i\}$ with $\pi_j(x) < \pi_i(x)$.

We observe that every entry of $\pi(x)$, that is larger than $\pi_i(x)$ only decreases under y, while every entry strictly smaller than $\pi_i(x)$ may only increase to a value strictly smaller than the threshold $\pi_i(x)$. Since the value $\pi_i(x)$ strictly decreases under y, we obtain $\pi(y) \prec \pi(x)$, contradicting the minimality of x. □

3.3 Price of stability and price of anarchy

To quantify the efficiency loss of selfish behavior with respect to a predefined social cost function, two notions have evolved. The *price of anarchy* has been introduced by Koutsoupias and Papadimitriou (1999) in the context of congestion games and is defined as the ratio of the cost of the worst PNE and that of the social optimum. A more optimistic performance index termed the *price of stability* measures the ratio of the cost of the best PNE and that of the social optimum (Anshelevich et al. 2004, 2008). Both concepts have been studied extensively in computer science and operations research, see Nisan et al. (2007, Part III) for a survey. More recently, they have also been studied in economics, see e.g. Johari and Tsitsiklis (2004), Moulin (2008).

Andelman et al. (2009) propose to study also the worst case ratio of the cost of an SE and that of a social optimum, which they term the *strong price of anarchy*. Clearly, the strong price of anarchy is not larger than the price of anarchy. For some classes of games this inequality is strict, see e.g. the results of Czumaj and Vöcking (2007) and Fiat et al. (2007) on the price of anarchy and strong price of anarchy of scheduling games on related machines, respectively. Andelman et al. (2009) also define the *strong price of stability* in the obvious way as the ratio of the cost of a best SE and that of a social optimum. Formally, given a game $G = (N, X, \pi)$ and a social cost function $C : X \to \mathbb{R}_+$, whose minimum is attained in a strategy profile $y \in X$, let $X^{\mathrm{SE}} \subseteq X$ denote the set of SE. Then, the strong price of anarchy for G with respect to C is defined as $\sup_{x\in X^{\mathrm{SE}}} C(x)/C(y)$ and the strong price of stability for G with respect to C is defined as $\inf_{x\in X^{\mathrm{SE}}} C(x)/C(y)$. We will consider the following natural social cost functions: the sum-objective or L_1-norm defined as $L_1(x) = \sum_{i\in N} \pi_i(x)$, the L_p-objective or L_p-norm, $p \in \mathbb{N}$, defined as $L_p(x) = (\sum_{i\in N} \pi_i(x)^p)^{1/p}$, and the min–max objective or L_∞-norm defined as $L_\infty(x) = \max_{i\in N}\{\pi_i(x)\}$.

Theorem 3 *Let G be a finite strategic game with the π-LIP. Then, the strong price of stability wrt L_∞ is 1, and, for any $p \in \mathbb{N}$, the strong price of stability wrt L_p is less than or equal to $n^{1/p}$.*

Proof To see that the strong price of stability wrt L_∞ is 1, note that a lexicographical minimum x^* of π is an SE. By construction, x^* minimizes L_∞.

For L_p we first show that for arbitrary $p, q \in \mathbb{N}$ with $p < q$ and $x \in \mathbb{R}^n_+$ we have $L_p(x) \le n^{1/p-1/q} L_q(x)$ and $L_p(x) \le n^{1/p} L_\infty(x)$. To see the first inequality, let $a = \frac{q}{p} > 1$ and $b > 0$ be such that $\frac{1}{a} + \frac{1}{b} = 1$. With Hölder's inequality we obtain

$$L_p(x) = \Big(\sum_{i=1}^{n} x_i^p\Big)^{1/p} \le \bigg(\Big(\sum_{i=1}^{n} x_i^{p\cdot a}\Big)^{1/a}\Big(\sum_{i=1}^{n} 1\Big)^{1/b}\bigg)^{1/p}$$
$$= n^{1/(pb)} L_q(x) = n^{1/p-1/q} L_q(x).$$

(a)

	l	r
u	$(k-\epsilon, k-\epsilon, k-\epsilon, \ldots, k-\epsilon)$	$(k, k, k, \ldots, k)$
d	$(k, 0, 0, \ldots, 0)$	$(k, \epsilon, k, \ldots, k)$

(b)

	l	r
u	$(0, 0)$	$(0, k)$
d	(k, k)	$(0, k)$

Fig. 1 **a** Private costs received by the players for strategy profiles $X_1 \times X_2$ of the game considered in Example 1. **b** A game with unbounded price of anarchy wrt any L_p-norm as considered in Example 2

For the L_∞-norm we have $a = \infty$ and $b = 1$, thus, we obtain $L_p(x) \leq n^{1/p} L_\infty$.

Next, let x^* be a lexicographical minimum of π. Fix $p \in N$ and let y be a strategy profile minimizing L_p. We derive $L_p(x^*) \leq n^{1/p} L_\infty(x^*) \leq n^{1/p} L_\infty(y) \leq n^{1/p} L_p(y)$, where we use for the second inequality that x^* minimizes L_∞ and for the third inequality that the L_p-norm is decreasing in p. □

We now provide an example of a class of games with the π-LIP whose parameters can be chosen in such a way that the strong price of stability wrt L_p is arbitrarily close to $n^{1/p}$, implying that the result of Theorem 3 is tight.

