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Strong Norlund Summability

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STRONG NÖRLUND SUMMABILITY

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ABSTRACT

In this thesis methods of strong Nörlund summability are defined and investigated. It is shown that the definition of strong Cesaro summability as a special case of strong Nörlund summability is equivalent to the standard definition of strong Cesaro summability. Such questions are answered as: "If one Nörlund method of summability includes another, is the same true of the associated strong methods?". A method of summability (C^*,α), known to be equivalent to the Cesaro method (C,α), for $\alpha>0$ is considered, and theorems established in this thesis are used to show that the associated strong methods of summability are equivalent. Relations are established between strong Nörlund, absolute Nörlund and Nörlund summability. Multiplication theorems for strong Nörlund summability are established, generalising known theorems for strong Cesaro summability. For a certain class of Nörlund methods of summability, associated one parameter families of Nörlund methods are constructed, the methods in each family increasing in strength as the parameter increases.

Finally generalisations are made of certain known theorems about strong Cesaro summability concerning inclusion of strong methods of summability with different indices.

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CHAPTER 0

The following inequalities are used frequently throughout this thesis.

1. Hölder's inequality.

If $\lambda>1$ *and* $\lambda' = \lambda/(\lambda-1)$, *so that* $\lambda'>1$ *and*

$$1/\lambda + 1/\lambda' = 1,$$

and if $a_o, a_1, a_2, \ldots, a_n$ *and* $b_o, b_1, b_2, \ldots, b_n$ *are two sets of non-negative numbers, then*

$$\sum_{m=o}^{n} a_m b_m \leq \{\sum_{m=o}^{n} a_m^{\lambda}\}^{1/\lambda} \{\sum_{m=o}^{n} b_m^{\lambda'}\}^{1/\lambda'}.$$

2. Minkowski's inequality.

If $\lambda>1$ *and* $a_o, a_1, a_2, \ldots, a_n$ *and* $b_o, b_1, b_2, \ldots, b_n$ *are two sets of non-negative numbers, then*

$$\{\sum_{m=o}^{n} (a_m + b_m)^{\lambda}\}^{1/\lambda} \leq \{\sum_{m=o}^{n} a_m^{\lambda}\}^{1/\lambda} + \{\sum_{m=o}^{n} b_m^{\lambda}\}^{1/\lambda}.$$

See Hardy [6].

Let $\{w_n\}$ be a sequence of positive numbers and $\{s_n\}$ be any sequence of complex numbers.

(i) We write

$$s_n = O(w_n)$$

if

$$|s_n| \leq Hw_n \quad \text{for } n = 0, 1, 2, \ldots$$

where H is a positive constant independent of n.

(ii) We write

$$s_n = o(w_n)$$

if

$$\lim_{n\to\infty}(s_n/w_n) = 0.$$

(iii) We write

$$s_n \sim w_n$$

if

$$\lim_{n\to\infty}(s_n/w_n) = 1.$$

We use the notation "$c_n \to L$" to mean "$\lim_{n\to\infty} c_n = L$".

CHAPTER 1

DEFINITIONS AND GENERALITIES

Throughout this thesis, H, H_1, etc. will denote positive constants, which will not necessarily take the same value at different occurrences.

Suppose throughout, that

$$s_n = \sum_{r=o}^{n} a_r, \text{ and } \varepsilon_n^\alpha = \binom{n+\alpha}{\alpha} \doteq \frac{(\alpha+1)(\ldots)(\alpha+n)}{n!}, \quad n = 0, 1, \ldots$$

Given an arbitrary sequence $\{w_n\}$, we define

$$\Delta w_n = w_n - w_{n-1}, \quad w_{-1} = 0.$$

Let T be a linear transformation of the sequence $\{s_n\}$ into the sequence $\{\sigma_n\}$ given by

$$(1.1) \qquad \sigma_n = \sum_{r=o}^{\infty} c_{n,r} s_r \quad n = 0, 1, 2, \ldots$$

where the $c_{n,r}$ are complex numbers. We shall have occasion to denote the transformation T by $(c_{n,r})$.

σ_n is called the n-th T transform of the sequence $\{s_n\}$.

The series $\sum_{r=o}^{\infty} a_r$ is said to be *summable* T to s, if $\sigma_n \to s$ as $n\to\infty$. We denote this by

$$(1.2) \qquad \sum_{r=o}^{\infty} a_r = s \ (T) \text{ or by } s_n \to s \ (T).$$

In this context we refer to T as a method of summability.

A method of summability is *regular*, if it sums every convergent series to its ordinary sum.

Theorem 2 in Hardy [5] states that the method of summability $T = (c_{n,r})$ is regular if and only if

$$(1.3) \qquad \mu_n = \sum_{r=o}^{\infty} |c_{n,r}| < H$$

where H is independent of n,

$$(1.4) \qquad c_{n,r} \to 0 \quad \text{for each } r \text{ as } n \to \infty$$

and

$$(1.5) \qquad c_n = \sum_{r=o}^{\infty} c_{n,r} \to 1.$$

Let $\{p_n\}$ and $\{q_n\}$ be arbitrary sequences of complex numbers, let

$$P_n = \sum_{r=o}^{n} p_r, \qquad Q_n = \sum_{r=o}^{n} q_r,$$

and assume throughout that P_n and Q_n are non-zero for all values of n. Also let

$$P_n^* = \sum_{r=0}^{n} |p_r|, \qquad Q_n^* = \sum_{r=0}^{n} |q_r|.$$

We give now the standard definitions of the Nörlund methods of summability (N,p_n) and (N,q_n).

(N,p_n) is the method of summability $T = (c_{n,r})$ with

$$c_{n,r} = p_{n-r}/P_n \quad \text{for } r \leq n$$

and

$$c_{n,r} = 0 \qquad \text{for } r > n.$$

We denote the n-th (N,p_n) transform of the sequence $\{s_n\}$ by t_n, i.e.

(1.6) $$t_n = \frac{1}{P_n} \sum_{r=0}^{n} p_r s_{n-r} = \frac{1}{P_n} \sum_{r=0}^{n} P_r a_{n-r}.$$

Similarly, (N,q_n) is the method of summability $T = (c_{n,r})$ with

$$c_{n,r} = q_{n-r}/Q_n \quad \text{for } r \leq n$$

and

$$c_{n,r} = 0 \qquad \text{for } r > n.$$

We denote the n-th (N,q_n) transform of the sequence $\{s_n\}$

by u_n, i.e.

(1.7) $$u_n = \frac{1}{Q_n}\sum_{r=o}^{n} q_r s_{n-r} = \frac{1}{Q_n}\sum_{r=o}^{n} Q_r a_{n-r}.$$

The appropriate forms of (1.2) in these special cases are:

(1.8) $$\sum_{r=o}^{\infty} a_r = s\ (N,p_n) \text{ or } s_n \to s\ (N,p_n)$$

if $t_n \to s$, and

(1.9) $$\sum_{r=o}^{\infty} a_r = s\ (N,q_n) \text{ or } s_n \to s\ (N,q_n)$$

if $u_n \to s$.

For the sake of completeness we prove the following well-known result.

PROPOSITION 1.1

(N,p_n) *is regular if and only if*

(1.10) $$P_n^* = O(|P_n|)$$

and

(1.11) $$p_n/P_n = o(1).$$

PROOF.

We have

$$c_{n,r} = p_{n-r}/P_n \quad \text{for } r \leq n$$

and

$$c_{n,r} = 0 \quad \text{for } r > n,$$

thus

$$c_n = \sum_{r=0}^{n} c_{n,r} = 1$$

so, a fortiori $c_n \to 1$. Further

$$\mu_n = \sum_{r=0}^{\infty} |c_{n,r}| = \frac{P^*_n}{|P_n|},$$

thus

$$\mu_n = O(1),$$

if and only if (1.10) holds.

It remains to show that in the presence of (1.10),

(1.12) $$p_{n-r}/P_n = o(1) \quad \text{for each } r \text{ as } n \to \infty$$

if and only if $p_n/P_n = o(1)$. Clearly, if (1.12) holds, then (1.11) holds. On the other hand, using (1.10), we have for $n > r \geq 0$

$$0 \leq |p_{n-r}/P_n| \leq H|p_{n-r}|/P^*_n \leq H|p_{n-r}|/P^*_{n-r}$$

$$\leq H|p_{n-r}/P_{n-r}|$$

so that, if (1.11) holds, then (1.12) must also hold. This completes the proof.

Thus for a regular Nörlund method (N,p_n), either,

$$(1.13) \qquad |P_n| \leq \sum_{r=o}^{\infty} |p_r| < \infty$$

or

$$(1.14) \qquad |P_n| \to \infty.$$

In the case $p_n \neq 0$ and $q_n \neq 0$ for all values of n, we shall use the notation

$$(1.15) \qquad t_n^{\Delta} = \frac{1}{p_n} \sum_{r=o}^{n} p_r a_{n-r}$$

and

$$(1.16) \qquad u_n^{\Delta} = \frac{1}{q_n} \sum_{r=o}^{n} q_r a_{n-r}.$$

We note that (1.15) and (1.16) are respectively the n-th $(N,\Delta p_n)$ and the n-th $(N,\Delta q_n)$ transforms of the sequence $\{s_n\}$.

If P and Q are methods of summability, Q is said to *include* P (written "P $\Longrightarrow$ Q") if every series summable by the method P is also summable by the method Q to the same sum. P and Q are said to be *equivalent* (written "P $\Longleftrightarrow$ Q") if each method includes the other.

Since p_o and q_o are both non-zero, there exist sequences $\{k_n\}$, $\{\ell_n\}$ and $\{\gamma_n\}$ such that

$$(1.17) \qquad k_o p_n + \dots + k_n p_o = q_n, \quad n = 0, 1, 2, \dots,$$

(1.18) $\ell_o q_n + \ldots + \ell_n q_o = p_n$, $n = 0, 1, 2, \ldots,$

(1.19) $\gamma_o = 1/p_o$,

(1.20) $\gamma_o p_n + \ldots + \gamma_n p_o = 0$, $n = 1, 2, 3, \ldots$.

To define the sequence $\{k_n\}$ by induction, set

$$k_o = q_o/p_o,$$

suppose that $k_o, \ldots, k_{n-1}$ have been defined, and let

$$k_n = \frac{q_n - k_o p_n - \ldots - k_{n-1}p_1}{p_o}.$$

The sequences $\{\ell_n\}$ and $\{\gamma_n\}$ are obtained in a similar fashion. Thus

$$t_n = \frac{1}{P_n} \sum_{r=o}^{n} p_r s_{n-r}$$

if and only if

$$s_n = \sum_{r=o}^{n} \gamma_{n-r} P_r t_r.$$

Therefore every Nörlund transformation is invertible.

The following propositions give necessary and sufficient conditions for inclusion or equivalence relations to hold between two Nörlund methods. Proposition 1.2 is given by Hayashi and Shin-ichi Izumi in [8]. Our proof of this proposition, however, is simpler than their proof. The proofs of these propositions are closely modelled on proofs given by Hardy in [5, Theorems 19 and 21], but are applicable to a larger class of Nörlund methods than are considered by Hardy. In connection with Proposition 1.3 see also Meisner [13, Corollary 1].

PROPOSITION 1.2.

For Nörlund methods (N,p_n) *and* (N,q_n), *(not necessarily regular), necessary and sufficient conditions that* $(N,p_n) \Longrightarrow (N,q_n)$ *are*

$$(1.21) \qquad |k_0||P_n| + \dots + |k_n||P_0| \leq H|Q_n| \quad (n = 0, 1, 2, \dots)$$

where H *is independent of* n, *and*

$$(1.22) \qquad k_{n-r}/Q_n \to 0 \text{ for each } r \text{ as } n \to \infty.$$

Further, if (N,q_n) *is regular and* $|P_n| \to \infty$, *then* (1.22) *may be omitted.*

PROOF.

Refering to (1.6) and (1.7), we have

(1.23) $$u_n = \sum_{r=0}^{\infty} c_{n,r} t_r,$$

with

$$c_{n,r} = k_{n-r} P_r / Q_n \quad \text{for } r \leq n$$

and

$$c_{n,r} = 0 \quad \text{for } r > n.$$

Since a Nörlund transformation is invertible, $(N,p_n) \Rightarrow (N,q_n)$ if and only if the sequence to sequence transformation (1.23) is regular. (1.3) and (1.4) now yield (1.21) and (1.22) respectively. (1.5) is satisfied, since

$$\sum_{r=0}^{\infty} k_{n-r} P_r = Q_n.$$

Suppose now that (N,q_n) is regular and that $|P_n| \to \infty$. Clearly (1.22) is equivalent to the condition

$$k_n / Q_n = o(1).$$

Since $|P_n| \to \infty$, then given $G > 0$, we can choose r so that $|P_r| > G$. If also (1.21) is satisfied, then

$$G|k_{n-r}| \leq H|Q_n|$$

so that

$$\limsup_{n\to\infty} \frac{|k_{n-r}|}{|Q_{n-r}|} \leq \frac{H}{G} \lim_{n\to\infty} \left| \frac{Q_n}{Q_{n-r}} \right| = \frac{H}{G},$$

and (1.22) follows. This completes the proof.

PROPOSITION 1.3.

For regular Nörlund methods (N,p_n) *and* (N,q_n), *necessary and sufficient conditions that* $(N,p_n) \Leftrightarrow (N,q_n)$ *are*

$$(1.24) \qquad \sum_{r=0}^{\infty} |k_r| < \infty \quad \text{and} \quad \sum_{r=0}^{\infty} |\ell_r| < \infty.$$

PROOF.

Necessity.

By Proposition 1.2, both $\{|P_n|/|Q_n|\}$ and $\{|Q_n|/|P_n|\}$ are bounded. Also by (1.21)

$$|k_0| + |k_1| \frac{|P_{n-1}|}{|P_n|} + \ldots + |k_r| \frac{|P_{n-r}|}{|P_n|} \leq H|Q_n|/|P_n|$$

for $r \leq n$. Now fixing r, and letting n tend to infinity, and using the fact that for a regular Nörlund method (N,p_n) $P_{n-r} \sim P_n$ for each r, we have

$$|k_0| + \ldots + |k_r| \leq H \limsup_{n\to\infty} (|Q_n|/|P_n|) = H_1 < \infty,$$

so that

$$\sum_{r=0}^{\infty} |k_r| < \infty.$$

Similarly we find that

$$\sum_{r=0}^{\infty} |\ell_r| < \infty,$$

and hence we have (1.24).

