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Strong Limit Theorems for Dependent Random Variables

Libin Wu, Bainian Li
School of Stat & Applied mathematics Anhui University of Finance and Economics, Bengbu 233030, PR China

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***Abstract*—**In this article We establish moment inequality of dependent random variables, furthermore some theorems of strong law of large numbers and complete convergence for sequences of dependent random variables. In particular, independent and identically distributed Marcinkiewicz Law of large numbers are generalized to the case of m_0 -dependent sequences..

***Keywords*—**Lacunary System, Generalized Gaussian, NA sequences, strong law of large numbers

I. Introduction

Let $X_1, X_2, \cdots,$ denote a sequence of random varibles defined on a fixed probability space (Ω,F,P) .The partial sums of the random variables are

$$S_n = \sum_{i=1}^{n} X_i, \text{for } n \geq 1 \text{ and } S_0 = 0,$$

$$S_{b,n} = \sum_{i=b+1}^{b+n} X_i, S(i,j) = \sum_{k=i}^{j} X_k, 1 \leq i \leq j.$$

Definition 1.1 (cf. Fazekas and Klesov, 2000, p. 447)[1]. A sequence of random variables $\{X_n, n \geq 1\}$ is said to have the rth (r>0) moment function of superadditive structure if there exists a non-negative function $g(i,j)$ of two arguments such that for all b≥0 and 1≤k≤k+l

$$g(b,k) + g(b+k,l) \leq g(b,k+l) \tag{1}$$

$$E|S_{b,n}|^r \leq g^{\alpha}(b,n), n \geq 1, \text{for some } \alpha > 1. \tag{2}$$

Definition 1.2. Let X be a real-valued random variable, we call a Locally Generalized Gaussian, If there exists $\alpha > 0$,such that

$$E(\exp(uX)\,|\,F) \leq \exp(u^2\alpha^2/2) \qquad a.s. \tag{3}$$

for any $u \in R$.

Definition 1.3. Given p>0, a sequence of real-valued random variables $\{X_n, n \geq 1\}$ is called a Lacunary System or an S_p system, if there exists a positive constant K_p such that

$$E\left|\sum_{i=m}^{n} C_i X_i\right|^p \leq K_p \left(\sum_{i=m}^{n} C_i^2\right)^{p/2}$$

for any sequence of real constant C_i and all n≥m.

Definition 1.4 The random variables $X_1, X_2, \cdots, X_n$ are said to be negatively associated if for every n and every pair of disjoint subsets A_1, A_2 of {1,2,...,n}

$$Cov(f_1(X_i, i \in A_1), f_2(X_j, j \in A_2)) \leq 0,$$

Whenever f_1 and f_2 are coordinatewise increasing and this covariance exists.

Definition 1.5 A stationary sequence $\{X_n\}$ is called m_0 -dependent for a given fixed m_0 if ($X_1, X_2, \cdots, X_i$) and ($X_j, X_{j+1}, \cdots,$) are independent whenever $j - i > m_0$.

Lemma 1.1 (Dini Theorem, Fikhtengolts, 1954, p. 286)[2]. Let $c_1, c_2, \cdots$ be non-negative numbers, $v_n = \sum_{k=n}^{\infty} c_k$, if $0 < v_n < \infty$ for n≥1,then

$$\sum_{n=1}^{\infty} \frac{c_k}{v_n^{\delta}} < \infty, 0 < \delta < 1. \tag{4}$$

Lemma 1.2 (cf. Hu, 2005,Theorem 2.1.)[3] We assume that $\{X_n, n \geq 1\}$ has the rth moment function of superadditive structure, $g(0,n) = g_n$ and g_n non-decreases, $\{b_n\}$ is a non-decreasing unbounded sequence of positive numbers and $\sum_{n=1}^{\infty} \frac{g_n^{\alpha} - g_{n-1}^{\alpha}}{b_n^r} < \infty$, then

$$\lim_{n\to\infty} \frac{S_n}{b_n} = 0 \quad a.s., \tag{5}$$

and with the growth rate

$$\frac{S_n}{b_n} = O\left(\frac{\beta_n}{b_n}\right) \quad a.s., \tag{6}$$

where $\beta_n = \max_{1\leq k\leq n} b_k v_k^{\delta/r}, 0 < \delta < 1, v_n = n\sum_{k=1}^{\infty} \frac{\alpha_k}{b_k^r}, \lim_{n\to\infty} \frac{\beta_n}{b_n} = 0.$

