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STRONG RENEWAL THEOREMS WITH INFINITE MEAN

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Abstract. Let F be a nonarithmetic probability distribution on $(0, \infty)$ and suppose $1-F(t)$ is regularly varying at ∞ with exponent α, $0<\alpha\leqq 1$. Let $U(t)=\sum F^{n*}(t)$ be the renewal function. In this paper we first derive various asymptotic expressions for the quantity $U(t+h)-U(t)$ as $t\to\infty$, $h>0$ fixed. Next we derive asymptotic relations for the convolution $U*z(t)$, $t\to\infty$, for a large class of integrable functions z. All of these asymptotic relations are expressed in terms of the truncated mean function $m(t)=\int_0^t [1-F(x)]\,dx$, t large, and appear as the natural extension of the classical strong renewal theorem for distributions with finite mean. Finally in the last sections of the paper we apply the special case $\alpha=1$ to derive some limit theorems for the distributions of certain waiting times associated with a renewal process.

1. **Principal theorems.** Let F be a probability measure concentrated on $[0, \infty)$(2) and let U be the associated renewal measure defined for any measurable set I by

$$U\{I\} = \sum_0^{\infty} F^{n*}\{I\} \tag{1.1}$$

where F^{n*} denotes the n-fold convolution of F with itself (F^{0*} is the probability measure concentrated at the origin). The series (1.1) converges to a finite number for every bounded I. (For this and other elementary properties of U see [3, VI. 6]; for a probabilistic interpretation of U see §9 in this paper.) We write $U(x)$ for $U\{[0, x]\}$ and we shall henceforth ignore the distinction between U the measure and U the function. (This convention applies to other measures as well.)

The main results of this paper deal primarily with the differences $U(t+h)-U(t)$ for $h>0$ fixed, and $t\to\infty$. The principal assumption is that F has the form

$$1-F(t) = t^{-\alpha}L(t), \qquad t>0, \tag{1.2}$$

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(2) We assume, however, that not all the mass is at the origin.

where $0 \leqq \alpha \leqq 1$ (fixed) and L is a slowly varying function(3). Unless otherwise indicated, we also assume F is nonarithmetic; that is, we exclude the possibility that F concentrates the entire mass on the multiples of some positive real number. For $\alpha \neq 1$, the arithmetic versions of Theorems 1 and 2 below were treated by A. Garsia and J. Lamperti, [5] (nothing was known in the case $\alpha = 1$). See §2(ii) for further discussion. Define the "truncated mean" function

$$m(t) = \int_0^t (1-F(x))\,dx = t(1-F(t)) + \int_0^t xF\{dx\}. \tag{1.3}$$

THEOREM 1. *Let F satisfy* (1.2) *with $\frac{1}{2} < \alpha \leqq 1$. Then for every $h > 0$ and as $t \to \infty$*

$$U(t+h) - U(t) \sim C_\alpha h / m(t) \tag{1.4}$$

where $C_\alpha = [\Gamma(\alpha)\Gamma(2-\alpha)]^{-1}$.

THEOREM 2. *If $0 < \alpha \leqq \frac{1}{2}$ then*

$$\liminf_{t \to \infty} m(t)(U(t+h) - U(t)) = C_\alpha h. \tag{1.5}$$

REMARK. When $\alpha \neq 1$, $m(t) \sim (1-\alpha)^{-1} t^{1-\alpha} L(t)$, $t \to \infty$ (see Lemma 1, §3) and $\Gamma(\alpha)\Gamma(2-\alpha) = \pi(1-\alpha) \csc \pi\alpha$. It follows that (1.4) is equivalent to

$$\lim_{t \to \infty} t^{1-\alpha} L(t)(U(t+h) - U(t)) = \frac{\sin \pi\alpha}{\pi} h. \tag{1.6}$$

The results of Theorems 2, 3, and 4 may be restated in an analogous fashion.

Let z be a nonnegative function on $[0, \infty)$. For $h > 0$ write

$$\sigma^- = h \sum_{k=1}^{\infty} \sup \{z(x) : (k-1)h \leqq x < kh\}$$

and similarly define σ_- with inf in place of sup. Following Feller [3, p. 348], we say that z is *directly Riemann integrable* (dri) if the series defining the upper sum σ^- converges and $\sigma^- - \sigma_- \to 0$ as $h \to 0$. It follows immediately that a dri function is bounded, measurable and (Lebesgue) integrable.

THEOREM 3. *Let z be a nonnegative dri function on* $[0, \infty)$ *which satisfies*

$$z(t) = O(1/t), \qquad t > 0. \tag{1.7}$$

If F has the form (1.2) *with $\frac{1}{2} < \alpha \leqq 1$ then*

$$\int_0^t z(t-y) U\{dy\} \sim \frac{C_\alpha}{m(t)} \int_0^\infty z(x)\,dx. \tag{1.8}$$

(3) A measurable ultimately positive function L on $[0, \infty)$ is regularly varying with exponent ρ if as $t \to \infty$, $L(xt)/L(t) \to x^\rho$ for all $x > 0$. When $\rho = 0$, i.e., $L(xt)/L(t) \to 1$, we also say L is slowly varying. We assume as known the various properties of slowly varying functions as described in [3, pp. 272–274], or in [6]. Note that the function L in (1.2) must be bounded on bounded subintervals of $[0, \infty)$.

THEOREM 4. *Let $z \geqq 0$ be a dri function* (*not necessarily satisfying* (1.7)). *If F satisfies* (1.2) *with $\alpha \neq 0$ then*

$$\liminf_{t\to\infty} m(t) \int_0^t z(t-y)U\{dy\} = C_\alpha \int_0^\infty z(x)\,dx. \tag{1.9}$$

REMARKS. 1. Define a complex valued z to be dri if $|z|$ is dri as defined above. With this definition it follows readily from Theorem 3 that (1.8) holds for any dri z satisfying (1.7).

2. Any piecewise continuous function on $[0, \infty)$ vanishing off a compact interval is dri and certainly satisfies (1.7). In particular, taking $z(x)=1$ for $0 \leqq x \leqq h$, and $z(x)=0$ elsewhere we have by (1.8)

$$U(t+h)-U(t) = \int_0^{t+h} z(t+h-x)U\{dx\} \sim \frac{C_\alpha h}{m(t+h)} \sim C_\alpha \frac{h}{m(t)}$$

as $t \to \infty$. (That $m(t+h) \sim m(t)$, $t \to \infty$, h fixed, follows easily from monotonicity and regular variation of m, see Lemma 1.) Thus Theorem 3 is equivalent to Theorem 1 (we use Theorem 1 to prove Theorem 3). Similarly Theorem 4 (with $0<\alpha \leqq \frac{1}{2}$) is equivalent to Theorem 2.

For a generalization of (1.8) to nonintegrable but regularly varying z see §2(iii).

§§3–8 of this paper are concerned with the proofs of Theorems 1–4. In §9 we give an application of the special case $\alpha=1$ to obtain some curious limit theorems for the spent and residual waiting times of a renewal process.

2. **Notes.** (i) Let m and U be defined as in §1 and let $\hat{m}$ and $\hat{U}$ be their Laplace transforms:

$$\hat{m}(\lambda) = \int_0^\infty e^{-\lambda x}(1-F(x))\,dx, \qquad \hat{U}(\lambda) = \int_0^\infty e^{-\lambda x} U\{dx\}.$$

If in addition $\hat{F}$ is the transform of F then by (1.1) and (1.3)

$$\hat{m}(\lambda) = \frac{1-\hat{F}(\lambda)}{\lambda}, \qquad \hat{U}(\lambda) = \frac{1}{1-\hat{F}(\lambda)}$$

and hence $\hat{U}(\lambda)\hat{m}(\lambda) = 1/\lambda$. Using this relation and Karamata's Tauberian theorem, [3, p. 420], we conclude the following:

THEOREM 5. *Let $0 \leqq \alpha \leqq 1$. Each of statements* (a) *and* (b) *which follow implies the other and both imply the asymptotic relation* (2.1).

(a) *m is regularly varying with exponent $1-\alpha$.*

(b) *U is regularly varying with exponent α.*

$$U(t) \sim [\Gamma(\alpha+1)\Gamma(2-\alpha)]^{-1}(t/m(t)). \tag{2.1}$$

By Lemma 1 statement (a) is true when F satisfies (1.2). (The converse is also true provided $\alpha \neq 1$; if (a) is true for some $0 \leqq \alpha < 1$, then (1.2) holds for some slowly varying L, cf. [3, p. 422].) When $\alpha \neq 1$ in (1.2) we see as in the remark following Theorem 2 that (2.1) is equivalent to

$$U(t) \sim \frac{\sin \pi\alpha}{\pi\alpha} \frac{t^\alpha}{L(t)}, \qquad t \to \infty, \tag{2.2}$$

(when $\alpha = 0$, $(\sin \pi\alpha)/\pi\alpha \equiv 1$). For a proof of (2.2) when $0 < \alpha < 1$ cf. [3, p. 446]. See also Teugels [10]. When $\frac{1}{2} < \alpha \leqq 1$ (2.1) may also be derived from Theorem 1 (1.4). We shall not do this however. Theorem 1 cannot be proved from (2.1).

(ii) Let F be an arithmetic distribution on $(0, \infty)$ which we suppose, without loss of generality, has span 1. (A distribution has span $b > 0$ if it is concentrated on the multiples of b and b is the largest such number.) The renewal measure U defined by (1.1) is also arithmetic with span 1. Denote by f_n and u_n the mass assigned to the integer n by F and U. If F satisfies (1.2), i.e.,

$$1 - F(n) = \sum_{n+1}^{\infty} f_k = n^{-\alpha} L(n)$$

for some $0 < \alpha < 1$ and slowly varying L, then (Lamperti-Garsia, 1962) for $\frac{1}{2} < \alpha < 1$

$$\lim_{n \to \infty} n^{1-\alpha} L(n) u_n = \frac{\sin \pi\alpha}{\pi} \tag{2.3}$$

while for $0 < \alpha \leqq \frac{1}{2}$ the lim must be replaced by lim inf. However (2.3) does hold when $0 < \alpha \leqq \frac{1}{2}$ provided the limit is taken excluding a set of intergers having density 0.

