Strong Implication-Form ISS-Lyapunov Functions for Discontinuous Discrete-Time Systems

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***Abstract*— Input-to-State Stability (ISS) and the ISS-Lyapunov function have proved to be useful tools for the analysis and design of nonlinear systems in a variety of contexts. Motivated by the fact that many feedback control laws, such as model predictive control or event-based control, lead to discontinuous discrete-time dynamics, we investigate ISS-Lyapunov functions for such systems. ISS-Lyapunov functions were originally introduced in a so-called *implication-form* and, in many cases, this has been shown to be equivalent to an ISS-Lyapunov function of *dissipation-form*. However, for discontinuous dynamics, we demonstrate via an example that this equivalence no longer holds. We therefore propose a stronger implication-form ISS-Lyapunov which re-establishes the equivalence to dissipation-form ISS-Lyapunov functions and to the ISS property for discontinuous systems.**

I. Introduction

The notion of input-to-state stability (ISS) was introduced by Sontag in [20] in order to formalize a Lyapunov type stability property of nonlinear systems taking into account persisting inputs. Soon after its introduction it was recognized as a versatile tool for analyzing stability properties of nonlinear systems and it has become one of the most influential concepts in nonlinear stability theory of the last decades.

One of the most important features of ISS is that it can be fully characterized by means of ISS-Lyapunov functions. To this end, two different concepts of ISS-Lyapunov functions have been extensively used in the literature: ISS-Lyapunov functions in dissipation-form and in implication-form, see Section II for the respective definitions. Both formulations have their own advantages and are useful in different contexts, so it is indeed useful to have both formulations available and to be able to switch from one concept to the other, if necessary, e.g., in proofs.

For continuous-time systems [22] as well as for continuous discrete-time systems [8] these two concepts of ISS-Lyapunov functions are indeed fully equivalent. In this paper, we consider discrete-time nonlinear systems given by

$$x^+ = f(x, w) \tag{1}$$

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where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. We take as inputs sequences with values in $\mathbb{R}^m$ and denote this space by $\mathcal{W}$. We denote solutions of (1) by $\phi : \mathbb{Z}_{\geq 0} \times \mathbb{R}^n \times \mathcal{W} \to \mathbb{R}^n$.

Contrary to [8], in this paper we do not assume any continuity properties of the dynamics $f(\cdot, \cdot)$. This is motivated by the use of discontinuous controllers $u : \mathbb{R}^n \to U$ leading to a discontinuous closed loop system $x^+ = g(x, u(x), w) =: f(x, w)$ of the form (1), even if $g(\cdot, \cdot, \cdot)$ is a continuous map. Among modern controller design techniques, optimization based techniques like model predictive control (MPC) naturally lead to discontinuous feedback laws and, in the presence of state constraints, even the corresponding Lyapunov function is typically discontinuous, cf. [3], [19] or [5, Sections 8.5–8.9]. Similarly, quantized [18], [4] or event-based [2], [17] feedback laws naturally lead to discontinuous closed loop dynamics.

For discontinuous discrete-time dynamics, the equivalence between the two types of ISS-Lyapunov functions fails to hold as the existence of an implication-form ISS-Lyapunov function may not imply the existence of a dissipation-form ISS Lyapunov function, which we will demonstrate by a simple example. It was already observed in [6] that additional regularity properties are needed in order to conclude ISS from the existence of an implication form Lyapunov function. More generally, it is known that discontinuities may affect the usual inherent robustness properties of, e.g., asymptotic [11] or exponential stability [13].

In this paper we do not introduce additional regularity properties in order to fix this problem. Rather, we propose a new "strong" definition of an implication-form ISS-Lyapunov function which we will prove to be fully equivalent to its dissipation-form counterpart but which will maintain the general implication-form structure. Thus, in proofs it can be used like conventional implication-form ISS-Lyapunov functions which we demonstrate in this paper by deriving results about nonlinear scalings of ISS-Lyapunov functions. The construction relies on the idea of including a second implication in the implication-form ISS-Lyapunov function formulation. This idea is not entirely new. Conditions of a similar form have appeared in [14], [15] for hybrid systems and in [16, Formula (7)] for continuous discrete time systems. However, in these references the conditions are introduced as sufficient conditions while here we introduce and systematically study a variant which yields a necessary and sufficient ISS characterization. Moreover, in contrast to some of these references we will not impose any continuity assumptions.

The paper is organized as follows. In Section II we recall the definitions of input-to-state stability (ISS) and ISS-Lyapunov functions and discuss the relation between these concepts for continuous and discontinuous dynamics. In Section III we present and analyze our new strong implication-form ISS-Lyapunov function. We show that the existence of such a function is equivalent to the ISS property for discontinuous systems and that any strong implication-form ISS-Lyapunov function is a dissipation-form ISS-Lyapunov function and vice versa. In Section IV we illustrate the usefulness of this concept by proving two properties for which our newly defined Lyapunov function concepts turns out beneficial. Conclusions are presented in Section V and proofs of the main results can be found in Section VI.

II. ISS AND ISS-LYAPUNOV FUNCTIONS

In the sequel, we will denote the class of continuous positive definite functions $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by $\mathcal{P}$. We will also make use of the standard function classes $\mathcal{K}$, $\mathcal{K}_\infty$, $\mathcal{L}$, and $\mathcal{KL}$ (see [7] or [10]).

Definition 2.1: The system (1) is input-to-state stable (ISS) if there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ such that

$$|\phi(k,x,w)| \leq \max\left\{\beta(|x|,k), \sup_{i\in\mathbb{Z}_{[0,k-1]}} \gamma(|w(i)|)\right\} \quad (2)$$

for all $x \in \mathbb{R}^n$, $w \in \mathcal{W}$, and $k \in \mathbb{Z}_{\geq 0}$.

One of the reasons for the success of the ISS notion is that it is fully compatible with the concept of Lyapunov functions. To this end, both in continuous and in discrete-time two types of ISS-Lyapunov functions are used. The first is the so called dissipation-form ISS-Lyapunov function.