Sufficiency.

We now have $k_{n-r} = o(1)$ for each r as n tends to infinity, and because of the regularity of (N,q_n),

$$|Q_n| \geq H\, Q_n^* \geq H\, Q_0^* > 0, \text{ for } n \geq 0$$

Thus it follows that (1.22) holds. Also by (1.10) and its analogue for the method (N,q_n),

$$|P_n| \leq |Q_0||\ell_n| + \ldots + |Q_n||\ell_0| \leq H|Q_n| \sum_{r=0}^{\infty} |\ell_r|,$$

and

$$|k_0||P_n| + \ldots + |k_n||P_0| \leq H_1|Q_n| \sum_{r=0}^{\infty} |k_r| \sum_{r=0}^{\infty} |\ell_r|.$$

Thus (1.21) holds, and hence $(N,p_n) \Longrightarrow (N,q_n)$. Similarly, we find that $(N,q_n) \Longrightarrow (N,p_n)$, and the proof is complete.

We give next the definition of the standard method of summability $(\overline{N},p_n)$.

$(\overline{N},p_n)$ is the method of summability $T = (c_{n,r})$ with

$$c_{n,r} = p_r/P_n \text{ for } r \leq n$$

and

$$c_{n,r} = 0 \quad \text{for } r > n.$$

We denote the n-th $(\overline{N},p_n)$ transform of the sequence $\{s_n\}$ by τ_n, i.e.

$$(1.25) \qquad \tau_n = \frac{1}{P_n} \sum_{r=o}^{n} p_r s_r.$$

We find, by using (1.3) and (1.4) with $c_{n,r}$ defined as above, that $(\overline{N},p_n)$ is regular if and only if

$$(1.10A) \qquad P_n^* = O(|P_n|)$$

and

$$(1.14A) \qquad |P_n| \to \infty.$$

We recall that the Cesaro method of summability (C,α) is the Nörlund method of summability (N,p_n) with $p_n = \varepsilon_n^{\alpha-1}$, where $\alpha > -1$.

PROPOSITION 1.4

If $(\overline{N},p_n)$ *is regular,* $p_n > 0$ *for all* n, *and either*

$\{p_n\}$ *is non-decreasing and* $P_n/p_n \geq H_1(n + 1)$

or

$\{p_n\}$ *is non-increasing and* $P_n/p_n \leq H_2(n + 1)$

then $(\overline{N},p_n) \Leftrightarrow (C,1)$.

PROOF.

This result is an immediate consequence of a theorem given by Hardy [5, Theorem 14].

Definitions of strong and absolute Nörlund summability methods.

1. *Strong summability.* $[N,p_n]_\lambda$, $\lambda > 0$.

Let (N,p_n) *be a Nörlund method with* $p_n \neq 0$ *for all values of* n. *We shall say that* $\sum_{r=o}^{\infty} a_r$ *is strongly summable* (N,p_n) *with index* λ *to* s, *if*

$$(1.26) \qquad \frac{1}{|P_n|}\sum_{r=o}^{n} |p_r|\,|t_r^\Delta - s|^\lambda = o(1).$$

We shall denote this by

$$\sum_{r=o}^{\infty} a_r = s\ [N,p_n]_\lambda \text{ or by } s_n \to s\ [N,p_n]_\lambda .$$

REMARK.

Whenever (1.10) holds, $\sum_{r=o}^{\infty} a_r = s\ [N,p_n]_\lambda$ if and only if

$$(1.27) \qquad \frac{1}{P_n^*}\sum_{r=o}^{n} |p_r|\,|t_r^\Delta - s|^\lambda = o(1).$$

We shall take advantage of this result without further comment.

2. *Absolute summability.* $|N,p_n|_\lambda$, $\lambda > 0$.

Let (N,p_n) *be a Nörlund method. We shall say that* $\sum_{r=o}^{\infty} a_r$ *is absolutely summable* (N,p_n) *with index* λ, *or summable* $|N,p_n|_\lambda$, *if*

$$\sum_{n=1}^{\infty} n^{\lambda-1}|t_n - t_{n-1}|^\lambda < \infty. \tag{1.28}$$

When $\lambda = 1$, this definition reduces to the customary definition of absolute Nörlund summability, as given by Mears in [11] for example. See also Borwein [1].

We recall now the standard definition of strong Cesaro summability $[C,\alpha+1]_\lambda$.

For $\lambda > 0$, $\alpha > -1$, the series $\sum_{r=o}^{\infty} a_r$ is said to be summable $[C,\alpha+1]_\lambda$ to s, if

$$\frac{1}{n+1}\sum_{r=o}^{n}|s_r^\alpha - s|^\lambda = o(1),$$

where

$$s_n^\alpha = \frac{1}{\varepsilon_n^\alpha}\sum_{r=o}^{n}\varepsilon_{n-r}^{\alpha-1}s_r.$$

THEOREM 1.1

The definition of strong Cesaro summability as strong Nörlund summability, is equivalent to the standard definition of strong Cesaro summability.

PROOF.

Since the Cesaro method of summability $(C,\alpha+1)$ is the method (N,p_n) with $p_n = \varepsilon_n^\alpha$, it suffices to show that $(C,1) \Longleftrightarrow (\overline{N},\varepsilon_n^\alpha)$ for $\alpha > -1$. This follows from Proposition 1.4, because $\{\varepsilon_n^\alpha\}$ is a non-increasing sequence when $-1 < \alpha \leq 0$, a non-decreasing sequence when $\alpha > 0$, and

$$\varepsilon_n^{\alpha+1}/\varepsilon_n^\alpha \sim \frac{n}{\alpha + 1} .$$

CHAPTER 2

INCLUSION THEOREMS

In this chapter we shall prove certain theorems giving sufficient conditions for one strong Nörlund method of summability to include another. Before doing so however, we make the following simplifying remark.

REMARK

If (N,p_n) is a Nörlund method with $p_n \neq 0$ for all values of n, and $\{s_n\}$ is any sequence, then $t_n^{\Delta} - s$ is the n-th $(N,\Delta p_n)$ transform of the sequence $\{s_n - s\}$. Thus we have $\sum_{r=o}^{\infty} a_r = s\ [N,p_n]_\lambda$ if and only if $s_n - s \to 0\ [N,p_n]_\lambda$. Hence in order to prove that $[N,p_n]_\lambda \Longrightarrow [N,q_n]_\lambda$, it is sufficient to prove that $s_n \to 0\ [N,p_n]_\lambda$ implies $s_n \to 0\ [N,q_n]_\lambda$. For the remainder of this chapter, we shall assume that if (N,p_n) is a Nörlund method, then $p_n \neq 0$ for all values of n, unless mention is made to the contrary.

THEOREM 2.1

If $(N,p_n) \Longrightarrow (N,q_n)$, *then* $[N,p_n]_1 \Longrightarrow [N,q_n]_1$.

PROOF.

Referring to (1.15),(1.17),(1.16), we have, for a given sequence $\{s_n\}$,

$$q_n u_n^{\Delta} = \sum_{r=o}^{n} k_{n-r} p_r t_r^{\Delta}.$$

Thus

$$|q_r||u_r^{\Delta}| \leq \sum_{\nu=o}^{r} |k_{r-\nu}||p_\nu||t_\nu^{\Delta}|$$

and hence

$$\sum_{r=o}^{n} |q_r||u_r^{\Delta}| \leq \sum_{r=o}^{n} \sum_{\nu=o}^{r} |k_\nu||p_{r-\nu}||t_{r-\nu}^{\Delta}|$$

$$= \sum_{\nu=o}^{n} |k_\nu| \sum_{r=\nu}^{n} |p_{r-\nu}||t_{r-\nu}^{\Delta}|.$$

Setting

$$\phi_n = \sum_{\nu=o}^{n} |p_\nu||t_\nu^{\Delta}|,$$

we see that

(a) $$\sum_{r=o}^{n} |q_r||u_r^{\Delta}| \leq \sum_{\nu=o}^{n} |k_\nu| \phi_{n-\nu}$$

$$= \sum_{\nu=o}^{n} |k_{n-\nu}||P_\nu| \frac{\phi_\nu}{|P_\nu|}.$$

Supposing now, that $s_n \to 0\ [N,p_n]_1$, we have

$$\phi_n/|P_n| = o(1).$$

Thus using (1.3) and (1.4) with

$$c_{n,r} = \frac{|k_{n-r}||P_r|}{|Q_n|} \quad \text{for } r \leq n$$

and

$$c_{n,r} = 0 \qquad\qquad \text{for } r > n,$$

we see that

(b) $$\frac{1}{|Q_n|} \sum_{\nu=o}^{n} |k_{n-\nu}||P_\nu| \frac{\phi_\nu}{|P_\nu|} = o(1)$$

provided

$$|k_o||P_n| + \dots + |k_n||P_o| = O(|Q_n|)$$

and

$$|k_{n-\nu}| = o(|Q_n|)$$

for each ν as n tends to infinity. But by Proposition 1.2 this is equivalent to our hypothesis $(N,p_n) \Longrightarrow (N,q_n)$. It follows from (a) and (b) that $s_n \to 0\ [N,q_n]_1$. Thus

$$[N,p_n]_1 \Longrightarrow [N,q_n]_1,$$

and the proof is complete.

COROLLARY 2.1.1

If $(N,p_n) \Longleftrightarrow (N,q_n)$ *then* $[N,p_n]_1 \Longleftrightarrow [N,q_n]_1$.

THEOREM 2.2

If $(N,p_n) \Longrightarrow (N,q_n)$ *and*

(2.1) $$\sum_{r=o}^{n} |k_{n-r}||p_r| = O(|q_n|),$$

then

$$[N,p_n]_\lambda \Longrightarrow [N,q_n]_\lambda \quad \textit{for } \lambda > 1.$$

PROOF. Since $\lambda > 1$,

$$|q_r|^\lambda |u_r^\Delta|^\lambda \leq \{\sum_{\nu=o}^{r} |k_{r-\nu}||p_\nu||t_\nu^\Delta|\}^\lambda$$

$$= \{\sum_{\nu=o}^{r} |k_{r-\nu}|^{1/\lambda}|p_\nu|^{1/\lambda}|t_\nu^\Delta||k_{r-\nu}|^{1/\lambda'}|p_\nu|^{1/\lambda'}\}^\lambda.$$

Using Hölder's inequality with index λ, we obtain

$$|q_r|^\lambda |u_r^\Delta|^\lambda \leq \{\sum_{\nu=o}^{r} |k_{r-\nu}||p_\nu||t_\nu^\Delta|^\lambda\}\{\sum_{\nu=o}^{r} |k_{r-\nu}||p_\nu|\}^{\lambda-1}$$

$$\leq H|q_r|^{\lambda-1} \sum_{\nu=o}^{r} |k_{r-\nu}||p_\nu||t_\nu^\Delta|^\lambda,$$

by (2.1). Thus

$$\sum_{r=o}^{n} |q_r||u_r^\Delta|^\lambda \leq H \sum_{r=o}^{n} \sum_{\nu=o}^{r} |k_{r-\nu}||p_\nu||t_\nu^\Delta|^\lambda.$$

To complete the proof we now set

$$\phi_n = \sum_{r=o}^{n} |p_r||t_r^\Delta|^\lambda,$$

and proceed as in the proof of Theorem 2.1.

COROLLARY 2.2.1

If $Q_n^* = O(|Q_n|)$,

(2.2) $$k_{n-r}/Q_n = o(1)$$

for each r *as* n *tends to infinity, and* (2.1) *holds, then*

$$[N,p_n]_\lambda \Longrightarrow [N,q_n]_\lambda \text{ for } \lambda > 1.$$

PROOF.

By summing both sides of (2.1) and using the fact that $Q_n^* = O(|Q_n|)$, we find that

$$\sum_{r=o}^{n} |k_{n-r}||P_r| = O(|Q_n|),$$

which together with (2.2) implies that $(N,p_n) \Longrightarrow (N,q_n)$. The desired result now follows from Theorem 2.2.

COROLLARY 2.2.2

If (N,q_n) *is regular,* $|P_n| \to \infty$ *and* (2.1) *holds then*

$$[N,p_n]_\lambda \Longrightarrow [N,q_n]_\lambda \text{ for } \lambda > 1.$$

PROOF.

This result follows from Proposition 1.2 and Corollary 2.2.1.

COROLLARY 2.2.3

If (N,q_n) *is regular,* $|P_n| \to \infty$ *and*

$$(N,\Delta p_n) \Longrightarrow (N,\Delta q_n)$$

then

$$(N,p_n) \Longrightarrow (N,q_n),$$

and

$$[N,p_n]_\lambda \Longrightarrow [N,q_n]_\lambda \text{ for } \lambda > 1.$$

PROOF.

Since $(N,\Delta p_n) \Longrightarrow (N,\Delta q_n)$, it follows from Proposition 1.2 that (2.1) holds. Thus, as in the proof of Corollary 2.2.1, we find that

$$\sum_{r=o}^{n} |k_{n-r}||P_r| = O(|Q_n|).$$

Now using Proposition 1.2 we obtain

$$(N,p_n) \Longrightarrow (N,q_n).$$

Finally using Theorem 2.2 we obtain

$$[N,p_n]_\lambda \Longrightarrow [N,q_n]_\lambda \text{ for } \lambda > 1.$$

COROLLARY 2.2.4

If (N,p_n) *and* (N,q_n) *are regular Nörlund methods with* $\{|p_n|\}$ *and* $\{|q_n|\}$ *non-decreasing, and if* $(N,p_n) = (N,q_n)$, *then*

$$[N,p_n]_\lambda \Longrightarrow [N,q_n]_\lambda \text{ for } \lambda > 1.$$

PROOF.