These authors did not consider the case $\alpha = 1$ (nor, for that matter, $\alpha = 0$). The appropriate and true conclusion for $\alpha = 1$ is

$$\lim_{n \to \infty} m(n) u_n = 1 \tag{2.4}$$

where, as before,

$$m(n) = \int_0^n (1 - F(x))\, dx = \sum_{k=1}^{n} \sum_{j=k}^{\infty} f_j \sim \sum_{1}^{n} j f_j, \qquad n \to \infty.$$

The proof of (2.3) and (2.4) starts with the following representation for u_n (see [5] or [8, pp. 98–99]): let $\phi(\theta) = \sum f_k e^{ik\theta}$ and put $W(\theta) = \operatorname{Re}\, [1 - \phi(\theta)]^{-1}$ then provided F has an infinite mean

$$u_n = \frac{1}{\pi} \operatorname{Re} \int_0^\pi \frac{e^{-in\theta}}{1 - \phi(\theta)}\, d\theta = \frac{2}{\pi} \int_0^\pi W(\theta) \cos n\theta \, d\theta \tag{2.5}$$

for $n \geqq 1$. (When the mean μ is finite (2.5) holds with u_n replaced by $u_n - 1/\mu$.) The lack of a similar formula for $U(t+h) - U(t)$ when F is nonarithmetic constitutes the chief difficulty in the proof of Theorem 1.

Here is a brief proof of (2.4): from (2.5)

$$\frac{\pi}{2} u_n = \left(\int_0^{B/n} + \int_{B/n}^{\pi/2} \right) W(\theta) \cos n\theta \, d\theta = J_1 + J_2.$$

As in the latter part of the proof of Theorem 1, see (5.10) and (5.11), we get

$$\lim_{n\to\infty} m(n)J_1 = \pi/2, \qquad \limsup_{n\to\infty} m(n)|J_1| = O(1/B).$$

(The first limit follows directly from Lemma 4, $\alpha=1$.) Hence

$$\lim_{n\to\infty} m(n)u_n = \lim_{B\to\infty}\lim_{n\to\infty} (2/\pi)m(n)(J_1+J_2) = 1.$$

J. A. Williamson [11] has extended the results of Lamperti and Garsia [5] to include distributions not necessarily restricted to the positive integers nor to 1-dimension. He does not, however, consider nonarithmetic distributions. He also gives examples showing that (2.3) and its generalization to d-dimensions cannot hold when $\alpha\leqq d/2$ without making further assumptions on F. In this connection, see also [5, §3.4].

(iii) Suppose the positive function z on $(0,\infty)$ is nondecreasing and regularly varying with exponent $\beta>0$. Consider the integral

$$U^*z(t) = \int_0^t z(t-x)U\{dx\} = \int_0^1 z(t(1-y))U\{tdy\}.$$

By Theorem 5 $U(ty)/U(t)\to y^\alpha$ and it follows that the measure $U\{tdy\}/U(t)$ converges weakly as $t\to\infty$ to the measure with density $\alpha y^{\alpha-1}$. Furthermore

$$f_t(y) = z(t(1-y))/z(t) \to (1-y)^\beta, \qquad t\to\infty \tag{2.6}$$

and the convergence is *uniform* in y, $0\leqq y\leqq 1$, since each $f_t(y)$ is monotone in y and the limit function $(1-y)^\beta$ is continuous. We see therefore that

$$\frac{U^*z(t)}{z(t)U(t)} = \int_0^1 \frac{z(t(1-y))}{z(t)}\cdot\frac{U\{tdy\}}{U(t)} \to \alpha\int_0^1 (1-y)^\beta y^{\alpha-1}\,dy \tag{2.7}$$

as $t\to\infty$. Now $tz(t)\sim(1+\beta)\int_0^t z(x)\,dx$ by Karamata's theorem on regular variation, [3, p. 273]. Hence using (2.1) we see that (2.7) may be put in the equivalent form

$$\int_0^t z(t-x)U\{dx\} \sim \frac{D(\alpha,\beta)}{m(t)}\int_0^t z(x)\,dx, \qquad t\to\infty, \tag{2.8}$$

where

$$D(\alpha,\beta) = \frac{\alpha(1+\beta)}{\Gamma(1+\alpha)\Gamma(2-\alpha)}\cdot\int_0^1 (1-y)^\beta y^{\alpha-1}\,dy = \frac{\Gamma(2+\beta)}{\Gamma(\alpha+\beta+1)\Gamma(2-\alpha)}.$$

Notice that the proof of (2.7) and (2.8) did not depend on the renewal nature, (1.1), of U; (2.8) remains true when $U>0$ is any nondecreasing function regularly varying with exponent α, $0<\alpha\leqq 1$, and m is any function satisfying (2.1).

J. Teugels [10] gave a proof of (2.8) when $z>0$ is nonincreasing and regularly varying with exponent β where $-1<\beta\leqq 0$. The proof is much complicated by the fact that convergence in (2.6) is no longer uniform: when $\beta<0$ the function

$(1-y)^\beta$ is not bounded at $y=1$. (Teugels imposes a supplementary and rather technical condition on U, in addition to regular variation, which seems to me to be unnecessary; compare the proof in Feller [3, p. 447], of a result where similar problems arise.) Again the proof makes no use of the renewal properties of U.

The regular variation of z with exponent $\beta > -1$ and to a lesser extent the monotonicity of z is clearly essential to the proof of (2.8). In particular, the condition $\beta > -1$ cannot be dropped. When $\beta > -1$, the integral $\int_0^t z(x)\,dx$ occurring in (2.8) diverges to ∞ as $t \to \infty$, while for $\beta < -1$, $\int_A^\infty z(x)\,dx$ is finite for all large enough A. In this case, $\beta < -1$, Theorem 3, §1, usually applies and leads to results directly opposed to (2.8). For example, let $z(t) = t^{-5}$, $t > 1$ and $z(t) = 1$, $t \leqq 1$ (z is regularly varying with exponent $\beta = -5$). Then $\int_0^\infty z(x)\,dx = 5/4$ and, provided $\alpha > \frac{1}{2}$, Theorem 3 gives $m(t)U^*z(t) \to C_\alpha 5/4 < \infty$ as $t \to \infty$. On the other hand, if (2.8) were true we would get $m(t)U^*z(t) \to D(\alpha, -5)5/4 = \infty$.

One last remark. As noted before, one could prove Theorem 5 from Theorem 1 (and Lemma 1) at least for $\frac{1}{2} < \alpha \leqq 1$. Since (2.8) depends only on Theorem 5 for the regular variation of U and since Theorem 3 is equivalent to Theorem 1, we see that (2.8) could be derived from Theorem 3, at least in principle, when the only data given, besides the function z, is that U is the renewal function of a distribution F of the form (1.2). In no way, however, can Theorem 3 be proved from (2.8).

(iv) The classical "strong" and "weak" renewal theorems assert respectively

$$U(t+h) - U(t) \to h/\mu \qquad (h > 0) \tag{2.9}$$

$$(1/t)U(t) \to 1/\mu \tag{2.10}$$

as $t \to \infty$, for any (nonarithmetic) distribution F on $(0, \infty)$ with mean $\mu \leqq \infty$ ($1/\mu$ is interpreted as 0 when $\mu = \infty$). Since $m(t) \to \mu$ as $t \to \infty$ we may rewrite (2.9) and (2.10) as

$$U(t+h) - U(t) \sim h/m(t), \qquad U(t) \sim t/m(t)$$

provided $\mu < \infty$. Thus apart from the constant C_α in (1.4) and $[\Gamma(\alpha+1)\Gamma(2-\alpha)]^{-1} = C_\alpha/\alpha$ in (2.1), Theorems 1 and 5 are the natural generalizations of these classical theorems.

(v) It should be pointed out that when $\alpha = 1$ in (1.2), i.e., if F has the form $1 - F(t) = L(t)/t$ for some slowly varying L, then F may or may not have a finite mean. For an example when $\mu < \infty$ consider $L(t) = [\log(t+2)]^{-3} \sim (\log t)^{-3}$. For $\mu = \infty$, consider $L(t) \sim \text{const} > 0$.

As noted in (iv), the classical theorems already imply Theorem 1 (and 5) when $\mu < \infty$. Hence we shall assume from now on that $\mu = \infty$ when $\alpha = 1$ in (1.2).

3. **Properties of distributions satisfying (1.2).** Let F be of the form (1.2) (when $\alpha = 1$ we assume in addition that F have infinite expectation, see §2). Let ϕ be the characteristic function of F:

$$\phi(\theta) = \int_0^\infty e^{ix\theta} F\{dx\}.$$

LEMMA 1. *The function m defined by* (1.3) *is regularly varying with exponent* $1-\alpha$, *and as* $t\to\infty$

$$t(1-F(t))/m(t) = t^{1-\alpha}L(t)/m(t) \to 1-\alpha. \tag{3.1}$$

We shall need the following immediate consequence of Lemma 1: *let* $\eta>0$, *then provided* $\alpha>1/2$ *and* $B>0$,

$$\lim_{t\to\infty} t^{-1}m^2(t)\int_\eta^{t/B} m^{-2}(x)\,dx = [(2\alpha-1)B^{2\alpha-1}]^{-1}. \tag{3.2}$$

NOTE. The restriction to $\alpha>1/2$ in (3.2) partly explains the failure (at least of the proof) of Theorems 1 and 3 when $\alpha\leqq 1/2$. See equation (5.11).

Proof. This lemma is a direct consequence of Karamata's theorem on regularly varying functions, see Feller [3, p. 273]. The relation (3.2) likewise follows from this theorem. To see this, define $Z(x)=m^{-2}(x)$ for $x\geqq\eta$, $Z(x)=0$, $0\leqq x<\eta$. Since m is regularly varying with exponent $1-\alpha$, Z varies regularly with exponent $-2(1-\alpha)=2\alpha-2$. Hence, according to the theorem,

$$\lim_{t\to\infty}\frac{tZ(t)}{\int_0^t Z(x)\,dx} = \lim_{t\to\infty}\frac{(t/B)Z(t/B)}{\int_0^{t/B} Z(x)\,dx} = 1+2\alpha-2 = 2\alpha-1.$$

But $Z(t/B)\sim(1/B)^{2\alpha-2}Z(t)$, $t\to\infty$ (by definition of regular variation). Therefore

$$\int_\eta^{t/B} m^{-2}(x)\,dx \sim (2\alpha-1)^{-1}(t/B)Z(t/B) \sim tm^{-2}(t)/(2\alpha-1)B^{2\alpha-1}$$

as $t\to\infty$ which proves (3.2).