Definition 2.2: A dissipation-form ISS-Lyapunov function for (1) is a function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that there exist $\alpha_1, \alpha_2, \alpha \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ so that, for all $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \quad (3)$$

$$V(f(x,w)) - V(x) \leq -\alpha(|x|) + \sigma(|w|). \quad (4)$$

The second type of ISS-Lyapunov function is the following implication form.

Definition 2.3: An implication-form ISS-Lyapunov function for (1) is a function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that there exist $\alpha_1, \alpha_2, \hat{\alpha} \in \mathcal{K}_\infty$ and $\chi \in \mathcal{K}$ so that, for all $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$, (3) and

$$|x| \geq \chi(|w|) \quad \Rightarrow \quad V(f(x,w)) - V(x) \leq -\hat{\alpha}(|x|) \quad (5)$$

holds.

We observe that an implicit constraint in these definitions is $\alpha(|x|) \leq V(x)$ and $\hat{\alpha}(|x|) \leq V(x)$, respectively.

For continuous dynamics f it was shown in [8] that the existence of either type of ISS-Lyapunov function is equivalent to system (1) being ISS. An inspection of the proofs in this reference reveals that only the proof for the implication "existence of a dissipation-form ISS Lyapunov function ⇒ system (1) is ISS" can be carried over to the discontinuous case. We state this in the following lemma.

Lemma 2.4: [8, Lemma 3.5] If there exists a dissipation-form ISS-Lyapunov function for (1) then the system is ISS.

The proof of the converse implication in [8] relies on the fact that the existence of a dissipation-form ISS-Lyapunov function is equivalent to the existence of an implication-form ISS-Lyapunov function. While the following proposition shows that one implication of this equivalence remains true in the discontinuous setting, the subsequent example demonstrates that the opposite implication fails to hold.

Proposition 2.5: If $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a dissipation-form ISS-Lyapunov function with $\alpha_1, \alpha_2, \alpha \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ then V is an implication-form ISS-Lyapunov function with $\chi \doteq \alpha^{-1} \circ 2\sigma \in \mathcal{K}$ and $\hat{\alpha} = \frac{1}{2}\alpha \in \mathcal{K}_\infty$.

Proof: We rewrite (4) as

$$V(f(x,w)) - V(x) \quad \leq \quad -\frac{1}{2}\alpha(|x|) - \frac{1}{2}\alpha(|x|) + \sigma(|w|).$$

Therefore, with $\chi \doteq \alpha^{-1} \circ 2\sigma \in \mathcal{K}$ and $\hat{\alpha} \doteq \frac{1}{2}\alpha \in \mathcal{K}_\infty$ we immediately see that

$$|x| \geq \chi(|w|) \quad \Rightarrow \quad V(f(x,w)) - V(x) \leq -\frac{1}{2}\alpha(|x|).$$

■

Remark 2.6: We observe that we can trade off the decrease rate, $\hat{\alpha}$ and the input-dependent level set defined by χ. In particular, for any $\hat{\alpha} \in \mathcal{K}_\infty$ and $\varphi \in \mathcal{K}_\infty$ such that $\hat{\alpha}(s) + \varphi(s) \leq \alpha(s)$, for all $s \in \mathbb{R}_{\geq 0}$, we see that V satisfies (5) with $\chi \doteq \varphi^{-1} \circ \sigma$.

The following example shows that the converse of Proposition 2.5 does not hold in general.

Example 2.7: Consider the system

$$x^+ = f(x,w) = \nu(w)\kappa(x) \quad (6)$$

where

$$\kappa(x) \doteq \begin{cases} 0 & , \quad x = 0 \\ \frac{1}{|x|} & , \quad |x| \in (0,1) \\ \frac{1}{2|x|} & , \quad |x| \geq 1 \end{cases} \quad (7)$$

and

$$\nu(w) \doteq \begin{cases} 0 & , \quad w = 0 \\ \frac{1}{2}|w|^2 & , \quad |w| \in (0,1) \\ 1 & , \quad |w| \geq 1. \end{cases} \quad (8)$$

Take $V(x) \doteq |x|$ so that both the upper and lower bounds of (3) can be trivially taken as $|x|$. We observe that if $|x| \geq |w|$ then for $|x| \in (0,1)$

$$|f(x,w)| = \frac{|w|^2}{2|x|} \leq \frac{|x|^2}{2|x|} = \frac{|x|}{2}$$

and for $|x| \geq 1$

$$|f(x,w)| = \nu(w)\frac{1}{2|x|} \leq \frac{1}{2|x|} \leq \frac{|x|}{2}.$$

Therefore, with $\alpha(s) \doteq \frac{1}{2}s$ we see that

$$|x| \geq |w| \quad \Rightarrow \quad V(f(x,w)) - V(x) \leq -\alpha(|x|).$$

However, it is straightforward to see that the system (6) is not ISS. Take $w \equiv 1$ and any initial condition $x \in (0,1)$. Then we see that

$$\phi(2k+1, x) = 2^{2k}\frac{1}{x}, \quad \forall k \in \mathbb{Z}_{\geq 0}.$$

In other words, every other time step the solution increases so that the ISS estimate (2) can never be satisfied.

According to Lemma 2.4, this implies that a dissipation-form ISS-Lyapunov function cannot exist for (6). Hence, the example also shows that the existence of an implication-form ISS-Lyapunov function does not imply the existence of a dissipation-form ISS-Lyapunov function.

Remark 2.8: We did not require continuity of V in any of our definitions because this yields additional flexibility in constructing V for discontinuous systems. Note, however, that V in Example 2.7 is continuous, hence the gap between ISS-Lyapunov functions in implication- and dissipation-form is not due to possible discontinuities in V but only due to the discontinuities in f.

III. The strong implication form ISS-Lyapunov function

As just demonstrated, in the discontinuous setting the existence of an ISS-Lyapunov function in the implication form (3), (5) does not imply ISS and is not equivalent to the existence of an ISS-Lyapunov function in dissipation form (3), (4). In this section, we propose the following stronger alternative to the implication (5) which fixes these problems.