By Proposition 1.3, we know that $\sum_{n=o}^{\infty} |k_n| < \infty$ and $\sum_{n=o}^{\infty} |\ell_n| < \infty$. Thus

$$\sum_{r=o}^{n} |k_{n-r}||p_r| \leq H|p_n|$$

and

$$|p_n| \leq \sum_{r=o}^{n} |\ell_{n-r}||q_r| \leq H_1|q_n|.$$

Hence

$$\sum_{r=o}^{n} |k_{n-r}||p_r| = O(|q_n|).$$

Similarly we find that

$$\sum_{r=o}^{n} |\ell_{n-r}||q_r| = O(|p_n|).$$

The conclusion now follows from Theorem 2.2.

As condition (2.1) is in general a difficult condition to check, we proceed now to obtain some theorems which obviate the necessity of checking (2.1). We shall suppose now that the sequences $\{p_n\}$ and $\{q_n\}$ are sequences of positive terms.

We shall write

$$\{p_n\} \in \Omega$$

if

$$p_o = 1,\ p_n > 0,\ p_{n+1}/p_n \geq p_n/p_{n-1} \quad \text{for } n > 0.$$

We now quote part of a result proved by Hardy [5, Theorem 22].

If (N, p_n) is a regular Nörlund method and $\{p_n\} \in \Omega$, then $\gamma_o = 1$ and $\gamma_n \leq 0$ for all $n > 0$, where the γ_n are defined by (1.19) and (1.20).

THEOREM 2.3

If (N, q_n) *is a regular Nörlund method,* $\{p_n\} \in \Omega$,

$$q_n > 0,\ q_n/q_{n-1} \leq p_n/p_{n-1} \quad \textit{for } n > 0,$$

and $p_n = O(q_n)$, *then* (N, p_n) *is regular and* $(N, p_n) \Longrightarrow (N, q_n)$.

PROOF.

$$(1/q_o)Q_n \leq P_n$$

and

$$p_n \leq Hq_n$$

so that

$$p_n/P_n \leq q_o(Hq_n/Q_n)$$

and hence, since (N, q_n) is regular,

$$p_n/P_n = o(1)$$

The condition $P_n^* = O(|P_n|)$ is satisfied because $p_n > 0$. Thus (N, p_n) is regular.

We write $\gamma_n = -c_n$ for $n > 0$, so that, in view of the above mentioned result of Hardy, $c_n \geq 0$ for $n > 0$.

Now

$$p_n - c_1 p_{n-1} - \cdots - c_n p_o = 0 \text{ for } n > 0$$

and by (1.17) and (1.18)

$$q_n - c_1 q_{n-1} - \cdots - c_n q_o = k_n \text{ for } n \geq 0.$$

Hence

$$k_n/q_n = 1 - c_1(q_{n-1}/q_n) - \cdots - c_n(q_o/q_n)$$

$$\leq 1 - c_1(p_{n-1}/p_n) - \cdots - c_n(p_o/p_n) = 0,$$

and thus

$$k_n \leq 0 \text{ for } n > 0.$$

So we have

$$|k_o| p_n + \cdots + |k_n| p_o$$

$$= k_o p_n - k_1 p_{n-1} - \cdots - k_n p_o$$

$$= 2k_o p_n - k_o p_n - \cdots - k_n p_o$$

$$= 2k_o p_n - q_n$$

$$\leq 2k_o p_n + q_n \leq H_1 q_n,$$

since $p_n = O(q_n)$. This proves (2.1). It follows by summing both sides of (2.1) that

$$\sum_{r=o}^{n} |k_{n-r}| P_r = O(Q_n).$$

Also we have

$$|k_n| p_o \le |k_o| p_n + \dots + |k_n| p_o$$

$$= O(q_n) = o(Q_n)$$

since (N,q_n) is regular. Further

$$0 < Q_n < Q_{n+r}$$

for $r > 0$ and so

$$k_{n-r}/Q_n = o(1) \quad \text{for each } r \text{ as } n \to \infty.$$

It follows now from Proposition 1.2 that $(N,p_n) \Longrightarrow (N,q_n)$.

COROLLARY 2.3.1

If (N,p_n) *and* (N,q_n) *satisfy the hypotheses of Theorem* 2.3, *then*

$$[N,p_n]_\lambda \Longrightarrow [N,q_n]_\lambda \textit{ for } \lambda > 1.$$

PROOF.

This result is an immediate consequence of Theorem 2.2 (page 20) and Theorem 2.3 (page 25).

THEOREM 2.4

If (N,p_n) *and* (N,q_n) *are regular Nörlund methods with* $\{p_n\} \varepsilon \Omega$, $q_n > 0$, *and*

$$p_n/p_{n-1} \le q_n/q_{n-1} \text{ for } n > n_o,$$

then

$$[N,p_n]_\lambda \Longrightarrow [N,q_n]_\lambda \textit{ for } \lambda > 1.$$

PROOF.

In the case $n_o = 0$ the result is an immediate consequence of Theorem 2.2 (page 20) and Hardy's Theorem 23 in [5], which yields $(N,p_n) \Longrightarrow (N,q_n)$ and $k_n \geq 0$ for all values of n, so that

$$|k_o|p_n + \dots + |k_n|p_o = k_o p_n + \dots + k_n p_o = q_n.$$

For the general case we modify the second part of Hardy's proof of Theorem 23 in [5]. We have

$$p_n/p_{n-1} \leq q_n/q_{n-1} \text{ for } n = n_o + 1,\ n_o + 2,\ \dots$$

Let

$$r_n = p_n \text{ for } n = n_o,\ n_o + 1,\ \dots$$

and define r_n recursively for $n = n_o - 1,\ n_o - 2,\ \dots,\ 0$, so that $r_n > 0$ and

$$r_{n+1}/r_n \leq \min\,(r_{n+2}/r_{n+1},\ q_{n+1}/q_n,\ p_{n+1}/p_n)$$

Let $\xi_n = r_n/r_o$, then $\{\xi_n\} \in \Omega$,

$$\xi_n/\xi_{n-1} \leq q_n/q_{n-1},$$

and

$$\xi_n/\xi_{n-1} \leq p_n/p_{n-1} \text{ for } n > 0.$$

Also we have $p_n = O(\xi_n)$. Thus we have for $\lambda > 1$,

$$[N,p_n]_\lambda \Longrightarrow [N,\xi_n]_\lambda$$

by Corollary 2.3.1 (page 27), and

$$[N,\xi_n]_\lambda \Longrightarrow [N,q_n]_\lambda$$

by the case $n_o = 0$. Thus

$$[N,p_n]_\lambda \Longrightarrow [N,q_n]_\lambda$$

for $\lambda > 1$, as required.

THEOREM 2.5

If (N,p_n) *and* (N,q_n) *are regular Nörlund methods,* $\{p_n\} \in \Omega$, $\{q_n\} \in \Omega$,

$$p_n/p_{n-1} \leq q_n/q_{n-1} \textit{ for } n > n_o$$

and $q_n = O(p_n)$, *then*

$$(N,p_n) \Longleftrightarrow (N,q_n)$$

and

$$[N,p_n]_\lambda \Longleftrightarrow [N,q_n]_\lambda \text{ for } \lambda \geq 1.$$

PROOF.

That $(N,p_n) \Longrightarrow (N,q_n)$ follows directly from Hardy's Theorem 23 in [5].

To prove that $(N,q_n) \Longrightarrow (N,p_n)$ let $\{\xi_n\}$ be defined as in the proof of Theorem 2.4 (page 27). Now

$$(N,q_n) \Longrightarrow (N,\xi_n)$$

by Theorem 2.3 (page 25),

$$(N,\xi_n) \Longrightarrow (N,p_n)$$

by Hardy's Theorem 23 in [5], and so

$$(N,q_n) \Longrightarrow (N,p_n).$$

Thus

$$(N,p_n) \Longleftrightarrow (N,q_n).$$

Consequently, by Corollary 2.1.1(page 21)

$$[N,p_n]_1 \Longleftrightarrow [N,q_n]_1.$$

To show that $[N,p_n]_\lambda \Longleftrightarrow [N,q_n]_\lambda$ for $\lambda > 1$, we observe that

$$[N,p_n]_\lambda \Longrightarrow [N,q_n]_\lambda \quad (\lambda>1)$$

by Theorem 2.4 (page 27),

$$[N,\xi_n]_\lambda \Longrightarrow [N,p_n]_\lambda \quad (\lambda>1)$$

by the case $n_o = 0$ of Theorem 2.4, and

$$[N,q_n]_\lambda \Longrightarrow [N,\xi_n]_\lambda \quad (\lambda>1)$$

by Corollary 2.3.1 (page 27), and hence

$$[N,q_n]_\lambda \Longrightarrow [N,p_n]_\lambda. \quad (\lambda>1)$$

This completes the proof.

CHAPTER 3

AN APPLICATION

The metnod (C^*,μ) is defined by Borwein [2] as follows:

let $\mu = m + \delta$, where m is a non-negative integer, and $0 \leq \delta < 1$, and let

$$\pi_\mu(x) = m!\varepsilon_m^x(x + m + 1)^\delta$$

Note $\pi_\mu(x) = (x + 1)(\ldots)(x + m)(x + m + 1)^\delta$. $(m \neq 0)$.

A series $\sum_{r=0}^{\infty} a_r$ is said to be summable (C^*,μ) to s if,

$$\sigma_n = \frac{1}{\pi_\mu(n)} \sum_{r=0}^{n} \pi_\mu(n - r)a_r \to s$$

The method (C^*,μ) is the Nörlund method (N,p_n) with

$$p_n = \pi_\mu(n) - \pi_\mu(n - 1). \tag{3.1}$$

Borwein [2] has proved that

(3.2) $$(C^*,\mu) \Longleftrightarrow (C,\mu) \text{ for } \mu \geq 0.$$

We now define the strong method $[C^*,\mu]_\lambda$ to be the method $[N,p_n]_\lambda$ with p_n given by (3.1), and prove the following theorem.

THEOREM 3.1

For $\mu > 0$, $\lambda \geq 1$,

$$[C^*,\mu]_\lambda \Longleftrightarrow [C,\mu]_\lambda.$$

PROOF.

The case $\lambda = 1$ follows immediately from (3.2) and Theorem 2.1 (page 18).

Suppose therefore that $\lambda > 1$, and let $q_n = \varepsilon_n^{\mu-1}$. We consider two cases, (i) $\mu \geq 1$ and (ii) $\mu < 1$.

Case (i) $\mu \geq 1$.

Now

$$\pi_\mu(n) \sim \Gamma(\mu + 1)\varepsilon_n^\mu.$$

Also

$$\frac{\pi_\mu(n) - \pi_\mu(n-1)}{\pi_{\mu-1}(n)} = (n + m)[(1 + 1/(n+m))^\delta - 1] + m$$

$$\rightarrow \delta + m = \mu, \text{ as } n \rightarrow \infty.$$

Thus

$$p_n = \pi_\mu(n) - \pi_\mu(n-1) \sim \mu\pi_{\mu-1}(n) \sim \Gamma(\mu + 1)\varepsilon_n^{\mu-1}$$

$$= \Gamma(\mu + 1)q_n.$$

Now $q_n = \varepsilon_n^{\mu-1} \leq q_{n+1}$ since $\mu \geq 1$, and by (3.2) and Proposition 1.3,

$$\sum_{n=0}^{\infty} |\ell_n| < \infty \text{ and } \sum_{n=0}^{\infty} |k_n| < \infty.$$

Hence

$$|k_o|p_n + \ldots + |k_n|p_o = O(|k_o|q_n + \ldots + |k_n|q_o)$$

$$= O(q_n)$$

and

$$|\ell_o|q_n + \ldots + |\ell_n|q_o = O(q_n) = O(p_n).$$

The desired result now follows from Theorem 2.2 (page 20).

Case (ii) $\mu < 1$. i.e. $\mu = \delta$ with $0 < \delta < 1$.

Now

(3.3) $$q_{n+1}/q_n \geq q_n/q_{n-1} \text{ for } n > 0,$$

because

$$\frac{q_{n+1}q_{n-1}}{q_n^2} = \frac{n^2 + n\delta}{n^2 + n\delta + \delta - 1} \geq 1$$

for $n > 0$. Also

(3.4) $$p_{n+1}/p_n \geq p_n/p_{n-1} \text{ for } n > 0$$

because

$$p_{n+1}/p_n = [(n + 1 + \theta_n)/(n + \theta_n)]^{\delta-1} \quad (0 < \theta_n < 1)$$

$$= (1 + 1/(n + \theta_n))^{\delta-1}$$

$$\geq (1 + 1/(n - 1 + \theta_{n-1})^{\delta-1}$$

$$= p_n/p_{n-1}.$$

We show next that there is an integer n_o such that

(3.5) $$p_{n+1}/p_n \geq q_{n+1}/q_n \text{ for } n > n_o.$$

Let

$$f(x) = \sum_{n=o}^{\infty} f_n x^n$$

$$= \frac{1}{2}(1 + 2x)^{1+\delta} + \frac{1}{2}(1 + 2x)^{\delta} + 1 + \delta x - (1 + \delta)(1 + x)^{1+\delta} -$$

$$- (1 - \delta)(1 + x)^{\delta}.$$

An elementary computation shows that,

$$f_o = f_1 = f_2 = 0 \text{ and } 2f_3 = \delta(1 - \delta)^2 > 0.$$

Hence for n sufficiently large,

$$n^{1+\delta} f(1/n) = (n + 1)[(n + 2)^{\delta} - (n + 1)^{\delta}] - (n + \delta)[(n + 1)^{\delta} - n^{\delta}]$$

$$> 0$$

and (3.5) follows, because

$$\frac{p_{n+1}q_n}{p_n q_{n+1}} = \frac{(n + 1)[(n + 2)^{\delta} - (n + 1)^{\delta}]}{(n + \delta)[(n + 1)^{\delta} - n^{\delta}]} \text{ for } \delta > 0$$

Since $p_o = q_o = 1$, we have $\{p_n\} \varepsilon \Omega$ $\{q_n\} \varepsilon \Omega$ by (3.3) and (3.4), also $p_n = O(q_n)$ and (3.5) holds, we obtain the desired conclusion in case (ii) by appealing to Theorem 2.5 (page 29).