Lemma 1.3 (Hu, 2005,Lemma1.2)[3] Let $b_1, b_2, \cdots$ be a non-decreasing unbounded sequence of positive numbers and $a_1, a_2, \cdots$ be nonnegative numbers. Let r and C be fixed positive numbers. Assume that for each n≥1

$$E(\max_{1\leq l\leq n} |S_l|)^r \leq C\sum_{l=1}^{n} \alpha_l, \tag{7}$$

$$\sum_{l=1}^{n} \frac{\alpha_l}{b_l^r} < \infty. \tag{8}$$

then (5) and (6) hold .

Lemma 1.4 (Yang, 2000, Corollary 3, Yang, 2001; Shao, 2000)[4][6][7]. Let $X_1, X_2, \cdots$ be negatively associated random variables with zero means and $E|X_l|^r < \infty$, where r>1. Then there exists a positive constant C, which does not depend on n, such that

$$E(\max_{1\leq k\leq n} |S_k|^r) \leq C\sum_{j=1}^{n} E|X_j|^r, \qquad 1 < r \leq 2$$

$$E(\max_{1\leq k\leq n} |S_k|^r) \leq C[\sum_{j=1}^{n} E|X_j|^r + (\sum_{j=1}^{n} EX_j^2)^{r/2}, r > 2$$

In this paper, we assume that $C, C_1, \cdots$ are some positive constants (not necessarily always the same) independent of n.

II. Main Results

Theorem 2.1 Assume that $\{X_n, n \ge 1\}$ be a Lacunary System, exists a positive constant K_p and p>2 , such that $\sup_j E|X_j|^p < \infty$,then for every $\delta > 0$,

$$\lim_{n\to\infty} \frac{S_n}{\sqrt{n}(\log n)^{1/p}(\log\log n)^{(1+\delta)/p}} = 0 \quad a.s., \tag{9}$$

and for $b_n = \sqrt{n}(\log n)^{1/p}(\log\log n)^{(1+\delta)/p}, n \ge n_0$,
$\alpha_n = K_p n^{p/2} - K_p(n-1)^{p/2}$, $\beta_n = \max_{1\le k\le n} b_k v_k^{\delta/p}, 0 < \delta < 1$,

$$\frac{S_n}{b_n} = O(\frac{\beta_n}{b_n}) \quad a.s., \lim_{n\to\infty}\frac{\beta_n}{b_n} = 0. \tag{10}$$

Proof From definition 1.3 ,for any sequence of real constant $\{C_i\}$

$$E|\sum_{i=b+1}^{b+n} C_i X_i|^p \le K_p(\sum_{i=b+1}^{b+n} C_i^2)^{p/2},$$

in particularly where $C_i \equiv 1$,we have

$$E|\sum_{i=b+1}^{b+n} C_i X_i|^p = E|S_{b,n}|^p \le K_p n^{p/2},$$

In Definition 1.1 take $g_n = g(b,n) = K_p^{2/p} n, \alpha = \frac{p}{2}$,then

$$g(b,k) = K_p^{2/p}k, g(b+k,l) = K_p^{2/p}l \ ,$$
$$g(b,k) + g(b+k,l) = K_p^{2/p}(k+l) \le g(b,k+l),$$
$$E|S_{b,n}|^p \le K_p n^{p/2} = g^{p/2}(b,n), n \ge 1, p > 2,$$

we know that $\{X_n, n \ge 1\}$ has the pth moment function of superadditive structure, and

$$\sum_{n=n_0}^{\infty} \frac{g_n^{p/2} - g_{n-1}^{p/2}}{b_n^p} = \sum_{n=n_0}^{\infty} \frac{K_p(n^{p/2} - (n-1)^{p/2})}{n^{p/2}(\log n)(\log\log n)^{1+\delta}}$$
$$\le C\sum_{n=n_0}^{\infty} \frac{1}{n(\log n)(\log\log n)^{1+\delta}} < \infty$$

thus (9) follows from Lemma1.2.