Example 1 (*Strong price of stability*) Consider the game $G = (N, X, \pi)$ with $N = \{1, \ldots, n\}$, $X_1 = \{u, d\}$, $X_2 = \{l, r\}$ and $X_i = \{z\}$ for $3 \leq i \leq n$. For $k > \epsilon$, the private costs are shown in Fig. 1a. It is straightforward to check that this game has the π-LIP. The unique SE is the strategy profile $(u, l, z, \ldots, z)$ realizing a private cost vector of $(k - \epsilon, \ldots, k - \epsilon)$. For any $p \in \mathbb{N}$, there is $\epsilon > 0$ such that $L_p(\cdot)$ is maximized in strategy profile $(d, l, z, \ldots, z)$ realizing a cost vector of $(k, 0, \ldots, 0)$. Hence the price of stability approaches $n^{1/p}$.

So far, our results concern the strong price of stability only. The next example shows that games with the π-LIP may have an unbounded strong price of anarchy.

Example 2 (*Strong price of anarchy*) Consider the game $G = (N, X, \pi)$ with $N = \{1, 2\}$, $X_1 = \{u, d\}$, $X_2 = \{l, r\}$ and private costs given in Fig. 1b for any $k > 0$. It is straightforward to check that this game has the π-LIP and that both (u, l) and (d, r) are SE. Hence, the price of anarchy wrt any L_p norm is unbounded from above.

4 Bottleneck congestion games

We now present a rich class of finite games satisfying the π-LIP. We call these games *bottleneck congestion games*. They are natural generalizations of variants of congestion games. In contrast to standard congestion games, we focus on *bottleneck-objectives*, that is, the cost of a player only depends on the highest cost of the facilities she uses. For the sake of a clean mathematical definition, we introduce the general notion of a congestion model.

Definition 5 (*Congestion model*) A tuple $\mathcal{M} = (N, F, X, (c_f)_{f \in F})$ is called a *congestion model* if $N = \{1, \ldots, n\}$ is a non-empty, finite set of players, $F = \{1, \ldots, m\}$ is a non-empty set of facilities, and $X = \times_{i \in N} X_i$ is the set of strategies. For each player $i \in N$, her collection of pure strategies X_i is a non-empty set of subsets of F. Given a strategy profile x, we define $\mathcal{N}_f(x) = \{i \in N : f \in x_i\}$ for all $f \in F$. Every facility $f \in F$ has a cost function $c_f : X \to \mathbb{R}_+$ satisfying

Non-negativity: $c_f(x) \geq 0$ for all $x \in X$,
Independence of Irrelevant Choices: $c_f(x) = c_f(y)$ for all $x, y \in X$ with $\mathcal{N}_f(x) = \mathcal{N}_f(y)$,
Monotonicity: $c_f(x) \leq c_f(y)$ for all $x, y \in X$ with $\mathcal{N}_f(x) \subseteq \mathcal{N}_f(y)$.

We now define bottleneck congestion games relative to a congestion model.

Definition 6 (*Bottleneck congestion game*) Let $\mathcal{M} = (N, F, X, (c_f)_{f \in F})$ be a congestion model. The corresponding *bottleneck congestion game* is the strategic game $G(\mathcal{M}) = (N, X, \pi)$ in which π is defined as $\pi = \bigtimes_{i \in N} \pi_i$ and $\pi_i(x) = \max_{f \in x_i} c_f\big(x\big)$.

A bottleneck congestion game with $|x_i| = 1$ for all $x_i \in X_i$ and $i \in N$ will be called a *singleton* bottleneck congestion game. Note that for singleton strategies, congestion games with bottleneck objective and congestion games with sum-objective coincide.

Our assumptions on the cost functions are weaker than in the load-based models often used in the congestion games literature, e.g., Banner and Orda (2007). In our approach, we only require that the cost function $c_f(x)$ of facility f for strategy profile x depends on the set of players using f in x and that costs are increasing with larger sets. Note that this may cover, e.g., dependencies on the identities of players using f. Our condition "Independence of Irrelevant Choices" is also weaker than the one frequently used in the literature. In Konishi et al. (1996, 1997a,b), the definition of "Independence of Irrelevant Choices" requires that the strategy spaces are symmetric and, given a strategy profile $x = (x_1, \ldots, x_n)$, the utility of a player i depends only on her own choice x_i and the cardinality of the set of other players who also choose x_i. On the one hand, our model is more general as it does neither require symmetry of strategies, nor that the utility of player i only depends on the set-cardinality of other players who also choose x_i. On the other hand, the model of Konishi et al. allows for player-specific facility cost functions, which our model does not. For the relationship between games considered by Konishi et al. (1996, 1997a,b) and congestion games, see the discussion in Voorneveld et al. (1999).

Before we prove that bottleneck congestion games have the π-LIP and thus possess an SE with the efficiency and fairness properties shown in the last section, we give a series of examples of games that fit into the rich class of bottleneck congestion games and show how they are related to the literature.

4.1 Scheduling games

Scheduling games model situations in which each player controls a task that needs to be processed by one machine out of a finite number of available machines, see Vöcking (2007) for a survey. In each strategy profile every player $i \in N$ selects a single machine on which her job is processed. In the most general machine model of unrelated machines each job is associated with a machine-dependent weight $w_{i,f} \in \mathbb{R}_+$. Scheduling games are singleton bottleneck congestion games where the cost function of machine f is defined as $c_f(x) = \sum_{i \in N: x_i = \{f\}} w_{i,f}$. This function satisfies non-negativity, independence of irrelevant choices and monotonicity. The existence of SE in scheduling games has been established before by Andelman et al. (2009) by arguing that the lexicographically minimal schedule is a SE. They also showed that the strong price of stability wrt L_∞ is 1. Note that our general framework of bottleneck congestion games allows more complex cost structures on the machines than in these classical load-based models. One such example are dependencies between the weights of jobs on the same machine.