CHAPTER 4

RELATIONS BETWEEN STRONG NORLUND, ABSOLUTE NORLUND AND NORLUND SUMMABILITY METHODS

If in Theorem 4.1 we take (N,p_n) to be the method $(N,\varepsilon_n^{\alpha-1})$, i.e. the method (C,α) we obtain the result

$$[C,\alpha]_1 \Longrightarrow (C,\alpha) \text{ for } \alpha > 0.$$

This has been proved by C.E.Winn in [16]. Winn's proof of his theorem is more complicated than our proof of Theorem 4.1, because the standard definition of strong Cesaro summability is not as "natural" as our definition of strong Nörlund summability. To establish his result, Winn has to prove our Proposition 1.4, whereas the essence of our proof rests in the observation of the validity of equation (4.1).

THEOREM 4.1

$$[N,p_n]_1 \Longrightarrow (N,p_n).$$

PROOF.

Suppose $s_n \to s\ [N,p_n]_1$.

Now

$$\frac{1}{|P_n|}\left|\sum_{r=o}^{n} p_r(t_r^\Delta - s)\right| \leq \frac{1}{|P_n|}\sum_{r=o}^{n} |p_r||t_r^\Delta - s|.$$

Thus

$$\frac{1}{P_n}\sum_{r=o}^{n} p_r t_r^\Delta \to s$$

but

$$(4.1) \qquad \frac{1}{P_n}\sum_{r=o}^{n} p_r t_r^\Delta = \frac{1}{P_n}\sum_{r=o}^{n} p_r s_{n-r},$$

and so,

$$s_n \to s\ (N,p_n).$$

The proof is thus complete.

THEOREM 4.2

If $P_n^* = O(|P_n|)$ *then*

$$[N,p_n]_\lambda \Longrightarrow (N,p_n) \textit{ for } \lambda > 1.$$

PROOF.

Using the fact that $P_n^* = O(|P_n|)$ in conjunction with Borwein's Theorem 1 in [1], we find that

$$[N,p_n]_\lambda \Longrightarrow [N,p_n]_1 \text{ for } \lambda > 1.$$

The result now follows from Theorem 4.1.

THEOREM 4.3

If $(\overline{N},p_n)$ *is regular, and* $\lambda \geq 1$, *then* $s_n \to s\ [N,p_n]_\lambda$ *if and only if,*

$$s_n \to s\ (N,p_n)$$

and

$$\frac{1}{|P_n|}\sum_{r=o}^{n} |p_r||t_r^\Delta - t_r|^\lambda = o(1).$$

The proof of this theorem follows closely the proof of Borwein's Theorem 7 in [1].

PROOF.

We have to prove that

$$(4.2) \qquad \frac{1}{|P_n|}\sum_{r=o}^{n}|p_r||t_r^{\Delta} - s|^{\lambda} = o(1)$$

if and only if

$$(4.3) \qquad t_n \to s$$

and

$$(4.4) \qquad \frac{1}{|P_n|}\sum_{r=o}^{n}|p_r||t_r^{\Delta} - t_r|^{\lambda} = o(1).$$

(i) Suppose that (4.2) holds. Then by Theorem 4.2, (4.3) holds, and so

$$(4.5) \qquad \frac{1}{|P_n|}\sum_{r=o}^{n}|p_r||t_r - s|^{\lambda} = o(1)$$

since $(\overline{N},p_n)$ is regular. Hence, using Minkowski's inequality and (4.2), we have for $\lambda \geq 1$,

$$\{\frac{1}{|P_n|}\sum_{r=o}^{n}|p_r||t_r^{\Delta} - t_r|^{\lambda}\}^{1/\lambda} \leq \{\frac{1}{|P_n|}\sum_{r=o}^{n}|p_r||t_r^{\Delta} - s|^{\lambda}\}^{1/\lambda} +$$

$$+ \{\frac{1}{|P_n|}\sum_{r=o}^{n}|p_r||t_r - s|^{\lambda}\}^{1/\lambda} = o(1)$$

and (4.4) follows.

(ii) Suppose that (4.3) and (4.4) hold. Again (4.5) holds. Hence, using Minkowski's inequality and (4.4),

$$\left\{\frac{1}{|P_n|}\sum_{r=o}^{n}|p_r||t_r^{\Delta} - s|^{\lambda}\right\}^{1/\lambda}$$

$$\leq \left\{\frac{1}{|P_n|}\sum_{r=o}^{n}|p_r||t_r^{\Delta} - t_r|^{\lambda}\right\}^{1/\lambda} + \left\{\frac{1}{|P_n|}\sum_{r=o}^{n}|p_r||t_r - s|^{\lambda}\right\}^{1/\lambda}$$

$$= o(1)$$

so that (4.2) holds. The proof is thus complete.

REMARK

If $\sum_{n=o}^{\infty} a_n$ is summable $|N,p_n|_1$, then

$$\sum_{n=o}^{\infty} a_n = s \ (N,p_n)$$

where

(4.6) $$s = \sum_{n=1}^{\infty}(t_n - t_{n-1}) + t_o$$

THEOREM 4.4

If $(\overline{N},p_n)$ *is regular and* $\sum_{n=o}^{\infty} a_n$ *is summable* $|N,p_n|_1$ *then*

$$\sum_{n=o}^{\infty} a_n = s \ [N,p_n]_1,$$

where s *is given by* (4.6).

PROOF. Since $(\overline{N},p_n)$ is regular, by (1.10A) and Theorem 4.3 it suffices to prove that,

$$(4.7) \qquad \frac{1}{P^*_n} \sum_{r=0}^{n} |p_r||t^{\Delta}_r - t_r| = o(1).$$

Now, for $r > 0$,

$$t^{\Delta}_r - t_r = \frac{1}{p_r} \sum_{\nu=0}^{r} \Delta p_\nu s_{r-\nu} - \frac{1}{P_r} \sum_{\nu=0}^{r} p_\nu s_{r-\nu}$$

$$= \frac{P_r \sum_{\nu=0}^{r} p_\nu s_{r-\nu} - P_r \sum_{\nu=0}^{r-1} p_\nu s_{r-\nu-1} - p_r \sum_{\nu=0}^{r} p_\nu s_{r-\nu}}{p_r P_r}$$

$$= \frac{P_{r-1} \sum_{\nu=0}^{r} p_\nu s_{r-\nu} - P_r \sum_{\nu=0}^{r-1} p_\nu s_{r-\nu-1}}{p_r P_r}$$

$$= \frac{P_{r-1} t_r - P_{r-1} t_{r-1}}{p_r},$$

so that

$$(4.8) \qquad p_r(t^{\Delta}_r - t_r) = P_{r-1}(t_r - t_{r-1}),$$

and hence

$$|t^{\Delta}_r - t_r| \leq \frac{P^*_{r-1}}{|p_r|} |t_r - t_{r-1}|.$$

Consequently, since $t^{\Delta}_o = t_o$,

$$\frac{1}{P^*_n} \sum_{r=o}^{n} |p_r||t^{\Delta}_r - t_r| \leq \frac{1}{P^*_n} \sum_{r=1}^{n} P^*_{r-1} |t_r - t_{r-1}|.$$

Let

$$b_r = |t_r - t_{r-1}|, \text{ and } B_n = \sum_{r=1}^{n} b_r;$$

then

$$\frac{1}{P_n^*}\sum_{r=1}^{n} P_{r-1}^* b_r = B_n - \frac{1}{P_n^*}\sum_{r=1}^{n} B_r |p_r|$$

$$= o(1)$$

by the regularity of $(\overline{N},p_n)$. The required conclusion follows.

THEOREM 4.5

If $\lambda > 1$, $(\overline{N},p_n)$ *is regular and*

(4.9) $$P_{n-1}^* = O(n|p_n|);$$

and if the series $\sum_{n=o}^{\infty} a_n$ *is summable* (N,p_n) *to* s *and is summable* $|N,p_n|_\lambda$, *then the series is summable* $[N,p_n]_\lambda$ *to* s.

PROOF.

Using (4.8), we find that, for $\lambda > 1$,

$$\frac{1}{P_n^*}\sum_{r=o}^{n} |p_r||t_r^\Delta - t_r|^\lambda \le \frac{1}{P_n^*}\sum_{r=1}^{n} \frac{P_{r-1}^*}{|p_r|^{\lambda-1}}|t_r - t_{r-1}|^\lambda$$

$$\le \frac{1}{P_n^*}\sum_{r=1}^{n} P_{r-1}^* r^{\lambda-1}|t_r - t_{r-1}|^\lambda$$

by (4.9). Using the same technique as in the proof of Theorem 4.4, we find that the final term is o(1), so that

$$\frac{1}{P_n^*} \sum_{r=o}^{n} |p_r| |t_r^{\Delta} - t_r|^{\lambda} = o(1).$$

Finally, using Theorem 4.3, we obtain the desired conclusion.

REMARK

Condition (4.9) is satisfied when $\{|p_n|\}$ is non-decreasing, and also if (N,p_n) is the (C,α) method of summability with $\alpha > -1$.

Flett in [4] has established some relations between absolute Cesaro and strong Cesaro methods of summability.

CHAPTER 5

MULTIPLICATION THEOREMS

In this chapter we shall suppose that $\{p_n\}$ and $\{q_n\}$ are sequences satisfying

$$(5.1) \qquad p_o > 0,\ p_n \geq 0 \text{ for } n > 0$$

$$q_o > 0,\ q_n \geq 0 \text{ for } n > 0.$$

Let

$$(5.2) \qquad r_n = \sum_{\nu=o}^{n} p_\nu q_{n-\nu}$$

and

$$R_n = \sum_{\nu=o}^{n} r_\nu .$$

Then $r_o > 0$ and $r_n \geq 0$ for $n > 0$.
For a given sequence $\{\zeta_n\}$, let $\zeta(x)$ denote the formal power series $\sum_{n=o}^{\infty} \zeta_n x^n$. Let $\sum_{n=o}^{\infty} c_n$ denote the Cauchy product of the series $\sum_{n=o}^{\infty} a_n$ and $\sum_{n=o}^{\infty} b_n$ i.e.

$$(5.3) \qquad c_n = \sum_{\nu=o}^{n} a_\nu b_{n-\nu} .$$

The following propositions (α), (β) and (γ) about

Cesaro summability have been established; the first two by Winn in [16], and the other by Boyd in [3].

(α) If $\sum_{n=o}^{\infty} a_n = A\ [C,k]_1$ and $\sum_{n=o}^{\infty} b_n = B\ (C,\ell)$ where $k > 0$ $\ell \geq 0$, then $\sum_{n=o}^{\infty} c_n = AB\ (C,k+\ell)$.

(β) If $\sum_{n=o}^{\infty} a_n = A\ [C,k]_1$ and $\sum_{n=o}^{\infty} b_n = B\ [C,\ell]_1$ where $k > 0$ and $\ell > 0$, then $\sum_{n=o}^{\infty} c_n = AB\ [C,\ k+\ell]_1$.

(γ) If $\sum_{n=o}^{\infty} a_n = A\ [C,k]_1$, where $k > 0$, $\sum_{n=o}^{\infty} b_n$ is absolutely convergent and $\sum_{n=o}^{\infty} b_n = B$, then $\sum_{n=o}^{\infty} c_n = AB\ [C,k]_1$.

Propositions (α), (β) and (γ) are special cases of our Theorems 5.2, 5.4 and 5.6 respectively. Multiplication theorems for Nörlund summability have been established by Mears in [10], [11] and [12], and by Polujanova in [14].

A series $\sum_{r=o}^{\infty} a_r$ is said to be *bounded* (N,p_n) if $|t_n| = O(1)$.

THEOREM 5.1

If $p_n > 0$ *for all values of* n, $\lambda \geq 1$, (N,q_n) *is regular,* $\sum_{n=o}^{\infty} a_n = O[N,p_n]_\lambda$ *and* $\sum_{n=o}^{\infty} b_n$ *is bounded* (N,q_n), *then* $\sum_{n=o}^{\infty} c_n = 0\ (N,r_n)$.

PROOF.

Since by Theorem 1 in [1], $[N,p_n]_\lambda \Longrightarrow [N,p_n]_1$ when $p_n > 0$ and $\lambda > 1$, it is sufficient to prove Theorem 5.1 for the case $\lambda = 1$. Let

$$w_n = \frac{1}{Q_n} \sum_{\nu=o}^{n} Q_{n-\nu} b_\nu$$

and

$$v_n = \frac{1}{R_n} \sum_{\nu=o}^{n} R_{n-\nu} c_\nu .$$

Now, $R_n v_n$ is the coefficient of x^n in the series defined by

$$p(x)q(x)(1 - x)^{-1}c(x) = p(x)a(x)(1 - x)^{-1}q(x)b(x)$$

$$= \sum_{n=o}^{\infty} p_n t_n^\Delta x^n \sum_{n=o}^{\infty} Q_n w_n x^n .$$

Thus

$$R_n v_n = \sum_{\nu=o}^{n} p_\nu t_\nu^\Delta Q_{n-\nu} w_{n-\nu}$$

and so,

$$R_n |v_n| \leq \sum_{\nu=o}^{n} p_\nu |t_\nu^\Delta| Q_{n-\nu} |w_{n-\nu}| .$$

We have by hypothesis that

$$\sum_{r=o}^{n} p_r |t_r^\Delta| = o(P_n)$$

and

$$|w_n| = O(1).$$

Thus

$$R_n|v_n| \leq H\sum_{\nu=o}^{n} p_\nu |t_\nu^\Delta| Q_{n-\nu}$$

$$= H\sum_{\nu=o}^{n} q_{n-\nu} \sum_{r=o}^{\nu} p_r |t_r^\Delta|$$

$$= H\sum_{\nu=o}^{n} q_{n-\nu} o(P_\nu).$$

Now,

$$\sum_{\nu=o}^{n} q_{n-\nu} o(P_\nu) = o(R_n),$$

for if $\{\phi_n\}$ is any sequence,

$$\sum_{r=o}^{n} q_{n-r}\phi_r = \sum_{r=o}^{n} q_{n-r} P_r \frac{\phi_r}{P_r}$$

and if $\phi_n = o(P_n)$, then

$$\sum_{r=o}^{n} q_{n-r} P_r \frac{\phi_r}{P_r} = o(R_n)$$

provided that the linear transformation $(c_{n,r})$ with

$$c_{n,r} = q_{n-r}P_r/R_n \quad \text{for } r \leq n$$

and

$$c_{n,r} = 0 \qquad\qquad \text{for } r > n$$

is regular. That this linear transformation is in fact regular, has been proved in Hardy [5] in the proof of his Theorem 17. In this proof Hardy requires (N,q_n) to be regular. Thus $|v_n| = o(1)$, and so

$$\sum_{n=o}^{\infty} c_n = 0 \ (N,r_n)$$

as required.