Definition 3.1: A strong implication-form ISS-Lyapunov function for (1) is a function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that there exist functions $\alpha_1, \alpha_2, \hat{\alpha} \in \mathcal{K}_\infty$ and $\varphi, \chi \in \mathcal{K}$, so that, for all $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$, V satisfies (3) and

$$|x| \geq \chi(|w|) \quad \Rightarrow \quad V(f(x,w)) - V(x) \leq -\hat{\alpha}(|x|) \qquad (9)$$

$$|x| < \chi(|w|) \quad \Rightarrow \quad V(f(x,w)) \leq \varphi(|w|). \qquad (10)$$

This definition is motivated by the ISS Lyapunov functions in implication form in continuous time, which will always satisfy the second implication on time intervals on which w is constant.

Before we turn to investigating the relation between the strong implication-form ISS-Lyapunov function and the notions introduced in the last section, we show a useful rescaling property of strong implication-form ISS-Lyapunov function (which in fact can be proved analogously also for conventional implication-form ISS-Lyapunov functions).

To this end, we observe that a weaker form of (9) is obtained when $\hat{\alpha} \in \mathcal{K}_\infty$ is replaced by a merely positive definite function $\hat{\rho} \in \mathcal{P}$ such that

$$|x| \geq \chi(|w|) \quad \Rightarrow \quad V(f(x,w)) - V(x) \leq -\hat{\rho}(|x|). \quad (11)$$

Conversely, (9) can be strengthened to requiring the existence of $\hat{\lambda} \in (0,1)$ such that

$$|x| \geq \chi(|w|) \quad \Rightarrow \quad V(f(x,w)) \leq \hat{\lambda} V(x). \qquad (12)$$

Note that the latter implies exponential decay of $k \mapsto V(\phi(k,x,0))$.

Theorem 3.2: The following are equivalent:

(i) There exists a $\hat{\rho} \in \mathcal{P}$ and a strong implication-form ISS-Lyapunov function V satisfying (11) instead of (9);

(ii) There exists a strong implication-form ISS-Lyapunov function $\widehat{V}$;

(iii) For any given $\hat{\lambda} \in (0,1)$ there exists a strong implication-form ISS-Lyapunov function $\widetilde{V}$ satisfying (12).

Moreover, for V satisfying (i) there exist $\bar{\alpha}$ and $\tilde{\alpha} \in \mathcal{K}_\infty$ such that $\widehat{V}$ in (ii) and $\widetilde{V}$ in (iii) can be written in the form $\widehat{V} = \bar{\alpha}(V)$ and $\widetilde{V} = \tilde{\alpha}(V)$.

The proof of this theorem can be found in Section VI-A.

The next theorem shows that the strong implication-form ISS-Lyapunov function is equivalent to the dissipative-form ISS-Lyapunov function (4). This then overcomes the gap observed between the dissipative-form ISS-Lyapunov function and the conventional implication-form ISS-Lyapunov function (5) when considering discontinuous systems and ISS-Lyapunov functions.

Theorem 3.3: Let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a function satisfying (3) for $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$.

(i) If V together with functions $\alpha \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ satisfies (4) then V also satisfies (9) with $\hat{\alpha} \doteq \alpha/2$, $\chi \doteq \alpha^{-1} \circ 2\sigma$ and satisfies (10) with $\varphi \doteq \gamma \circ \chi + \sigma$, where $\gamma \in \mathcal{K}_\infty$ is arbitrary with $\gamma \geq \alpha_2 - \alpha$.

(ii) If V together with functions $\hat{\alpha} \in \mathcal{K}_\infty$ and $\chi, \varphi \in \mathcal{K}$ satisfies (9) and (10), then V also satisfies (4) with $\alpha \doteq \min\{\hat{\alpha}, \alpha_1\}$ and $\sigma = \varphi$.

Proof: (i) We rewrite (4) as

$$\begin{aligned} V(f(x,w)) - V(x) \quad &\leq \quad -\frac{1}{2}\alpha(|x|) - \frac{1}{2}\alpha(|x|) \\ &\qquad +\sigma(|w|). \end{aligned}$$

Therefore, with $\chi \doteq \alpha^{-1} \circ 2\sigma \in \mathcal{K}$ and $\hat{\alpha} \doteq \frac{1}{2}\alpha \in \mathcal{K}_\infty$ we immediately see that

$$\begin{aligned} &|x| \geq \chi(|w|) \\ &\quad \Rightarrow \quad V(f(x,w)) - V(x) \leq -\frac{1}{2}\alpha(|x|) = -\hat{\alpha}(|x|) \end{aligned}$$

and

$$\begin{aligned} &|x| < \chi(|w|) \\ &\Rightarrow \quad V(f(x,w)) \quad \leq \quad V(x) - \alpha(|x|) + \sigma(|w|) \\ &\qquad\qquad\qquad \leq \quad \alpha_2(|x|) - \alpha(|x|) + \sigma(|w|) \\ &\qquad\qquad\qquad \leq \quad \gamma(|x|) + \sigma(|w|) \\ &\qquad\qquad\qquad \leq \quad \gamma(\chi(|w|)) + \sigma(|w|) \\ &\qquad\qquad\qquad = \quad \varphi(|w|). \end{aligned}$$

(ii) If $|x| \geq \chi(|w|)$ then we get

$$V(f(x,w)) - V(x) \leq -\hat{\alpha}(|x|) \leq -\alpha(|x|) + \sigma(|w|).$$

In case $|x| < \chi(|w|)$, using (3) we obtain

$$V(f(x,w)) - V(x) \leq \varphi(|w|) - \alpha_1(|x|) \leq -\alpha(|x|) + \sigma(|w|).$$

■

Hence, the stronger implication form (9) is equivalent to the dissipation form (4). This enables us to carry over the proof of the equivalence between ISS and the existence of a (strong implication-form) ISS-Lyapunov function to the discontinuous setting. This leads to the following theorem whose proof can be found in Section VI-B.

Theorem 3.4: System (1) is ISS if and only if there exists a strong implication-form ISS-Lyapunov function in the sense of Definition 3.1.

IV. Two consequences

In this section we illustrate the usefulness of the newly introduced strong implication-form ISS-Lyapunov function by proving two immediate consequences from the new concept. The first states that by nonlinear rescaling of a dissipation-form ISS-Lyapunov function one can always obtain an exponentially decaying dissipation-form ISS Lyapunov function. This means that there exists $\lambda \in (0,1)$ such that the inequality

$$V(f(x,w)) \le \lambda V(x) + \sigma(|w|) \tag{13}$$

holds for all $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$.