LEMMA 2. *As* $\theta\to 0+$

$$1-\phi(\theta) \sim e^{-i\pi\alpha/2}\Gamma(2-\alpha)\theta m(1/\theta) \qquad (\alpha\neq 0). \tag{3.3}$$

When $\alpha=1$ *we have in addition to* (3.3)

$$\operatorname{Re}(1-\phi(\theta)) \sim \tfrac{1}{2}\pi\theta L(1/\theta), \qquad \theta\to 0+. \tag{3.4}$$

Proof. Suppose $0<\alpha<1$. Then by (3.1) $m(1/\theta)\sim(1-\alpha)^{-1}\theta^{\alpha-1}L(1/\theta)$, $\theta\to 0+$. Since $\Gamma(2-\alpha)/(1-\alpha)=\Gamma(1-\alpha)$ we see that (3.3) is equivalent to

$$1-\phi(\theta) \sim e^{-i\pi\alpha/2}\Gamma(1-\alpha)\theta^\alpha L(1/\theta), \qquad \theta\to 0+. \tag{3.5}$$

Stated in this form (3.3) is well known so we omit the proof. See Garsia and Lamperti [5], or Feller [3, Problems 12 and 13, p. 562]. (There is a slight misprint in the latter reference.)

When $\alpha=1$, (3.3) and (3.4) do not seem to be as well known. Here then is a brief proof. For any A, $\theta>0$, write

$$1-\phi(\theta) = \left(\int_0^{A/\theta}+\int_{A/\theta}^\infty\right)(1-e^{iy\theta})F\{dy\} = J_1+J_2$$

then

$$|J_2| = \left|\int_{A/\theta}^{\infty}(1-e^{iy\theta})F\{dy\}\right| \leqq 2(1-F(A/\theta)),$$

$$J_1 = \int_0^{A/\theta}(1-e^{iy\theta})F\{dy\} = -(1-e^{iA})(1-F(A/\theta))-i\int_0^A e^{ix}(1-F(x/\theta))\,dx.$$

But $1-F(t)=L(t)/t$ with L slowly varying. Hence

$$1-\phi(\theta) = O\left(\frac{\theta L(A/\theta)}{A}\right)-i\int_0^A e^{ix}(1-F(x/\theta))\,dx. \tag{3.6}$$

(The bound in the 0 term is $\leqq 4$ in magnitude.)

We prove (3.3) first. From (3.1) and slow variation of L we get

$$L(A/\theta) \sim L(1/\theta) = o(m(1/\theta)), \qquad \theta \to 0+.$$

Hence from (3.6)

$$\lim_{\theta\to 0+}\frac{1-\phi(\theta)}{\theta m(1/\theta)} = -i\lim_{\theta\to 0+}\int_0^A e^{ix}\left(\frac{1-F(x/\theta)}{\theta m(1/\theta)}\right)dx \tag{3.7}$$

provided the latter limit exists. Now by Lemma 1 m is slowly varying ($\equiv$regularly varying with exponent 0); also $m(0)=0$. Hence, the measure Q_θ on $[0, A]$ with distribution function $Q_\theta(y)=m(y/\theta)/m(1/\theta)$ converges weakly as $\theta\to 0+$ to the measure which assigns unit mass to the origin. Whence, for any continuous g on $[0, A]$

$$\int_0^A g(x)Q_\theta\{dx\} = \int_0^A g(x)\left(\frac{1-F(x/\theta)}{\theta m(1/\theta)}\right)dx \to g(0)$$

as $\theta\to 0+$. Taking $g(x)=e^{ix}$ we see that the right-hand side of (3.7) equals $-i$. This proves (3.3).

Note. The preceding proof requires only minor changes to apply in the case $0<\alpha<1$. In particular, a term $O(1/A^\alpha)$ must be added to the right side of (3.7); also Q_θ converges to the measure with density $(1-\alpha)x^{-\alpha}$. In (3.7) one lets $\theta\to 0+$ followed by $A\to\infty$. The remainder of the proof is then an evaluation of an improper integral.

To prove (3.4), take real parts in (3.6). Then

$$\frac{\operatorname{Re}(1-\phi(\theta))}{\theta L(1/\theta)} = O\left(\frac{1}{A}\right)+\int_0^A\frac{\sin x}{x}\cdot\frac{L(x/\theta)}{L(1/\theta)}\,dx.$$

(The bound in the 0 term is $\leqq 8$ for all $0<\theta\leqq\theta_A$ sufficiently small.) Letting $\theta\to 0+$ and then $A\to\infty$ we see that

$$\lim_{\theta\to 0}\frac{\operatorname{Re}(1-\phi(\theta))}{\theta L(1/\theta)} = \lim_{A\to\infty}\lim_{\theta\to 0}\int_0^A\frac{\sin x}{x}\cdot\frac{L(x/\theta)}{L(1/\theta)}\,dx \tag{3.8}$$

provided the iterated limit exists. Since L is slowly varying, we get from the Karamata theorem mentioned earlier

$$\int_0^t L(u)\,du \sim tL(t), \qquad t \to \infty.$$

Hence, for every $y \geqq 0$,

$$\lim_{\theta\to 0} \int_0^y \frac{L(x/\theta)}{L(1/\theta)}\,dx = \lim_{\theta\to 0} \frac{\theta}{L(1/\theta)} \int_0^{y/\theta} L(u)\,du = y.$$

That is, the measure with density $L(x/\theta)/L(1/\theta)$, $x \geqq 0$, converges weakly as $\theta \to 0$ to Lebesgue measure. Hence for any continuous function f and any compact interval $[0, A]$, say,

$$\lim_{\theta\to 0} \int_0^A f(x)\left(\frac{L(x/\theta)}{L(1/\theta)}\right) dx = \int_0^A f(x)\,dx.$$

Letting $f(x) = (\sin x)/x$ and returning to (3.8) we have

$$\lim_{\theta\to 0+} \frac{\operatorname{Re}(1-\phi(\theta))}{\theta L(1/\theta)} = \lim_{A\to\infty} \int_0^A \frac{\sin x}{x}\,dx = \frac{\pi}{2},$$

which proves (3.4).

For the purposes of the next two lemmas put

$$W(x) = \operatorname{Re}\left(\frac{1}{1-\phi(x)}\right) = \frac{\operatorname{Re}(1-\phi(x))}{|1-\phi(x)|^2}. \tag{3.9}$$

Note that W is positive since $\operatorname{Re}(1-\phi(x)) = \int_0^\infty (1-\cos xt)F\{dt\} > 0$, and symmetric: $W(-x) = W(x)$. Also, W is unbounded (hence undefined) at all x for which $\phi(x) = 1$ (in particular at $x = 0$); at all other x W is continuous.

LEMMA 3. *As* $\theta \to 0+$

$$\int_0^\theta W(x)\,dx \sim \frac{\cos(\pi\alpha/2)}{(1-\alpha)\Gamma(2-\alpha)} \cdot \frac{1}{m(1/\theta)}. \tag{3.10}$$

When $\alpha = 1$ *the constant on the right is replaced by*

$$\frac{\pi}{2} \quad \left(= \lim_{\alpha\to 1} \frac{\cos(\pi\alpha/2)}{(1-\alpha)\Gamma(2-\alpha)}\right).$$

REMARK. The integrability of W over bounded intervals containing the origin is, of course, part of the conclusion. This fact, however, is true for any distribution on $(0, \infty)$ (and for some distributions on the entire line); see [3, p. 578].

Proof. A simple calculation using (3.9) and the asymptotic relations (3.3), (3.4) and (3.5) gives

$$W(x) \sim \frac{k_\alpha L(1/x)}{x^{2-\alpha} m^2(1/x)}, \qquad x \to 0+, \tag{3.11}$$

where k_α is the constant occurring on the right in (3.10) ($k_1=\pi/2$). Next note that the function $1/m(1/x)$, $x>0$ is absolutely continuous on any interval bounded away from 0 and ∞. So, by the chain rule and (1.2)

$$\frac{d}{dx}\left(\frac{1}{m(1/x)}\right)=\frac{1-F(1/x)}{x^2m^2(1/x)}=\frac{L(1/x)}{x^{2-\alpha}m^2(1/x)} \tag{3.12}$$

for almost all x. (The exceptional set is at most countable.)

Consider $0<\varepsilon<1$ fixed but arbitrary. By (3.11) there is a $\lambda=\lambda(\varepsilon)>0$ such that

$$W(x) \gtrless (1\pm\varepsilon)k_\alpha\cdot\frac{L(1/x)}{x^{2-\alpha}m^2(1/x)}$$

whenever $0<x\leqq\lambda$. Integrating these inequalities from $x=\delta$ to $x=\theta$ and using (3.12) yields

$$\int_\delta^\theta W(x)\,dx \lessgtr (1\pm\varepsilon)k_\alpha\left(\frac{1}{m(1/\theta)}-\frac{1}{m(1/\delta)}\right)$$

for $0<\delta\leqq\theta\leqq\lambda$. Now let $\delta\to 0$, then $m(1/\delta)\to\infty$ ($\mu=\infty$ recall), hence

$$(1-\varepsilon)\frac{k_\alpha}{m(1/\theta)} < \int_0^\theta W(x)\,dx < (1+\varepsilon)\frac{k_\alpha}{m(1/\theta)}$$

whenever $0<\theta\leqq\lambda$. This concludes the proof.