THEOREM 5.2

If $p_n > 0$ *for all* n, $\lambda \geq 1$, (N,p_n) *and* (N,q_n) *are regular*, $\sum_{n=o}^{\infty} a_n = s \ [N,p_n]_\lambda$ *and* $\sum_{n=o}^{\infty} b_n = t \ (N,q_n)$, *then* $\sum_{n=o}^{\infty} c_n = st \ (N,r_n)$.

PROOF.

If $s = 0$ the result is an immediate consequence of Theorem 5.1, since $\sum_{n=o}^{\infty} b_n$ is a fortiori bounded (N,q_n). Suppose then, that $s \neq 0$. Let

$$a'_o = a_o - s, \ a'_n = a_n \text{ for } n > 0$$

and

$$c'_n = \sum_{\nu=o}^{n} a'_\nu b_{n-\nu},$$

then

$$c'_n = c_n - sb_n.$$

Thus, since

$$\sum_{n=o}^{\infty} a'_n = 0 \ [N,p_n]_1$$

we have

$$\sum_{n=o}^{\infty} c'_n = 0 \ (N,r_n)$$

by Theorem 5.1. Furthermore $(N,q_n) \Longrightarrow (N,r_n)$. This fact is established in Hardy [5] in the proof of his Theorem 17, under the hypothesis that (N,p_n) is regular. Thus

$$\sum_{n=o}^{\infty} b_n = t \ (N,r_n)$$

hence, since

$$\sum_{n=o}^{m} c_n = \sum_{n=o}^{m} c'_n + s \sum_{n=o}^{m} b_n \text{ for } m = 0, 1, 2, \ldots$$

$$\sum_{n=o}^{\infty} c_n = st \ (N,r_n).$$

This completes the proof.

THEOREM 5.3

If $p_n > 0$, $q_n > 0$ *for all values of* n, (N,q_n) *is regular,* $\sum_{n=o}^{\infty} a_n = 0 \ [N,p_n]_\lambda$ *and* $\sum_{\nu=o}^{n} q_\nu |w_\nu^\Delta|^\lambda = O(Q_n)$, *then* $\sum_{n=o}^{\infty} c_n = 0 \ [N,r_n]_\lambda$ *for* $\lambda \geq 1$, *where* $w_n^\Delta = \frac{1}{q_n} \sum_{\nu=o}^{n} q_{n-\nu} b_\nu$.

PROOF.

$$\sum_{n=o}^{\infty} r_n v_n^\Delta x^n = \sum_{n=o}^{\infty} p_n t_n^\Delta x^n \sum_{n=o}^{\infty} q_n w_n^\Delta x^n$$

where

$$v_n^{\Delta} = \frac{1}{r_n} \sum_{\nu=0}^{n} r_{n-\nu} c_\nu .$$

Thus

$$\{r_n |v_n^{\Delta}|\}^{\lambda} \leq \{\sum_{\nu=0}^{n} p_\nu |t_\nu^{\Delta}| q_{n-\nu} |w_{n-\nu}^{\Delta}|\}^{\lambda}.$$

Using Hölder's inequality, we find that for $\lambda \geq 1$

$$\{r_n |v_n^{\Delta}|\}^{\lambda} \leq \{\sum_{\nu=0}^{n} p_\nu |t_\nu^{\Delta}|^{\lambda} q_{n-\nu} |w_{n-\nu}^{\Delta}|^{\lambda}\}\{\sum_{\nu=0}^{n} p_\nu q_{n-\nu}\}^{\lambda-1}.$$

Hence

$$\sum_{n=0}^{m} r_n |v_n^{\Delta}|^{\lambda} \leq \sum_{n=0}^{m} \sum_{\nu=0}^{n} p_\nu |t_\nu^{\Delta}|^{\lambda} q_{n-\nu} |w_{n-\nu}^{\Delta}|^{\lambda}$$

$$= \sum_{\nu=0}^{m} p_\nu |t_\nu^{\Delta}|^{\lambda} \sum_{n=\nu}^{m} q_{n-\nu} |w_{n-\nu}^{\Delta}|^{\lambda}, \text{ for } \lambda \geq 1.$$

Now, by hypothesis,

$$\sum_{\nu=0}^{m} p_\nu |t_\nu^{\Delta}|^{\lambda} = o(P_m)$$

and

$$\sum_{\nu=0}^{m} q_\nu |w_\nu^{\Delta}|^{\lambda} = O(Q_m), \text{ for } \lambda \geq 1.$$

Thus for $\lambda \geq 1$,

$$\sum_{n=0}^{m} r_n |v_n^{\Delta}|^{\lambda} \leq H \sum_{\nu=0}^{m} p_\nu |t_\nu^{\Delta}|^{\lambda} Q_{m-\nu}$$

$$= H \sum_{\nu=0}^{m} q_{m-\nu} \sum_{r=0}^{\nu} p_r |t_r^{\Delta}|^{\lambda}$$

$$= H \sum_{\nu=0}^{m} q_{m-\nu} o(P_\nu)$$

$$= o(R_m)$$

by the regularity of (N,q_n), as in the proof of Theorem 5.1.

Thus

$$\sum_{n=o}^{\infty} c_n = 0 \ [N,r_n]_\lambda, \text{ for } \lambda \geq 1$$

as required.

THEOREM 5.4

If $p_n > 0$, $q_n > 0$ *for all values of* n, (N,p_n) *and* (N,q_n) *are regular,* $\sum_{n=o}^{\infty} a_n = s \ [N,p_n]_\lambda$ *and* $\sum_{n=o}^{\infty} b_n = t \ [N,q_n]_\lambda$, *then* $\sum_{n=o}^{\infty} c_n = st \ [N,r_n]_\lambda$, *for* $\lambda \geq 1$.

PROOF.

When s = t = 0, the result is an immediate consequence of Theorem 5.3. Suppose, therefore that $s \neq 0$, and $t = 0$. Let $a_o' = a_o - s$, $a_n' = a_n$ for $n > 0$ and $c_n' = \sum_{\nu=o}^{n} a_\nu' b_{n-\nu}$, then

$$\sum_{n=o}^{\infty} a_n' = 0 \ [N,p_n]_\lambda. \quad (\lambda \geq 1)$$

So by Theorem 5.3

$$\sum_{n=o}^{\infty} c_n' = 0 \ [N,r_n]_\lambda. \quad (\lambda \geq 1)$$

We now show that

$$\sum_{n=o}^{\infty} b_n = 0 \ [N,r_n]_\lambda. \quad (\lambda \geq 1)$$

Now

$$[N,q_n]_\lambda \Rightarrow [N,r_n]_\lambda \text{ for } \lambda \geq 1$$

because $(N,q_n) \Longrightarrow (N,r_n)$ as was observed in the proof of Theorem 5.2. Now p_n is positive, so since

$$p_o q_n + \dots + p_n q_o = r_n$$

the appropriate form of (2.1) is satisfied, and Theorem 2.2 (page 20) applies yielding $[N,q_n]_\lambda \Longrightarrow [N,r_n]_\lambda$ for $\lambda > 1$. Theorem 2.1 (page 18) yields the case $\lambda = 1$. Thus for $\lambda \geq 1$

$$\sum_{n=o}^{\infty} c_n = 0 \; [N,r_n]_\lambda \text{ as required.}$$

The case $s \neq 0$, $t \neq 0$ is proved similarly by defining $b'_o = b_o - t$, $b'_n = b_n$ for $n > 0$. The proof of Theorem 5.4 is now complete.

THEOREM 5.5

If $p_n > 0$ *for all* n, $\lambda \geq 1$, $\sum_{n=o}^{\infty} a_n = 0 \; [N,p_n]_\lambda$ *and* $\sum_{n=o}^{\infty} b_n$ *is absolutely convergent, then* $\sum_{n=o}^{\infty} c_n = 0 \; [N,p_n]_1$.

PROOF.

Since $[N,p_n]_\lambda \Longrightarrow [N,p_n]_1$ for $\lambda \geq 1$, we need only consider the case $\lambda = 1$. Let

$$v_n^\Delta = \frac{1}{p_n} \sum_{\nu=o}^{n} p_{n-\nu} c_\nu$$

then $p_n v_n^\Delta$ is the coefficient of x^n in the series defined by

$$p(x)c(x) = p(x)a(x)b(x),$$

thus

$$p_n |v_n^\Delta| \leq \sum_{\nu=o}^{n} p_\nu |t_\nu^\Delta| |b_{n-\nu}|$$

hence

$$\sum_{n=o}^{m} p_n |v_n^\Delta| \leq \sum_{n=o}^{m} \sum_{\nu=o}^{n} p_\nu |t_\nu^\Delta| |b_{n-\nu}|$$

$$\leq \sum_{\nu=o}^{m} p_\nu |t_\nu^\Delta| \sum_{n=\nu}^{m} |b_{n-\nu}|$$

$$\leq H \sum_{\nu=o}^{m} p_\nu |t_\nu^\Delta|$$

since $\sum_{n=o}^{\infty} b_n$ is absolutely convergent. The final term is $o(P_m)$ by hypothesis, hence

$$\sum_{n=o}^{m} p_n |v_n^\Delta| = o(P_m)$$

i.e.

$$\sum_{n=o}^{\infty} c_n = 0 \ [N,p_n]_1.$$

This completes the proof.

DEFINITION. *ABSOLUTE REGULARITY.*

We say that a Nörlund method (N,p_n) *is absolutely regular, if* (N,p_n) *is regular and whenever* $\sum_{n=o}^{\infty} a_n$ *is absolutely convergent,* $\sum_{n=o}^{\infty} a_n$ *is summable* $|N,p_n|_1$.

We now state necessary and sufficient conditions for the absolute regularity of the method (N,p_n) as given by Miesner in [13].

(N,p_n) is absolutely regular if and only if

$$P_n^* = O(|P_n|),$$

$$p_n/P_n = o(1)$$

and

$$\sum_{n=k}^{\infty} \left| \frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} \right| \leq H \quad (P_{-1} = 0).$$

where H is independent of k ($k = 1, 2, \ldots$). See also Mears [11].

THEOREM 5.6

If (N,p_n) *is absolutely regular,* $p_n > 0$ *for all* n, $P_n \to \infty$, $\lambda \geq 1$, $\sum_{n=0}^{\infty} a_n = s\ [N,p_n]_\lambda$ *and* $\sum_{n=0}^{\infty} b_n$ *is absolutely convergent, then* $\sum_{n=0}^{\infty} c_n = st\ [N,p_n]_1$, *where* $t = \sum_{n=0}^{\infty} b_n$.

PROOF.

If $s = 0$ the result is an immediate consequence of Theorem 5.5. Suppose therefore that $s \neq 0$, and let $a_0' = a_0 - s$, $a_n' = a_n$ for $n > 0$, then

$$\sum_{n=0}^{\infty} a'_n = 0 \; [N,p_n]_1 .$$

Let

$$c'_n = \sum_{\nu=0}^{n} a'_\nu b_{n-\nu}$$

then

$$c'_n = c_n - sb_n$$

and

$$\sum_{n=0}^{\infty} c'_n = 0 \; [N,p_n]_1$$

by Theorem 5.5. Further $\sum_{n=0}^{\infty} b_n$ is summable $|N,p_n|_1$ by the absolute regularity of (N,p_n) and thus

$$\sum_{n=0}^{\infty} b_n = t \; [N,p_n]_1$$

by Theorem 4.4, since when $p_n > 0$, $P_n \to \infty$ is necessary and sufficient for the regularity of $(\overline{N},p_n)$. Thus

$$\sum_{n=0}^{\infty} c_n = st \; [N,p_n]_1 .$$

This completes the proof.

CHAPTER 6

CONSTRUCTION OF A SCALE OF NORLUND METHODS

We shall restrict ourselves to Nörlund methods (N,p_n) for which $p_o > 0$ and $p_n \geq 0$ for $n > 0$. Given any sequence $\{v_n\}$ we use the notation

(6.1) $$v_n^\alpha = \sum_{r=o}^{n} \varepsilon_r^{\alpha-1} v_{n-r}$$

so that

(6.2) $$\Delta v_n = v_n^{-1}.$$

The following identities are immediate:

(6.3) $$\sum_{r=o}^{n} \varepsilon_r^{\beta-1} v_{n-r}^{\alpha} = v_n^{\alpha+\beta},$$

(6.4) $$P_n^\alpha = p_n^{\alpha+1} = \sum_{r=o}^{n} p_r^\alpha$$

We are going to consider the family of Nörlund methods (N,p_n^α) for $\alpha > -1$, and, when $p_n \neq 0$ for all values of n, we shall allow $\alpha = -1$. In the special case $p_o = 1$, $p_n = 0$ for $n > 0$ we have $p_n^\alpha = \varepsilon_n^{\alpha-1}$, so that (N,p_n^α) is the

Cesaro method (C,α). The methods of summability (N,p_n^{α}) were first introduced by D.Russell in [15]. For the sake of completeness we prove the following theorem given by Russell in [15].

THEOREM 6.1

If either (i) $\beta > \alpha > -1$ *or* (ii) $\beta > \alpha = -1$, $p_n > 0$ *for all values of* n *and* $P_n \to \infty$, *then*

$$(N,p_n^{\alpha}) \Longrightarrow (N,p_n^{\beta}).$$

PROOF.

