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STRONG INSTABILITY OF STANDING WAVES FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH ATTRACTIVE INVERSE POWER POTENTIAL

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Abstract

We study the strong instability of standing waves $e^{i\omega t}\phi_\omega(x)$ for nonlinear Schrödinger equations with an L^2-supercritical nonlinearity and an attractive inverse power potential, where $\omega \in \mathbb{R}$ is a frequency, and $\phi_\omega \in H^1(\mathbb{R}^N)$ is a ground state of the corresponding stationary equation. Recently, for nonlinear Schrödinger equations with a harmonic potential, Ohta (2018) proved that if $\partial_\lambda^2 S_\omega(\phi_\omega^\lambda)|_{\lambda=1} \le 0$, then the standing wave is strongly unstable, where S_ω is the action, and $\phi_\omega^\lambda(x) := \lambda^{N/2}\phi_\omega(\lambda x)$ is the scaling, which does not change the L^2-norm. In this paper, we prove the strong instability under the same assumption as the above-mentioned in inverse power potential case. Our proof is applicable to nonlinear Schrödinger equations with other potentials such as an attractive Dirac delta potential.

1. Introduction

In this paper, we consider the nonlinear Schrödinger equation with an attractive inverse power potential

$$(\text{NLS}) \qquad i\partial_t u = -\Delta u - \frac{\gamma}{|x|^\alpha}u - |u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N,$$

where

$$(1.1) \qquad N \in \mathbb{N}, \quad \gamma > 0, \quad 0 < \alpha < \min\{2, N\}, \quad 1 + \frac{4}{N} < p < 1 + \frac{4}{N-2},$$

and $u \colon \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$ is an unknown function of $(t, x) \in \mathbb{R} \times \mathbb{R}^N$. Here, $1 + 4/(N-2)$ stands for ∞ if $N = 1$ or 2.

Let us consider the Cauchy problem for (NLS). Since the potential $V(x) := -\gamma|x|^{-\alpha}$ belongs to $(L^r + L^\infty)(\mathbb{R}^N)$ for some $r > \max\{1, N/2\}$ under the assumption (1.1), the multiplication operator $v \mapsto V(x)v$ is continuous from $H^1(\mathbb{R}^N)$ to $(L^{\rho'} + L^2)(\mathbb{R}^N)$ for some $\rho \in [2, 2N/(N-2))$, and thus, the potential energy $\int_{\mathbb{R}^N} V(x)|v(x)|^2\,dx$ is well-defined on $H^1(\mathbb{R}^N)$. Therefore, the local well-posedness of (NLS) in the energy space $H^1(\mathbb{R}^N)$ follows from the standard theory, e.g., [3, Theorems 3.3.5, 3.3.9, Proposition 4.2.3]. More precisely, for each $u_0 \in H^1(\mathbb{R}^N)$, there exist a maximal interval $I_{\max} = [0, T^+) \subset \mathbb{R}$ with $T^+ = T^+(u_0) \in (0, \infty]$ and a unique solution $u \in C(I_{\max}, H^1(\mathbb{R}^N))$ of (NLS) with $u(0) = u_0$ such that if $T^+ < \infty$, then $\lim_{t \nearrow T^+} \|u(t)\|_{H^1} = \infty$. Here, if $T^+ < \infty$, we say that the solution $u(t)$ *blows up in finite time*. Moreover, the solution $u(t)$ of (NLS) satisfies the two

conservation laws

$$E(u(t)) = E(u_0), \quad \|u(t)\|_{L^2} = \|u_0\|_{L^2}$$

for all $t \in I_{\max}$, where

$$E(v) := \frac{1}{2}\|\nabla v\|_{L^2}^2 - \frac{\gamma}{2}\int_{\mathbb{R}^N} \frac{|v(x)|^2}{|x|^\alpha}\, dx - \frac{1}{p+1}\|v\|_{L^{p+1}}^{p+1}$$

is the energy.

By a *standing wave*, we mean a solution of (NLS) with the form $e^{i\omega t}\phi(x)$, where $\omega \in \mathbb{R}$ is a frequency, and $\phi \in H^1(\mathbb{R}^N)$ is a nontrivial solution of the stationary equation

$$(1.2) \qquad\qquad -\Delta\phi + \omega\phi - \frac{\gamma}{|x|^\alpha}\phi - |\phi|^{p-1}\phi = 0, \quad x \in \mathbb{R}^N.$$

Eq. (1.2) can be written as $S'_\omega(\phi) = 0$, where

$$S_\omega(v) := E(v) + \frac{\omega}{2}\|v\|_{L^2}^2$$

is the action. The following existence and variational characterization of ground states by using the Nehari functional

$$K_\omega(v) := \partial_\lambda S_\omega(\lambda v)|_{\lambda=1} = \langle S'_\omega(v), v\rangle$$

$$= \|\nabla v\|_{L^2}^2 + \omega\|v\|_{L^2}^2 - \gamma\int_{\mathbb{R}^N}\frac{|v(x)|^2}{|x|^\alpha}\,dx - \|v\|_{L^{p+1}}^{p+1}$$

are known (see [6, Remarks 1.2 and 1.3]), where a *ground state* is a nontrivial solution of (1.2) with the least action.

Proposition 1.1. *Assume* (1.1) *and*

$$(1.3) \qquad \omega > \omega_0 := -\inf\left\{\|\nabla v\|_{L^2}^2 - \gamma\int_{\mathbb{R}^N}\frac{|v(x)|^2}{|x|^\alpha}\,dx\ \middle|\ v \in H^1(\mathbb{R}^N),\ \|v\|_{L^2} = 1\right\}.$$

Then the set of ground states

$$\mathcal{G}_\omega := \{\phi \in \mathcal{F}_\omega \mid S_\omega(\phi) \le S_\omega(v)\ \text{for all}\ v \in \mathcal{F}_\omega\}$$

is not empty, where

$$\mathcal{F}_\omega := \{\phi \in H^1(\mathbb{R}^N)\setminus\{0\} \mid S'_\omega(\phi) = 0\}$$

is the set of all nontrivial solutions of (1.2). *Moreover, if* $\phi \in \mathcal{G}_\omega$, *then*

$$(1.4) \qquad\qquad S_\omega(\phi) = \inf\{S_\omega(v) \mid v \in H^1(\mathbb{R}^N)\setminus\{0\},\ K_\omega(v) = 0\}.$$

For the sake of completeness, we give a proof of Proposition 1.1 in Section 2 by using the argument in [8, Section 3].

In the present paper, we study the strong instability of the standing wave solution $e^{i\omega t}\phi_\omega$ of (NLS), where $\omega > \omega_0$ and $\phi_\omega \in \mathcal{G}_\omega$. We recall the definitions of stability and instability of standing waves.

DEFINITION 1.2. Let $e^{i\omega t}\phi$ be a standing wave solution of (NLS).