By Lemmas 1 and 3, as $t\to\infty$

$$\frac{m(t)}{t}\int_0^\theta W(y/t)\,dy = m(t)\int_0^{\theta/t} W(x)\,dx\to k_\alpha\theta^{1-\alpha} \tag{3.13}$$

for all $\theta>0$ and it follows that the measure with density $q_t(y)=(m(t)/t)W(y/t)$ converges weakly as $t\to\infty$ to a measure which when $\alpha=1$ is concentrated at the origin with total mass $k_1=\pi/2$ and when $0<\alpha<1$ is absolutely continuous with density $(1-\alpha)k_\alpha y^{-\alpha}$. Denote the limit measure by E_α. Then for any function f continuous on a compact interval, $[0, B]$, say,

$$m(t)\int_0^{B/t} f(t\theta)W(\theta)\,d\theta = \int_0^B f(y)q_t(y)\,dy\to\int_0^B f(y)E_\alpha\{dy\},\qquad t\to\infty.$$

Taking $f(y)=\cos y$ we have

Lemma 4. *Let W be given by* (3.9). *Then for any* $B>0$

$$\lim_{t\to\infty} m(t)\int_0^{B/t} W(\theta)\cos t\theta\,d\theta = \frac{\cos(\pi\alpha/2)}{\Gamma(2-\alpha)}\int_0^B\frac{\cos y}{y^\alpha}\,dy,\qquad \alpha\neq 1, \tag{3.14}$$
$$= \pi/2,\qquad \alpha=1.$$

Lemma 5. (i) *For all* $\theta_1\neq\theta_2$

$$|\phi(\theta_2)-\phi(\theta_1)| \leqq 2|\theta_2-\theta_1|m(1/|\theta_2-\theta_1|). \tag{3.15}$$

(ii) *If F is nonarithmetic, then for each $A>0$, there is a number $k>0$, which may depend on A, such that*

$$\theta m(1/\theta) \leqq k|1-\phi(\theta)| \quad \text{for } 0 < \theta \leqq A. \tag{3.16}$$

If F is arithmetic with span h, (3.16) *is true provided $A<2\pi/h=$period of ϕ.*

Proof. (i) Fix $B>0$. Then

$$\begin{aligned} |\phi(\theta_2)-\phi(\theta_1)| &= \left|\left(\int_0^B+\int_B^\infty\right)(e^{ix\theta_2}-e^{ix\theta_1})F\{dx\}\right| \\ &\leqq \int_0^B |e^{ix\theta_2}-e^{ix\theta_1}|F\{dx\}+2(1-F(B)) \\ &\leqq |\theta_2-\theta_1| \int_0^B xF\{dx\}+2(1-F(B)). \end{aligned}$$

But $0\leqq\int_0^B xF\{dx\}=m(B)-B(1-F(B))$ by (1.3). Hence setting $B=|\theta_2-\theta_1|^{-1}$ we get $|\phi(\theta_2)-\phi(\theta_1)|\leqq B^{-1}[m(B)-B(1-F(B))]+2(1-F(B))=B^{-1}m(B)+1-F(B)\leqq 2B^{-1}m(B)$ which proves (3.15). (Note that (1.2) was not used; (3.15) holds for any F on $[0,\infty)$.)

(ii) If F is nonarithmetic then $|1-\phi(\theta)|>0$ for all $\theta\neq 0$. By Lemma 2 as $\theta\to 0+$

$$\theta m(1/\theta)/|1-\phi(\theta)| \to 1/\Gamma(2-\alpha)$$

and it follows that the function

$$\begin{aligned} \beta(\theta) &= \theta m(1/\theta)|1-\phi(\theta)|^{-1}, && \theta\neq 0 \\ &= (\Gamma(2-\alpha))^{-1}, && \theta=0 \end{aligned}$$

is continuous on $[0, A]$. Taking $k=\max\{\beta(\theta) : 0\leqq\theta\leqq A\}$ gives (3.16).

4. An inversion formula for the renewal measure. Define the symmetric renewal measure

$$V\{I\} = \tfrac{1}{2}(U\{I\}+U\{-I\})$$

where U is given by (1.1) and $-I=\{x : -x\in I\}$. In this section we establish the following

FORMULA. Suppose F is nonarithmetic and has an infinite mean. Then for any continuous function g with compact support whose Fourier transform

$$\gamma(x) = \int_{-\infty}^{\infty} e^{ix\theta}g(\theta)\,d\theta \tag{4.1}$$

satisfies

$$\gamma(x) = O(1/x^2), \qquad |x|\to\infty, \tag{4.2}$$

we have

$$\int_{-\infty}^{\infty} e^{-ix\lambda}\gamma(x)V\{t+dx\} = \int_{-\infty}^{\infty} e^{-it\theta}g(\theta+\lambda)\,\mathrm{Re}\left(\frac{1}{1-\phi(\theta)}\right)d\theta \tag{4.3}$$

for all real λ and t. Here, as elsewhere, ϕ is the characteristic function of F. Note that the integral on the right in (4.3) only extends over a bounded interval. For examples of g and γ see §5.

LEMMA 6. *Let γ be any continuous function satisfying* (4.2). *Then for every t the integral*

$$\int_{-\infty}^{\infty} |\gamma(x-t)|\, V\{dx\}$$

is finite.

Proof. Since $\int_{-1}^{1} |\gamma(x-t)|\, V\{dx\}<\infty$ and since $|\gamma(x-t)|$ is bounded by a constant (which may depend on t but not x) times $1/x^2$, it suffices to show

$$\int_{|x|\geq 1} \frac{1}{x^2}\, V\{dx\} = \int_1^{\infty} \frac{1}{x^2}\, U\{dx\} < \infty. \tag{4.4}$$

From (2.10) it follows that $U(x)\leqq k_1 x$ for some constant $k_1<\infty$ and all $x\geqq 1$. Therefore integrating by parts in (4.4) we get

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^2}\, U\{dx\} &= \lim_{A\to\infty} \left(\frac{U(A)}{A^2} - U(1) + 2\int_1^{A} \frac{U(x)}{x^3}\, dx\right) \\ &= -U(1) + 2\int_1^{\infty} \frac{U(x)}{x^3}\, dx \leqq 2k_1 \int_1^{\infty} \frac{1}{x^2}\, dx < \infty\end{aligned}$$

which proves (4.4) and the lemma.

For $0\leqq s<1$ let V_s be the finite symmetric measure

$$V_s\{dx\} = \frac{1}{2}\sum_{n=0}^{\infty} s^n(F^{n*}\{dx\}+F^{n*}\{-dx\})$$

and note that

$$V_s\{I\}\uparrow V\{I\} \quad \text{as } s\uparrow 1 \tag{4.5}$$

for every measurable I bounded or not.

Since

$$\phi(-\theta) = \overline{\phi(\theta)}$$

we have

$$\int_{-\infty}^{\infty} e^{ix\theta} V_s\{dx\} = \frac{1}{2}\sum_{0}^{\infty} s^n(\phi^n(\theta)+\phi^n(-\theta)) = \operatorname{Re}\left(\frac{1}{1-s\phi(\theta)}\right)$$

and an application of Fubini's theorem gives

$$\int_{-\infty}^{\infty} \gamma(x) V_s\{dx\} = \int_{-\infty}^{\infty} g(\theta) \operatorname{Re}\left(\frac{1}{1-s\phi(\theta)}\right) d\theta \qquad (0\leqq s<1)$$

for any (Lebesgue) integrable function g with γ given by (4.1). Replacing g by

$$g_1(\theta) = e^{-it\theta} g(\theta+\lambda)$$

and γ by

$$\gamma_1(x) = \int_{-\infty}^{\infty} e^{ix\theta} g_1(\theta)\, d\theta = e^{-i\lambda(x-t)}\gamma(x-t)$$

we get

$$\int_{-\infty}^{\infty} e^{-i\lambda(x-t)}\gamma(x-t) V_s\{dx\} = \int_{-\infty}^{\infty} e^{-it\theta} g(\theta+\lambda) \operatorname{Re}\left(\frac{1}{1-s\phi(\theta)}\right) d\theta. \tag{4.6}$$

Lemma 7. *For any continuous function h with compact support*

$$\lim_{s\to 1-} \int_{-\infty}^{\infty} h(\theta) \operatorname{Re}\left(\frac{1}{1-s\phi(\theta)}\right) d\theta = \int_{-\infty}^{\infty} h(\theta) \operatorname{Re}\left(\frac{1}{1-\phi(\theta)}\right) d\theta \tag{4.7}$$

provided F is nonarithmetic and has infinite expectation.

Proof. We base the proof on the following proposition due to Feller and Orey [4]:

Proposition. *The measure whose density is*

$$\frac{1}{1+\theta^2} \operatorname{Re}\left(\frac{1}{1-s\phi(\theta)}\right)$$

converges weakly and in variation to a finite measure as $s \to 1-$. In every interval excluding the origin the limit measure is automatically absolutely continuous with density given by

$$\frac{1}{1+\theta^2} \operatorname{Re}\left(\frac{1}{1-\phi(\theta)}\right).$$

If β is the mass assigned to the origin by the limit then $\beta=\pi/\mu>0$ when μ (the mean of F) is finite and $\beta=0$ in case $\mu=\infty$.

We omit the proof. (Besides the Feller-Orey paper, see also Breimann [1, p. 221], and Feller [3, p. 578].) The proposition implies, among other things, that

$$\lim_{s\to 1-} \int_{-\infty}^{\infty} \frac{f(\theta)}{1+\theta^2} \operatorname{Re}\left(\frac{1}{1-s\phi(\theta)}\right) d\theta = \beta f(0) + \int_{-\infty}^{\infty} \frac{f(\theta)}{1+\theta^2} \operatorname{Re}\left(\frac{1}{1-\phi(\theta)}\right) d\theta$$

for every continuous function f with compact support. In our case $\beta=0$, and (4.7) follows by setting $f(\theta)=(1+\theta^2)h(\theta)$.

Proof of formula (4.3). The very strong convergence (4.5) of the measures V_s to V implies

$$\lim_{s\to 1-} \int_{-\infty}^{\infty} f(x) V_s\{dx\} = \int_{-\infty}^{\infty} f(x) V\{dx\} \tag{4.8}$$

for every f integrable with respect to V. (In fact, if f is nonnegative the integral on the left is nondecreasing as a function of s and one can show (4.8) holds even if f is not integrable.)

Suppose now g and γ satisfy (4.1) and (4.2) with g continuous and vanishing off a compact set. Then by Lemma 6

$$e^{-i\lambda(x-t)}\gamma(x-t)$$

is integrable with respect to $V\{dx\}$ for every t and λ. Hence by (4.6) and (4.8)

$$\int_{-\infty}^{\infty} e^{-i\lambda x}\gamma(x)V\{t+dx\} \equiv \int_{-\infty}^{\infty} e^{-i\lambda(x-t)}\gamma(x-t)V\{dx\}$$
$$= \lim_{s\to 1-}\int_{-\infty}^{\infty} e^{-it\theta}g(\theta+\lambda)\operatorname{Re}\left(\frac{1}{1-s\phi(\theta)}\right)d\theta.$$

Formula (4.3) now follows from Lemma 7.