Before formulating the actual rescaling result, we first note that we can prove a result similar to Theorem 3.3 for the relationship between dissipative-form and strong implication-form ISS Lyapunov functions when we have an exponential decrease.

Theorem 4.1: Let $V : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ be a function satisfying (3) for $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$.

(i) If V together with $\lambda \in (0,1)$ and $\sigma \in \mathcal{K}$ satisfies (13), then V also satisfies (10) and (12) with $\hat{\lambda} \doteq \lambda + \varepsilon$ satisfying $\lambda + \varepsilon < 1$, $\chi \doteq \alpha_1^{-1}(\frac{1}{\varepsilon}\sigma)$, and $\varphi \doteq \lambda\alpha_2 \circ \chi + \sigma$.

(ii) If V together with $\hat{\lambda} \in (0,1)$, $\chi, \varphi \in \mathcal{K}$ satisfies (10) and (12), then V also satisfies (13) with $\lambda \doteq \hat{\lambda}$ and $\sigma \doteq \varphi$.

Proof: (i) Since $\lambda \in (0,1)$ there exists $\varepsilon > 0$ such that $\lambda + \varepsilon \in (0,1)$. We may then rewrite (13) as

$$\begin{aligned} V(f(x,w)) &\le (\lambda+\varepsilon)V(x) - \varepsilon V(x) + \sigma(|w|) \\ &\le \hat{\lambda}V(x) - \varepsilon\alpha_1(|x|) + \sigma(|w|) \end{aligned}$$

which yields the implication (12). The implication (10) follows from the upper bound on V and the condition $|x| < \chi(|w|)$ as

$$V(f(x,w)) \le \lambda\alpha_2(|x|) + \sigma(|w|) \le \lambda\alpha_2 \circ \chi(|w|) + \sigma(|w|).$$

The proof of (ii) is immediate by inspection. ■

The actual rescaling result is now formulated in the following corollary.

Corollary 4.2: For any dissipation-form ISS-Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ and any $\lambda \in (0,1)$, there exists a function $\tilde{\alpha} \in \mathcal{K}_\infty$ such that $\widetilde{V} \doteq \tilde{\alpha} \circ V : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ is a dissipation-form ISS-Lyapunov function satisfying the exponential decay inequality (13).

Proof: Theorem 3.3 implies that V is a strong implication-form ISS-Lyapunov function. Theorem 3.2 then implies the existence of $\tilde{\alpha} \in \mathcal{K}_\infty$ such that $\tilde{\alpha} \circ V$ has the exponential decrease (12) and thus Theorem 4.1(ii) implies (13). ■

The second consequence of our setting is related to conventional implication-form ISS-Lyapunov functions as defined in Definition 2.2. As discussed after this definition, it was already observed in [8] that the existence of such a Lyapunov function implies ISS if the discrete-time dynamics f and the Lyapunov function V are continuous.

Using the framework of strong implication-form ISS-Lyapunov functions we can now show that continuity of f is actually only needed in $w = 0$, in the following uniform sense.

Definition 4.3: We say that f is continuous in $w = 0$ uniformly in x, if for each $r > 0$ there is $\gamma_r \in \mathcal{K}_\infty$ such that for all $|x| \le r$, $|w| \le r$ the inequality

$$|f(x,w) - f(x,0)| \le \gamma_r(|w|)$$

holds.

Moreover, we need continuity of V in $x = 0$. Since the latter is a consequence of the bounds (3), only the continuity condition on f is explicitly demanded in the following proposition.

Proposition 4.4: Let V be a (conventional) implication-form ISS-Lyapunov function for appropriate $\alpha_1, \alpha_2, \hat{\alpha} \in \mathcal{K}_\infty$ and $\chi \in \mathcal{K}$. Assume that f is continuous in $w = 0$ uniformly in x in the sense of Definition 4.3. Then there exists $\varphi \in \mathcal{K}$ so that V is a strong implication-form ISS-Lyapunov function and thus also a dissipation-form ISS-Lyapunov function.

Proof: First, consider $w \equiv 0$. Then we observe that

$$\begin{aligned} |f(x,0)| &\le \alpha_1^{-1}(V(f(x,0))) \le \alpha_1^{-1}(V(x) - \hat{\alpha}(|x|)) \\ &\le \alpha_1^{-1}(\alpha_2(|x|) - \hat{\alpha}(|x|)). \end{aligned} \tag{14}$$

Since by (5) $\hat{\alpha}(|x|) \le V(x) \le \alpha_2(|x|)$ for all $x \in \mathbb{R}^n$, with equality if and only if $x = 0$, the function $\alpha_1^{-1}(\alpha_2(s) - \hat{\alpha}(s))$ is positive definite. Define $\bar{\gamma} \in \mathcal{K}_\infty$ by

$$\bar{\gamma}(s) \doteq \max\left\{s, \alpha_1^{-1}(\alpha_2(s) - \hat{\alpha}(s))\right\}, \quad \forall s \in \mathbb{R}_{\ge 0} \tag{15}$$

so that

$$|f(x,0)| \le \bar{\gamma}(|x|), \quad \forall x \in \mathbb{R}^n. \tag{16}$$

Now, if for all $r > 0$ we define

$$\hat{\gamma}(r) \doteq \sup\{|f(x,w) - f(x,0)| : |x| \le r, |w| \le r\},$$

then for all $r_1 \ge r$ we obtain $\hat{\gamma}(r) \le \gamma_{r_1}(r)$ which implies $\hat{\gamma}(r) \to 0$ as $r \to 0$. Moreover, $\hat{\gamma}(r)$ is finite for all $r > 0$. Hence, we may overbound $\hat{\gamma}$ with a function $\gamma \in \mathcal{K}_\infty$.