4.2 Resource allocation in wireless networks

Interference games are motivated by resource allocation problems in wireless networks. Consider a set of n terminals that want to connect to one out of m available base stations. Terminals assigned to the same base station impose interferences among each other as they use the same frequency band. We model the interference relations by an undirected interference graph $\mathcal{D} = (V, E)$, where $V = \{1, \dots, n\}$ is the set of vertices/terminals and an edge $e = (v, w)$ between terminals v, w has a non-negative weight $w_e \geq 0$ representing the level of pair-wise interference. We assume that the service quality of a base station j is proportional to the total interference $w(j)$, which is defined as $w(j) = \sum_{(v, w) \in E : x_v = x_w = j} w_{(v, w)}$.

We now obtain an interference game as follows. The nodes of the graph are the players, the set of strategies is given by $X_i = \{\{1\}, \dots, \{m\}\}$, $i = 1, \dots, n$, that is, the set of base stations, and the private cost function for every player is defined as $\pi_i(x) = w(x_i)$, $i = 1, \dots, n$. Interference games fit into the framework of singleton bottleneck congestion games with m facilities.

Note that in interference games, we crucially exploit the property that facility cost functions depend on the *set* of players using the facility, that is, their *identity* determines the resulting cost. The existence of an SE in all interference games follows from our main theorem, while most previous game-theoretic works addressing wireless networks only considered Nash equilibria, see for instance Liu and Wu (2008) and Etkin et al. (2007).

4.3 Bottleneck routing in networks

A special case of bottleneck congestion games are bottleneck routing games. Here, the set of facilities are the edges of a directed or undirected graph $\mathcal{D} = (V, E)$. Every edge $e \in E$ has a load dependent cost function c_e. Every player is associated with a pair of vertices (s_i, t_i) and a fixed demand $d_i > 0$ that she wishes to send along the chosen path in $\mathcal{D}$ connecting s_i to t_i. The private cost for every player is the maximum arc cost along the path, which is a common assumption for data routing in computer networks, see Keshav (1997), Banner and Orda (2007), Cole et al. (2006), Qiu et al. (2006). The existence of PNE in bottleneck routing games has been studied before by Banner and Orda (2007). They, however, did not study the existence of SE. To the best of our knowledge, our main result (Theorem 4) establishes for the first time that bottleneck routing games have the FIP, while Banner and Orda (2007) only proved that best-response dynamics converge.

4.4 Existence of SE

We are now ready to state our main result for bottleneck congestion games, providing a large class of games that satisfies the π-LIP.

Theorem 4 *Every bottleneck congestion game has the π-LIP.*

Proof For an arbitrary improving move $(x, (y_S, x_{-S})) \in I$, let $j \in S$ be a member of the coalition with highest cost before the improvement step, i.e., $j \in \arg\max_{i \in S} \pi_i(x)$. We set $N^+ = \{i \in -S : \pi_i(x) \geq \pi_j(x)\}$ and claim that $\pi_i(x) \geq \pi_i(y_S, x_{-S})$ for all $i \in N^+$. To see this, suppose there is $i \in N^+$ such that $\pi_i(x) < \pi_i(y_S, x_{-S})$. The independence of irrelevant choices and the monotonicity of the cost functions imply that there is a member $k \in S$ of the coalition with $y_k \cap x_i \neq \emptyset$. We obtain

$$\pi_j(x) \geq \pi_k(x) > \pi_k(y_S, x_{-S}) \geq \pi_i(y_S, x_{-S}) > \pi_i(x),$$

which contradicts $i \in N^+$. Next, we define $N^- = \{i \in -S : \pi_i(x) < \pi_j(x)\}$ and claim that $\pi_i(y_S, x_{-S}) < \pi_j(x)$ for all $i \in N^-$. To see this, suppose there is $i \in N^-$ such that $\pi_i(y_S, x_{-S}) \geq \pi_j(x)$. Because $\pi_j(x) \geq \pi_i(x)$, the independence of irrelevant choices and the monotonicity of the cost functions, there is a member $k \in S$ of the coalition with $y_k \cap x_i \neq \emptyset$ giving rise to

$$\pi_j(x) \geq \pi_k(x) > \pi_k(y_S, x_{-S}) \geq \pi_i(y_S, x_{-S}) \geq \pi_j(x),$$

which is a contradiction. Note that $N = N^+ \cup N^- \cup S$ and that we have shown $\pi_i(x) \geq \pi_i(y_S, x_{-s})$ for all $i \in N^+$ and $\pi_i(y_S, x_{-S}) < \pi_j(x)$ for all $i \in N^-$. As the private cost of the players with cost larger than $\pi_j(x)$ does not increase, the private cost of player j strictly decreases, and the private costs of all other players may only increase up to a value strictly smaller than $\pi_j(x)$, we have $\pi(x) \succ \pi(y_S, x_{-S})$ as claimed. □

As a corollary of Theorem 4 we obtain that bottleneck congestion games possess SE with the efficiency and fairness properties shown in Sect. 3. Note that our existence result holds for *arbitrary* strategy spaces. This contrasts a result of Holzman and Law-Yone (1997) who have shown that, for standard congestion games (with sum-objective), a certain combinatorial property of the players' strategy spaces (called *good configuration*) is necessary and sufficient for the existence of SE.

In bottleneck congestion games, the vector-valued potential function need not be unique. In fact, one can prove with similar arguments as in the proof of Theorem 4 that the function $\psi : X \to \mathbb{R}_+^{mn}$ defined as $\psi_{i,f}(x) = c_f(x)$, if $f \in x_i$, and $\psi_{i,f} = 0$, otherwise, decreases lexicographically along any improvement path. Moreover, if cost functions are strictly monotonic, one can show along the same lines that also the function $\upsilon : X \to \mathbb{R}^m$ defined as $\upsilon(x) = (c_f(x))_{f \in F}$ has this property. Interestingly, the lexicographical minima of the functions π, ψ, and υ need not coincide, as illustrated in the following example.