- We say that $e^{i\omega t}\phi$ is *stable* if for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $u_0 \in H^1(\mathbb{R}^N)$ satisfies $\|u_0 - \phi\|_{H^1} < \delta$, then the solution $u(t)$ of (NLS) with $u(0) = u_0$ exists globally in time and satisfies

$$\sup_{t\geq 0} \inf_{\theta\in\mathbb{R}} \|u(t) - e^{i\theta}\phi\|_{H^1} < \varepsilon.$$

- We say that $e^{i\omega t}\phi$ is *unstable* if $e^{i\omega t}\phi$ is not stable.
- We say that $e^{i\omega t}\phi$ is *strongly unstable* if for each $\varepsilon > 0$, there exists $u_0 \in H^1(\mathbb{R}^N)$ such that $\|u_0 - \phi\|_{H^1} < \varepsilon$, and the solution $u(t)$ of (NLS) with $u(0) = u_0$ blows up in finite time.

Now, we state some known results related to our works. The stability and instability of standing waves with a ground state profile for nonlinear Schrödinger equations have been studied by many researchers. For (NLS) in the nonpotential case $\gamma = 0$, Berestycki and Cazenave [1] proved the strong instability for any $\omega > 0$ when $1 + 4/N \leq p < 1 + 4/(N-2)$ (for the case $p = 1 + 4/N$, see also [22]). On the other hand, Cazenave and Lions [4] proved the stability for any $\omega > 0$ if $1 < p < 1 + 4/N$. For abstract Hamiltonian systems including nonlinear Schrödinger equations, Grillakis, Shatah, and Strauss [10, 11] gave sufficient conditions for the stability and instability, that is, if $\partial_\omega \|\phi_\omega\|_{L^2}^2 > 0$, the standing wave is stable, and if $\partial_\omega \|\phi_\omega\|_{L^2}^2 < 0$, the standing wave is unstable (see also [20, 21, 23]).

For the nonlinear Schrödinger equation with a general potential

$$(1.5) \qquad i\partial_t u = -\Delta u + \tilde{V}(x)u - |u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N,$$

Rose and Weinstein [19] proved the stability for $\omega > \tilde{\omega}_0$ sufficiently closed to $\tilde{\omega}_0$ even when $1 + 4/N \leq p < 1 + 4/(N-2)$ by using the criteria of Grillakis, Shatah, and Strauss [10], where $-\tilde{\omega}_0$ is the smallest eigenvalue of the Schrödinger operator $-\Delta + \tilde{V}$. In [6], Ohta and Fukuizumi improved the stability results of Rose and Weinstein, and in [7], they proved the instability for sufficiently large ω when $1 + 4/N < p < 1 + 4/(N-2)$ by using the sufficient condition of Ohta [15], that is, if $\partial_\lambda^2 \tilde{S}_\omega(\tilde{\phi}_\omega^\lambda)|_{\lambda=1} < 0$, the standing wave $e^{i\omega t}\tilde{\phi}_\omega$ is unstable, where $\tilde{S}_\omega$ is the action corresponding to (1.5), and $v^\lambda(x) := \lambda^{N/2}v(\lambda x)$ is the scaling, which does not change the L^2-norm (see also [8, 9] in the Dirac delta potential case and [5] in the harmonic potential case).

For the nonlinear Schrödinger equation with an attractive Dirac delta potential

$$(1.6) \qquad i\partial_t u = -\partial_x^2 u - \tilde{\gamma}\delta(x)u - |u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R},$$

Ohta and Yamaguchi [18] proved the strong instability of the standing wave with positive energy $\tilde{E}(\tilde{\phi}_\omega) > 0$ when $\tilde{\gamma} > 0$ and $p > 5$, and as a corollary, they proved the strong instability for sufficiently large ω (see also [17] for related works). Recently, for the nonlinear Schrödinger equation with a harmonic potential

$$(1.7) \qquad i\partial_t u = -\Delta u + |x|^2 u - |u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N,$$

Ohta [16] proved the strong instability under the same assumption $\partial_\lambda^2 \tilde{S}_\omega(\tilde{\phi}_\omega^\lambda)|_{\lambda=1} \leq 0$ as in [7, 15] when $1 + 4/N < p < 1 + 4/(N-2)$.

In view of the graph of $\lambda \mapsto \tilde{S}_\omega(\tilde{\phi}_\omega^\lambda)$, we see that $\tilde{E}(\tilde{\phi}_\omega) > 0$ implies $\partial_\lambda^2 \tilde{S}_\omega(\tilde{\phi}_\omega^\lambda)|_{\lambda=1} < 0$.

Therefore, the question naturally arises whether the standing wave is strongly unstable or not in the case $\tilde{E}(\tilde{\phi}_\omega) \leq 0$ and $\partial_\lambda^2 \tilde{S}_\omega(\tilde{\phi}_\omega^\lambda)|_{\lambda=1} \leq 0$ for (1.6). However, the proof for (1.7) in [16] is not applicable to (1.6).

Now we turn to (NLS). In order to treat more general potentials with suitable properties related to the scaling $\lambda \mapsto v^\lambda$, we study the nonlinear Schrödinger equation with an inverse power potential. To the standing wave solution $e^{i\omega t}\phi_\omega$ of (NLS), we can apply the results of Fukuizumi and Ohta [6, 7] and Ohta and Yamaguchi [18]. More precisely, if ω is sufficiently closed to ω_0, then the standing wave is stable, if $\partial_\lambda^2 S_\omega(\phi_\omega^\lambda)|_{\lambda=1} < 0$, then it is unstable, and if $E(\phi_\omega) > 0$, then it is strongly unstable.

In this paper, we consider the strong instability of standing waves under the same assumption as in [7, 15, 16]. Now, we state our main result.

Theorem 1.3. *Assume* (1.1), $\omega > \omega_0$, *and that* $\phi_\omega \in \mathcal{G}_\omega$ *satisfies* $\partial_\lambda^2 S_\omega(\phi_\omega^\lambda)|_{\lambda=1} \leq 0$, *where* $\phi_\omega^\lambda(x) = \lambda^{N/2}\phi_\omega(\lambda x)$. *Then the standing wave solution* $e^{i\omega t}\phi_\omega$ *of* (NLS) *is strongly unstable.*

It is proven in [7, Section 2] that the assumption $\partial_\lambda^2 S_\omega(\phi_\omega^\lambda)|_{\lambda=1} \leq 0$ is satisfied for sufficiently large ω. Therefore, we have the following corollary.

Corollary 1.4. *Assume* (1.1). *Then there exists* $\omega_1 > \omega_0$ *such that if* $\omega \geq \omega_1$ *and* $\phi_\omega \in \mathcal{G}_\omega$, *the standing wave solution* $e^{i\omega t}\phi_\omega$ *of* (NLS) *is strongly unstable.*

REMARK 1.5. Theorem 1.3 can be extended to more general settings. The important feature used in the proof of Theorem 1.3 is that the energy satisfies

$$(1.8) \qquad E(v) = \frac{1}{2}\|\nabla v\|_{L^2}^2 - \frac{1}{2}G(v) - \frac{1}{p+1}\|v\|_{L^{p+1}}^{p+1},$$

$$(1.9) \qquad G(v) \geq 0, \quad G(\lambda v) = \lambda^2 G(v), \quad G(v^\lambda) = \lambda^\alpha G(v), \quad \|v^\lambda\|_{L^{p+1}}^{p+1} = \lambda^\beta \|v\|_{L^{p+1}}^{p+1}$$

with $0 < \alpha < 2 < \beta$. Since the energy of (1.6) satisfies (1.8) and (1.9) with $G(v) = \gamma|v(0)|^2$, $\alpha = 1$, and $\beta = (p-1)/2$, the proof is applicable to (1.6) for $p > 5$. This gives an improvement of the result of Ohta and Yamaguchi [18].