It is now sufficient to show that there exists $\varphi \in \mathcal{K}_\infty$ such that the implication in (10) holds. To this end, let $|x| < \chi(|w|)$. Then we have

$$\begin{aligned} |f(x,w)| &= |f(x,w) - f(x,0) + f(x,0)| \\ &\le \gamma(\max\{|w|, \chi(|w|)\}) + \bar{\gamma}(|x|) \\ &\le \gamma(\max\{|w|, \chi(|w|)\}) + \bar{\gamma}(\chi(|w|)) =: \tilde{\gamma}(|w|) \end{aligned}$$

implying

$$V(f(x,w)) \le \alpha_2(|f(x,w)|) \le \alpha_2(\tilde{\gamma}(|w|)).$$

This shows the desired inequality with $\varphi(r) \doteq \alpha_2(\tilde{\gamma}(|w|))$. ■

We note that the map $f(x,w) = \nu(w)\kappa(x)$ in (6) of Example 2.7 does not satisfy the required continuity property of Proposition 4.4. To see this, we first observe that

$$|f(x,w) - f(x,0)| = |f(x,w)|.$$

Choose $r = 1$ and any $\gamma_1 \in \mathcal{K}_\infty$. Then, with $w = 1$, we see that

$$|f(x,1)| = \frac{1}{|x|}, \quad \forall x \in (-1,1)\backslash\{0\}$$

so that $|f(x,1)| > \gamma_1(1)$ for some $x \in (0,1)$.

V. Conclusions

We have shown that the equivalence between ISS-Lyapunov functions in dissipation-form and in implication-form known for continuous time systems [22] and continuous discrete-time systems [8] fails to hold for discrete-time discontinuous systems. More precisely, for discontinuous dynamics the implication-form ISS-Lyapunov function turns out to be a weaker concept and does not necessarily guarantee ISS.

As a remedy, we proposed a new "strong" implication-form ISS-Lyapunov function. This fixes the problem because any strong implication-form ISS-Lyapunov function is also a dissipation-form ISS-Lyapunov function and vice versa. We demonstrated that the newly defined Lyapunov function is useful for performing nonlinear scalings of ISS-Lyapunov functions and for deriving weakened continuity conditions under which the conventional implication-form ISS-Lyapunov function guarantees ISS.

VI. Proofs

A. Proof of Theorem 3.2

We observe that the implications $(iii) \Rightarrow (ii) \Rightarrow (i)$ are trivial. It thus suffices to prove the converse implications.

1) Positive Definite to $\mathcal{K}_\infty$: $(i) \Rightarrow (ii)$: We start from an ISS-Lyapunov function with a positive definite decrease rate; i.e., $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\chi, \varphi \in \mathcal{K}$, and $\rho \in \mathcal{P}$ satisfying (3) and (11).

For $\rho \in \mathcal{P}$, [1, Lemma IV.1] ([10, Lemma 12]) yields $\alpha \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{L}$ so that

$$\rho(s) \geq \alpha(s)\sigma(s), \quad \forall s \in \mathbb{R}_{\geq 0}. \tag{17}$$

Using the bounds (3) we see that for all $x \in \mathbb{R}^n$ the inequality $|x| \geq \chi(|w|)$ implies

$$\begin{aligned} V(f(x,w)) - V(x) &\leq -\rho(|x|) \\ &\leq -\alpha(|x|)\sigma(|x|) \\ &\leq -\alpha \circ \alpha_2^{-1}(V(x))\sigma \circ \alpha_1^{-1}(V(x)) \\ &= -\hat{\rho}(V(x)) \end{aligned} \tag{18}$$

where $\hat{\rho}(s) \doteq \alpha \circ \alpha_2^{-1}(s)\sigma \circ \alpha_1^{-1}(s)$ for all $s \in \mathbb{R}_{\geq 0}$ is positive definite.

From here, we follow [9, Lemma 2.8]. Let $\bar{\alpha} \in \mathcal{K}_\infty$ be such that

$$\bar{\alpha}\left(\frac{s}{2}\right)\hat{\rho}(s) \geq s, \quad \forall s \geq 1 \tag{19}$$

and define $\hat{\alpha} \in \mathcal{K}_\infty$ by

$$\hat{\alpha}(s) \doteq s + \int_0^s \bar{\alpha}(r)dr, \quad \forall s \in \mathbb{R}_{\geq 0}. \tag{20}$$

We observe that $\hat{\alpha} \in \mathcal{K}_\infty$ and

$$\hat{\alpha}'(s) = 1 + \bar{\alpha}(s), \quad \forall s \in \mathbb{R}_{>0} \tag{21}$$

so that $\hat{\alpha}'$ is strictly increasing.

Define $\widehat{V}(x) \doteq \hat{\alpha}(V(x))$ for all $x \in \mathbb{R}^n$ and observe that with the $\mathcal{K}_\infty$ functions $\hat{\alpha}_1 \doteq +\hat{\alpha} \circ \alpha_1$ and $\hat{\alpha}_2 \doteq \hat{\alpha} \circ \alpha_2$ we have

$$\hat{\alpha}_1(|x|) \leq \widehat{V}(x) \leq \hat{\alpha}_2(|x|). \tag{22}$$

Additionally, let $\hat{\varphi} \in \mathcal{K}$ be given by $\hat{\varphi} \doteq \hat{\alpha} \circ \varphi$ so that (10) implies, for $|x| < \chi(|w|)$,

$$\widehat{V}(f(x,w)) = \hat{\alpha}(V(f(x,w))) \leq \hat{\alpha} \circ \varphi(|w|) = \hat{\varphi}(|w|).$$

To simplify the notation, we use $\widehat{V}^+ \doteq \widehat{V}(f(x,w))$, $\widehat{V} \doteq \widehat{V}(x)$, $V^+ \doteq V(f(x,w))$, and $V \doteq V(x)$. In what follows we assume $|x| \geq \chi(|w|)$.

Since $\hat{\alpha}$ is differentiable, the mean value theorem yields the existence of $\theta \in (0,1)$ so that

$$\hat{\alpha}(V^+) - \hat{\alpha}(V) = \hat{\alpha}'(V^+ + \theta(V - V^+))(V^+ - V). \tag{23}$$

Note that, as a consequence of (18), $V^+ - V \leq 0$.