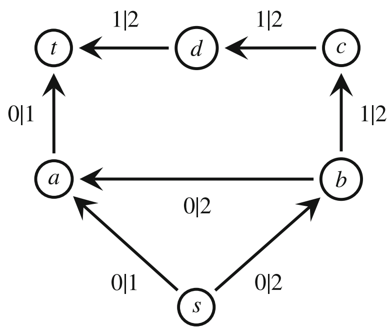

Fig. 2 Bottleneck routing game with multiple SE

Example 3 Consider the symmetric bottleneck routing game with two players $N = \{1, 2\}$ depicted in Fig. 2. Here, edges correspond to facilities; the cost of each edge depends only on the number of players using it and is given explicitly for the two possible values. The strategy set X_i of each player $i \in N$ comprises all paths from s to t, that are $P_1 = \{(sa), (at)\}$, $P_2 = \{(sb), (bc), (cd), (dt)\}$ and $P_3 = \{(sb), (ba), (at)\}$. There are three types of SE. In the first type, one player plays P_1 and the other player plays P_2. Here, the player on P_1 experiences a cost of 0 while the player on P_2 experiences a cost of 1. It is easy to see, that (upon permutation of the two players) this strategy profile is the unique lexicographical minimum of π. In the second type of SE one player chooses P_1 while the other player chooses P_3. Here, both players experience a cost of 1, thus this SE is not strictly Pareto efficient. It is easy to see that this equilibrium minimizes lexicographically both ψ and ν. There is a third SE where both players choose P_1. This profile minimizes none of the functions π, ψ, and υ. These different SE have also different efficiency properties. While the lexicographical minimum x^{π} of π is strictly Pareto efficient and min–max fair (as show in Theorems 1 and 2), the lexicographical minimum x^{υ} of υ has the property that it is strictly Pareto efficient with respect to using the resources, i.e., there is no strategy profile $y \in X$ such that $c_f(y) \leq c_f(x^{\upsilon})$ for all $f \in F$ where at least one of these inequalities is strict.

5 Infinite strategic games

We now consider *infinite* strategic games in which the players' strategy sets are topological spaces and the private cost functions are defined on the product topology. Formally, an infinite game is a tuple $G = (N, X, \pi)$, where $N = \{1, \dots, n\}$ is a set of players, and $X = X_1 \times \cdots \times X_n$ is the set of pure strategies, where we assume that $X_i \subseteq \mathbb{R}^{n_i}$, $n_i \in \mathbb{N}$, $i \in N$ are compact sets. The cost function of player i is defined by a non-negative real-valued function $\pi_i : X \to \mathbb{R}_+$, $i \in N$. Turning from finite games to infinite games, it becomes more complicated to characterize structural properties of games having the LIP. First, Proposition 1 is no longer valid, that is, infinite games with the LIP need not possess a generalized strong potential.[2] Also the existence of

[2] This observation resembles Debreu's (1954) result showing that the lexicographical ordering on an uncountable subset of $\mathbb{R}^2$ cannot be represented by a real-valued function.

an SE does not immediately follow. The global minimum of the function ϕ associated with the LIP need not exist as the strategy space is not finite. We will show that continuity of ϕ is sufficient for the existence of an SE. However, this assumption may be too strong for many classes of games. For instance, the splittable version of bottleneck congestion games (formally defined in Sect. 5.1) has the π-LIP but the function π may be discontinuous in general.

To obtain existence results for SE also for splittable bottleneck congestion games, we slightly generalize the LIP. Let $G = (N, X, \pi)$ be an infinite game and let $\phi : X \to \mathbb{R}_+^q \times \mathbb{R}_+^q$ be a function that associates with each strategy profile a pair $\phi(x) = (\phi^{(1)}(x), \phi^{(2)}(x))$. For two indices $i, j \in \{1, \ldots, q\}$ and two strategy profiles $x, y \in X$, let $\phi_i(x) \leq \phi_j(y)$ if and only if $\phi_i^{(1)}(x) < \phi_j^{(1)}(y)$ or $\phi_i^{(1)}(x) = \phi_j^{(1)}(y)$ and $\phi_i^{(2)}(x) \leq \phi_j^{(2)}(y)$. Let $\phi_i(x) < \phi_j(y)$ if and only if $\phi_i(x) \leq \phi_j(y)$ and $\phi_i(x) \neq \phi_j(y)$. Moreover, let $\preceq$ denote the sorted lexicographical order, where $\phi_i(x)$ is sorted according to $\leq$. Then, we say that ϕ is a *pairwise strong vector-valued potential* if $\phi(y) \prec \phi(x)$ for all $(x, y) \in I$. G has the *pairwise LIP* if it admits a pairwise strong vector-valued potential.

Clearly, every game with the LIP has also the pairwise LIP, as we may simply set the second component of the pairwise strong vector-valued potential equal to the first component (or, alternatively, equal to zero). We show below that every game with a continuous pairwise strong vector-valued potential admits an SE.

Theorem 5 *Every infinite game with a continuous pairwise strong vector-valued potential ϕ possesses an SE.*

Proof By assumption, there exists $q \in \mathbb{N}$ and a function $\phi : X \to \mathbb{R}_+^q \times \mathbb{R}_+^q$ such that $\phi(y_S, x_{-S}) \prec \phi(x)$ for all $(x, (y_S, x_{-S})) \in I$.

