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STRONG LAWS FOR WEIGHTED SUMS OF I.I.D. RANDOM VARIABLES

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ABSTRACT. Strong laws are established for linear statistics that are weighted sums of a random sample. We show extensions of the Marcinkiewicz-Zygmund strong laws under certain moment conditions on both the weights and the distribution. The result obtained extends and sharpens the result of Sung ([12]).

1. Introduction

Let $\{X, X_i, i \geq 1\}$ be a sequence of i.i.d. random variables and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of constants. Throughout this paper, we assume that $\phi(x)$ is a positive increasing function on $(0, \infty)$ satisfying

$$(1) \qquad \phi(x) \uparrow \infty \quad \text{and} \quad \phi(Cx) = O(\phi(x)), \forall C > 0.$$

We also assume that $EX = 0$ and $E[\phi(|X|)] < \infty$.

Many useful linear statistics based on a random sample are weighted sums of i.i.d. random variables. Examples include least-squares estimators, nonparametric regression function estimators and jackknife estimates, among others. In this respect, studies of strong laws for these weighted sums have demonstrated significant progress in probability theory with applications in mathematical statistics. Let $\{X, X_i, i \geq 1\}$ be a sequence of i.i.d. random variables and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of constants. The almost sure (a.s.) limiting behavior of weighted sums $\sum_{i=1}^{n} a_{ni} X_i$ was studied by many authors (see, Sung [12], Sung [11], Bai and Cheng [1], Choi and Sung [2], Cuzick [4], Li et al, 1995, Wu [3]). Recently Sung [12] proved the following strong laws of large numbers (see Theorem A and Theorem B).

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THEOREM A. *Let $\{X, X_i, i \geq 1\}$ be a sequence of i.i.d. random variables satisfying $EX = 0$ and $E[\phi(|X|)] < \infty$. Assume that the inverse function $\psi(x)$ of $\phi(x)$ satisfies*

$$\psi(n)\sum_{i=1}^{n}\frac{1}{\psi(i)} = O(n), \psi^2(n)\sum_{i=n}^{\infty}\frac{1}{\psi^2(i)} = O(n). \tag{2}$$

Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of constants such that

(i) $\max_{1\leq i\leq n}|a_{ni}| = O(\frac{1}{\psi(n)})$,

(ii) $\max_{1\leq j\leq n}\frac{\psi^2(j)}{j}\sum_{i=j}^{n}a_{ni}^2 = O(\frac{1}{n^\alpha})$ *for some $\alpha > 0$. Then*

$$\sum_{i=1}^{n}a_{ni}X_i \to 0 \quad a.s.$$

THEOREM B. *Let $\{X, X_i, i \geq 1\}$ be a sequence of i.i.d. random variables satisfying $EX = 0$, $EX^2 < \infty$ and $E[\phi(|X|)] < \infty$. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of constants such that*

(i) $\max_{1\leq i\leq n}|a_{ni}| = O(\frac{1}{\psi(n)})$,

(ii) $\sum_{i=1}^{n}a_{ni}^2 = O(\frac{1}{n^\alpha})$ *for some $\alpha > 0$. Then*

$$\sum_{i=1}^{n}a_{ni}X_i \to 0 \quad a.s.$$

As for negatively associated (NA) random variables, Joag [7] gave the following definition.

DEFINITIONS 2.2 ([7]). A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated (NA) if for every pair of disjoint subsets T_1 and T_2 of $\{1, 2, \ldots, n\}$, we have

$$Cov(f_1(X_i, i \in T_1), f_2(X_j, j \in T_2)) \leq 0,$$

whenever f_1 and f_2 are coordinatewise increasing and the covariance exists. An infinite family is negatively associated if every finite subfamily is negatively associated.

The main purpose of this paper is to establish the Marcinkiewicz-Zygmund strong laws for linear statistics of i.i.d. sequences and NA sequences of random variables. The results obtained (see Theorem 2.1, Theorem 2.2, Corollary 2.1 and Corollary 2.2) extends and sharpens the result of Sung [12].