The proof of blowup for nonlinear Schrödinger equations relies on the virial identity

$$(1.10) \qquad \frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 = 8Q(u(t)),$$

where Q is the functional on $H^1(\mathbb{R}^N)$ defined by

$$Q(v) = \|\nabla v\|_{L^2}^2 - \frac{\gamma\alpha}{2}\int_{\mathbb{R}^N}\frac{|v(x)|^2}{|x|^\alpha}\,dx - \frac{N(p-1)}{2(p+1)}\|v\|_{L^{p+1}}^{p+1}.$$

Note that

$$S_\omega(v^\lambda) = \frac{\lambda^2}{2}\|\nabla v\|_{L^2}^2 + \frac{\omega}{2}\|v\|_{L^2}^2 - \frac{\gamma\lambda^\alpha}{2}\int_{\mathbb{R}^N}\frac{|v(x)|^2}{|x|^\alpha}\,dx - \frac{\lambda^{N(p-1)/2}}{p+1}\|v\|_{L^{p+1}}^{p+1},$$

$$Q(v) = \partial_\lambda S_\omega(v^\lambda)|_{\lambda=1}.$$

Since $x \cdot \nabla V(x) = \gamma\alpha|x|^{-\alpha} \in (L^q + L^\infty)(\mathbb{R}^N)$ for some $q > \max\{1, N/2\}$ under the assumption (1.1), from the standard theory [3, Proposition 6.5.1], we obtain the local well-posedness of the Cauchy problem for (NLS) in the weighted space

$$\Sigma := \{ v \in H^1(\mathbb{R}^N) \mid \|xv\|_{L^2} < \infty \},$$

and the virial identity (1.10) holds for all $t \in I_{\max}$.

To prove Theorem 1.3, we introduce the set

$$\mathcal{B}_\omega = \left\{ v \in H^1(\mathbb{R}^N) \left| \begin{array}{c} S_\omega(v) < S_\omega(\phi_\omega),\ \|v\|_{L^2} \leq \|\phi_\omega\|_{L^2}, \\ \|v\|_{L^{p+1}} > \|\phi_\omega\|_{L^{p+1}},\ Q(v) < 0 \end{array} \right. \right\}.$$

Then we have the following blowup result.

Theorem 1.6. *Assume* (1.1)*,* $\omega > \omega_0$*, and that* $\phi_\omega \in \mathcal{G}_\omega$ *satisfies* $\partial_\lambda^2 S_\omega(\phi_\omega^\lambda)|_{\lambda=1} \leq 0$*. If* $u_0 \in \mathcal{B}_\omega \cap \Sigma$*, then the solution* $u(t)$ *of* (NLS) *with* $u(0) = u_0$ *blows up in finite time.*

Theorem 1.3 follows from Theorem 1.6 and the fact that the ground state ϕ_ω belongs to the closure of $\mathcal{B}_\omega \cap \Sigma$ in H^1-topology.

The key to the proof of Theorem 1.6 is Lemma 3.2 below. The same assertion of Lemma 3.2 is proven in [16, Lemma 4] for (1.7). In [16, Lemma 4], the proof is divided into two cases $\|x\tilde{\phi}_\omega\|_{L^2}^2 \leq \|xv\|_{L^2}^2$ and $\|xv\|_{L^2}^2 \leq \|x\tilde{\phi}_\omega\|_{L^2}^2$. Although the first case is easy to treat, the second case is more complicated. In the second case, the inequality $\|xv\|_{L^2}^2 \leq \|x\tilde{\phi}_\omega\|_{L^2}^2$ is used to obtain upper bounds for the potential energy. However, in our case, this argument does not work well because the sign of the potential is different from that of (1.7). In our proof here, to obtain upper bounds for the potential energy, we use the inequality coming out of the variational characterization of the ground state (see Lemma 2.6 (i) below).

We remark that in [16, 18], they consider

$$\left\{ v \in H^1(\mathbb{R}^N) \left| \begin{array}{c} \tilde{E}(v) < \tilde{E}(\tilde{\phi}_\omega),\ \|v\|_{L^2} = \|\tilde{\phi}_\omega\|_{L^2}, \\ \|v\|_{L^{p+1}} > \|\tilde{\phi}_\omega\|_{L^{p+1}},\ \tilde{Q}(v) < 0 \end{array} \right. \right\} \cap \Sigma$$

as the set of initial data of blowup solutions. On the other hand, in our definition of $\mathcal{B}_\omega$, we use the action S_ω instead of the energy E in order to treat more general initial data.

We finally remark that the assumption $\partial_\lambda^2 S_\omega(\phi_\omega^\lambda)|_{\lambda=1} \leq 0$ is not a necessary condition for the instability because it is known for (1.6) that there exist unstable standing waves satisfying $\partial_\lambda^2 \tilde{S}_\omega(\tilde{\phi}_\omega^\lambda)|_{\lambda=1} > 0$ (see [18, Section 4]). It is an open problem whether the standing wave is strongly unstable or not in this case.

This paper is organized as follows. In Section 2, we give a proof of Proposition 1.1 and prove a useful lemma (Lemma 2.6 below). In Section 3, we prove Theorem 1.6. In Section 4, we prove Theorem 1.3.

2. Existence and variational characterization of ground states

The aim of this section is to prove Proposition 1.1 and Lemma 2.6 below. Here, we assume (1.1) and $\omega > \omega_0$, where ω_0 is defined in (1.3). Hereafter, we denote

$$(2.1) \qquad\qquad G(v) = \gamma \int_{\mathbb{R}^N} \frac{|v(x)|^2}{|x|^\alpha}\, dx.$$

We define

$$d(\omega) = \inf\{ S_\omega(v) \mid v \in H^1(\mathbb{R}^N) \setminus \{0\},\ K_\omega(v) = 0 \},$$
$$\mathcal{M}_\omega = \{ v \in H^1(\mathbb{R}^N) \setminus \{0\} \mid K_\omega(v) = 0,\ S_\omega(v) = d(\omega) \}.$$

Note that since $-\omega_0$ is the smallest eigenvalue of the Schrödinger operator $-\Delta - \gamma|x|^{-\alpha}$, under the assumption $\omega > \omega_0$, we have the equivalence of norms

$$(2.2) \qquad \sqrt{L_\omega(v)} \simeq \|v\|_{H^1},$$

where

$$L_\omega(v) = \|\nabla v\|_{L^2}^2 + \omega\|v\|_{L^2}^2 - G(v).$$

First, we show that ground states of (1.2) are characterized as the minimizers for S_ω under the constraint $K_\omega = 0$.