5. **Proof of Theorem 1.**

1°. Introduce measures μ_t, $t>0$, by

$$\mu_t\{I\} = 2m(t)V\{I+t\} = m(t)(U\{I+t\}+U\{-I-t\}) \tag{5.1}$$

where I is measurable and $I+t=\{x : x-t\in I\}$. Since U is concentrated on $[0, \infty)$ it follows by taking $I=[0, h]$ in (5.1) that

$$U(t+h)-U(t) = (1/m(t))\mu_t\{I\}.$$

Therefore to prove Theorem 1 it suffices to show

$$\mu_t\{I\}\to C_\alpha|I|, \qquad t\to\infty, \tag{5.2}$$

for every bounded interval I where $|I|$ denotes the length of I and

$$C_\alpha = [\Gamma(\alpha)\Gamma(2-\alpha)]^{-1}.$$

For each $a>0$ put $\gamma_a(0)=1$ and

$$\gamma_a(x) = 2(1-\cos(ax))/a^2x^2. \tag{5.3}$$

Lemma 8. *Let $\{\mu_t\}$, $t>0$, be a family of measures such that $\mu_t\{I\}<\infty$ for every compact set I and all t. Suppose for some constant C*

$$\lim_{t\to\infty}\int_{-\infty}^{\infty} e^{-i\lambda x}\gamma_a(x)\mu_t\{dx\} = C\int_{-\infty}^{\infty} e^{-i\lambda x}\gamma_a(x)\,dx \tag{5.4}$$

for every $a>0$ and all real λ. Then $C^{-1}\mu_t$ converges weakly to Lebesgue measure: $\mu_t\{I\}\to C|I|$ for every bounded interval I.

(We defer the proof until §6.)

Now γ_a is the Fourier transform (4.1) of the function

$$\begin{aligned} g_a(\theta) &= (1/a)(1-|\theta|/a), && \text{when } |\theta|\leqq a\\ &= 0, && \text{when } |\theta| > a. \end{aligned} \tag{5.5}$$

Whence by the Fourier inversion theorem

$$\int_{-\infty}^{\infty} e^{-i\lambda x}\gamma_a(x)\,dx = 2\pi g_a(\lambda). \tag{5.6}$$

Clearly we may also apply our inversion formula (4.3) to obtain

$$(5.7) \qquad \int_{-\infty}^{\infty} e^{-i\lambda x}\gamma_a(x)\mu_t\{dx\} = 2m(t)\int_{-\infty}^{\infty} e^{-it\theta}g_a(\theta+\lambda)W(\theta)\,d\theta$$

where $W(\theta)=\operatorname{Re}[1-\phi(\theta)]^{-1}$. Note that the integral on the right extends from $\theta=-a-\lambda$ to $\theta=a-\lambda$. From (5.6) and (5.7) we see that (5.4) in our case is equivalent to

$$(5.8) \qquad \lim_{t\to\infty} m(t)\int_{-\infty}^{\infty} e^{-it\theta}g_a(\theta+\lambda)W(\theta)\,d\theta = \pi C g_a(\lambda)$$

and, by Lemma 8, the proof of (5.2) (and Theorem 1) will be completed when we establish (5.8), with $C=C_\alpha$ for every $a>0$ and all real λ.

2°. Let $B>1$ be fixed but otherwise arbitrary, and write the integral in (5.8) as the sum J_1+J_2 where

$$(5.9) \qquad \begin{aligned} J_1(t,b) &= \int_{-B/t}^{B/t} e^{-it\theta}g_a(\theta+\lambda)W(\theta)\,d\theta \quad \text{and} \\ J_2(t,B) &= \int_{|\theta|>B/t} e^{-it\theta}g_a(\theta+\lambda)W(\theta)\,d\theta \\ &= \int_{B/t}^{A} [e^{-it\theta}g_a(\theta+\lambda)+e^{it\theta}g_a(\theta-\lambda)]W(\theta)\,d\theta, \\ A &= \max\{a+\lambda, a-\lambda\}. \end{aligned}$$

(The last integral follows by making the substitution $\theta\to-\theta$ in the integral $\int_{-\infty}^{-B/t}$, using the evenness of the functions g_a and W and noting that g_a vanishes outside the interval $(-a,a)$.) We will show

$$(5.10) \qquad \begin{aligned} \lim_{t\to\infty} m(t)J_1(t,B) &= g_a(\lambda)\frac{2\cos\pi\alpha/2}{\Gamma(2-\alpha)}\int_0^B \frac{\cos x}{x^\alpha}\,dx, \qquad \alpha\neq 1 \\ &= \pi g_a(\lambda), \qquad \alpha=1 \end{aligned}$$

and

$$(5.11) \qquad \limsup_{t\to\infty} m(t)|J_2(t,B)| = O\left(\frac{1}{B^{2\alpha-1}}\right), \qquad \frac{1}{2}<\alpha\leqq 1$$

which lead directly to (5.8).

3°. **Proof of (5.10).** It is clear from (5.5) that

$$(5.12) \qquad |g_a(\theta_2)-g_a(\theta_1)| \leqq (1/a^2)|\theta_2-\theta_1|$$

for all θ_1, θ_2. Hence

$$\begin{aligned} m(t)\left|J_1(t,B)-g_a(\lambda)\int_{-B/t}^{B/t} e^{-it\theta}W(\theta)\,d\theta\right| &\leqq m(t)\int_{-B/t}^{B/t}|g_a(\theta+\lambda)-g_a(\lambda)|W(\theta)\,d\theta \\ &\leqq \frac{2B}{a^2}\cdot\frac{m(t)}{t}\int_0^{B/t} W(\theta)\,d\theta = O\left(\frac{1}{t}\right) \end{aligned}$$

where the $O(1/t)$ follows from (3.10) and Lemma 1. Thus

$$\lim_{t\to\infty} m(t)J_1(t,B) = g_a(\lambda)\lim_{t\to\infty} m(t)\int_{-B/t}^{B/t} e^{-it\theta}W(\theta)\,d\theta$$
$$= 2g_a(\lambda)\lim_{t\to\infty} m(t)\int_0^{B/t} W(\theta)\cos t\theta\,d\theta$$

and (5.10) now follows from Lemma 4.

4°. **Proof of (5.11).** Let

$$h_1(\theta) = e^{-it\theta}g_a(\theta+\lambda)+e^{it\theta}g_a(\theta-\lambda),$$
$$h_2(\theta) = e^{-it\theta}g_a(\theta+\pi/t+\lambda)+e^{it\theta}g_a(\theta+\pi/t-\lambda).$$

Then $h_1(\theta+\pi/t)=-h_2(\theta)$ and making the change of variables $\theta\to\theta+\pi/t$ in (5.9) gives

$$J_2(t,B) = \int_{B/t}^{A} h_1(\theta)W(\theta)\,d\theta = \int_{(B-\pi)/t}^{A} -h_2(\theta)W(\theta+\pi/t)\,d\theta$$

(note that the integrand in the last written integral vanishes for $A-\pi/t\leqq\theta$). Adding these integrals we get

$$(5.13)\quad 2J_2 = -\int_{(B-\pi)/t}^{B/t} h_2(\theta)W\left(\theta+\frac{\pi}{t}\right)d\theta+\int_{B/t}^{A}\left[h_1(\theta)W(\theta)-h_2(\theta)W\left(\theta+\frac{\pi}{t}\right)\right]d\theta.$$

Now $|h_j(\theta)|\leqq 2/a$ and from (5.12) we have

$$|h_1(\theta)-h_2(\theta)| \leqq \left|g_a(\theta+\lambda)-g_a\left(\theta+\lambda+\frac{\pi}{t}\right)\right|+\left|g_a(\theta-\lambda)-g_a\left(\theta-\lambda+\frac{\pi}{t}\right)\right| \leqq \frac{2\pi}{a^2t}.$$

Thus

$$\left|h_1(\theta)W(\theta)-h_2(\theta)W\left(\theta+\frac{\pi}{t}\right)\right| \leqq |h_1(\theta)-h_2(\theta)|\,W(\theta)+\left|W(\theta)-W\left(\theta+\frac{\pi}{t}\right)\right||h_2(\theta)|$$
$$\leqq \frac{2\pi}{a^2t}W(\theta)+\frac{2}{a}\left|W\left(\theta+\frac{\pi}{t}\right)-W(\theta)\right|.$$

Applying these inequalities in (5.13) gives

$$(5.14)\quad \begin{aligned}|J_2| &\leqq \frac{1}{a}\int_{(B-\pi)/t}^{B/t} W\left(\theta+\frac{\pi}{t}\right)d\theta+\frac{\pi}{a^2t}\cdot\int_{B/t}^{A} W(\theta)\,d\theta\\ &\quad+\frac{1}{a}\int_{B/t}^{A}\left|W\left(\theta+\frac{\pi}{t}\right)-W(\theta)\right|d\theta.\end{aligned}$$

From Lemma 3 it is clear that

$$\lim_{t\to\infty} m(t)\int_{(B-\pi)/t}^{B/t} W\left(\theta+\frac{\pi}{t}\right)d\theta = k_\alpha[(B+\pi)^{1-\alpha}-B^{1-\alpha}] = O\left(\frac{1}{B^\alpha}\right).$$

Also, since W is integrable on $[0, A]$, $A<\infty$,

$$\frac{\pi}{a^2}\cdot\frac{m(t)}{t}\cdot\int_{B/t}^{A} W(\theta)\,d\theta = O\left(\frac{m(t)}{t}\right)\to 0 \quad\text{as } t\to\infty.$$

(That $m(t)/t \to 0$, $t \to \infty$, follows from Lemma 1, §3, in our case, but is true for any F on $[0, \infty)$ with m given by (1.3).) Hence from (5.14)

$$\limsup_{t\to\infty} m(t)|J_2(t, B)| = a^{-1} \limsup_{t\to\infty} m(t) \int_{B/t}^{A} \left| W\left(\theta+\frac{\pi}{t}\right) - W(\theta) \right| d\theta + O\left(\frac{1}{B^\alpha}\right).$$

But $O(B^{-\alpha}) = O(B^{1-2\alpha})$ ($B > 1$, $0 \leqq \alpha \leqq 1$), so the proof of (5.11) will be complete when we show

$$\limsup_{t\to\infty} m(t) \int_{B/t}^{A} \left| W\left(\theta+\frac{\pi}{t}\right) - W(\theta) \right| d\theta = O\left(\frac{1}{B^{2\alpha-1}}\right). \tag{5.15}$$

By Lemma 5 (i) we get

$$\left| W\left(\theta+\frac{\pi}{t}\right) - W(\theta) \right| = \left| \operatorname{Re} \frac{\phi(\theta+\pi/t)-\phi(\theta)}{[1-\phi(\theta+\pi/t)][1-\phi(\theta)]} \right| \leqq \frac{2(\pi/t)m(t/\pi)}{|1-\phi(\theta+\pi/t)||1-\phi(\theta)|}.$$