To get the desired result, we will show by induction over $q \in \mathbb{N}$ that for each $q \in \mathbb{N}$, each compact $X \neq \emptyset$ and each continuous function $\phi : X \to \mathbb{R}_+^q \times \mathbb{R}_+^q$ there is a strategy profile $x_{\min} \in X$ with $\phi(x_{\min}) \preceq \phi(x)$ for all $x \in X$.

For the base case $q = 1$, let $Y = \{x \in X : \phi^{(1)}(x) = \min_{x \in X} \phi^{(1)}(x)\}$ be the subset of those $x \in X$ for which the first component $\phi^{(1)}$ is minimized. Note that Y is non-empty and compact as ϕ is continuous and X is compact. Next, let $Y' = \{x \in Y : \phi^{(2)}(x) = \min_{x \in Y} \phi^{(2)}(x)\}$. With the same arguments, $Y' \neq \emptyset$ and by construction, Y' contains all vectors that minimize ϕ.

For the inductive step, suppose that the statement holds for all continuous functions $\phi' : X' \to \mathbb{R}_+^q \times \mathbb{R}_+^q$ with $q \leq k-1$ and consider an arbitrary compact X and an arbitrary continuous function $\phi : X \to \mathbb{R}_+^k \times \mathbb{R}_+^k$. In order to construct a lexicographical minimum of ϕ, we set $K = \{1, \ldots, k\}$ and solve the minimization problem

$$\min_{x \in X} \max_{i \in K} \phi_i^{(1)}(x) \tag{2}$$

of minimizing the maximum value within the first component of ϕ. Let α be the optimal value of (2). For arbitrary $\emptyset \neq J \subseteq K$, we set

$$Y^J = \{x \in X : \phi_i^{(1)}(x) \leq \alpha \quad \forall i \in K \setminus J \quad \phi_j^{(1)}(x) = \alpha \quad \forall j \in J\}.$$

Then, we define $\mathcal{J} = \{J \subseteq K : J \neq \emptyset, Y^J \neq \emptyset\}$. Note that because ϕ is continuous and X is compact, the optimal value of (2) is attained, and thus $\mathcal{J}$ is non-empty. For each $J \in \mathcal{J}$, we solve the minimization problem

$$\alpha^J = \min_{x \in Y^J} \max_{j \in J} \phi_j^{(2)}(x). \tag{3}$$

For each $J \in \mathcal{J}$ and $j \in J$, we set

$$Y^{J,j} = \{x \in Y^J : \phi_i^{(2)} \leq \alpha^J \quad \forall i \in J \setminus \{j\}, \quad \phi_j^{(2)}(x) = \alpha^J\}.$$

We define $\mathcal{J}' = \{(J, j) \in \mathcal{J} \times K : j \in J, Y^{J,j} \neq \emptyset\}$. Again, $\mathcal{J}'$ is non-empty as ϕ is continuous and X is compact. For each pair $(J, j) \in \mathcal{J}'$, we consider the function $\phi^{J,j} : Y^{J,j} \to \mathbb{R}_+^{k-1}$ that arises from ϕ by deleting the j-th index, i.e., $\phi_i^{J,j}(y) = (\phi_i^{(1)}(y), \phi_i^{(2)}(y))$ for all $i < j$ and $\phi_i^{J,j}(y) = (\phi_{i+1}^{(1)}(y), \phi_{i+1}^{(2)}(y))$ for all $i \in \{j, \dots, k-1\}$. Clearly, for all $(J, j) \in \mathcal{J}'$, the function $\phi^{J,j}$ is continuous and its domain $Y^{J,j}$ is compact and non-empty. For each $(J, j) \in \mathcal{J}'$, we apply the induction hypothesis and obtain $|\mathcal{J}'|$ vectors $y_{\min}^{J,j}$ minimizing $\phi^{J,j}$ on $Y^{J,j}$. We claim that the lexicographically minimal vector among the vectors $\big((\alpha, \alpha^J), \phi^{J,j}(y_{\min}^{J,j})\big) \in \mathbb{R}_+^k \times \mathbb{R}_+^k$ for each pair $(J, j) \in \mathcal{J}'$ is also a lexicographical minimum of the original function ϕ on X. For a contradiction, suppose that there is a vector $z \in X$ with $\phi(z) \prec \big((\alpha, \alpha^J), \phi^{J,j}(y_{\min}^{J,j})\big)$ for all $(J, j) \in \mathcal{J}'$. First, we observe that there is a set $\emptyset \neq J^* \subseteq K$ such that $\phi_i^{(1)}(z) = \alpha$ for all $i \in J^*$ and $\phi_i^{(1)}(z) < \alpha$ for all $i \in K \setminus J^*$ as otherwise we obtain a contradiction to the fact that α is the optimal value of (2). This implies in particular that $J^* \in \mathcal{J}$. Because α^{J^*} is the optimal value of (3), for at least one index $j^* \in J^*$, we have $\phi_{j^*}^{(2)}(z) = \alpha^{J^*}$. Using this fact together with the induction hypothesis that $y_{\min}^{J^*, j^*}$ minimizes ϕ among the vectors with $\phi_i^{(1)}(z) = \alpha$ for all $i \in J^*$ and $\phi_{j^*}^{(2)} = \alpha^{J^*}$ leads to a contradiction. □