We first restrict attention to $V \geq 1$ and consider two cases. First, we assume $V^+ \leq \frac{V}{2}$ and note that $\hat{\alpha}'(s) \geq 1$ for all $s \in \mathbb{R}_{\geq 0}$. Then

$$\widehat{V}^+ - \widehat{V} \leq V^+ - V \leq -\frac{V}{2}. \tag{24}$$

Now suppose that $V^+ \geq \frac{V}{2}$. In this case, using $V - V^+ \geq 0$ and (21), we have

$$\begin{aligned} \hat{\alpha}'(V^+ + \theta(V - V^+)) &\geq \hat{\alpha}'(V^+) \\ &\geq \hat{\alpha}'\left(\frac{V}{2}\right) > \bar{\alpha}\left(\frac{V}{2}\right). \end{aligned} \tag{25}$$

Therefore, for $V \geq 1$, using (25), (18), and (19) we obtain

$$\begin{aligned} \widehat{V}^+ - \widehat{V} &\leq \bar{\alpha}\left(\frac{V}{2}\right)(V^+ - V) \\ &\leq -\bar{\alpha}\left(\frac{V}{2}\right)\hat{\rho}(V) \leq -V. \end{aligned} \tag{26}$$

Combining (24) and (26) we see that, for $V \geq 1$,

$$\widehat{V}^+ - \widehat{V} \leq -\frac{V}{2}. \tag{27}$$

For $V \leq 1$, we note that by definition (20) and (18) we have

$$\begin{aligned} \widehat{V}^+ - \widehat{V} &= V^+ + \int_0^{V^+} \bar{\alpha}(r)dr - V - \int_0^V \bar{\alpha}(r)dr \\ &\leq V^+ - V \leq -\hat{\rho}(V). \end{aligned} \tag{28}$$

Take $\breve{\alpha} \in \mathcal{K}_\infty$ so that

$$\begin{aligned} \breve{\alpha}(s) &\leq \hat{\rho}(s), \quad s \in [0,1] \\ \breve{\alpha}(s) &\leq \frac{s}{2}, \quad s \geq 1. \end{aligned}$$

Finally, let $\alpha \in \mathcal{K}_\infty$ be defined as $\alpha \doteq \breve{\alpha} \circ \alpha_1$ so that

$$|x| \geq \chi(|w|) \quad \Rightarrow \quad \widehat{V}(f(x,w)) - \widehat{V}(x) \leq -\breve{\alpha}(V(x)) \\ \leq -\breve{\alpha} \circ \alpha_1(|x|) = -\alpha(|x|). \qquad (29)$$

2) $\mathcal{K}_\infty$ to Exponential: $(ii) \Rightarrow (iii)$: Since every $\mathcal{K}_\infty$-function is also positive definite, we can follow the first part of the proof to conclude (22) and (29) which imply

$$|x| \geq \chi(|w|) \quad \Rightarrow \quad \widehat{V}(f(x,w)) - \widehat{V}(x) \leq -\alpha(|x|) \\ \leq -\alpha \circ \hat{\alpha}_2^{-1}(\widehat{V}(x)). \qquad (30)$$

Define $\mu \in \mathcal{K}_\infty$ by

$$\mu(s) \doteq \min\left\{\alpha \circ \hat{\alpha}_2^{-1}(s), \frac{s}{2}\right\} \qquad (31)$$

and note that $\mathrm{id} - \mu \in \mathcal{K}_\infty$ and

$$|x| \geq \chi(|w|) \quad \Rightarrow \quad \widehat{V}(f(x,w)) \leq \widehat{V}(x) - \mu(\widehat{V}(x)) \\ = (\mathrm{id} - \mu)(\widehat{V}(x)). \qquad (32)$$

Select any $\lambda \in (0,1)$. Then [10, Corollary 1] yields $\hat{\mu} \in \mathcal{K}_\infty$ so that

$$\hat{\mu} \circ (\mathrm{id} - \mu)(s) = \lambda \hat{\mu}(s), \quad \forall s \in \mathbb{R}_{\geq 0}. \qquad (33)$$

Define $\widetilde{V} \doteq \hat{\mu}(\widehat{V})$ and note that, with $\mathcal{K}_\infty$ functions $\bar{\alpha}_1 \doteq \hat{\mu} \circ \hat{\alpha}_1$ and $\bar{\alpha}_2 \doteq \hat{\mu} \circ \hat{\alpha}_2$,

$$\bar{\alpha}_1(|x|) \leq \widetilde{V}(x) \leq \bar{\alpha}_2(|x|). \qquad (34)$$

Furthermore,

$$|x| \geq \chi(|w|) \quad \Rightarrow \quad \widetilde{V}(f(x,w)) = \hat{\mu}(\widehat{V}(f(x,w))) \\ \leq \hat{\mu} \circ (\mathrm{id} - \mu)(\widehat{V}(x)) = \lambda \hat{\mu}(\widehat{V}(x)) \\ = \lambda \widetilde{V}(x).$$

The form $\widetilde{V} = \tilde{\alpha}(V)$ follows by combining both parts of the proof and setting $\tilde{\alpha} = \hat{\mu} \circ \hat{\alpha}$. Finally, we define $\bar{\varphi} \in \mathcal{K}$ by $\bar{\varphi} \doteq \hat{\mu} \circ \hat{\varphi}$ so that, for $|x| < \chi(|w|)$,

$$\widetilde{V}(f(x,w)) = \hat{\mu}(\widehat{V}(f(x,w))) \leq \hat{\mu} \circ \hat{\varphi}(|w|) = \bar{\varphi}(|w|). \qquad (35)$$

■

B. Proof of Theorem 3.4

"Existence of strong implication-form $V \Rightarrow$ ISS": This follows immediately from Theorem 3.3(ii) followed by Lemma 2.4.

"ISS $\Rightarrow$ Existence of strong implication-form V": We show the existence of a Lyapunov function in exponential form, i.e., satisfying (12) (implying (9)) and (10). Our proof relies on a converse Lyapunov theorem for difference inclusions. We denote the set of solutions to the difference inclusion

$$x^+ \in F(x), \quad x \in \mathbb{R}^n \qquad (36)$$

defined by the set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and from an initial condition $x \in \mathbb{R}^n$ by $\mathcal{S}(x)$. A solution $\phi \in \mathcal{S}(x)$ is a function $\phi : \mathbb{Z}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ such that $\phi(0,x) = x$ and $\phi(k+1,x) \in F(\phi(k,x))$ for all $k \in \mathbb{Z}_{\geq 0}$.