We use Proposition 1.2 with p_n^{α} and p_n^{β} in place of p_n and q_n. Let $\beta - \alpha = \delta > 0$. Now the appropriate $k_n = \varepsilon_n^{\delta-1} > 0$, and by (6.3),

$$\varepsilon_n^{\delta-1} p_o^{\alpha} + \ldots + \varepsilon_o^{\delta-1} p_n^{\alpha} = p_n^{\beta}$$

So the appropriate form of (1.21) in Proposition 1.2 holds. Since in this case $k_{n-r} \sim k_n$ for each r as n tends to infinity, in order to verify the appropriate form of (1.22) it suffices to show that

$$\varepsilon_n^{\delta-1}/P_n^{\beta} = o(1). \tag{6.5}$$

Now for $\alpha > -1$,

$$\varepsilon_n^{\delta-1}/P_n^{\alpha+\delta} = \varepsilon_n^{\delta-1}/\sum_{r=o}^{n} \varepsilon_r^{\alpha+\delta} p_{n-r} \leq \varepsilon_n^{\delta-1}/\varepsilon_n^{\alpha+\delta} p_o$$

$$= O(1/n^{1+\alpha}) = o(1).$$

Finally when $\alpha = -1$ and $P_n \to \infty$, we may suppose without loss in generality that $0 < \delta \leq 1$ and then obtain

$$\varepsilon_n^{\delta-1}/P_n^{\alpha+\delta} \leq \varepsilon_n^{\delta-1}/\varepsilon_n^{\delta-1}P_n = 1/P_n = o(1).$$

This completes the proof of the theorem.

COROLLARY 6.1.1

If (N,p_n) *is a regular Norlund method, then so also is* (N,p_n^{α}) *for* $\alpha > 0$.

THEOREM 6.2

If either (i) $\alpha > 0$ *or* (ii) $\alpha = 0$, $p_n > 0$ *for all values of* n *and* $P_n \to \infty$, *then*

$$(N,p_n^{\alpha-1}) \Longrightarrow [N,p_n^{\alpha}]_{\lambda} \text{ for } \lambda > 0.$$

PROOF.

Let $s_n \to s$ $(N,p_n^{\alpha-1})$, i.e. let

$$w_n = \frac{1}{P_n^{\alpha-1}} \sum_{r=o}^{n} p_{n-r}^{\alpha-1} s_r \to s$$

then

$$\frac{1}{P_n^{\alpha}} \sum_{r=o}^{n} p_r^{\alpha} |w_r - s|^{\lambda} = o(1) \quad (\lambda > 0) \tag{6.6}$$

if $(\overline{N},p_n^{\alpha})$ is regular, which is the case when either (i) or (ii) is satisfied. Since (6.6) is equivalent to

$$s_n \to s \; [N, p_n^{\alpha}]_{\lambda}, \; (\lambda > 0)$$

the proof is complete.

THEOREM 6.3

For $\alpha \geq 0$ *and* $\lambda \geq 1$,

$$[N, p_n^{\alpha}]_{\lambda} \Longrightarrow (N, p_n^{\alpha})$$

provided $p_n > 0$ *when* $\alpha = 0$.

PROOF.

This result follows immediately from Theorems 4.1 (page 36) and 4.2 (page 37) with p_n^{α} in place of p_n.

THEOREM 6.4

If $(N, p_n) \Longrightarrow (N, q_n)$ *then*

$$(N, p_n^{\alpha}) \Longrightarrow (N, q_n^{\alpha}) \text{ for } \alpha > 0.$$

PROOF.

By hypothesis and Proposition 1.2,

$$|k_o| P_n + \dots + |k_n| P_o \leq HQ_n.$$

Now

$$|k_o| P_n^{\alpha} + \dots + |k_n| P_o^{\alpha}$$

$$= \sum_{r=o}^{n} |k_{n-r}| \sum_{\nu=o}^{r} \varepsilon_{\nu}^{\alpha-1} P_{r-\nu}$$

$$= \sum_{\nu=o}^{n} \varepsilon_\nu^{\alpha-1} \sum_{r=\nu}^{n} |k_{n-r}| P_{r-\nu}$$

$$\leq H \sum_{\nu=o}^{n} \varepsilon_\nu^{\alpha-1} Q_{n-\nu}$$

$$= HQ_n^\alpha .$$

Also

$$|k_{n-r}|/Q_n^\alpha \leq |k_{n-r}|/Q_n = o(1),$$

for each r as $n \to \infty$ by hypothesis and Proposition 1.2. The required result now follows from Proposition 1.2.

COROLLARY 6.4.1

If (N,p_n) *is regular, then*

$$(C,\alpha) \Longrightarrow (N,p_n^\alpha) \text{ for } \alpha > 0.$$

COROLLARY 6.4.2

If (N,p_n) *is regular, then*

$$[C,\alpha]_\lambda \Longrightarrow [N,p_n^\alpha]_\lambda \text{ for } \alpha > 0 \text{ and } \lambda \geq 1.$$

PROOF.

This result follows from Corollary 6.4.1 and Theorem 2.1 (page 18) when $\lambda = 1$ and from Corollary 6.4.1 and Theorem 2.2 (page 20) when $\lambda > 1$, since the appropriate k_n is p_n which is positive.

THEOREM 6.5

If $\beta > \alpha \geq 0$ *and* $\lambda \geq 1$ *then*

$$[N,p_n^{\alpha}]_{\lambda} \Longrightarrow [N,p_n^{\beta}]_{\lambda},$$

provided $p_n > 0$ *when* $\alpha = 0$.

PROOF.

This follows immediately from Theorems 2.1 (page 18) and 6.1 (page 56) in the case $\lambda = 1$, and from Theorems 2.2 (page 20) and 6.1 in the case $\lambda > 1$, because

$$\sum_{r=o}^{n} \varepsilon_r^{\beta-\alpha-1} p_{n-r}^{\alpha} = p_n^{\beta} \text{ and } \varepsilon_r^{\beta-\alpha-1} > 0.$$

The following four corollaries are corollaries to Theorems 5.2, 5.4 and 5.6 (pages 47, 50 and 53 respectively).

COROLLARY 5.2.1

If $\alpha \geq 0$, $\beta > 0$, (N,p_n^{α}) *is regular,*

$$\sum_{n=o}^{\infty} a_n = s\ [N,p_n^{\alpha}]_{\lambda},$$

and

$$\sum_{n=o}^{\infty} b_n = t\ (C,\beta),$$

then

$$\sum_{n=o}^{\infty} c_n = st\ (N,p_n^{\alpha+\beta}) \textit{ for } \lambda \geq 1$$

provided $p_n > 0$ *when* $\alpha = 0$, *where* c_n *is defined by* (5.3).

COROLLARY 5.2.2[1]

If (N, p_n^{β}) *is regular,* $p_n^{\beta} \geq 0$ *for all* n, $\alpha > 0$

$$\sum_{n=0}^{\infty} a_n = s \; [C, \alpha]_{\lambda}$$

and

$$\sum_{n=0}^{\infty} b_n = t \; (N, p_n^{\beta})$$

then

$$\sum_{n=0}^{\infty} c_n = st \; (N, p_n^{\alpha+\beta}) \text{ for } \lambda \geq 1.$$

COROLLARY 5.4.1[1]

If (N, p_n^{β}) *is regular,* $p_n^{\beta} > 0$ *for all* n, $\alpha > 0$

$$\sum_{n=0}^{\infty} a_n = s \; [N, p_n^{\beta}]_{\lambda}$$

and

$$\sum_{n=0}^{\infty} b_n = t \; [C, \alpha]_{\lambda}$$

then

$$\sum_{n=0}^{\infty} c_n = st \; [N, p_n^{\alpha+\beta}]_{\lambda} \text{ for } \lambda \geq 1.$$

COROLLARY 5.6.1

If (N, p_n) *is absolutely regular,* $(\overline{N}, p_n^{\alpha})$ *is regular,*

$\alpha \geq 0$

$$\sum_{n=0}^{\infty} a_n = s \; [N, p_n^{\alpha}]_1$$

and

$$\sum_{n=0}^{\infty} b_n \text{ is absolutely convergent,}$$

1) p_n is real but not necessarily positive, β is real.

then

$$\sum_{n=o}^{\infty} c_n = st\ [N,p_n^{\alpha}]_1,$$

where $t = \sum_{n=o}^{\infty} b_n$, *provided* $p_n \neq 0$ *when* $\alpha = 0$.

PROOF.

We begin by quoting a result proved by McFadden in [9]. It is his Theorem 2.19.

If (1) q_n is non-negative, (2) P_n is non-negative, (3) $P_n/Q_n \leq C < \infty$ and (4) there exists a positive integer N such that k_n is non-negative and non-increasing whenever $n > N$, then every series summable $|N,p_n|_1$ is also summable $|N,q_n|_1$.

By Theorem 5.6 (page 53), it suffices to show that (N,p_n^{α}) is absolutely regular. To do this we show, using the above quoted result of McFadden, that $|N,p_n|_1 \Longrightarrow |N,p_n^{\alpha}|_1$ for $\alpha > 0$. Suppose $0 < \alpha \leq 1$

(1) p_n^{α} is non-negative for $n \geq 0$.

(2) P_n is non-negative for $n \geq 0$.

(3) $P_n/P_n^{\alpha} \leq 1 < \infty$ for $\alpha \geq 0$,

and $\{\varepsilon_n^{\alpha-1}\}$ is non-negative and non-decreasing. Thus, using McFadden's result with p_n^{α}, P_n and P_n^{α} in place of q_n, P_n and Q_n respectively, we find that "k_n" $= \varepsilon_n^{\alpha-1}$, and so if $\sum_{n=o}^{\infty} b_n$ is summable $|N,p_n|_1$, it is also summable $|N,p_n^{\alpha}|_1$ for $0 < \alpha \leq 1$. It follows by induction, that if $\sum_{n=o}^{\infty} b_n$ is summable $|N,p_n|_1$

then it is also summable $|N,p_n^\alpha|_1$ for all $\alpha > 0$. Since (N,p_n) is absolutely regular by hypothesis, if $\sum_{n=o}^{\infty} b_n$ is absolutely convergent then it is also summable $|N,p_n|_1$ and hence is summable $|N,p_n^\alpha|_1$. Thus (N,p_n^α) is absolutely regular for all $\alpha > 0$. This completes the proof.

Flett, in [4] established the following three propositions:

(ρ) If $\alpha > -1$, $\mu > \lambda \geq 1$, $\delta > 1/\lambda - 1/\mu$ then

$$[C,\alpha+1]_\lambda \Longrightarrow [C,\alpha+\delta+1]_\mu .$$

(σ) If $\alpha > -1$, $\lambda \geq 1$ and $\delta > 1/\lambda$ then

$$[C,\alpha+1]_\lambda \Longrightarrow (C,\alpha+\delta).$$

(τ) If $\alpha > -1$, $\mu > \lambda > 1$ and $\delta = 1/\lambda - 1/\mu$ then

$$[C,\alpha+1]_\lambda \Longrightarrow [C,\alpha+\delta+1]_\mu .$$

We shall now proceed to generalise these results to the methods (N,p_n^α). We are unable, however to extend them in complete generality. The technique employed in their proof requires some restriction on the sequence $\{p_n^\alpha\}$. Among the restrictions we shall impose are the following:

(6.7) $$P_n^\alpha/P_n = O(n^\alpha) \text{ for } \alpha \geq -1$$

and

(6.8) $$P_n/P_n^\alpha = O(n^{-\alpha}) \text{ for } \alpha \geq -1.$$

Where $p_n \neq 0$ for all n when $\alpha = -1$. Since the conditions (6.7) and (6.8) may be difficult to verify in a given case we first proceed to show that the conditions

(6.9) $\{p_n\}$ is monotonic

and

(6.10) $$P_{2n} = O(P_n)$$

are sufficient to yield (6.7) and (6.8).

We note that the (C,α) method satisfies (6.9) and (6.10) for $\alpha \geq 0$, so that the above quoted results of Flett will be special cases of our results. Also we note that (6.10) is satisfied whenever $\{p_n\}$ is a monotonic non-increasing sequence. It will emerge from the following lemmas that (6.9) is only required when working in the range $-1 \leq \alpha < 0$.

LEMMA 6.1

If (6.10) *is satisfied and* $\{p_n\}$ *is monotonic non-decreasing, then*

$$P_n \geq H(n + 1)p_{2n}.$$

PROOF.

$$P_n \geq HP_{3n} \geq H(P_{3n} - P_{2n-1})$$

$$= H(p_{3n} + \ldots + p_{2n})$$

$$\geq H(n + 1)p_{2n}.$$

LEMMA 6.2.

If $\{p_n\}$ *is monotonic non-increasing, then*

$$P_n \geq (n + 1)p_n.$$

LEMMA 6.3

If (6.10) *is satisfied, then*

(i) $$P_n^\alpha \geq H\varepsilon_n^\alpha P_n \text{ for } \alpha > -1$$

(ii) $$P_n^\alpha \leq \varepsilon_n^\alpha P_n \text{ for } \alpha \geq 0.$$

(iii) *If further* $\{p_n\}$ *is monotonic, then*

$$P_n^\alpha \leq H_1\varepsilon_n^\alpha P_n \text{ for } \alpha > -1,$$

~~*provided* $p_n \neq 0$ *when* $\alpha = -1$.~~

PROOF.

$$P_n^\alpha = \sum_{\nu=o}^{n} \epsilon_{n-\nu}^\alpha p_\nu .$$

Thus

$$P_n^\alpha \leq \epsilon_n^\alpha P_n \text{ for } \alpha \geq 0,$$

establishing (ii).

(6.11) $$P_n^\alpha \geq \epsilon_n^\alpha P_n \text{ for } -1 < \alpha \leq 0$$

$$P_{2n}^\alpha \geq p_o \epsilon_{2n}^\alpha + \dots + p_n \epsilon_n^\alpha$$

$$\geq \epsilon_n^\alpha P_n \text{ for } \alpha \geq 0$$

$$\geq H_2 \epsilon_{2n}^\alpha P_{2n} .$$

Thus

(6.12) $$P_n^\alpha \geq H \epsilon_n^\alpha P_n \text{ for } \alpha \geq 0.$$

(6.11) and (6.12) establish (i).