Applying this estimate and the Cauchy-Schwarz inequality to the integral in (5.15) gives

$$\begin{aligned} &\int_{B/t}^{A} \left| W\left(\theta+\frac{\pi}{t}\right) - W(\theta) \right| d\theta \\ &\leqq \frac{2\pi}{t} m\left(\frac{t}{\pi}\right) \left(\int_{B/t}^{A} \frac{d\theta}{|1-\phi(\theta+\pi/t)|^2}\right)^{1/2} \left(\int_{B/t}^{A} \frac{d\theta}{|1-\phi(\theta)|^2}\right)^{1/2} \\ &< 8\frac{m(t)}{t} \int_{B/t}^{2A} \frac{d\theta}{|1-\phi(\theta)|^2} \qquad (\pi/t \leqq A). \end{aligned} \tag{5.16}$$

Again by Lemma 5(ii) there is a constant $k < \infty$ such that

$$1/|1-\phi(\theta)| \leqq k/\theta m(1/\theta)$$

for $0 < \theta \leqq 2A$. Consequently

$$\int_{B/t}^{2A} \frac{d\theta}{|1-\phi(\theta)|^2} \leqq k^2 \int_{B/t}^{2A} \frac{d\theta}{\theta^2 m^2(1/\theta)} = k^2 \int_{\eta}^{t/B} \frac{dx}{m^2(x)} \tag{5.17}$$

where $\eta = 1/2A$. Combining (5.16) and (5.17) we get

$$\begin{aligned} \limsup_{t\to\infty} m(t)\cdot \int_{B/t}^{A} \left| W\left(\theta+\frac{\pi}{t}\right) - W(\theta) \right| d\theta &\leqq 8k^2 \lim_{t\to\infty} \frac{m^2(t)}{t} \int_{\eta}^{t/B} \frac{dx}{m^2(x)} \\ &= \frac{1}{(2\alpha-1)B^{2\alpha-1}} \qquad (\alpha > \tfrac{1}{2}) \end{aligned}$$

where the last equality comes from (3.2). This completes the proof of (5.15) and hence of (5.11).

5°. The proof of (5.8) with $C = C_\alpha = [\Gamma(\alpha)\Gamma(2-\alpha)]^{-1}$ is now almost immediate. Let

$$\begin{aligned} \Delta(t) &= \left| m(t) \int_{-\infty}^{\infty} e^{-it\theta} g_a(\theta+\lambda) W(\theta)\, d\theta - \pi C_\alpha g_a(\lambda) \right| \\ &= |m(t)(J_1+J_2) - \pi C_\alpha g_a(\lambda)| \end{aligned}$$

and suppose $\alpha \neq 1$. Then by (5.10) and (5.11)

$$(5.18)\quad \begin{aligned} \limsup_{t\to\infty} \Delta(t) &\leqq \lim_{t\to\infty} \left| m(t)J_1 - \frac{\pi g_a(\lambda)}{\Gamma(\alpha)\Gamma(2-\alpha)} \right| + \limsup_{t\to\infty} m(t)|J_2| \\ &= \frac{g_a(\lambda)}{\Gamma(2-\alpha)} \cdot \left| 2\cos\left(\frac{\pi\alpha}{2}\right) \int_0^B \frac{\cos x}{x^\alpha}\,dx - \frac{\pi}{\Gamma(\alpha)} \right| + O\left(\frac{1}{B^{2\alpha-1}}\right). \end{aligned}$$

Now as $B \to \infty$, $\int_0^B x^{-\alpha}\cos x\,dx \to \sin(\pi\alpha/2)\Gamma(1-\alpha)$, hence

$$\lim_{B\to\infty} \left| 2\cos\left(\frac{\pi\alpha}{2}\right) \int_0^B \frac{\cos x}{x^\alpha}\,dx - \frac{\pi}{\Gamma(\alpha)} \right| = \left| \sin(\pi\alpha)\Gamma(1-\alpha) - \frac{\pi}{\Gamma(\alpha)} \right| = 0.$$

Therefore taking the limit in (5.18) as $B \to \infty$ we get

$$\limsup_{t\to\infty} \Delta(t) = \lim_{B\to\infty} \limsup_{t\to\infty} \Delta(t) = 0$$

which proves (5.8) when $\alpha \neq 1$. When $\alpha = 1$ the proof of (5.8), with $C = C_1 = 1$, from (5.10) and (5.11) is even simpler so we omit it. Theorem 1 now follows from Lemma 8.

6. **Proof of Lemma 8.** There is no loss in generality in supposing $C=1$. Taking $\lambda = 0$ in (5.4) and (5.6) we see that as $t \to \infty$

$$\Delta_t(a) = \int_{-\infty}^{\infty} \gamma_a(x)\mu_t\{dx\} \to \int_{-\infty}^{\infty} \gamma_a(x)\,dx = \frac{2\pi}{a} > 0.$$

Hence (5.4) implies that the characteristic function of the probability measure

$$P_t\{dx\} = \frac{1}{\Delta_t(a)} \gamma_a(x)\mu_t\{dx\}$$

converges pointwise to the characteristic function of the probability measure

$$P\{dx\} = (a/2\pi)\gamma_a(x)\,dx.$$

Consequently, by the continuity theorem for characteristic functions P_t converges weakly to P as $t \to \infty$. Whence

$$(6.1)\qquad \lim_{t\to\infty} \int_{-\infty}^{\infty} B(x)\gamma_a(x)\mu_t\{dx\} = \int_{-\infty}^{\infty} B(x)\gamma_a(x)\,dx$$

for every bounded continous function B on R^1 and for every $a > 0$.

For any continuous function f with compact support, write

$$\lambda_t(f) = \int_{-\infty}^{\infty} f(x)\mu_t\{dx\}, \qquad \lambda(f) = \int_{-\infty}^{\infty} f(x)\,dx.$$

Let I be a bounded interval and let $\varepsilon > 0$ be arbitrary but fixed. We can find continuous functions f^+ and f^- both with compact support such that

(i) $0 \leqq f^- \leqq 1$, $f^-(x) = 0$ for $x \notin I$,

(ii) $|I| \leqq \lambda(f^-) + \varepsilon$,

(iii) $f^+ \geqq 0, f^+(x)=1$ for x in I,

(iv) $\lambda(f^+) \leqq |I|+\varepsilon$.

Now choose $a>0$ so small that

$$f^+(x) = f^-(x) = 0 \quad \text{when } |x| \geqq \pi/4a.$$

Then since

$$\gamma_a(x) = 2\left(\frac{1-\cos ax}{a^2x^2}\right) > 0 \quad \text{for } |x| < \pi/2a$$

it follows that $B^+ = f^+/\gamma_a$ and $B^- = f^-/\gamma_a$ are continuous functions on R^1 with compact support (hence bounded). Therefore by (6.1)

$$\text{(6.2)} \qquad \lambda_t(f^\pm) = \int_{-\infty}^{\infty} B^\pm(x)\gamma_a(x)\mu_t\{dx\} \to \int_{-\infty}^{\infty} B^\pm(x)\gamma_a(x)\,dx = \lambda(f^\pm).$$

From (i) and (iii) it is clear that

$$\lambda_t(f^-) \leqq \mu_t\{I\} \leqq \lambda_t(f^+)$$

for all $t>0$. Letting $t \to \infty$ and using (6.2) we get

$$\lambda(f^-) \leqq \liminf \mu_t\{I\} \leqq \limsup \mu_t\{I\} \leqq \lambda(f^+),$$

and hence by (ii) and (iv)

$$|I|-\varepsilon \leqq \liminf \mu_t\{I\} \leqq \limsup \mu_t\{I\} \leqq |I|+\varepsilon.$$

Since this holds for every $\varepsilon>0$ it follows that

$$\mu_t\{I\} \to |I|, \qquad t \to \infty,$$

which completes the proof.

7. **Proof of Theorem 2.**

1°. Our first task is to show

$$\text{(7.1)} \qquad \liminf_{t\to\infty} m(t)(U(t+h)-U(t)) \geqq C_\alpha h \qquad (h > 0),$$

or, equivalently,

$$\text{(7.2)} \qquad \liminf_{t\to\infty} t^{1-\alpha}L(t)(U(t+h)-U(t)) \geqq \frac{\sin \pi\alpha}{\pi} h.$$

(See remark following the statement of Theorem 2.)

Condition (1.2) with $0<\alpha<1$ is necessary and sufficient for F to be in the domain of attraction of the unique (apart from a scale factor) stable distribution with exponent α concentrated on $[0, \infty)$. Thus if a sequence $\{B_n\}$ is chosen so that $0<B_n \uparrow \infty$ and

$$n(1-F(B_n)) \equiv nB_n^{-\alpha}L(B_n) \to 1$$

as $n \to \infty$, then

$$\text{(7.3)} \qquad F^{n*}(B_n x) \to \int_0^x q_\alpha(y)\,dy \qquad (n \to \infty,\ x \geqq 0)$$

where $q_\alpha > 0$ and satisfies

$$\int_0^\infty e^{-\lambda y} q_\alpha(y)\, dy = \exp[-\lambda^\alpha \Gamma(1-\alpha)], \qquad \lambda \geqq 0.$$

In addition to (7.3) a local limit theorem for nonarithmetic distributions due to C. Stone [9] implies the somewhat stronger result

$$F^{k*}(t+h) - F^{k*}(t) = (h/B_k) q_\alpha(t/B_k) + \delta_k/B_k \tag{7.4}$$

where $\delta_k \to 0$ as $k \to \infty$ *uniformly* in $t > 0$ ((7.3) only allows $F^{k*}(t+h) - F^{k*}(t) \sim hB_k^{-1} q_\alpha(tB_k^{-1})$ for t and h fixed). Using (7.4) we prove (7.2) almost exactly as Garsia and Lamperti [5] prove the analogous inequality in the arithmetic case. Thus from (1.1) and (7.4)

$$\begin{aligned} U(t+h) - U(t) &> \sum_{k=n}^{r} (F^{k*}(t+h) - F^{k*}(t)) \\ &= h \sum_n^r \frac{1}{B_k} q_\alpha\left(\frac{t}{B_k}\right) + \sum_n \frac{\delta_k}{B_k}. \end{aligned}$$

Let $0 < A < C < \infty$, and choose $n = [At^\alpha/L(t)]$, $r = [Ct^\alpha/L(t)]$. Then, as in [5], we have both

$$t^{1-\alpha} L(t) \sum_n^r \frac{\delta_k}{B_k} = o(1), \qquad t \to \infty$$

and, writing $x_k = kL(t)/t^\alpha$, $n \leqq k \leqq r$,

$$\begin{aligned} t^{1-\alpha} L(t) \sum_n^r \frac{1}{B_k} q_\alpha\left(\frac{t}{B_k}\right) &\sim \sum_{A \leqq x_k \leqq C} x_k^{-1/\alpha} q_\alpha(x_k^{-1/\alpha})(x_{k+1} - x_k) \\ &\to \int_A^C x^{-1/\alpha} q_\alpha(x^{-1/\alpha})\, dx \end{aligned}$$

as $t \to \infty$. Hence for any $\varepsilon > 0$

$$t^{1-\alpha} L(t)(U(t+h) - U(t)) \geqq \int_A^C x^{-1/\alpha} q_\alpha(x^{-1/\alpha})\, dx - \varepsilon$$

for all t sufficiently large. In other words

$$\liminf_{t \to \infty} t^{1-\alpha} L(t)(U(t+h) - U(t)) \geqq \int_A^C x^{-1/\alpha} q_\alpha(x^{-1/\alpha})\, dx,$$

and (7.2) now follows by letting $A \to 0$, $C \to \infty$ and noting

$$\int_0^\infty x^{-1/\alpha} q_\alpha(x^{-1/\alpha})\, dx = \alpha \int_0^\infty y^{-\alpha} q_\alpha(y)\, dy = \frac{\sin \pi\alpha}{\pi}.$$

2°. To complete the proof of Theorem 2 we need the following lemma (also needed in the proof of Theorem 3).