Definition 6.1: The difference inclusion (36) is said to be $\mathcal{KL}$-stable if there exists $\beta \in \mathcal{KL}$ so that

$$|\phi(k,x)| \leq \beta(|x|,k), \quad \forall x \in \mathbb{R}^n, \ \phi \in \mathcal{S}(x), \ k \in \mathbb{Z}_{\geq 0}. \qquad (37)$$

Theorem 6.2: If the difference inclusion (36) is $\mathcal{KL}$-stable then, for any given $\lambda \in (0,1)$ there exists an exponential-decrease Lyapunov function; i.e., there exist functions $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ so that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \qquad (38)$$

$$V(\phi(1,x)) \leq \lambda V(x) \qquad (39)$$

for all $x \in \mathbb{R}^n$ and $\phi(1,x) \in F(x)$.

Proof: The proof follows that of [12, Theorem 2.7] where, here, we need not worry about regularity of the Lyapunov function.

Given $\beta \in \mathcal{KL}$ and $\lambda \in (0,1)$, Sontag's lemma on $\mathcal{KL}$-estimates [21, Proposition 7] yields $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ so that

$$\alpha_1(\beta(s,k)) \leq \alpha_2(s)\lambda^k, \quad \forall s \in \mathbb{R}_{\geq 0}, \ k \in \mathbb{Z}_{\geq 0}. \qquad (40)$$

For all $x \in \mathbb{R}^n$, define

$$V(x) \doteq \sup_{k \in \mathbb{Z}_{\geq 0}} \sup_{\phi \in \mathcal{S}(x)} \alpha_1(|\phi(k,x)|)\lambda^{-k}. \qquad (41)$$

Then

$$V(x) \geq \sup_{\phi \in \mathcal{S}(x)} \alpha_1(|\phi(0,x)|)\lambda^0 = \alpha_1(|x|)$$

and

$$\begin{aligned} V(x) &\leq \sup_{k \in \mathbb{Z}_{\geq 0}} \alpha_1(\beta(|x|,k))\lambda^{-k} \\ &\leq \sup_{k \in \mathbb{Z}_{\geq 0}} \alpha_2(|x|)\lambda^k \lambda^{-k} = \alpha_2(|x|) \end{aligned}$$

so that $V(x)$ satisfies the desired upper and lower bounds (38). The desired decrease condition follows as

$$\begin{aligned} V(\phi(1,x)) &= \sup_{k \in \mathbb{Z}_{\geq 0}} \sup_{\psi \in \mathcal{S}(\phi(1,x))} \alpha_1(|\psi(k,\phi(1,x))|)\lambda^{-k} \\ &\leq \sup_{k \in \mathbb{Z}_{\geq 1}} \sup_{\phi \in \mathcal{S}(x)} \alpha_1(|\phi(k,x)|)\lambda^{-k+1} \\ &\leq \sup_{k \in \mathbb{Z}_{\geq 0}} \sup_{\phi \in \mathcal{S}(x)} \alpha_1(|\phi(k,x)|)\lambda^{-k+1} \\ &= \lambda V(x) \end{aligned}$$

for all $x \in \mathbb{R}^n$. ■

In order to demonstrate that ISS implies the existence of an ISS-Lyapunov function, we follow the standard argument as in [22] and [8]. Denote the closed unit ball in $\mathbb{R}^m$ by $\mathcal{B}^m$. We show that there exists a $\mu \in \mathcal{K}_\infty$ such that the differential inclusion defined by

$$x(k+1) \in f(x(k), \mu(|x(k)|)\mathcal{B}^m) \qquad (42)$$

is $\mathcal{KL}$-stable, allowing us to appeal to Theorem 6.2 to obtain an ISS-Lyapunov function. We denote the solution set of (42) from an initial condition $x \in \mathbb{R}^n$ by $\mathcal{S}_\mu(x)$.

Proposition 6.3: [12, Proposition 2.2.] The following are equivalent:

1) The difference inclusion $x(k+1) \in F(x(k))$ is $\mathcal{KL}$-stable.
2) The following hold:

a) (Uniform stability): There exists $\gamma \in \mathcal{K}_\infty$ so that, for each $x \in \mathbb{R}^n$, all solutions $\phi \in \mathcal{S}(x)$ satisfy

$$|\phi(k,x)| \leq \gamma(|x|), \quad \forall k \in \mathbb{Z}_{\geq 0}.$$

b) (Uniform global attractivity): For each $r, \varepsilon \in \mathbb{R}_{>0}$, there exists $K(r,\varepsilon) > 0$ so that, for each $x \in \mathbb{R}^n$, all solutions $\phi \in \mathcal{S}(x)$ satisfy

$$|x| \leq r, \quad k \in \mathbb{Z}_{\geq K(r,\varepsilon)} \quad \Rightarrow \quad |\phi(k,x)| \leq \varepsilon.$$

Lemma 6.4: If (1) is ISS then there exists $\mu \in \mathcal{K}_\infty$ such that the difference inclusion (42) is $\mathcal{KL}$-stable.

Proof: Without loss of generality, we assume that $\gamma \in \mathcal{K}$ from (2) satisfies $\gamma(r) \geq r$. Define $\alpha, \mu \in \mathcal{K}_\infty$ as

$$\alpha(s) \doteq \max\left\{\gamma(\beta(s,0)), \gamma\left(\tfrac{1}{2}s\right)\right\},$$

$$\mu(s) \doteq \tfrac{1}{2}\gamma^{-1}\left(\tfrac{1}{4}\alpha^{-1}(s)\right)$$

for all $s \in \mathbb{R}_{\geq 0}$.