Lemma 2.1. $\mathcal{M}_\omega \subset \mathcal{G}_\omega$.

Proof. Let $\phi \in \mathcal{M}_\omega$. Then by $L_\omega(\phi) - \|\phi\|_{L^{p+1}}^{p+1} = K_\omega(\phi) = 0$, we have

$$(2.3) \qquad \langle K'_\omega(\phi), \phi \rangle = 2L_\omega(\phi) - (p+1)\|\phi\|_{L^{p+1}}^{p+1} = -(p-1)\|\phi\|_{L^{p+1}}^{p+1} < 0.$$

Therefore, there exists a Lagrange multiplier $\eta \in \mathbb{R}$ such that $S'_\omega(\phi) = \eta K'_\omega(\phi)$. Moreover, since

$$\eta\langle K'_\omega(\phi), \phi \rangle = \langle S'_\omega(\phi), \phi \rangle = K_\omega(\phi) = 0,$$

it follows from (2.3) that $\eta = 0$, which implies $S'_\omega(\phi) = 0$.

Furthermore, if $v \in H^1(\mathbb{R}^N)$ satisfies $v \neq 0$ and $S'_\omega(v) = 0$, then by $K_\omega(v) = \langle S'_\omega(v), v \rangle = 0$ and the definition of $\mathcal{M}_\omega$, we have $S_\omega(\phi) \leq S_\omega(v)$. Thus, we obtain $\phi \in \mathcal{G}_\omega$. This completes the proof. □

Lemma 2.2. *If $\mathcal{M}_\omega$ is not empty, then $\mathcal{G}_\omega \subset \mathcal{M}_\omega$.*

Proof. Let $\phi \in \mathcal{G}_\omega$. Since $\mathcal{M}_\omega$ is not empty, we take $\psi \in \mathcal{M}_\omega$. Then by Lemma 2.1, we have $\psi \in \mathcal{G}_\omega$. Therefore, if $v \in H^1(\mathbb{R}^N)$ satisfies $v \neq 0$ and $K_\omega(v) = 0$, then $S_\omega(\phi) = S_\omega(\psi) \leq S_\omega(v)$. This implies $\phi \in \mathcal{M}_\omega$. This completes the proof. □

Next, we show that $\mathcal{M}_\omega$ is not empty. By using the expressions

$$(2.4) \qquad S_\omega(v) = \frac{1}{2}K_\omega(v) + \frac{p-1}{2(p+1)}\|v\|_{L^{p+1}}^{p+1}$$

$$= \frac{1}{p+1}K_\omega(v) + \frac{p-1}{2(p+1)}L_\omega(v),$$

we rewrite $d(\omega)$ as

$$(2.5) \qquad d(\omega) = \inf\left\{ \frac{p-1}{2(p+1)}\|v\|_{L^{p+1}}^{p+1} \;\middle|\; v \in H^1(\mathbb{R}^N) \setminus \{0\}, \; K_\omega(v) = 0 \right\}$$

$$(2.6) \qquad = \inf\left\{ \frac{p-1}{2(p+1)}L_\omega(v) \;\middle|\; v \in H^1(\mathbb{R}^N) \setminus \{0\}, \; K_\omega(v) = 0 \right\}.$$

Lemma 2.3. *If $K_\omega(v) < 0$, then*

$$\frac{p-1}{2(p+1)}\|v\|_{L^{p+1}}^{p+1} > d(\omega), \quad \frac{p-1}{2(p+1)}L_\omega(v) > d(\omega).$$

In particular,

$$(2.7) \qquad d(\omega) = \inf \left\{ \frac{p-1}{2(p+1)} \|v\|_{L^{p+1}}^{p+1} \, \middle| \, v \in H^1(\mathbb{R}^N) \setminus \{0\}, \ K_\omega(v) \le 0 \right\}$$

$$= \inf \left\{ \frac{p-1}{2(p+1)} L_\omega(v) \, \middle| \, v \in H^1(\mathbb{R}^N) \setminus \{0\}, \ K_\omega(v) \le 0 \right\}.$$

Proof. Let

$$\lambda_1 = \left(\frac{L_\omega(v)}{\|v\|_{L^{p+1}}^{p+1}} \right)^{1/(p-1)},$$

where note that $L_\omega(v) > 0$ by (2.2). Then since $K_\omega(\lambda v) = \lambda^2 L_\omega(v) - \lambda^{p+1} \|v\|_{L^{p+1}}^{p+1}$ and $K_\omega(v) < 0$, we have $K_\omega(\lambda_1 v) = 0$ and $0 < \lambda_1 < 1$. Therefore, by (2.5),

$$d(\omega) \le \frac{p-1}{2(p+1)} \|\lambda_1 v\|_{L^{p+1}}^{p+1} = \lambda_1^{p+1} \frac{p-1}{2(p+1)} \|v\|_{L^{p+1}}^{p+1} < \frac{p-1}{2(p+1)} \|v\|_{L^{p+1}}^{p+1}.$$

Similarly, by using (2.6), we obtain $d(\omega) < \frac{p-1}{2(p+1)} L_\omega(v)$. This completes the proof. $\qquad \square$

It is well known that in the nonpotential case $\gamma = 0$, the set of all minimizers

$$\mathcal{M}_\omega^0 := \{ v \in H^1(\mathbb{R}^N) \setminus \{0\} \mid K_\omega^0(v) = 0, \ S_\omega^0(v) = d^0(\omega) \}$$

is not empty (see e.g., [12, 14]), where

$$S_\omega^0(v) = \frac{1}{2} \|\nabla v\|_{L^2}^2 + \frac{\omega}{2} \|v\|_{L^2}^2 - \frac{1}{p+1} \|v\|_{L^{p+1}}^{p+1},$$

$$K_\omega^0(v) = \|\nabla v\|_{L^2}^2 + \omega \|v\|_{L^2}^2 - \|v\|_{L^{p+1}}^{p+1},$$

$$d^0(\omega) = \inf \{ S_\omega^0(v) \mid v \in H^1(\mathbb{R}^N) \setminus \{0\}, \ K_\omega^0(v) = 0 \}$$

$$= \inf \left\{ \frac{p-1}{2(p+1)} \|v\|_{L^{p+1}}^{p+1} \, \middle| \, v \in H^1(\mathbb{R}^N) \setminus \{0\}, \ K_\omega^0(v) = 0 \right\}.$$

Lemma 2.4. $d^0(\omega) > d(\omega) > 0$.