2. The Marcinkiewicz-Zygmund strong laws

Throughout this paper, C will represent a positive constant though its value may change from one appearance to the next, and $a_n = O(b_n)$ will mean $a_n \leq Cb_n$, and $a_n \ll b_n$ will mean $a_n = O(b_n)$. Let $\psi(x)$ be the inverse function of $\phi(x)$. Since $\phi(x) \uparrow \infty$, it follows that $\psi(x) \uparrow \infty$. For easy notation, we let $\phi(0) = 0$ and $\psi(0) = 0$.

In order to prove our results, we need the following lemmas and the concept of complete convergence. As for complete convergence, let $\{X, X_n, n \geq 1\}$ be a sequence of independent identically distribution random variables (i.i.d) random variables and denote $S_n = \sum_{i=1}^{n} X_i$. The Hsu-Robbins-Erdös law of large numbers (Hsu and Robbins [6]; Erdös [5]) states that

$$\forall \varepsilon > 0, \sum_{n=1}^{\infty} P(|S_n| > \varepsilon n) < \infty$$

is equivalent to $EX = 0$ and $EX^2 < \infty$.

This is a fundamental theorem in probability theory and has been intensively investigated by many authors in the past decades. We can see in Petrov [8], Chow [3] and Stout [10]. There have been many extensions in various directions for Hsu-Robbins-Erdös law of large numbers.

LEMMA 2.1 (PETROV [8]). *Let $\{X_i, i \geq 1\}$ be a sequence of i.i.d. random variables, $EX_i = 0, E|X_i|^p < \infty$ for some $p \geq 2$ and for every $i \geq 1$. Then there exists $C = C(p)$, such that*

$$E \max_{1 \leq k \leq n} |\sum_{i=1}^{k} X_i|^p \leq C\{\sum_{i=1}^{n} E|X_i|^p + (\sum_{i=1}^{n} EX_i^2)^{p/2}\}.$$

LEMMA 2.2 (SUNG [12]). *Assume that the inverse function $\psi(x)$ of $\phi(x)$ satisfies (2). If $E[\phi(|X|)] < \infty$, then the followings hold.*

(i) $\sum_{n=1}^{\infty} \frac{1}{\psi(n)} E|X|I(|X| > \psi(n)) < \infty$,

(ii) $\sum_{n=1}^{\infty} \frac{1}{\psi^2(n)} EX^2 I(|X| > \psi^2(n)) < \infty$.

LEMMA 2.3. *Assume that the inverse function $\psi(x)$ of $\phi(x)$ satisfies*

$$\psi(n) \sum_{i=1}^{n} \frac{1}{\psi(i)} = O(n). \tag{3}$$

If $E[\phi(|X|)] < \infty$, *then* $\sum\limits_{n=1}^{\infty} \frac{1}{\psi(n)} E|X|I(|X| > \psi(n)) < \infty$.

PROOF. The proof of Lemma 2.3 is similar to the proof of Lemma 2.2. □

LEMMA 2.4 (SHAO [9]). *Let* $\{X_i, i \geq 1\}$ *be a sequence of negatively associated mean zero random variables with* $E|X_i|^p < \infty$. *Then for any* $p \geq 2$, *there exists* $C=C(p)$ *such that*

$$E \max_{1\leq k\leq n} |\sum_{i=1}^{k} X_i|^p \leq C\{(\sum_{i=1}^{n} EX_i^2)^{\frac{p}{2}} + \sum_{i=1}^{n} E|X_i|^p\}.$$

THEOREM 2.1. *Let* $\{X, X_i, i \geq 1\}$ *be a sequence of i.i.d. random variables satisfying* $EX = 0$, $EX^2 < \infty$ *and* $E[\phi(|X|)] < \infty$. *Let* $T_n = \sum\limits_{i=1}^{n} a_{ni}X_i, n \geq 1$. *Assume that the inverse function* $\psi(x)$ *of* $\phi(x)$ *satisfies*

$$\psi(n) \sum_{i=1}^{n} \frac{1}{\psi(i)} = O(n).$$

Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ *be a triangular array of constants such that*