Claim 6.5: For any $x \in \mathbb{R}^n$ and $\phi \in \mathcal{S}_\mu(x)$ we have

$$\gamma \circ \mu(|\phi(k,x)|) \leq \tfrac{1}{2}|x|, \quad \forall k \in \mathbb{Z}_{\geq 0}. \tag{43}$$

Proof: By the definition of α we have $|x| \leq \beta(|x|,0) \leq \alpha(|x|)$ so that

$$\gamma \circ \mu(|x|) \leq \tfrac{1}{4}\alpha^{-1}(|x|) \leq \tfrac{1}{4}|x|. \tag{44}$$

Let

$$k_1 \doteq \min\left\{k \in \mathbb{Z}_{\geq 0} : \gamma \circ \mu(|\phi(k,x)|) > \tfrac{1}{2}|x|\right\}$$

and note that (44) implies $k_1 \in \mathbb{Z}_{\geq 1}$. In order to obtain a contradiction, assume $k_1 < \infty$. Then (43) holds for all $k \in \mathbb{Z}_{[0,k_1-1]}$. Therefore, $\gamma(|\mu(|\phi(k,x)|)\mathcal{B}^m|) \leq \frac{1}{2}|x|$ for all $\phi \in \mathcal{S}_\mu(x)$ and $k \in \mathbb{Z}_{[0,k_1-1]}$. Applying $\gamma \in \mathcal{K}_\infty$ to both sides of the ISS-estimate (2) in conjunction with this fact yields

$$\gamma(|\phi(k,x)|) \leq \max\left\{\gamma(\beta(|x|,0)), \gamma\left(\tfrac{1}{2}|x|\right)\right\} = \alpha(|x|), \tag{45}$$

for all $\phi \in \mathcal{S}_\mu(x)$, $k \in \mathbb{Z}_{[0,k_1-1]}$.

Then, using the definition of μ, the ISS-estimate (2), and (45), we have

$$\begin{aligned}
&\gamma \circ \mu(|\phi(k_1,x)|) \ \leq \ \tfrac{1}{4}\alpha^{-1}(|\phi(k_1,x)|) \\
&\leq \ \tfrac{1}{4}\max\Big\{\alpha^{-1}(\beta(|x|,k)) \\
&\qquad\qquad \times \max_{j \in \mathbb{Z}_{[0,k_1-1]}} \alpha^{-1} \circ \gamma(|\phi(j,x)|)\Big\} \\
&\leq \ \tfrac{1}{4}\max\{|x|,|x|\} = \tfrac{1}{4}|x|
\end{aligned}$$

which contradicts the definition of k_1 and hence proves the claim. ■

We now prove $\mathcal{KL}$-stability of difference inclusion (42) by proving uniform stability and uniform global attractivity and then appeal to Proposition 6.3.

Uniform stability follows using (2), (43), and the fact that $\gamma(s) \geq s$ as

$$\begin{aligned}
&|\phi(k,x)| \\
&\leq \ \max\Big\{\beta(|x|,k), \max_{i \in \mathbb{Z}_{[0,k-1]}} \gamma(|\mu(|\phi(i,x)|)\mathcal{B}^m)\Big\} \\
&\leq \ \max\Big\{\beta(|x|,0), \max_{i \in \mathbb{Z}_{[0,k-1]}} \gamma(|\mu(|\phi(i,x)|))\Big\} \\
&\leq \ \max\left\{\beta(|x|,0), \tfrac{1}{2}|x|\right\} \\
&\leq \ \max\left\{\beta(|x|,0), \tfrac{1}{2}\gamma(|x|)\right\} \ = \ \alpha(|x|).
\end{aligned}$$

To establish uniform global attractivity, as above we note that, for all $x \in \mathbb{R}^n$, $\phi \in \mathcal{S}_\mu(x)$, and $k \in \mathbb{Z}_{\geq 0}$,

$$|\phi(k,x)| \leq \max\left\{\beta(|x|,k), \tfrac{1}{2}|x|\right\}.$$

Since $\beta \in \mathcal{KL}$, for each $r \in \mathbb{R}_{\geq 0}$ there exists a finite $T(r) \in \mathbb{Z}_{\geq 1}$ so that $\beta(r,k) \leq \frac{1}{2}r$ for all $k \in \mathbb{Z}_{\geq T(r)}$. Therefore, for all $|x| \leq r$ we have $|\phi(k,x)| \leq \frac{1}{2}r$ for all $\phi \in \mathcal{S}_\mu(x)$ and $k \in \mathbb{Z}_{\geq T(r)}$.

Fix any $\varepsilon \in \mathbb{R}_{>0}$ and let $k \in \mathbb{Z}_{\geq 1}$ be such that $2^{-k}r \leq \varepsilon$. Define $r_1 \doteq r$, $r_i \doteq \frac{1}{2}r_{i-1}$ for all $i \in \mathbb{Z}_{\geq 2}$, and $K(r,\varepsilon) \doteq \sum_{i=1}^{k} T(r_i)$. Then

$$|\phi(k,x)| \leq 2^{-k}r \leq \varepsilon$$

holds for all $|x| \leq r$, $\phi \in \mathcal{S}_\mu(x)$ and $k \in \mathbb{Z}_{\geq K(r,\varepsilon)}$. Therefore, the difference inclusion (42) is $\mathcal{KL}$-stable. ■

We now complete the proof of Theorem 3.4. Since (1) is ISS, the difference inclusion (42) is $\mathcal{KL}$-stable, and by Theorem 6.2, for any $\lambda \in \mathbb{R}_{(0,1)}$, there exist functions $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ so that (38) and (39) hold for the difference inclusion given by (42). This then implies

$$|w| \leq \mu(|x|) \quad \Rightarrow \quad V(\phi(1,x,w)) \leq \lambda V(x) \tag{46}$$

for all $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, and $\phi(1,x,w) = f(x,w)$, i.e., (12).

For proving (10), let $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ come from the ISS-estimate (2) and $\alpha_2 \in \mathcal{K}_\infty$ be the upper bound in (38). Define $\varphi \in \mathcal{K}$ by

$$\varphi(s) \doteq \alpha_2(\beta(\mu^{-1}(s),1) + \gamma(s)), \quad \forall s \in \mathbb{R}_{\geq 0}.$$

Then for $|w| > \mu(|x|)$ we have

$$\begin{aligned}
V(f(x,w)) \ &\leq \ \alpha_2(|f(x,w)|) \leq \alpha_2\left(\beta(|x|,1) + \gamma(|w|)\right) \\
&\leq \ \alpha_2\left(\beta(\mu^{-1}(|w|),1) + \gamma(|w|)\right) = \varphi(|w|),
\end{aligned}$$

i.e., (10). ■

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