5.1 Splittable bottleneck congestion games

In this section, we introduce the splittable counterpart of bottleneck congestion games. We start with a congestion model $\mathcal{M} = (N, F, X, (c_f)_{f \in F})$ with $X_i = \{x_{i,1}, \dots, x_{i,n_i}\}$, $n_i \in \mathbb{N}$, $i \in N$, where as usual every $x_{i,j}$ is a subset of facilities of F. From $\mathcal{M}$ we derive a corresponding *splittable congestion model* $\mathcal{M}_s = (N, F, X, d, \Delta, (c_f)_{f \in F})$, where $d \in \mathbb{R}_+^n$, $\Delta = \Delta_1 \times \dots \times \Delta_n$, and

$$\Delta_i = \left\{ \xi_i = (\xi_{i,1}, \dots, \xi_{i,n_i}) : \xi_{i,k} \geq 0 \quad \forall k \in \{1, \dots, n_i\}, \sum_{k=1}^{n_i} \xi_{i,k} = d_i \right\}.$$

The strategy profile $\xi_i = (\xi_{i,1}, \dots, \xi_{i,n_i})$ of player i can be interpreted as a distribution of non-negative *intensities* over the elements in X_i satisfying $\sum_{k=1}^{n_i} \xi_{i,k} = d_i$ for $d_i \in \mathbb{R}_+$, $i \in N$. Clearly, Δ_i is a compact subset of $\mathbb{R}_+^{n_i}$ for all $i \in N$. For a profile $\xi = (\xi_1, \dots, \xi_n)$, we define $\xi_{i,f} = \sum_{k \in \{1,\dots,n_i : f \in x_{i,k}\}} \xi_{i,k}$ as the total intensity put on facility f by player i; the set of used facilities of player i is defined as $F_i(\xi) = \{f \in F : \xi_{i,f} > 0\}$. We assume that for all $f \in F$ the cost function $c_f : \Delta \to \mathbb{R}_+$ satisfies the assumptions

Non-negativity: $c_f(\xi) \geq 0$ for all $\xi \in \Delta$,
Independence of Irrelevant Choices:
$c_f(\xi) = c_f(\xi')$ for all $\xi, \xi' \in \Delta$ with $\xi_{i,f} = \xi'_{i,f}$ for all $i \in N$,
Monotonicity: $c_f(\xi) \leq c_f(\xi')$ for all $\xi, \xi' \in \Delta$ with $\xi_{i,f} \leq \xi'_{i,f}$ for all $i \in N$,
Continuity: $c_f(\xi)$ is continuous in ξ.

Up to continuity, we basically impose the same assumptions as in the case of finite bottleneck congestion games.

Definition 7 (*Splittable bottleneck congestion game*) For the splittable congestion model $\mathcal{M}_s = (N, F, X, d, \Delta, (c_f)_{f \in F})$, we define the corresponding *splittable bottleneck congestion game* as the infinite strategic game $G(\mathcal{M}_s) = (N, \Delta, \pi)$, where π is defined as $\pi = \times_{i \in N} \pi_i$ and $\pi_i(\xi) = \max_{f \in F_i(\xi)} c_f(\xi)$.

The following examples fit into this model.

5.1.1 Bottleneck routing games with splittable demands

The facilities correspond to the edges of a directed graph $\mathcal{D} = (V, E)$. Each player i is associated with a source-sink pair $(s_i, t_i) \in V \times V$ and a positive demand d_i that she wishes to route from s_i to t_i. The private cost of each player equals the maximum cost over all facilities she uses with positive demand. The fundamental difference to non-splittable bottleneck congestion games is that each player i is allowed to distribute her demand among all paths connecting s_i and t_i, thus, bottleneck routing games with splittable demands serve as a model of multi-path routing protocols in telecommunication networks, see Banner and Orda (2007). They, however, study only existence of PNE. In addition to being more general, our result gives also an alternative and constructive proof for the existence of PNE in bottleneck routing games with splittable demands compared to the involved proof by Banner and Orda (2007).

5.1.2 Scheduling of malleable jobs

In the scheduling literature jobs are called *malleable* if they can be distributed among multiple machines (Feitelson and Rudolph 1996; Carroll and Grosu 2010). In a scheduling game with malleable jobs, each player i controls a job with weight w_i that she distributes over an arbitrary *subset* of allowable machines. The private cost is determined by the makespan, which is a non-decreasing function of the total load of the machine that finishes latest among the chosen machines. To the best of our knowledge, our work investigates for the first time the existence of equilibria (PNE or SE) in such games.

5.2 Existence of SE

As mentioned earlier, using similar arguments as in the proof of Theorem 4 one can prove that splittable bottleneck congestion games have the π-LIP. However, the function π may be discontinuous even if cost functions are continuous. To see this, consider the bottleneck congestion game with one player having access to two facilities $X_1 = \{\{f_1\}, \{f_2\}\}$ over which she has to assign a demand of size 1. The facility f_1 has a cost function equal to the load, while facility f_2 has a constant cost function equal to 2. Let $\xi_{1,2}(\epsilon) = \epsilon > 0$ be assigned to facility f_2 and the remaining demand $\xi_{1,1}(\epsilon) = 1 - \epsilon$ be assigned to f_1. Then, for *any* $\epsilon > 0$ we have $\pi(\xi(\epsilon)) = 2$, while $\pi(\xi(0)) = 1$.

To resolve this difficulty, we define the load of facility f under strategy profile ξ as $\ell_f(\xi) = \sum_{i\in N} \xi_{i,f}$ and show that $\nu : \Delta \to \mathbb{R}^m_+ \times \mathbb{R}^m_+,\ \xi \mapsto (c_f(\xi), \ell_f(\xi))_{f\in F}$ is a continuous pairwise strong vector-valued potential.