We assume that $\alpha_n > 0$ for infinitely many n. By (8) and Lemma 1.3, we know that $\sum_{n=1}^{\infty} \frac{\alpha_n}{b_n^r v_n^\delta} < \infty$, ,it is easy to see that $0 < \beta_k \le \beta_{k+1}$ for k≥1,and

$$\sum_{n=1}^{\infty} \frac{\alpha_n}{\beta_n^r} \le \sum_{n=1}^{\infty} \frac{\alpha_n}{\beta_n^r v_n^\delta} < \infty,$$

$$\frac{\beta_k}{b_k} \le \frac{\max_{1\le l\le k_1} b_l v_l^{\delta/r}}{b_k} + \frac{\max_{k_1\le l\le k} b_l v_l^{\delta/r}}{b_k} \le \frac{\max_{1\le l\le k_1} b_l v_l^{\delta/r}}{b_k} + v_{k_1}^{\delta/r},$$

from (8) and $\lim_{n\to\infty} b_n = 0$, we get $\lim_{n\to\infty}\frac{\beta_n}{b_n} = 0$, Eq. (7) and Theorem 1.1 of Fazekas and Klesov (2000) imply that

$$E(\max_{1\le l\le n}|\frac{S_l}{\beta_l}|)^r \le 4C\sum_{l=1}^{n}\frac{\alpha_l}{\beta_l^r} \le 4C\sum_{l=1}^{\infty}\frac{\alpha_l}{\beta_l^r} < \infty,$$

hence by monotone convergence theorem, we have

$$E(\sup_{n\ge1}|\frac{S_n}{\beta_n}|)^r = \lim_{n\to\infty} E(\max_{1\le l\le n}|\frac{S_l}{\beta_l}|)^r \le 4C\sum_{n=1}^{\infty}\frac{\alpha_n}{\beta_n^r} < \infty,$$

so that $\sup_{n\ge1}|\frac{S_n}{\beta_n}| < \infty$, a.s., and

$$0 \le |\frac{S_n}{\beta_n}| \le \frac{\beta_n}{b_n}\sup_{n\ge1}|\frac{S_n}{\beta_n}| = O(\frac{\beta_n}{b_n}), \quad a.s.,$$

this completes the proof.

Remark 1 Theorem 2.1 improve result of Ryozo, Y.(Corollary 2) [8], and from strictly stationary strong mixing sequence to S_p system.

Theorem 2.2 Let (X_n, F_n) be a Locally Generalized Gaussian sequence, if $\sup_n X_n = k < \infty$, then for any r≥2

$$\sum_{i=a+1}^{a+n} E|C_i X_i|^r \le K_r(\sum_{i=a+1}^{a+n} C_i^2)^{r/2}, \tag{11}$$

Furthermore (9) and (10) hold.

Proof Let $A_n = \sum_{i=a+1}^{a+n} C_i^2, u = x/k^2 A_n$,by lemma 1 in [9], then

$$E(\exp(u\sum_{i=a+1}^{a+n} C_i X_i)) = E(\exp(uS_n)) \le \exp(u^2 k^2 A_n/2),$$

where $S_n = \sum_{i=a+1}^{a+n} C_i X_i$,for r≥2, by Chebyshev's inequality, we get

$$\begin{aligned} E|\sum_{i=a+1}^{a+n} C_i X_i|^r &= r\int_0^\infty x^{r-1}P(|S_n| > x)dx \\ &\le 2r\int_0^\infty x^{r-1}\exp(-x^2/2k^2A_n)dx \\ &= 2^{r/2}rk^r A_n^{r/2}\int_0^\infty x^{r/2-1}\exp(-x)dx \\ &= K_r(\sum_{i=a+1}^{a+n} C_i^2)^{r/2} \end{aligned}$$

where $K_r = 2^{r/2}rk^r\int_0^\infty x^{r/2-1}\exp(-x)dx$.

Therefore, Locally Generalized Gaussian sequence is a Lacunary system, by Theorem 2.1, (9) and (10) hold.

Theorem 2.3 Let $\{X_n, n \ge 1\}$ be a NA sequence, satisfying $\sup_n |X_n|^p < \infty$,then for any 0<p<2, $\alpha p \ge 1, p/2 \le \delta \le 1$, x>0,

$$\sum_{n=1}^{\infty} n^{\alpha p-(1+\delta)} P\{|S_n| \ge xn^\alpha\} < \infty, \tag{12}$$

where $S_n = \sum_{i=1}^n X_i$.