(i) $\max_{1\leq i\leq n} |a_{ni}| = O(\frac{1}{\psi(i)})$,

(ii) $\sum_{i=1}^{n} a_{ni}^2 = O(\log^{-1-\alpha} n)$ *for some* $\alpha > 0$. *Then*

$$\forall \varepsilon > 0, \sum_{n=1}^{\infty} n^{-1} P(\max_{1\leq j\leq n} |T_j| > \varepsilon) < \infty. \tag{4}$$

PROOF. $\forall i \geq 1$, define $X_i^{(n)} = X_i I(|X_i| \leq \psi(n))$, $T_j^{(n)} = \sum\limits_{i=1}^{j} (a_{ni}X_i^{(n)} - Ea_{ni}X_i^{(n)})$, then $\forall \varepsilon > 0$,

$$\begin{aligned} & P(\max_{1\leq j\leq n} |T_j| > \varepsilon) \\ \leq\ & P(\max_{1\leq j\leq n} |X_j| > \psi(n)) + P(\max_{1\leq j\leq n} |T_j^{(n)}| \\ >\ & \varepsilon - \max_{1\leq j\leq n} |\sum_{i=1}^{j} Ea_{ni}X_i^{(n)}|). \end{aligned} \tag{5}$$

First we show that

$$\max_{1\leq j\leq n} |\sum_{i=1}^{j} Ea_{ni}X_i^{(n)}| \to 0, \quad \text{as} \quad n \to \infty. \tag{6}$$

By Lemma 2.3 and Kronecker Lemma, then

$$\frac{1}{\psi(n)}\sum_{i=1}^{n} E|X|I(|X|>\psi(i))\to 0. \tag{7}$$

By $EX=0$, $\max_{1\le i\le n}|a_{ni}|=O(\frac{1}{\psi(i)})$, (7) and $\psi(n)\uparrow\infty$, when $n\to\infty$, then

$$\begin{aligned}
&\max_{1\le j\le n}|\sum_{i=1}^{j} Ea_{ni}X_i^{(n)}| \\
= &\max_{1\le j\le n}|\sum_{i=1}^{j} Ea_{ni}X_iI(|X_i|>\psi(n))| \\
\le &\sum_{i=1}^{n} E|a_{ni}X_i|I(|X_i|>\psi(n)) \\
= &\sum_{i=1}^{n} |a_{ni}|E|X|I(|X|>\psi(n)) \\
\ll &\frac{1}{\psi(n)}\sum_{i=1}^{n} E|X|I(|X|>\psi(n)) \\
\le &\frac{1}{\psi(n)}\sum_{i=1}^{n} E|X|I(|X|>\psi(i))\to 0. \qquad (8)
\end{aligned}$$

From (8), then (6) is true.

From (5) and (6), it follows that for n large enough

$$P(\max_{1\le j\le n}|T_j|>\varepsilon)\le\sum_{j=1}^{n}P(|X_j|>\psi(n))+P(\max_{1\le j\le n}|T_j^{(n)}|>\frac{\varepsilon}{2}).$$

Hence we need only to prove that

$$\begin{aligned}
&I=:\sum_{n=1}^{\infty}n^{-1}\sum_{j=1}^{n}P(|X_j|>\psi(n))<\infty, \\
&II=:\sum_{n=1}^{\infty}n^{-1}P(\max_{1\le j\le n}|T_j^{(n)}|>\frac{\varepsilon}{2})<\infty. \qquad (9)
\end{aligned}$$

From the fact that $E[\phi(|X|)] < \infty$, it follows easily that

$$\begin{aligned} I &= \sum_{n=1}^{\infty} n^{-1} n P(|X| > \psi(n)) \\ &= \sum_{n=1}^{\infty} P(|X| > \psi(n)) \\ &= \sum_{n=1}^{\infty} P(\phi(|X|) > n) \\ &\ll E[\phi(|X|)] < \infty. \end{aligned} \tag{10}$$