Theorem 6 *Every splittable bottleneck congestion game possesses an SE.*

Proof We show that the function $\nu : \Delta \to \mathbb{R}^m \times \mathbb{R}^m,\ \xi \mapsto (c_f(\xi), \ell_f(\xi))_{f\in F}$ is a pairwise strong vector-valued potential. Because ν is continuous, Theorem 5 gives then the desired result. Let $S \subseteq N$ be an arbitrary coalition and let $(\xi, (\xi'_S, \xi_{-S})) \in I(S)$ be an arbitrary improving move of coalition S. Choose a deviating player $j \in \arg\max_{i\in S} \pi_i(\xi)$ with highest cost before the improving move and one of the facilities $g \in \arg\max_{f\in F_j(\xi)} c_f(\xi)$ at which $\pi_j(\xi)$ is attained. Decompose F into F^+ and F^- defined as $F^+ = \{f \in F : c_f(\xi) \geq c_g(\xi)\}$ and $F^- = \{f \in F : c_f(\xi) < c_g(\xi)\}$.

We first claim that $c_f(\xi'_S, \xi_{-S}) \leq c_f(\xi)$ for all $f \in F^+$. Assume by contradiction that there is $f \in F^+$ with $c_f(\xi'_S, \xi_{-S}) > c_f(\xi)$. The independence of irrelevant choices and the monotonicity of the cost functions imply that there is a player $k \in S$ with $\xi'_{k,f} > 0$. We obtain $\pi_k(\xi'_S, \xi_{-S}) \geq c_f(\xi'_S, \xi_{-S}) > c_f(\xi) \geq c_g(\xi) = \pi_j(\xi) \geq \pi_k(\xi)$, which contradicts that k must improve.

Next we show that $\ell_f(\xi'_S, \xi_{-S}) \leq \ell_f(\xi)$ for all $f \in F^+$ with $c_f(\xi'_S, \xi_{-S}) = c_f(\xi)$. For a contradiction, assume that there is $f \in F^+$ with $\ell_f(\xi'_S, \xi_{-S}) > \ell_f(\xi)$ and $c_f(\xi'_S, \xi_{-S}) = c_f(\xi)$. Again this implies the existence of a player $k \in S$ with $\xi'_{k,f} > 0$. Using $c_f(\xi'_S, \xi_{-S}) = c_f(\xi)$, we obtain the same contradiction to the fact that k improves as before.

Finally, we claim that $c_f(\xi'_S, \xi_{-S}) < c_g(\xi)$ for all $f \in F^-$. To see this, assume that there is $f \in F^-$ with $c_f(\xi'_S, \xi_{-S}) \geq c_g(\xi)$. This again implies that there is a player $k \in S$ with $\xi'_{k,f} > 0$, thus, $\pi_k(\xi'_S, \xi_{-S}) \geq c_f(\xi'_S, \xi_{-S}) \geq c_g(\xi) = \pi_j(\xi) \geq \pi_k(\xi)$, and player k did not improve, contradiction!

To complete the proof, we show that $(c_g(\xi'_S, \xi_{-S}), \ell_g(\xi'_S, \xi_{-S})) < (c_g(\xi), \ell_g(\xi))$. We distinguish two cases. If $\xi'_{j,g} > 0$, we obtain $c_g(\xi'_S, \xi_{-S}) < c_g(\xi)$ using the fact that player j improves. For the second case, let $\xi'_{j,g} = 0$ and assume by contradiction that $(c_g(\xi'_S, \xi_{-S}), \ell_g(\xi'_S, \xi_{-S})) \geq (c_g(\xi), \ell_g(\xi))$. If $c_g(\xi'_S, \xi_{-S}) > c_g(\xi)$, we immediately derive the existence of a player $k \in S$ with $\xi'_{k,g} > 0$. On the other hand, if $c_g(\xi'_S, \xi_{-S}) = c_g(\xi)$ and $\ell_g(\xi'_S, \xi_{-S}) \geq \ell_g(\xi)$, we obtain the existence of $k \in S$ with $\xi'_{k,g} > 0$ using that $\xi'_{j,g} = 0$. In both cases, we calculate $\pi_k(\xi'_S, \xi_{-S}) \geq c_g(\xi'_S, \xi_{-S}) \geq c_g(\xi) = \pi_j(\xi) \geq \pi_k(\xi)$, a contradiction to the fact that k improves. □

5.3 Existence of approximate SE

We now relax the continuity assumption on the facility cost functions by assuming that they are only bounded from above. We will prove that bottleneck congestion games with bounded cost functions possess an α-approximate SE for every $\alpha > 0$. An *α-approximate* SE is stable only against (coalitional) improving moves that decrease the private cost of every moving player by at least $\alpha > 0$. More formally, we denote by $I^{\alpha}(S) \subset X \times X$ the set of tuples $(x, (y_S, x_{-S}))$ of α-improving moves for $S \subseteq N$ and define as I^{α} their union. Then a strategy profile x is an *α-approximate SE* if no coalition $\emptyset \neq S \subseteq N$ has an alternative strategy profile y_S such that $\pi_i(x) - \pi_i(y_S, x_{-S}) > \alpha$, for all $i \in S$. We call a function $P : X \to \mathbb{R}$ an *α-generalized strong potential* if $(x, y) \in I^{\alpha}$ implies $P(x) > P(y)$.

Theorem 7 *Every splittable bottleneck congestion game with bounded cost functions possesses an α-approximate SE for every $\alpha > 0$.*

We prove the theorem by stating a useful lemma.