Proof. First, we show $d^0(\omega) > d(\omega)$. Since $\mathcal{M}_\omega^0$ is not empty, we take $\psi \in \mathcal{M}_\omega^0$. Since

$$K_\omega(\psi) = K_\omega^0(\psi) - G(\psi) = -G(\psi) < 0,$$

by Lemma 2.3, we have

$$d(\omega) < \frac{p-1}{2(p+1)} \|\psi\|_{L^{p+1}}^{p+1} = d^0(\omega).$$

Next, we show that $d(\omega) > 0$. Let $v \in H^1(\mathbb{R}^N)$ satisfy $v \ne 0$ and $K_\omega(v) = 0$. By the Sobolev embedding, (2.2), and $L_\omega(v) = \|v\|_{L^{p+1}}^{p+1}$, we have

$$\|v\|_{L^{p+1}}^2 \le C_1 \|v\|_{H^1}^2 \le C_2 L_\omega(v) = C_2 \|v\|_{L^{p+1}}^{p+1}$$

for some positive constants C_1 and C_2. Since $v \ne 0$, we have $\|v\|_{L^{p+1}} \ge C_2^{-1/(p-1)}$. Taking the infimum over v, we obtain $d(\omega) > 0$. This completes the proof. $\qquad \square$

Lemma 2.5. *Let $(v_n)_n \subset H^1(\mathbb{R}^N)$ be a minimizing sequence for $d(\omega)$, that is,*

$$v_n \neq 0, \quad K_\omega(v_n) = 0, \quad S_\omega(v_n) \to d(\omega).$$

Then there exist a subsequence $(v_{n_k})_k$ of $(v_n)_n$ and $v_0 \in H^1(\mathbb{R}^N)$ such that $v_{n_k} \to v_0$ in $H^1(\mathbb{R}^N)$, $K_\omega(v_0) = 0$, and $S_\omega(v_0) = d(\omega)$. In particular, $\mathcal{M}_\omega$ is not empty.

Proof. First, by $K_\omega(v_n) = 0$, $S_\omega(v_n) \to d(\omega)$, and (2.4), we have

$$(2.8) \qquad \frac{p-1}{2(p+1)} L_\omega(v_n) = \frac{p-1}{2(p+1)} \|v_n\|_{L^{p+1}}^{p+1} \to d(\omega).$$

Therefore, it follows from (2.2) that $(v_n)_n$ is bounded in $H^1(\mathbb{R}^N)$. This implies that there exist a subsequence of $(v_n)_n$, which is still denoted by $(v_n)_n$, and $v_0 \in H^1(\mathbb{R}^N)$ such that $v_n \rightharpoonup v_0$ weakly in $H^1(\mathbb{R}^N)$.

Next, we show $v_0 \neq 0$. Since $v_n \neq 0$, letting

$$\lambda_n = \left(\frac{\|\nabla v_n\|_{L^2}^2 + \omega \|v_n\|_{L^2}^2}{\|v_n\|_{L^{p+1}}^{p+1}} \right)^{1/(p-1)} = \left(\frac{L_\omega(v_n) + G(v_n)}{\|v_n\|_{L^{p+1}}^{p+1}} \right)^{1/(p-1)},$$

then we have $\lambda_n > 0$ and $K_\omega^0(\lambda_n v_n) = 0$. Moreover, by (2.8) and the weak continuity of the potential energy (cf. [13, Theorem 11.4]), we obtain

$$(2.9) \qquad \lim_{n\to\infty} \lambda_n = \left(\frac{d(\omega) + \frac{p-1}{2(p+1)} G(v_0)}{d(\omega)} \right)^{1/(p-1)}.$$

By Lemma 2.4, $K_\omega^0(\lambda_n v_n) = 0$, and the definition of $d^0(\omega)$, it follows that

$$d(\omega) < d^0(\omega) \leq \frac{p-1}{2(p+1)} \|\lambda_n v_n\|_{L^{p+1}}^{p+1} = \lambda_n^{p+1} \frac{p-1}{2(p+1)} \|v_n\|_{L^{p+1}}^{p+1}$$

for all $n \in \mathbb{N}$. Therefore, taking the limit, by (2.8) and (2.9), we have

$$d(\omega) < \left(\frac{d(\omega) + \frac{p-1}{2(p+1)} G(v_0)}{d(\omega)} \right)^{(p+1)/(p-1)} d(\omega).$$

Since $d(\omega) > 0$, we obtain $G(v_0) > 0$. This implies $v_0 \neq 0$.

Finally, we show the strong convergence of $(v_n)_n$ in $H^1(\mathbb{R}^N)$. Taking a subsequence of $(v_n)_n$ if necessary, we may assume that $v_n \to v_0$ a.e. in $\mathbb{R}^N$. Then by using the Brezis–Lieb lemma [2], we have

$$(2.10) \qquad L_\omega(v_n) - L_\omega(v_n - v_0) \to L_\omega(v_0),$$

$$(2.11) \qquad -K_\omega(v_n - v_0) \to K_\omega(v_0),$$

where we used $K_\omega(v_n) = 0$ in (2.11). Since $L_\omega(v_0) > 0$ by $v_0 \neq 0$, it follows from (2.10) and (2.8) that

$$\frac{p-1}{2(p+1)} \lim_{n\to\infty} L_\omega(v_n - v_0) < \frac{p-1}{2(p+1)} \lim_{n\to\infty} L_\omega(v_n) = d(\omega).$$

From this and (2.7), we have $K_\omega(v_n - v_0) > 0$ for large n. Therefore, by (2.11), we obtain $K_\omega(v_0) \leq 0$, and thus, by (2.7) and the weak lower semicontinuity of norms,

$$d(\omega) \le \frac{p-1}{2(p+1)} L_\omega(v_0) \le \frac{p-1}{2(p+1)} \lim_{n\to\infty} L_\omega(v_n) = d(\omega).$$

This and (2.10) imply that $L_\omega(v_n - v_0) \to 0$, and therefore, $v_n \to v_0$ in $H^1(\mathbb{R}^N)$. This completes the proof. $\qquad\square$

Finally, we give a useful lemma for the proof of Theorem 1.6.

Lemma 2.6. *Let $\phi \in \mathcal{G}_\omega$. If $v \in H^1(\mathbb{R}^N)$ satisfies $\|v\|_{L^{p+1}} = \|\phi\|_{L^{p+1}}$, then the following hold.*

(i) $K_\omega(v) \ge 0$,

(ii) $S_\omega(v) \ge S_\omega(\phi)$.