LEMMA 9. *Let z be any nonnegative integrable (but not necessarily dri) function on* $[0, \infty)$. *Then*

$$\liminf_{t \to \infty} m(t) \int_0^t z(t-y) U\{dy\} \leqq C_\alpha \int_0^\infty z(x)\, dx \qquad (0 < \alpha \leqq 1). \tag{7.5}$$

To finish the proof of Theorem 2 we set $z(x)=1$ for $0 \leqq x \leqq h$, $z(x)=0$ elsewhere. Noting that $m(t+h) \sim m(t)$ as $t \to \infty$ we get from (7.5)

$$\begin{aligned} \liminf_{t\to\infty} m(t)(U(t+h)-U(t)) &= \liminf_{t\to\infty} m(t+h)U^*z(t+h) \\ &\leqq C_\alpha \int_0^\infty z(x)\,dx = C_\alpha h. \end{aligned} \tag{7.6}$$

Together (7.1) and (7.6) give (1.5).

Proof of Lemma 9. Let $v(t)=U^*z(t)=\int_0^t z(t-x)U\{dx\}$. Then

$$\hat{v}(\lambda) = \int_0^\infty e^{-\lambda x} v(x)\,dx = \left(\int_0^\infty e^{-\lambda x} z(x)\,dx\right)\hat{U}(\lambda) = \hat{z}(\lambda)\hat{U}(\lambda)$$

where $\hat{U}$ is defined as in §2(i). Since U is regularly varying with exponent α we have

$$\hat{U}(\lambda) \sim \Gamma(\alpha+1)U(1/\lambda) \quad \text{as } \lambda \to 0+$$

by Theorem 1 in [3, p. 420]. Now $\hat{z}(0)=\int_0^\infty z(x)\,dx<\infty$ and it follows that

$$\hat{v}(\lambda) \sim \hat{z}(0)\Gamma(\alpha+1)U(1/\lambda), \qquad \lambda \to 0+$$

which, by the converse of the same Theorem 1 in [3], is the same as

$$\int_0^t v(x)\,dx \sim \hat{z}(0)U(t), \qquad t \to \infty. \tag{7.7}$$

Now by Theorem 5 in §2

$$U(t) \sim (\Gamma(\alpha+1)\Gamma(2-\alpha))^{-1} t/m(t) = (C_\alpha/\alpha)t/m(t) \tag{7.8}$$

as $t \to \infty$; also, since $1/m$ is regularly varying with exponent $\alpha-1>-1$ we have for fixed $\eta>0$

$$\frac{1}{\alpha}\frac{t}{m(t)} \sim \int_\eta^t \frac{dx}{m(x)}, \qquad t \to \infty \tag{7.9}$$

(cf. [3, p. 273]). From (7.7), (7.8), and (7.9) it follows that

$$\int_0^t v(x)\,dx \sim C_\alpha \hat{z}(0) \int_\eta^t \frac{dx}{m(x)}, \qquad t \to \infty. \tag{7.10}$$

Suppose, contrary to (7.5),

$$\liminf_{t\to\infty} m(t)v(t) > C_\alpha \hat{z}(0).$$

Then for some $\varepsilon>0$ and all $x \geqq \eta$ sufficiently large

$$v(x) \geqq (1+\varepsilon)C_\alpha \hat{z}(0)(1/m(x)).$$

Hence

$$\int_0^t v(x)\,dx \geqq \int_\eta^t v(x)\,dx \geqq (1+\varepsilon)C_\alpha \hat{z}(0) \int_\eta^t \frac{dx}{m(x)}$$

for all $t \geqq \eta$. But this contradicts (7.10).

8. Proof of Theorems 3 and 4.

1°. Let $h>0$. Throughout this section put $z_k(x)=1$ when $(k-1)h\leqq x<kh$, $z_k(x)=0$ elsewhere, and let

$$v_k(t) = U^*z_k(t) = U(t-(k-1)h)-U(t-kh).$$

Since $m(t-kh)\sim m(t)$ for fixed kh, $t\to\infty$, we have by Theorems 1 and 2

$$\begin{aligned}(8.1)\qquad &\liminf_{t\to\infty} m(t)v_k(t) = C_\alpha h \qquad (0<\alpha\leqq \tfrac{1}{2}),\\ &\lim_{t\to\infty} m(t)v_k(t) = C_\alpha h \qquad (\tfrac{1}{2}<\alpha\leqq 1);\quad k=1,2,\ldots.\end{aligned}$$

2°. Let $z\geqq 0$ be any dri function on $[0,\infty]$. Then

$$(8.2)\qquad \liminf_{t\to\infty} m(t)\int_0^t z(t-y)U\{dy\} \geqq C_\alpha\int_0^\infty z(x)\,dx \qquad (0<\alpha\leqq 1).$$

Theorem 4 follows immediately from (8.2) and Lemma 9.

To prove (8.2) let $\varepsilon>0$ be arbitrary. We suppose $h>0$ is so small that

$$\int_0^\infty z(x)\,dx-\frac{\varepsilon}{C_\alpha} < \sum_1^\infty a_k h$$

where $a_k=\inf\{z(x) : (k-1)h\leqq x<kh\}$. Then by (8.1) and Fatou's lemma

$$\begin{aligned}C_\alpha\int_0^\infty z(x)\,dx-\varepsilon &< \sum_1^\infty a_k \liminf_{t\to\infty} m(t)v_k(t)\\ &\leqq \liminf_{t\to\infty} m(t)\sum_1^\infty a_k U^*z_k(t)\\ &\leqq \liminf_{t\to\infty} m(t)U^*z(t)\end{aligned}$$

which implies (8.2) as $\varepsilon>0$ is arbitrary.

3°. From now on in addition to being dri we assume z satisfies (1.7). That is for some constant $b<\infty$

$$(8.3)\qquad 0\leqq z(x)\leqq b/x, \qquad x>0.$$

We also assume $\frac{1}{2}<\alpha\leqq 1$ in (1.2). Obviously our goal now is to show

$$(8.4)\qquad \limsup_{t\to\infty} m(t)\int_0^t z(t-y)U\{dy\} \leqq C_\alpha\int_0^\infty z(x)\,dx.$$

4°. Fix $0<\theta<1$. Then

$$(8.5)\qquad \limsup_{t\to\infty} m(t)\int_0^{t\theta} z(t-y)U\{dy\} \leqq \frac{bC_\alpha\theta^\alpha}{\alpha(1-\theta)}$$

and

$$(8.6)\qquad \limsup_{t\to\infty} m(t)\int_{t\theta}^t z(t-y)U\{dy\} \leqq C_\alpha\int_0^\infty z(x)\,dx.$$

Proof of (8.5). From (8.3)

$$\int_0^{t\theta} z(t-y)U\{dy\} \leqq b\int_0^{t\theta} \frac{1}{t-y}\, U\{dy\} \leqq \frac{b}{(1-\theta)t}\, U(t\theta).$$

But $U(t\theta)\sim\theta^\alpha U(t)\sim\alpha^{-1}C_\alpha\theta^\alpha(t/m(t))$ as $t\to\infty$ by Theorem 5 and Lemma 1. Hence

$$\limsup_{t\to\infty} m(t)\int_0^{t\theta} z(t-y)U\{dy\} \leqq \frac{b}{1-\theta}\lim_{t\to\infty}\frac{m(t)}{t}\, U(t\theta) = \frac{bC_\alpha\theta^\alpha}{\alpha(1-\theta)}.$$

Proof of (8.6). Let $\varepsilon>0$ be arbitrary and put $b_k=\sup\{z(x) : (k-1)h\leqq x<kh\}$. We assume h is so small that

$$\sum_1^\infty b_k h < \int_0^\infty z(x)\,dx+\frac{\varepsilon}{C_\alpha}. \tag{8.7}$$

Let n be the largest integer satisfying $(n-1)h\leqq t(1-\theta)$. Then $z_k(t-y)=0$ for $k\geqq n+1$ and all $t\theta\leqq y\leqq t$, hence

$$\int_{t\theta}^t z(t-y)U\{dy\} \leqq \sum_1^n b_k\int_{t\theta}^t z_k(t-y)U\{dy\} \leqq \sum_1^n b_k v_k(t). \tag{8.8}$$

Suppose for the moment that

$$\lim_{t\to\infty} m(t)\sum_1^n b_k v_k(t) = C_\alpha\sum_1^\infty b_k h. \tag{8.9}$$

Then by (8.8) and (8.7)

$$\limsup_{t\to\infty} m(t)\int_{t\theta}^t z(t-y)U\{dy\} \leqq C_\alpha\sum_1^\infty b_k h < C_\alpha\int_0^\infty z(x)\,dx+\varepsilon$$

which yields (8.6) on letting $\varepsilon\to 0$.