Proof From lemma 1.3, when $C_i \equiv 1$,we have

$$E|\sum_{i=a+1}^{a+n} X_i|^p \le K_p n^{p/2},$$

By Markov's inequality we get

$$\sum_{n=1}^{\infty} n^{\alpha p-(1+\delta)} P\{|S_n| \ge xn^\alpha\} \le \sum_{n=1}^{\infty}\frac{n^{\alpha p-(1+\delta)}E|S_n|^p}{x^p n^{\alpha p}}$$
$$\le K_p x^{-p}\sum_{n=1}^{\infty}\frac{n^{\alpha p-(1+\delta)}n^{p/2}}{x^p n^{\alpha p}}$$
$$\le K_p x^{-p}\sum_{n=1}^{\infty}\frac{1}{n^{1+(\delta-p/2)}} < \infty.$$

Theorem 2.4 Let $\{X_n, n \ge 1\}$ be a m_0 dependent sequence with zero mean, if $\sup_n |X_n|^r < \infty$,then for any $1\le r<2$,

$$\frac{S_n}{n^{1/r}} \to 0, \quad a.s., \tag{13}$$

where $S_n = \sum_{i=1}^{n} X_i$.

Proof for every $t = 0,1,2,\cdots,m_0-1, \{X_{t+m+m_0k}, k = 1,2,\cdots\}$ be a independent sequence, and

$$\sum_{j=m}^{n} C_j X_j = \sum_{t=0}^{m_0-1} \sum_{k=0}^{n_0} C_{t+m+m_0k} X_{t+m+m_0k},$$

by C_r inequality ,we get

$$E(|\sum_{j=m}^{n} C_j X_j|^2 \le m_0 \sum_{t=0}^{m_0-1} E(\sum_{k=0}^{n_0} C_{t+m+m_0k} X_{t+m+m_0k})^2$$
$$= m_0 \sum_{t=0}^{m_0-1} \sum_{k=0}^{n_0} C_{t+m+m_0k}^2 E(X_{t+m+m_0k})^2$$
$$= m_0 n \sum_{j=m}^{n} C_j^2 E(X_j)^2$$
$$\le k m_0 \sum_{j=m}^{n} C_j^2,$$

$\{X_n, n \ge 1\}$ be a S_2 system, by Jensen's inequality , for $1\le r<2$,

$$E(|\sum_{j=m}^{n} C_j X_j|)^r \le C(\sum_{j=m}^{n} C_j^2)^{r/2},$$

in particular,when $C_i \equiv 1$,we obtaned $E|S_n|^r \le Cn^{r/2}$.

In Definition 1.1 take

$$g_n = g(b,n) = K^{2/r} n, \alpha = \frac{r}{2},\ b_n = n^{1/r}$$

then

$$g(b,k) = K^{2/r} k, g(b+k,l) = K^{2/r} l,$$
$$g(b,k) + g(b+k,l) = K^{2/r}(k+l) \le g(b,k+l),$$
$$E|S_{b,n}|^r \le Kn^{r/2} = g^{r/2}(b,n), n \ge 1, 1 \le r < 2,$$

we know that $\{X_n, n \ge 1\}$ has the pth moment function of superadditive structure, and

$$\sum_{n=1}^{\infty} \frac{g_n^{r/2} - g_{n-1}^{r/2}}{b_n^r} = \sum_{n=1}^{\infty} \frac{K(n^{r/2} - (n-1)^{r/2})}{n}$$
$$\le C\sum_{n=1}^{\infty} \frac{n^{r/2-1}}{n} = C\sum_{n=1}^{\infty} \frac{1}{n^{2-r/2}} < \infty,$$

thus (13) follows from Lemma1.2.

Remark 2 This result extends independent and identically distributed Marcinkiewicz Law of large numbers for m_0 -dependent sequences.

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LiBin Wu (1962 -), male, associate professor, master tutor, In 1984, graduated from the department of anhui normal university, bachelor. Research direction: random variable limit theory.

Bainian Li is Professor in the Department of information and science computer of School of Stat & Applied mathematics Anhui University of Finance and Economics, China. His research area is strong limit theorem for dependent random variables.