Now suppose $-1 < \alpha \leq 0$ and that $\{p_n\}$ is monotonic.

$$P_{2n}^\alpha = \epsilon_{2n}^\alpha p_o + \dots + \epsilon_n^\alpha p_n + \dots + \epsilon_o^\alpha p_{2n}$$

$$\leq \epsilon_{2n}^\alpha p_o + \dots + \epsilon_n^\alpha p_n + q_n \epsilon_n^{\alpha+1}$$

where $q_n = p_{2n}$ if $\{p_n\}$ is monotonic non-decreasing, and $q_n = p_n$ if $\{p_n\}$ is monotonic non-increasing. Thus

$$P_{2n}^\alpha \leq \epsilon_n^\alpha P_n + H_3 q_n (n + 1) \epsilon_n^\alpha$$

since

$$\varepsilon_n^{\alpha+1} = O((n + 1)\varepsilon_n^{\alpha}), \text{ for } \alpha > -1$$

hence

$$P_{2n}^{\alpha} \leq \varepsilon_n^{\alpha}(P_n + H_4 P_n)$$

$$= \varepsilon_n^{\alpha} P_n (1 + H_4)$$

$$\leq \varepsilon_{2n}^{\alpha} P_{2n} H_5,$$

thus

(6.13) $$P_n^{\alpha} \leq H_1 \varepsilon_n^{\alpha} P_n \text{ for } -1 \leq \alpha \leq 0.$$

(6.13) in conjunction with (ii), establishes (iii).

COROLLARY A

If $P_{2n} = O(P_n)$ *and* $\{p_n\}$ *is monotonic, then*

$$P_n^{\alpha}/P_n = O(n^{\alpha})$$

and

$$P_n/P_n^{\alpha} = O(n^{-\alpha})$$

for $\alpha > -1$, ~~*provided* $p_n \neq 0$ *when* $\alpha = -1$.~~

COROLLARY B

If $P_{2n} = O(P_n)$, *then*

$$P_n^{\alpha}/P_n = O(n^{\alpha})$$

and

$$P_n/P_n^{\alpha} = O(n^{-\alpha})$$

for $\alpha \geq 0$.

THEOREM 6.6

If (6.7) ($P_n^{\alpha}/P_n = O(n^{\alpha})$ for $\alpha \geq -1$) *and* (6.8) ($P_n/P_n^{\alpha} = O(n^{-\alpha})$ for $\alpha \geq -1$) *are satisfied, then*

$$[N, p_n^{\alpha+1}]_{\lambda} \Rightarrow [N, p_n^{\alpha+\delta+1}]_{\mu} \text{ for } \alpha \geq -1$$

$\mu > \lambda \geq 1$ *and* $\delta > 1/\lambda - 1/\mu$, *provided* $p_n \neq 0$ *when* $\alpha = -1$.

PROOF.

If $\delta \geq 1$, the result follows without (6.7) and (6.8) from Theorems 6.2 (page 57), 6.3 (page 58) and 6.5 (page 60). We suppose now that $0 < \delta < 1$ and $\mu > \lambda \geq 1$. Now

$$(6.14) \qquad \{P_n^{\alpha+\delta}|t_n^{\alpha+\delta}|\}^{\lambda} \leq \{\sum_{\nu=0}^{n} \varepsilon_{n-\nu}^{\delta-1} P_{\nu}^{\alpha}|t_{\nu}^{\alpha}|\}^{\lambda}$$

$$= \{\sum_{\nu=0}^{n} (\varepsilon_{n-\nu}^{\delta-1})^{p/\mu} (\varepsilon_{n-\nu}^{\delta-1})^{p(1-1/\lambda)} (P_{\nu}^{\alpha})^{1/\lambda} (P_{\nu}^{\alpha})^{1-1/\lambda} |t_{\nu}^{\alpha}|\}^{\lambda}$$

where $1/p = 1 + 1/\mu - 1/\lambda$.

Applying Hölder's inequality in the indicated manner with index λ we obtain

$$\{P_n^{\alpha+\delta}|t_n^{\alpha+\delta}|\}^{\lambda} \leq \{\sum_{\nu=0}^{n} (\varepsilon_{n-\nu}^{\delta-1})^{p\lambda/\mu} P_{\nu}^{\alpha}|t_{\nu}^{\alpha}|^{\lambda}\}\{\sum_{\nu=0}^{n} (\varepsilon_{n-\nu}^{\delta-1})^{p} P_{\nu}^{\alpha}\}^{\lambda-1}$$

$$\leq H\{\sum_{\nu=o}^{n}(\epsilon_{n-\nu}^{\delta-1})^{p\lambda/\mu}P_\nu^\alpha|t_\nu^\alpha|^\lambda\}\{P_n^{\alpha+p(\delta-1)+1}\}^{\lambda-1}$$

since $(\epsilon_{n-\nu}^{\delta-1})^p \leq H\epsilon_{n-\nu}^{p(\delta-1)}$ if $p(\delta-1) > -1$ i.e. if

$$\delta > 1/\lambda - 1/\mu.$$

Therefore

$$\{P_n^{\alpha+\delta}|t_n^{\alpha+\delta}|\}^\mu \leq H_1\{\sum_{\nu=o}^{n}(\epsilon_{n-\nu}^{\delta-1})^{p\lambda/\mu}P_\nu^\alpha|t_\nu^\alpha|^\lambda\}^{\mu/\lambda}\{P_n^{\alpha+p(\delta-1)+1}\}^{\mu-\mu/\lambda}.$$

Now apply Hölder's inequality with index μ/λ and obtain

$$\{P_n^{\alpha+\delta}|t_n^{\alpha+\delta}|\}^\mu$$

$$\leq H_1\{\sum_{\nu=o}^{n}(\epsilon_{n-\nu}^{\delta-1})^p P_\nu^\alpha|t_\nu^\alpha|^\lambda\}\{\sum_{\nu=o}^{n}P_\nu^\alpha|t_\nu^\alpha|^\lambda\}^{\mu/\lambda\ -1}\{P_n^{\alpha+p(\delta-1)+1}\}^{\mu-\mu/\lambda}.$$

Suppose now that $s_n \to 0\ [N,p_n^{\alpha+1}]_\lambda$ i.e.

$$\sum_{\nu=o}^{n}P_\nu^\alpha|t_\nu^\alpha|^\lambda = o(P_n^{\alpha+1}).$$

Then

$$P_n^{\alpha+\delta}|t_n^{\alpha+\delta}|^\mu$$

$$= o\left(\{\sum_{\nu=o}^{n}(\epsilon_{n-\nu}^{\delta-1})^p P_\nu^\alpha|t_\nu^\alpha|^\lambda\}\{\frac{P_n^{\alpha+1}}{P_n^{\alpha+\delta}}\}^{\mu/\lambda\ -1}\{\frac{P_n^{\alpha+p(\delta-1)+1}}{P_n^{\alpha+\delta}}\}^{\mu-\mu/\lambda}\right)$$

Now by (6.7) and (6.8),

$$\{P_n^{\alpha+1}/P_n^{\alpha+\delta}\}^{\mu/\lambda\ -1}\{P_n^{\alpha+p(\delta-1)+1}/P_n^{\alpha+\delta}\}^{\mu-\mu/\lambda}$$

$$\leq H_2 n^{(1-\delta)(\mu/\lambda\ -1)+(p(\delta-1)+1-\delta)(\mu-\mu/\lambda)}$$

$$= H_2 n^{(1-\delta)(\mu/\lambda\ -1+(1-p)(\mu-\mu/\lambda))}.$$

Also

$$\mu/\lambda - 1 + (1 - p)(\mu - \mu/\lambda) = p - 1 > 0.$$

Since $(1 - \delta) > 0$ we have the sequence $\{n^{(1-\delta)(p-1)}\}$ monotonic non-decreasing, thus

$$(6.15)\quad \sum_{n=o}^{m} P_n^{\alpha+\delta}|t_n^{\alpha+\delta}|^{\mu} = o[m^{(1-\delta)(p-1)} \sum_{n=o}^{m} \sum_{\nu=o}^{n} P_\nu^{\alpha}|t_\nu^{\alpha}|^{\lambda}\varepsilon_{n-\nu}^{(\delta-1)p}]$$

$$= o[m^{(1-\delta)(p-1)} \sum_{\nu=o}^{m} P_\nu^{\alpha}|t_\nu^{\alpha}|^{\lambda} \sum_{n=\nu}^{m} \varepsilon_{n-\nu}^{p(\delta-1)}]$$

$$= o[m^{(1-\delta)(p-1)+p(\delta-1)+1}P_m^{\alpha+1}]$$

since $p(\delta-1)+1 > 0$. Thus

$$\frac{1}{P_m^{\alpha+\delta+1}} \sum_{n=o}^{m} P_n^{\alpha+\delta}|t_n^{\alpha+\delta}|^{\mu} = o[m^{(1-\delta)(p-1)+p(\delta-1)+1-\delta}]$$

$$= o(1),$$

i.e.

$$s_n \to 0\ [N,p_n^{\alpha+\delta+1}]_\mu.$$

It remains to show that (6.15) is valid.

We have to show that

$$\sum_{n=o}^{m} o\left(n^{(1-\delta)(p-1)} \sum_{\nu=o}^{n} \varepsilon_{n-\nu}^{p(\delta-1)} P_\nu^\alpha |t_\nu^\alpha|^\lambda\right)$$

$$= o\left(\sum_{n=o}^{m} n^{(1-\delta)(p-1)} \sum_{\nu=o}^{n} \varepsilon_{n-\nu}^{p(\delta-1)} P_\nu^\alpha |t_\nu^\alpha|^\lambda\right).$$

For this, it is enough to show that

$$\sum_{n=o}^{m} n^{(1-\delta)(p-1)} \sum_{\nu=o}^{n} \varepsilon_{n-\nu}^{p(\delta-1)} P_\nu^\alpha |t_\nu^\alpha|^\lambda \to \infty \quad \text{as } m \to \infty.$$

Now

$$\sum_{n=o}^{m} n^{(1-\delta)(p-1)} \sum_{\nu=o}^{n} \varepsilon_{n-\nu}^{p(\delta-1)} P_\nu^\alpha |t_\nu^\alpha|^\lambda$$

$$\geq \sum_{\nu=o}^{m} P_\nu^\alpha |t_\nu^\alpha|^\lambda \sum_{n=\nu}^{m} \varepsilon_{n-\nu}^{p(\delta-1)}$$

$$= \sum_{\nu=o}^{m} P_\nu^\alpha |t_\nu^\alpha|^\lambda \varepsilon_{m-\nu}^{p(\delta-1)+1}$$

$$\geq P_{\nu_o} |t_{\nu_o}^\alpha|^\lambda \varepsilon_{m-\nu_o}^{p(\delta-1)+1},$$

where ν_o is the first value of ν for which $|t_\nu^\alpha| \neq 0$. (If there is no such value of ν then referring to (6.14) we see that $|t_\nu^{\alpha+\delta}| = 0$ for all values of ν and there is nothing further to prove.) We see then that the final term in the above calculation tends to infinity as m tends to infinity, since $p(\delta-1)+1 > 0$. It follows now that if

$$s_n \to s\ [N,p_n^{\alpha+1}]_\lambda,$$

then

$$s_n \to s\ [N,p_n^{\alpha+\delta+1}]_\mu,$$

by a trivial modification of the Remark on page 18. The proof is thus complete.

THEOREM 6.7

If (6.7) ($P_n^\alpha/P_n = O(n^\alpha)$ for $\alpha \geq -1$) *and* (6.8) ($P_n/P_n^\alpha = O(n^{-\alpha})$ for $\alpha \geq -1$) *are satisfied, and* $\alpha \geq -1$, *then*

$$[N,p_n^{\alpha+1}]_\lambda \Longrightarrow (N,p_n^{\alpha+\delta})$$

for $\lambda \geq 1$, *and* $\delta > 1/\lambda$ *provided* $p_n \neq 0$ *when* $\alpha = -1$.

PROOF.

We may suppose that $0 < \delta < 1$.

$$|t_n^{\alpha+\delta}|^\lambda \leq \left(\frac{1}{P_n^{\alpha+\delta}} \sum_{\nu=o}^{n} \varepsilon_{n-\nu}^{\delta-1} P_\nu^\alpha |t_\nu^\alpha|\right)^\lambda.$$

Applying Hölder's inequality, we find that

$$|t_n^{\alpha+\delta}|^\lambda \leq \left(\frac{1}{P_n^{\alpha+\delta}} \sum_{\nu=o}^{n} P_\nu^\alpha |t_\nu^\alpha|^\lambda\right)\left(\frac{1}{P_n^{\alpha+\delta}} \sum_{\nu=o}^{n} (\varepsilon_{n-\nu}^{\delta-1})^{\lambda'} P_\nu^\alpha\right)^{\lambda-1},$$

where $1/\lambda + 1/\lambda' = 1$. Now

$$\frac{1}{(P_n^{\alpha+\delta})^\lambda}\left[\sum_{\nu=o}^{n}(\varepsilon_{n-\nu}^{\delta-1})^{\lambda'}P_\nu^\alpha\right]^{\lambda-1}$$

$$\leq \frac{H}{(P_n^{\alpha+\delta})^\lambda}\left[\sum_{\nu=o}^{n}\varepsilon_{n-\nu}^{\lambda'(\delta-1)}P_\nu^\alpha\right]^{\lambda-1},$$

if $\lambda'(\delta-1) > -1$, i.e. $\delta > 1/\lambda$. Now

$$\frac{1}{(P_n^{\alpha+\delta})^\lambda}\left[\sum_{\nu=o}^{n}\varepsilon_{n-\nu}^{\lambda'(\delta-1)}P_\nu^\alpha\right]^{\lambda-1}$$

$$= (P_n^{\alpha+\lambda'(\delta-1)+1})^{\lambda-1}/(P_n^{\alpha+\delta})^\lambda$$

$$\leq \frac{Hn^{\lambda'(\lambda-1)(\delta-1)+\lambda-1-\lambda\delta-\alpha}}{P_n}$$

$$= \frac{Hn^{(\delta-1)(\lambda-\lambda)-(\alpha+1)}}{P_n}$$

$$\leq H_1/P_n^{\alpha+1}.$$

Thus

$$|t_n^{\alpha+\delta}|^\lambda \leq \frac{1}{P_n^{\alpha+1}}\sum_{\nu=o}^{n}P_\nu^\alpha|t_\nu^\alpha|^\lambda$$

Supposing now that $s_n \to 0\ [N,p_n^{\alpha+1}]_\lambda$, it follows that

$$s_n \to 0\ (N,p_n^{\alpha+\delta}).$$

Thus if $s_n \to s\ [N,p_n^{\alpha+1}]_\lambda$, then $s_n \to s\ (N,p_n^{\alpha+\delta})$ for $\lambda \geq 1$ and $\delta > 1/\lambda$. The proof is thus complete.