Lemma 1 *Let the function $\psi : \Delta \to \mathbb{R}^{mn}_{+}$ be defined as*

$$\psi_{i,f}(\xi) = \begin{cases} c_f(\xi), & \text{if } f \in F_i(\xi) \\ 0, & \text{else} \end{cases} \quad \text{for all } i \in N,\ f \in F.$$

Moreover, let $\alpha > 0$ and define $P_M(\xi) = \sum_{f \in F, i \in N} \psi_{i,f}(\xi)^M$, where $M \geq (2\,\psi_{\max}/\alpha + 1)\log(n\,m)$ and $\psi_{\max} = \sup_{\xi \in \Delta, f \in F} c_f(\xi)$. Then, P_M is an α-generalized strong potential satisfying $P_M(\xi) - P_M(\xi') \geq (\alpha/2)^M$ for all $(\xi, \xi') \in I^{\alpha}$.

Proof We must show that $P_M(\xi) - P_M(\xi'_S, \xi_{-S}) \geq (\alpha/2)^M$ for an arbitrary α-improving move $(\xi, (\xi'_S, \xi_{-S})) \in I^{\alpha}$. Let $j \in \arg\max_{i \in S} \pi_i(\xi'_S, \xi_{-S})$. We define $\Psi^{+} = \{(i, f) \in -S \times F : \psi_{i,f}(\xi) \geq \pi_j(\xi'_S, \xi_{-S})\}$ and $\Psi^{-} = \{(i, f) \in -S \times F : \psi_{i,f}(\xi) < \pi_j(\xi'_S, \xi_{-S})\}$. We claim that

$$\psi_{i,f}(\xi'_S, \xi_{-S}) \leq \psi_{i,f}(\xi) \quad \text{for all } (i, f) \in \Psi^{+}, \tag{4}$$

$$\psi_{i,f}(\xi'_S, \xi_{-S}) \leq \pi_j(\xi'_S, \xi_{-S}) \quad \text{for all } (i, f) \in \Psi^{-}. \tag{5}$$

To prove (4), suppose there is $(i, g) \in \Psi^{+}$ such that $\psi_{i,g}(\xi) < \psi_{i,g}(\xi'_S, \xi_{-S})$. Because of the independence of irrelevant choices and the monotonicity of cost functions there exists $k \in S$ with $g \in F_k(\xi'_S, \xi_S)$ implying

$$\pi_j(\xi'_S, \xi_{-S}) \leq \psi_{i,g}(\xi) < \psi_{i,g}(\xi'_S, \xi_{-S}) \leq \pi_k(\xi'_S, \xi_{-S}) \leq \pi_j(\xi'_S, \xi_{-S}),$$

which is a contradiction. For proving (5), suppose there is $(i, g) \in \Psi^{-}$ such that $\psi_{i,g}(\xi'_S, \xi_{-S}) > \pi_j(\xi'_S, \xi_{-S})$. Again, independence of irrelevant choices and monotonicity of cost functions implies that there is $k \in S$ with $g \in F_k(\xi'_S, \xi_{-S})$ giving rise to

$$\pi_k(\xi'_S, \xi_{-S}) \geq \psi_{i,g}(\xi'_S, \xi_{-S}) > \pi_j(\xi'_S, \xi_{-S}) \geq \pi_k(\xi'_S, \xi_{-S}),$$

which is a contradiction. To complete the proof, we observe that $N \times F = \Psi^{+} \cup \Psi^{-} \cup (S \times F)$. Then,

$$P_M(\xi) - P_M(\xi'_S, \xi_{-S}) = \sum_{(i,f)\in\Psi^{+}\cup\Psi^{-}\cup(S\times F)} \psi_{i,f}(\xi)^M - \psi_{i,f}(\xi'_S, \xi_{-S})^M$$
$$\geq \sum_{(i,f)\in\Psi^{-}\cup(S\times F)} \psi_{i,f}(\xi)^M - \psi_{i,f}(\xi'_S, \xi_{-S})^M.$$

The inequality follows from the first claim. We further derive

$$\sum_{(i,f)\in\Psi^{-}\cup(S\times F)} \psi_{i,f}(\xi)^M - \psi_{i,f}(\xi'_S, \xi_{-S})^M$$
$$\geq \sum_{f\in(S\times F)} \psi_{i,f}(\xi)^{M(\alpha)} - \sum_{(i,f)\in\Psi^{-}\cup(S\times F)} \psi_{i,f}(\xi'_S, \xi_{-S})^M$$
$$\geq (\pi_j(\xi'_S, \xi_{-S}) + \alpha)^M - n\,m\,\pi_j(\xi'_S, \xi_{-S})^M,$$

where the first inequality follows from the non-negativity of ψ. The second inequality follows from $\pi_j(\xi) \geq \pi_j(\xi'_S, \xi_{-S}) + \alpha$ and the second claim. Finally

$$P_M(\xi) - P_M(\xi'_S, \xi_{-S}) \geq (\alpha/2)^M + \left(\pi_j(\xi'_S, \xi_{-S}) + \alpha/2\right)^M$$
$$- n\,m\,\pi_j(\xi'_S, \xi_{-S})^M \geq (\alpha/2)^M,$$

where the last inequality follows from the choice of M. □

Proof (*Proof of Theorem 7*) Fix $\alpha > 0$. Since Δ is compact and P_M (as defined in Lemma 1) is bounded, there is a strategy profile z satisfying $P_M(z) \leq \inf_{\xi\in\Delta} P_M(\xi) - \epsilon$ with $0 < \epsilon < (\alpha/2)^M$. We claim that z is an α-approximate SE. Suppose not. Then by Lemma 1 there exists a profitable deviation $(z, (\xi'_S, z_{-S})) \in I^{\alpha}(S)$ with $P_M(z) - P_M(v_S, z_{-S}) \geq (\alpha/2)^M > \epsilon$, which contradicts the approximation guarantee of z. □

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