By Lemma 2.1 and $EX^2 < \infty$, it follows that

$$\begin{aligned} II &\leq C \sum_{n=1}^{\infty} n^{-1} E \max_{1\leq j\leq n} |T_j^{(n)}|^2 \\ &\leq C \sum_{n=1}^{\infty} n^{-1} \sum_{j=1}^{n} E|a_{nj} X_j^{(n)}|^2 \\ &= C \sum_{n=1}^{\infty} n^{-1} \sum_{j=1}^{n} a_{nj}^2 EX^2 I(|X| \leq \psi(n)) \\ &\ll \sum_{n=1}^{\infty} n^{-1} \sum_{j=1}^{n} a_{nj}^2 \\ &\ll \sum_{n=1}^{\infty} n^{-1} \log^{-1-\alpha} n < \infty. \end{aligned} \tag{11}$$

Now we complete the prove of Theorem 2.1. □

COROLLARY 2.1. *Under the conditions of Theorem 2.1, then*

$$\lim_{n\to\infty} T_n = 0 \quad a.s.$$

PROOF. By (4), we have

$$\infty > \sum_{n=1}^{\infty} n^{-1} P(\max_{1\leq j\leq n} |T_j| > \varepsilon)$$

$$= \sum_{i=0}^{\infty} \sum_{n=2^i}^{2^{i+1}-1} n^{-1} P(\max_{1 \le j \le n} |T_j| > \varepsilon)$$

$$\geq \frac{1}{2} \sum_{i=1}^{\infty} P(\max_{1 \le j \le 2^i} |T_j| > \varepsilon).$$

By Borel-Cantelli Lemma, we have

$$P(\max_{1 \le j \le 2^i} |T_j| > \varepsilon, i.o.) = 0.$$

Hence

$$\lim_{i \to \infty} \max_{1 \le j \le 2^i} |T_j| = 0 \quad a.s.$$

and using

$$\max_{2^{i-1} \le n < 2^i} |T_n| \le \max_{1 \le j \le 2^i} |T_j|,$$

then

$$\lim_{n \to \infty} |T_n| \le \lim_{i \to \infty} \max_{2^{i-1} \le n < 2^i} |T_n| \le \lim_{i \to \infty} \max_{1 \le j \le 2^i} |T_j| = 0 \quad a.s.$$

Then we have

$$\lim_{n \to \infty} T_n = 0 \quad a.s.$$

Now we complete the prove of Corollary 2.1. □

THEOREM 2.2. *Let $\{X, X_i, i \ge 1\}$ be a sequence of NA random variables satisfying $EX = 0$, $EX^2 < \infty$ and $E[\phi(|X|)] < \infty$. Let $T_n = \sum_{i=1}^{n} a_{ni} X_i, n \ge 1$. Assume that the inverse function $\psi(x)$ of $\phi(x)$ satisfies*

$$\psi(n) \sum_{i=1}^{n} \frac{1}{\psi(i)} = O(n).$$

Let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be a triangular array of constants such that

(i) $\max_{1 \le i \le n} |a_{ni}| = O(\frac{1}{\psi(i)})$,

(ii) $\sum_{i=1}^{n} a_{ni}^2 = O(\log^{-1-\alpha} n)$ *for some $\alpha > 0$. Then*

$$\forall \varepsilon > 0, \sum_{n=1}^{\infty} n^{-1} P(\max_{1 \le j \le n} |T_j| > \varepsilon) < \infty. \tag{12}$$

PROOF. The proof of Theorem 2.2 is similar to the proof of Theorem 2.1. □

COROLLARY 2.2. *Under the conditions of Theorem 2.2, then*

$$\lim_{n \to \infty} T_n = 0 \quad a.s.$$

PROOF. The proof of Corollary 2.2 is similar to the proof of Corollary 2.1. □

REMARK. Theorem 2.2 and Corollary 2.2 generalize the results of Sung [12] to NA sequences.

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