LEMMA 6.4

If (6.7) *and* (6.8) *are satisfied, then*

$$\text{(i) } (\overline{N},P_n^{\alpha}) \Longleftrightarrow (C,1) \text{ for } \alpha \geq 0.$$

If further $\{p_n\}$ *is monotonic,* $p_n \neq 0$ *for all values of* n, $P_n \to \infty$ *and* $s_n \geq 0$ *for all* n, *then*

$$\text{(ii) } s_n \to 0 \ (\overline{N},P_n^{\alpha}) \text{ if and only if}$$

$s_n \to 0$ (C,1) *for* $\alpha \geq -1$.

PROOF.

(i) $\alpha \geq 0$. We have $\{P_n^{\alpha}\}$ is monotonic non-decreasing and

$$P_n^{\alpha+1}/P_n^{\alpha} \geq H(n + 1)$$

thus

$$(\overline{N},P_n^{\alpha}) \Longleftrightarrow (C,1)$$

by Proposition 1.4 (page 14).

(ii) $-1 \leq \alpha < 0$. The case $\alpha = -1$ follows from Proposition 1.4. Suppose now that $-1 < \alpha < 0$, and that $s_n \geq 0$ for all n, then

$$\frac{1}{P_n^{\alpha+1}} \sum_{\nu=0}^{n} P_\nu^{\alpha} s_\nu \geq \frac{H}{\varepsilon_n^{\alpha} P_n^{1}} \sum_{\nu=0}^{n} \varepsilon_\nu^{\alpha} P_\nu s_\nu$$

$$\geq \frac{H}{P_n^1} \sum_{\nu=o}^{n} P_\nu s_\nu .$$

Thus $s_n \to 0\ (\overline{N}, P_n^\alpha)$ implies that $s_n \to 0\ (\overline{N}, P_n)$ which implies $s_n \to 0$ (C.1) by result (i).

Finally,

$$\frac{1}{P_n^{\alpha+1}} \sum_{\nu=o}^{n} P_\nu^\alpha s_\nu \leq \frac{H}{\varepsilon_n^{\alpha+1} P_n} \sum_{\nu=o}^{n} \varepsilon_\nu^{\alpha+1} p_\nu s_\nu$$

$$\leq \frac{H}{P_n} \sum_{\nu=o}^{n} p_\nu s_\nu .$$

Thus $s_n \to 0\ (\overline{N}, p_n)$ implies that $s_n \to 0\ (\overline{N}, P_n^\alpha)$. Since also $(C,1) \Longrightarrow (\overline{N}, p_n)$, we have $s_n \to 0$ (C,1) implies that $s_n \to 0\ (\overline{N}, P_n^\alpha)$. This completes the proof.

LEMMA 6.5

Let $0 \leq \mu < 1$, *let* $s_n \geq 0$, $t_n = (n+1)^{-1} \sum_{\nu=o}^{n} s_\nu$ *and* $t = \sup t_n$, *then*

$$\sum_{\nu=o}^{n} (\nu+1)^{-\mu} s_\nu \leq H(n+1)^{1-\mu} t .$$

If further $t_n = o(1)$, *then*

$$\sum_{\nu=o}^{n} (\nu+1)^{-\mu} s_\nu = o(n^{1-\mu}) \text{ as } n \to \infty .$$

PROOF.

Let $S_n = \sum_{\nu=o}^{n} s_\nu$, $S_{-1} = 0$.

$$\sum_{\nu=o}^{n}(\nu + 1)^{-\mu}s_{\nu} = (n + 1)^{-\mu}S_{n} + \sum_{\nu=o}^{n-1}S_{\nu}[(\nu + 1)^{-\mu} - (\nu + 2)^{-\mu}]$$

$$\leq (n + 1)^{1-\mu}t_{n} + \mu\sum_{\nu=o}^{n-1}S_{\nu}(\nu + \theta_{\nu})^{-\mu-1}$$

where $0 < \theta_{\nu} < 1$.

Thus

$$\sum_{\nu=o}^{n}(\nu + 1)^{-\mu}s_{\nu} \leq (n + 1)^{1-\mu}t_{n} + H\mu\sum_{\nu=o}^{n-1}S_{\nu}(\nu + 1)^{-\mu-1}$$

$$\leq (n + 1)^{1-\mu}t_{n} + H\mu\sum_{\nu=o}^{n-1}t_{\nu}(\nu + 1)^{-\mu} \quad \ldots \text{(a)}$$

$$\leq (n + 1)^{1-\mu}t(1 + H_{1}\mu),$$

completing the first part of the proof.

We also have, since $-\mu > -1$,

$$\sum_{\nu=o}^{n-1}t_{\nu}(\nu + 1)^{-\mu} \leq H\sum_{\nu=o}^{n-1}\varepsilon_{\nu}^{-\mu}t_{\nu}$$

$$\leq H\sum_{\nu=o}^{n}\varepsilon_{\nu}^{-\mu}t_{\nu}.$$

Thus

$$\frac{1}{(1 + n)^{1-\mu}}\sum_{\nu=o}^{n-1}t_{\nu}(\nu + 1)^{-\mu} \leq \frac{H}{\varepsilon_{n}^{1-\mu}}\sum_{\nu=o}^{n}\varepsilon_{\nu}^{-\mu}t_{\nu}.$$

Since $(\overline{N},\varepsilon_{\nu}^{-\mu})$ is regular, we have, if $t_{n} = o(1)$,

$$\sum_{\nu=o}^{n-1}t_{\nu}(\nu + 1)^{-\mu} = o(n^{1-\mu})$$

thus

$$\sum_{\nu=0}^{n}(\nu+1)^{-\mu}s_\nu = o(n^{1-\mu})$$

by (a). This completes the proof of Lemma 6.5.

Lemma 6.5 is stated by Flett in [4], although he does not offer a proof of the result.

LEMMA 6.6

If $1 < \lambda < \mu < \infty$, $\delta = 1/\lambda - 1/\mu$, $c_n \geq 0$, *and*

$$C_n = \sum_{\nu<n}(n-\nu)^{\delta-1}c_\nu,$$

then

$$\{\sum C_n^\mu\}^{1/\mu} \leq H\{\sum c_n^\lambda\}^{1/\lambda}.$$

This inequality is given by Hardy, Littlewood and Polya in [7]. See also Flett [4].

The proof of Theorem 6.8 is modelled on Flett's proof of his Theorem 2 in [4].

THEOREM 6.8

If (6.7) *and* (6.8) *are satisfied*, $p_n \neq 0$ *for all values of* n, $P_n \to \infty$ $\{p_n\}$ *is monotonic*, *then for* $\mu > \lambda > 1$ *and* $\delta = 1/\lambda - 1/\mu$,

$$[N, p_n^{\alpha+1}]_\lambda \Longrightarrow [N, p_n^{\alpha+\delta+1}]_\mu,$$

for $\alpha > -1$.

PROOF.

By Lemma 6.4, it is sufficient to show that if

$$\sum_{n=o}^{m} |t_n^\alpha|^\lambda = o(m)$$

then

$$\sum_{n=o}^{m} |t_n^{\alpha+\delta}|^\mu = o(m).$$

Now

$$|t_n^{\alpha+\delta}| \leq \frac{1}{P_n^{\alpha+\delta}} \sum_{\nu=o}^{n} \varepsilon_{n-\nu}^{\delta-1} P_\nu^\alpha |t_\nu^\alpha|$$

$$\leq \frac{1}{P_n^{\alpha+\delta}} \sum_{o\leq\nu\leq \frac{n}{2}} \varepsilon_{n-\nu}^{\delta-1} P_\nu^\alpha |t_\nu^\alpha| + \frac{1}{P_n^{\alpha+\delta}} \sum_{\frac{n}{2} \leq\nu<n} \varepsilon_{n-\nu}^{\delta-1} P_\nu^\alpha |t_\nu^\alpha| + \frac{P_n^\alpha |t_n^\alpha|}{P_n^{\alpha+\delta}}$$

$$= Q_n + R_n + S_n \text{ say.}$$

Thus

$$\sum_{n=o}^{m} |t_n^{\alpha+\delta}|^\mu \leq \sum_{n=o}^{m} (Q_n + R_n + S_n)^\mu.$$

Applying Minkowski's inequality we obtain

$$\left(\frac{1}{m+1} \sum_{n=o}^{m} |t_n^{\alpha+\delta}|^\mu\right)^{1/\mu}$$

$$\leq \left(\frac{1}{m+1} \sum_{n=o}^{m} Q_n^\mu\right)^{1/\mu} + \left(\frac{1}{m+1} \sum_{n=o}^{m} R_n^\mu\right)^{1/\mu} + \left(\frac{1}{m+1} \sum_{n=o}^{m} S_n^\mu\right)^{1/\mu}.$$

To complete the proof it will be sufficient to show that

$$\left(\frac{1}{m+1} \sum_{n=o}^{m} W_n^\mu\right)^{1/\mu} = o(1)$$

in each of the cases where W_n is replaced by Q_n, R_n or S_n.

$$Q_n \leq \frac{H}{\varepsilon_n^{\alpha+\delta} P_n} \sum_{o \leq \nu \leq \frac{n}{2}} \varepsilon_{n-\nu}^{\delta-1} P_\nu |t_\nu^\alpha| \varepsilon_\nu^\alpha$$

$$\leq \frac{H\varepsilon_{n/2}^{\delta-1}}{\varepsilon_n^{\alpha+\delta}} \sum_{o \leq \nu \leq \frac{n}{2}} \varepsilon_\nu^\alpha |t_\nu^\alpha|$$

since $0 < \delta < 1$ and $\{P_n\}$ is a monotonic non-decreasing sequence. Thus

$$Q_n \leq \frac{H}{\varepsilon_n^{\alpha+1}} \sum_{\nu=o}^{n} \varepsilon_\nu^\alpha |t_\nu^\alpha|.$$

Now by Theorem 1 in [1]

$$\sum_{\nu=o}^{n} |t_\nu^\alpha|^\lambda = o(n),$$

implies that

$$\sum_{\nu=o}^{n} |t_\nu^\alpha| = o(n),$$

and thus, since $(C,1) \Longrightarrow (\overline{N}, \varepsilon_n^\alpha)$ for $\alpha > -1$, we have

$$\frac{1}{\varepsilon_n^{\alpha+1}} \sum_{\nu=o}^{n} \varepsilon_\nu^\alpha |t_\nu^\alpha| = o(1)$$

under the hypothesis $s_n \to 0\ [N, p_n^{\alpha+1}]_\lambda$. So

$$Q_n = o(1)$$

hence

$$Q_n^{\mu} = o(1)$$

So by the regularity of the (C,1) method we obtain,

$$\left\{\frac{1}{m+1}\sum_{n=0}^{m} Q_n^{\mu}\right\}^{1/\mu} = o(1) \quad \ldots\ldots\ldots\ldots\ldots\ldots(a)$$

Now

$$R_n \leq \frac{H}{\varepsilon_n^{\alpha+\delta} P_n} \sum_{\frac{n}{2} \leq \nu < n} \varepsilon_{n-\nu}^{\delta-1} \varepsilon_{\nu}^{\alpha} P_{\nu} |t_{\nu}^{\alpha}|$$

$$\leq \frac{H\varepsilon_n^{\alpha}}{\varepsilon_n^{\alpha+\delta}} \sum_{\frac{n}{2} \leq \nu < n} (n-\nu)^{\delta-1} |t_{\nu}^{\alpha}|$$

$$\leq \frac{H}{\varepsilon_n^{\delta}} \sum_{0 \leq \nu < n} (n-\nu)^{\delta-1} |t_{\nu}^{\alpha}|$$

$$\leq H \sum_{0 \leq \nu < n} (n-\nu)^{\delta-1} (\nu+1)^{-\delta} |t_{\nu}^{\alpha}|,$$

since $\{\varepsilon_n^{\delta}\}$ is a monotonic increasing sequence. Now let

$$c_n = (n+1)^{-\delta} |t_n^{\alpha}| \quad \text{for } n \leq m$$

and

$$c_n = 0 \qquad\qquad \text{for } n > m,$$

also let

$$C_n = \sum_{0 \leq \nu < n} (n-\nu)^{\delta-1} c_{\nu}.$$

Then

$$R_n \leq HC_n,$$

whence by Lemma 6.6

$$\left(\sum_{n=o}^{m} R_n^{\mu}\right)^{1/\mu} \leq H\left(\sum_{n=o}^{m} C_n^{\mu}\right)^{1/\mu}$$

$$\leq H\left(\sum_{n=o}^{m} c_n^{\lambda}\right)^{1/\lambda}$$

$$= H\left(\sum_{n=o}^{m} (n+1)^{-\lambda\delta} |t_n^{\alpha}|^{\lambda}\right)^{1/\lambda}.$$

Since $\lambda\delta < 1$ we may apply Lemma 6.5 to obtain that the final term is

$$o(m^{(1-\lambda\delta)(1/\lambda)})$$

$$= o(m^{1/\mu}) \text{ as } m \to \infty \quad \ldots\ldots\ldots\ldots\ldots\ldots(b)$$

Finally

$$S_n \leq H(n+1)^{-\delta} |t_n^{\alpha}|,$$

so

$$\{\sum_{n=o}^{m} S_n^{\mu}\}^{1/\mu} \leq H\{\sum_{n=o}^{m} (n+1)^{-\mu\delta} |t_n^{\alpha}|^{\mu}\}^{1/\mu}$$

$$\leq H\{\sum_{n=o}^{m} (n+1)^{-\lambda\delta} |t_n^{\alpha}|^{\lambda}\}^{1/\lambda}$$

and again applying Lemma 6.5 we obtain

$$\left(\frac{1}{m+1} \sum_{n=o}^{m} S_n^{\mu}\right)^{1/\mu} = o(1) \quad \ldots\ldots\ldots\ldots\ldots\ldots(c)$$

Combining (a), (b) and (c) we obtain the desired conclusion.

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