Proof. Inequality (i) follows from Lemma 2.3 and $d(\omega) = \frac{p-1}{2(p+1)}\|\phi\|_{L^{p+1}}^{p+1}$. Inequality (ii) follows from (2.4) and (i). $\qquad\square$

3. Blowup solutions

In this section, we prove Theorem 1.6. Throughout this section, we impose the same assumption as in Theorem 1.6, that is, we assume (1.1), $\omega > \omega_0$, and

$$(3.1) \qquad \partial_\lambda^2 S_\omega(\phi_\omega^\lambda)|_{\lambda=1} = \|\nabla\phi_\omega\|_{L^2}^2 - \frac{\alpha(\alpha-1)}{2}G(\phi_\omega) - \frac{\beta(\beta-1)}{p+1}\|\phi_\omega\|_{L^{p+1}}^{p+1} \le 0,$$

where $v^\lambda(x) = \lambda^{N/2}v(\lambda x)$, G is defined in (2.1), and

$$\beta = \frac{N(p-1)}{2}.$$

By using this notation, we have

$$(3.2) \qquad S_\omega(v^\lambda) = \frac{\lambda^2}{2}\|\nabla v\|_{L^2}^2 + \frac{\omega}{2}\|v\|_{L^2}^2 - \frac{\lambda^\alpha}{2}G(v) - \frac{\lambda^\beta}{p+1}\|v\|_{L^{p+1}}^{p+1},$$

$$(3.3) \qquad Q(v^\lambda) = \lambda^2\|\nabla v\|_{L^2}^2 - \frac{\alpha\lambda^\alpha}{2}G(v) - \frac{\beta\lambda^\beta}{p+1}\|v\|_{L^{p+1}}^{p+1} = \lambda\partial_\lambda S_\omega(v^\lambda),$$

$$(3.4) \qquad K_\omega(v^\lambda) = \lambda^2\|\nabla v\|_{L^2}^2 + \omega\|v\|_{L^2}^2 - \lambda^\alpha G(v) - \lambda^\beta\|v\|_{L^{p+1}}^{p+1}.$$

Now, we define

$$\mathcal{A}_\omega = \{\, v \in H^1(\mathbb{R}^N) \mid S_\omega(v) < S_\omega(\phi_\omega), \ \|v\|_{L^2} \le \|\phi_\omega\|_{L^2}, \ \|v\|_{L^{p+1}} > \|\phi_\omega\|_{L^{p+1}} \,\}.$$

Recall that

$$\mathcal{B}_\omega = \{\, v \in \mathcal{A}_\omega \mid Q(v) < 0 \,\}.$$

Lemma 3.1. *If $u_0 \in \mathcal{A}_\omega$, then the solution $u(t)$ of (NLS) with $u(0) = u_0$ satisfies $u(t) \in \mathcal{A}_\omega$ for all $t \in I_{\max}$.*

Proof. Since E and $\|\cdot\|_{L^2}$ are conserved quantities of (NLS), we have $\|u(t)\|_{L^2} \le \|\phi_\omega\|_{L^2}$ and $S_\omega(u(t)) < S_\omega(\phi_\omega)$ for all $t \in I_{\max}$. By Lemma 2.6 (ii), it follows that $\|u(t)\|_{L^{p+1}} \ne \|\phi_\omega\|_{L^{p+1}}$ for all $t \in I_{\max}$. Therefore, by $\|u_0\|_{L^{p+1}} > \|\phi_\omega\|_{L^{p+1}}$ and the continuity of the solution $u(t)$, we obtain $\|u(t)\|_{L^{p+1}} > \|\phi_\omega\|_{L^{p+1}}$ for all $t \in I_{\max}$. This completes the proof. $\qquad\square$

The following is the key lemma for our proof.

Lemma 3.2. *Let $v \in H^1(\mathbb{R}^N)$ satisfy*

$$\|v\|_{L^2} \le \|\phi_\omega\|_{L^2}, \quad \|v\|_{L^{p+1}} \ge \|\phi_\omega\|_{L^{p+1}}, \quad Q(v) \le 0.$$

Then

$$(3.5) \qquad \qquad \frac{Q(v)}{2} \le S_\omega(v) - S_\omega(\phi_\omega).$$

In particular, if $u_0 \in \mathcal{B}_\omega$, then the solution $u(t)$ of (NLS) with $u(0) = u_0$ satisfies $u(t) \in \mathcal{B}_\omega$ for all $t \in I_{\max}$.

Proof. Let

$$\lambda_0 = \left(\frac{\|\phi_\omega\|_{L^{p+1}}^{p+1}}{\|v\|_{L^{p+1}}^{p+1}} \right)^{1/\beta}.$$

Then we have

$$0 < \lambda_0 \le 1, \quad \|v^{\lambda_0}\|_{L^2} = \|v\|_{L^2} \le \|\phi_\omega\|_{L^2}, \quad \|v^{\lambda_0}\|_{L^{p+1}}^{p+1} = \lambda_0^\beta \|v\|_{L^{p+1}}^{p+1} = \|\phi_\omega\|_{L^{p+1}}^{p+1}.$$

Now, we define

$$\begin{aligned}
f(\lambda) &= S_\omega(v^\lambda) - \frac{\lambda^2}{2} Q(v) \\
&= -\frac{1}{2} \left(\lambda^\alpha - \frac{\alpha \lambda^2}{2} \right) G(v) + \frac{\omega}{2} \|v\|_{L^2}^2 - \frac{1}{p+1} \left(\lambda^\beta - \frac{\beta \lambda^2}{2} \right) \|v\|_{L^{p+1}}^{p+1}
\end{aligned}$$

for $\lambda \in (0, 1]$. If we have $f(\lambda_0) \le f(1)$, then it follows from Lemma 2.6 (ii), $Q(v) \le 0$, and $f(\lambda_0) \le f(1)$ that

$$(3.6) \qquad S_\omega(\phi_\omega) \le S_\omega(v^{\lambda_0}) \le S_\omega(v^{\lambda_0}) - \frac{\lambda_0^2}{2} Q(v) \le S_\omega(v) - \frac{Q(v)}{2},$$

which is the desired inequality (3.5).

In what follows, we prove $f(\lambda_0) \le f(1)$, which is rewritten as

$$(3.7) \qquad G(v) \le \frac{2(2\lambda_0^\beta - \beta \lambda_0^2 - 2 + \beta)}{(p+1)(\alpha \lambda_0^2 - 2\lambda_0^\alpha - \alpha + 2)} \|v\|_{L^{p+1}}^{p+1}.$$

By $\alpha K_\omega(\phi_\omega) - (\alpha + 1) Q(\phi_\omega) = 0$ and (3.1), we have

$$\begin{aligned}
\alpha \omega \|\phi_\omega\|_{L^2}^2 &= \|\nabla \phi_\omega\|_{L^2}^2 - \frac{\alpha(\alpha-1)}{2} G(\phi_\omega) + \left(\alpha - \frac{\beta(\alpha+1)}{p+1} \right) \|\phi_\omega\|_{L^{p+1}}^{p+1} \\
&\le \left(\alpha + \frac{\beta(\beta - \alpha - 2)}{p+1} \right) \|\phi_\omega\|_{L^{p+1}}^{p+1}.
\end{aligned}$$

Therefore, it follows from $\|v\|_{L^2} \le \|\phi_\omega\|_{L^2}$ and $\|\phi_\omega\|_{L^{p+1}}^{p+1} = \lambda_0^\beta \|v\|_{L^{p+1}}^{p+1}$ that

$$(3.8) \qquad \omega \|v\|_{L^2}^2 \le \left(1 + \frac{\beta(\beta - \alpha - 2)}{(p+1)\alpha} \right) \lambda_0^\beta \|v\|_{L^{p+1}}^{p+1}.$$

By using Lemma 2.6 (i) for v^{λ_0}, (3.4), (3.8), and $Q(v) \le 0$, we have

$$G(v) \leq \lambda_0^{2-\alpha} \|\nabla v\|_{L^2}^2 + \lambda_0^{-\alpha} \omega \|v\|_{L^2}^2 - \lambda_0^{\beta-\alpha} \|v\|_{L^{p+1}}^{p+1}$$

$$\leq \lambda_0^{2-\alpha} \|\nabla v\|_{L^2}^2 + \frac{\beta(\beta - \alpha - 2)}{(p+1)\alpha} \lambda_0^{\beta-\alpha} \|v\|_{L^{p+1}}^{p+1}$$

$$\leq \frac{\alpha}{2} \lambda_0^{2-\alpha} G(v) + \frac{\beta}{p+1} \left(\lambda_0^{2-\alpha} + \frac{\beta - \alpha - 2}{\alpha} \lambda_0^{\beta-\alpha} \right) \|v\|_{L^{p+1}}^{p+1},$$

and thus,

$$(3.9) \qquad G(v) \leq \frac{2\beta}{(p+1)(2 - \alpha\lambda_0^{2-\alpha})} \left(\lambda_0^{2-\alpha} + \frac{\beta - \alpha - 2}{\alpha} \lambda_0^{\beta-\alpha} \right) \|v\|_{L^{p+1}}^{p+1}.$$

In view of (3.7) and (3.9), we only have to show that

$$\frac{\beta}{2 - \alpha\lambda^{2-\alpha}} \left(\lambda^{2-\alpha} + \frac{\beta - \alpha - 2}{\alpha} \lambda^{\beta-\alpha} \right) \leq \frac{2\lambda^\beta - \beta\lambda^2 - 2 + \beta}{\alpha\lambda^2 - 2\lambda^\alpha - \alpha + 2}$$

for all $\lambda \in (0, 1)$, which is equivalent to

$$g_1(\lambda) := \frac{(2 - \alpha\lambda^{2-\alpha})(2\lambda^\beta - \beta\lambda^2 - 2 + \beta)}{\beta\lambda^{\beta-\alpha}(\alpha\lambda^2 - 2\lambda^\alpha - \alpha + 2)} - \frac{1}{\lambda^{\beta-2}} - \frac{\beta - \alpha - 2}{\alpha} \geq 0.$$

Since $\lim_{\lambda \nearrow 1} g_1(\lambda) = 0$, it suffices to show that

$$g_1'(\lambda) = \frac{2(1 - \lambda^{2-\alpha})}{\beta\lambda^{\beta-\alpha+1}(\alpha\lambda^2 - 2\lambda^\alpha - \alpha + 2)^2} \left(2\alpha(2 - \alpha)\lambda^\beta - \alpha\beta(\beta - \alpha)\lambda^2 \right.$$

$$\left. + 2\beta(\beta - 2)\lambda^\alpha - (\beta - \alpha)(\beta - 2)(2 - \alpha) \right) \leq 0$$

for all $\lambda \in (0, 1)$, which holds if we have

$$g_2(\lambda) := 2\alpha(2 - \alpha)\lambda^\beta - \alpha\beta(\beta - \alpha)\lambda^2 + 2\beta(\beta - 2)\lambda^\alpha - (\beta - \alpha)(\beta - 2)(2 - \alpha) \leq 0.$$

Since $g_2(1) = 0$, it is enough to show that

$$g_2'(\lambda) = 2\alpha\beta\lambda^{\alpha-1} \left((2 - \alpha)\lambda^{\beta-\alpha} - (\beta - \alpha)\lambda^{2-\alpha} + \beta - 2 \right) \geq 0$$

for all $\lambda \in (0, 1)$. This is equivalent to

$$g_3(\lambda) := (2 - \alpha)\lambda^{\beta-\alpha} - (\beta - \alpha)\lambda^{2-\alpha} + \beta - 2 \geq 0.$$

Since $g_3(1) = 0$ and

$$g_3'(\lambda) = -(\beta - \alpha)(2 - \alpha)\lambda^{1-\alpha}(1 - \lambda^{\beta-2}) \leq 0,$$

we have $g_3(\lambda) \geq 0$ for all $\lambda \in (0, 1)$. Therefore, we obtain $f(\lambda_0) \leq f(1)$, and thus, the inequality (3.6) follows.

The last claim of Lemma 3.2 follows from Lemma 3.1 and (3.5). This completes the proof. □

Proof of Theorem 1.6. Let $u_0 \in \mathcal{B}_\omega \cap \Sigma$ and let $u(t)$ be the solution of (NLS) with $u(0) = u_0$. Then by the virial identity (1.10), Lemma 3.2, and the conservation of S_ω, we have

$$(3.10) \qquad \frac{d^2}{dt^2} \|xu(t)\|_{L^2}^2 = 8Q(u(t)) \leq 16(S_\omega(u(t)) - S_\omega(\phi_\omega)) = 16(S_\omega(u_0) - S_\omega(\phi_\omega)) < 0$$

for all $t \in I_{\max}$.

If $T^+ = \infty$, then it follows from (3.10) that $\|xu(t)\|_{L^2}$ becomes negative for large t. This is a contradiction. Thus, the solution $u(t)$ blows up in finite time. □

4. Strong instability of standing waves

In this section, we prove Theorem 1.3. We impose the same assumption as in Theorem 1.3.

Lemma 4.1. $\phi_\omega^\lambda \in \mathcal{B}_\omega$ *for all* $\lambda > 1$.

Proof. By the definition of the scaling $\lambda \mapsto \phi_\omega^\lambda$, we have $\|\phi_\omega^\lambda\|_{L^2} = \|\phi_\omega\|_{L^2}$ and $\|\phi_\omega^\lambda\|_{L^{p+1}} = \lambda^{\beta/(p+1)}\|\phi_\omega\|_{L^{p+1}} > \|\phi_\omega\|_{L^{p+1}}$ for all $\lambda > 1$, where $\beta = N(p-1)/2 > 2$.

Now, we show that $S_\omega(\phi_\omega^\lambda) < S_\omega(\phi_\omega)$ and $Q(\phi_\omega^\lambda) < 0$ for all $\lambda > 1$. In view of (3.2), the function $S_\omega(\phi_\omega^\lambda)$ of λ has the form $S_\omega(\phi_\omega^\lambda) = A\lambda^2 + B - C\lambda^\alpha - D\lambda^\beta$ with some positive constants A, B, C, and D. By $\partial_\lambda S_\omega(\phi_\omega^\lambda)|_{\lambda=1} = 0$ and the assumption $\partial_\lambda^2 S_\omega(\phi_\omega^\lambda)|_{\lambda=1} \le 0$, we have $-\beta(\beta-2)D \le -\alpha(2-\alpha)C$. This leads to

$$\partial_\lambda^3 S_\omega(\phi_\omega^\lambda) = \alpha(\alpha-1)(2-\alpha)C\lambda^{\alpha-3} - \beta(\beta-1)(\beta-2)D\lambda^{\beta-3}$$
$$\le -\alpha(2-\alpha)\lambda^{\alpha-3}\left((\beta-1)\lambda^{\beta-\alpha} - (\alpha-1)\right)C < 0$$

for all $\lambda > 1$. Therefore, for $\lambda > 1$, it follows that $\partial_\lambda^2 S_\omega(\phi_\omega^\lambda) < \partial_\lambda^2 S_\omega(\phi_\omega^\lambda)|_{\lambda=1} \le 0$, $\partial_\lambda S_\omega(\phi_\omega^\lambda) < \partial_\lambda S_\omega(\phi_\omega^\lambda)|_{\lambda=1} = 0$, and thus $S_\omega(\phi_\omega^\lambda) < S_\omega(\phi_\omega)$. Moreover, by differentiating (3.3), we have $\partial_\lambda Q(\phi_\omega^\lambda) = \partial_\lambda S_\omega(\phi_\omega^\lambda) + \lambda\partial_\lambda^2 S_\omega(\phi_\omega^\lambda) < 0$. This implies $Q(\phi_\omega^\lambda) < Q(\phi_\omega) = 0$. This completes the proof. □

Now, we prove the main theorem.

Proof of Theorem 1.3. Let $\varepsilon > 0$. Then since $\phi_\omega^\lambda \to \phi_\omega$ in $H^1(\mathbb{R}^N)$ as $\lambda \searrow 1$, there exists $\lambda_0 > 1$ such that $\|\phi_\omega - \phi_\omega^{\lambda_0}\|_{H^1} < \varepsilon/2$. Let $\chi \in C^\infty[0,\infty)$ be a function satisfying $0 \le \chi \le 1$, $\chi(r) = 1$ if $0 \le r \le 1$, and $\chi(r) = 0$ if $r \ge 2$. For $M > 0$, we define a cutoff function $\chi_M \in C_c^\infty(\mathbb{R}^N)$ by $\chi_M(x) = \chi(|x|/M)$. Then we see that $\chi_M\phi_\omega^{\lambda_0} \to \phi_\omega^{\lambda_0}$ in $H^1(\mathbb{R}^N)$ as $M \to \infty$. Moreover, we have $\chi_M\phi_\omega^{\lambda_0} \in \Sigma$ and $\|\chi_M\phi_\omega^{\lambda_0}\|_{L^2} \le \|\phi_\omega^{\lambda_0}\|_{L^2} = \|\phi_\omega\|_{L^2}$ for all $M > 0$. Therefore, by Lemma 4.1 and the continuity of S_ω, $\|\cdot\|_{L^{p+1}}$, and Q, there exists $M_0 > 0$ such that $\|\phi_\omega^{\lambda_0} - \chi_{M_0}\phi_\omega^{\lambda_0}\|_{H^1} < \varepsilon/2$ and $\chi_{M_0}\phi_\omega^{\lambda_0} \in \mathcal{B}_\omega \cap \Sigma$. Thus, we obtain $\|\chi_{M_0}\phi_\omega^{\lambda_0} - \phi_\omega\|_{H^1} < \varepsilon$, and by Theorem 1.6, the solution $u(t)$ with $u(0) = \chi_{M_0}\phi_\omega^{\lambda_0}$ blows up in finite time. Hence, the standing wave solution $e^{i\omega t}\phi_\omega$ of (NLS) is strongly unstable. □

References

[1] H. Berestycki and T. Cazenave: *Instabilité des états stationaires dans les équations de Schrödinger et de Klein–Gordon non linéaires*, C. R. Acad. Sci. Paris Sér. I Math. **293** (1981), 489–492.

[2] H. Brézis and E. Lieb: *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **88** (1983), 486–490.

[3] T. Cazenave: Semilinear Schrödinger equations, Courant Lecture Notes in Mathematics **10**, American Mathematical Society, Providence, RI, 2003.

[4] T. Cazenave and P.-L. Lions: *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Comm. Math. Phys. **85** (1982), 549–561.

[5] R. Fukuizumi: *Stability and instability of standing waves for the nonlinear Schrödinger equation with harmonic potential*, Discrete Contin. Dynam. Systems **7** (2001), 525–544.

[6] R. Fukuizumi and M. Ohta: *Stability of standing waves for nonlinear Schrödinger equations with potentials*, Differential Integral Equations **16** (2003), 111–128.

[7] R. Fukuizumi and M. Ohta: *Instability of standing waves for nonlinear Schrödinger equations with potentials*, Differential Integral Equations **16** (2003), 691–706.

[8] R. Fukuizumi, M. Ohta and T. Ozawa: *Nonlinear Schrödinger equation with a point defect*, Ann. Inst. H. Poincaré Anal. Non Linéaire **25** (2008), 837–845.

[9] R.H. Goodman, P.J. Holmes and M.I. Weinstein: *Strong NLS soliton-defect interactions*, Phys. D **192** (2004), 215–248.

[10] M. Grillakis, J. Shatah and W. Strauss: *Stability theory of solitary waves in the presence of symmetry I*, J. Funct. Anal. **74** (1987), 160–197.

[11] M. Grillakis, J. Shatah and W. Strauss: *Stability theory of solitary waves in the presence of symmetry II*, J. Funct. Anal. **94** (1990), 308–348.

[12] S. Le Coz: *Standing waves in nonlinear Schrödinger equations*; in Analytical and Numerical Aspects of Partial Differential Equations, de Gruyter, Berlin, 151–192, 2009.

[13] E.H. Lieb and M. Loss: Analysis, Second edition, Graduate Studies in Mathematics **14**, American Mathematical Society, Providence, RI, 2001.

[14] P.-L. Lions: *The concentration-compactness principle in the calculus of variations. The locally compact case, II*, Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), 223–283.

[15] M. Ohta: *Instability of standing waves for the generalized Davey–Stewartson system*, Ann. Inst. H. Poincaré Phys. Théor. **62** (1995), 69–80.

[16] M. Ohta: *Strong instability of standing waves for nonlinear Schrödinger equations with harmonic potential*, Funkcial. Ekvac. **61** (2018), 135–143.

[17] M. Ohta and T. Yamaguchi: *Strong instability of standing waves for nonlinear Schrödinger equations with double power nonlinearity*, SUT J. Math. **51** (2015), 49–58.

[18] M. Ohta and T. Yamaguchi: *Strong instability of standing waves for nonlinear Schrödinger equations with a delta potential*; in Harmonic analysis and nonlinear partial differential equations, RIMS Kôkyûroku Bessatsu, B56, Res. Inst. Math. Sci. (RIMS), Kyoto, 79–92, 2016.

[19] H. A. Rose and M.I. Weinstein: *On the bound states of the nonlinear Schrödinger equation with a linear potential*, Phys. D **30** (1988), 207–218.

[20] J. Shatah: *Stable standing waves of nonlinear Klein–Gordon Equations*, Comm. Math. Phys. **91** (1983), 313–327.

[21] J. Shatah and W. Strauss: *Instability of nonlinear bound states*, Comm. Math. Phys. **100** (1985), 173–190.

[22] M.I. Weinstein: *Nonlinear Schrödinger equations and sharp interpolation estimates*, Comm. Math. Phys. **87** (1982/83), 567–576.

[23] M.I. Weinstein: *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure Appl. Math. **39** (1986), 51–67.

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