Let $\beta_t(k)=b_k m(t)v_k(t)$ for $k=1,2,\dots, n$ and $\beta_t(k)=0$ for $k\geqq n+1$ then $m(t)\sum_1^n b_k v_k(t)=\sum_{k=1}^\infty \beta_t(k)$, and since, by (8.1), $\beta_t(k)\to C_\alpha hb_k$, $k=1,2,\dots, t\to\infty$, we see that to establish (8.9) it will suffice to find numbers T and B so that

$$\beta_t(k) \leqq Bb_k \quad \text{for all } k \geqq 1 \text{ and all } t \geqq T. \tag{8.10}$$

First choose s_0 so that $s\geqq s_0$ implies

$$U(s+h)-U(s) < 2C_\alpha h/m(s).$$

Next from $m(t\theta-h)\sim m(t\theta)\sim\theta^{1-\alpha}m(t)$ as $t\to\infty$, we find a t_0 so that for all $t\geqq t_0$

$$m(t) < 2\theta^{\alpha-1}m(t\theta-h).$$

Suppose now that $t\geqq t_0$, $t\theta-h\geqq s_0$ and $1\leqq k\leqq n$. Noting that $t\theta-h\leqq t-kh$, by definition of n, we get

$$m(t) < 2\theta^{\alpha-1}m(t\theta-h) \leqq 2\theta^{\alpha-1}m(t-kh)$$

and

$$v_k(t) = U(t-kh+h)-U(t-kh) < 2C_\alpha h/m(t-kh),$$

that is, $m(t)v_k(t) < 4C_\alpha h\theta^{\alpha-1}$. Since $\beta_t(k) = 0$ for $k > n$ we see that (8.10) holds with $T = \max\{(s_0+h)/\theta, t_0\}$ and $B = 4C_\alpha h\theta^{\alpha-1}$. This completes the proof of (8.6).

5°. From (8.5) and (8.6) we have

$$\limsup_{t\to\infty} m(t)U^*z(t) = \limsup_{t\to\infty} m(t)\left(\int_0^{t\theta} + \int_{t\theta}^{t}\right) z(t-y)U\{dy\}$$
$$= O\left(\frac{\theta^\alpha}{1-\theta}\right) + C_\alpha \int_0^\infty z(x)\, dx$$

whenever $0 < \theta < 1$. Letting $\theta \to 0$ gives (8.4).

Theorem 3 is evident from (8.2) and (8.4).

9. **An application.** In this section we study the asymptotic behavior of the spent and residual waiting times associated with a renewal process whose waiting time distribution has the form (1.2) with $\alpha = 1$.

A renewal process with waiting time distribution F is any sequence $\{S_n\}$, $n \geqq 0$ of the form $S_0 = 0$, $S_n = X_1 + \cdots + X_n$, $n \geqq 1$, where the X_n are positive mutually independent random variables with common distribution F. The S_n are usually interpreted as consecutive points on a time axis and are called renewal epochs. The X_n are then called waiting times. In this context $U\{I\} = \sum F^{n*}\{I\} = \sum P\{S_n \in I\}$ is clearly the expected number of renewal epochs falling in I.

Our interest here is in two auxiliary random variables Y_t and Z_t called, respectively, the spent and residual (or excess) waiting time at epoch t defined as follows: let $N_t = \max\{n : S_n \leqq t\}$ (= the number of renewal epochs in $(0, t]$). Then

$$Y_t = t - S_{N_t}, \qquad Z_t = S_{N_t+1} - t.$$

When the distribution F has a finite mean, Y_t and Z_t have nondegenerate limit distributions:

$$\lim_{t\to\infty} P\{Y_t > y, Z_t > z\} = \frac{1}{\mu}\int_{y+z}^{\infty} [1-F(u)]\, du \tag{9.1}$$

(see [3, p. 371, problem 3], or [2, Theorem 1]).

In general when $\mu = \infty$ the most one can say is $Y_t \to \infty$ and $Z_t \to \infty$ in probability. However, if F has the form (1.2) with $0 < \alpha < 1$, then Lamperti [7] and Dynkin [2] have shown that Y_t/t and Z_t/t have nontrivial limit distributions:

$$\lim_{t\to\infty} P\left\{\frac{Y_t}{t} > y, \frac{Z_t}{t} > z\right\} = \frac{\sin \pi\alpha}{\pi}\int_y^1 (z+u)^{-\alpha}(1-u)^{\alpha-1}\, du,$$

for $0 \leqq z < \infty$ and $0 \leqq y \leqq 1$. See also Feller [3, p. 447]. These writers show that (1.2) with $0 < \alpha < 1$ is in fact necessary and sufficient for Y_t/t and Z_t/t to have nontrivial limit distributions. (Dynkin proves that if $Y_t/\beta(t)$ (or $Z_t/\beta(t)$) has a nontrivial limit distribution where $\beta(t)$ is regularly varying and approaches infinity as $t \to \infty$, then (1.2) holds for some $0 < \alpha < 1$ and $\beta(t)/t \to$ const.)

When $\alpha = 1$ in (1.2) F may or may not have a finite mean (see §2(v)), but in either case it is quite straightforward to show that $Y_t/t \to 0$ and $Z_t/t \to 0$ in probability (see (9.4) for the precise rate). But as noted above if $\mu=\infty$ we also have Y_t and $Z_t\to\infty$ (in probability) so one might expect that some nonlinear normalization such as $\lambda(Y_t)/\beta(t)$ where $\lambda(t), \beta(t)\to\infty$ will yet produce a nontrivial limit distribution.

THEOREM 6. *Let F have the form*

$$1-F(t) = L(t)/t, \qquad t > 0,$$

where L is slowly varying at ∞ and suppose the mean of F is infinite. Then for $0\leqq a\leqq 1$, $b\geqq 0$

$$\lim_{t\to\infty} P\left\{\frac{m(Y_t)}{m(t)} \leqq a, \frac{m(Z_t)}{m(t)} \leqq b\right\} = \min\{a, b\} \tag{9.2}$$

where m is the function defined by (1.3).

The limit distribution in (9.2) is just the uniform distribution concentrated on the diagonal of the unit square, consequently we have the following.

COROLLARY. $(m(Y_t)-m(Z_t))/m(t)\to 0$ *in probability as $t\to\infty$, and for* $0<\theta<1$

$$\lim_{t\to\infty} P\left\{\frac{m(Y_t)}{m(t)} \leqq \theta\right\} = \lim_{t\to\infty} P\left\{\frac{m(Z_t)}{m(t)} \leqq \theta\right\} = \theta. \tag{9.3}$$

REMARKS. 1. Since Z_t and $Y_t\to\infty$ in probability it is clear that the function m in these results may be replaced by any function m_1 such that $m_1(t)\uparrow\infty$ and $m_1(t)/m(t)\to k\neq 0$ as $t\to\infty$.

2. It should be pointed out that for any F on $(0,\infty)$ with a finite mean (9.3) (but *not* (9.2)) is still valid. To see this consider for example Y_t. Let ρ be the continuous inverse of m: $\rho(m(t))=t$, $m(\rho(x))=x$, $0\leqq x<\mu$. From (9.1),

$$\lim_{t\to\infty} P\{Y_t \leqq y\} = \mu^{-1}\int_0^y [1-F(x)]\,dx = m(y)/\mu;$$

hence

$$\lim_{t\to\infty} P\{m(Y_t)/m(t) \leqq \theta\} = \lim_{t\to\infty} P\{Y_t \leqq \rho(\theta\mu)\} = m(\rho(\theta\mu))/\mu = \theta \qquad (0<\theta<1).$$

Our last result gives precise information about the distribution of Y_t/t and Z_t/t for large t.

THEOREM 7. *Let F be as in Theorem* 6 *and let* $0\leqq a\leqq 1$, $b\geqq 0$, $a+b\neq 0$. *Then as* $t\to\infty$

$$P\left\{\frac{Y_t}{t} > a, \frac{Z_t}{t} > b\right\} \sim \frac{L(t)}{m(t)}\cdot\log\left(\frac{1+b}{a+b}\right). \tag{9.4}$$

(*Note that* $L(t)/m(t)\to 0$ *as* $t\to\infty$ *by Lemma* 1.)

Proof. From (9.7) it follows that

$$G_t(a,b) = P\{Y_t > ta, Z_t > tb\} = \int_0^{t-at} [1-F(t+tb-x)]U\{dx\}$$
$$= \int_0^{1-a} [1-F(t(1+b-y))]U\{tdy\}.$$

We now argue as in the proof of (2.8): By Lemma 1 and Theorem 5 (with $\alpha=1$)

$$[1-F(t)]U(t) \sim L(t)/m(t), \qquad t\to\infty,$$

so

$$G_t(a,b)\frac{m(t)}{L(t)} \sim \int_0^{1-a} \frac{1-F(t(1+b-y))}{1-F(t)}\cdot\frac{U\{tdy\}}{U(t)}, \qquad t\to\infty.$$

Now

$$f_t(y) = \frac{1-F(t(1+b-y))}{1-F(t)} \to \frac{1}{1+b-y} \quad \text{as } t\to\infty$$

and the convergence is uniform in $0\leqq y\leqq 1-a$ (provided $a+b\neq 0$) since each $f_t(y)$ is monotone in y and since the limit $1/(1+b-y)$ is continuous on $0\leqq y\leqq 1-a$. Also, since $U(ty)/U(t)\to y$, the measure $U\{tdy\}/U(t)$ converges weakly to Lebesgue measure as $t\to\infty$.

From these remarks we see that

$$P\{Y_t > ta, Z_t > tb\}\frac{m(t)}{L(t)} \to \int_0^{1-a}\frac{1}{1+b-y}\,dy, \qquad t\to\infty,$$

and (9.4) follows.

Proof of Theorem 6. Since we use Theorem 1 we shall assume F is nonarithmetic. Theorem 6 is still true when F is arithmetic, and, though certain of the details in the present proof must be slightly modified, the essential points are the same. (Of course one uses (2.4) rather than Theorem 1 in the arithmetic case.)

Let ρ be the strictly increasing continuous inverse of the function m: $\rho(m(t)) = m(\rho(t)) = t$. Since F has infinite expectation, $m(t)\to\infty$ as $t\to\infty$ so ρ is defined on $[0,\infty)$. Fix $0<a<1$, $b>0$ and let

(9.5) $$a_t = \rho(am(t)), \qquad b_t = \rho(bm(t)).$$

We will prove

(9.6) $$\lim_{t\to\infty} P\{Y_t \leqq a_t, Z_t > b_t\} = \max\{a,b\}-b$$

which is evidently the same as (9.2).

Our starting point in proving (9.6) is the following equation

(9.7) $$P\{Y_t\leqq a, Z_t > b\} = \int_{t-a}^{t}[1-F(t+b-y)]U\{dy\}.$$

Here is a probabilistic derivation: By definition $Y_t = t-S_{N_t}$, $Z_t = S_{N_t+1}-t$ where $N_t = n$ if and only if $S_n\leqq t<S_{n+1}$. Hence the joint event $\{Y_t\leqq a, Z_t>b\}$ occurs if and only if for some (unique) n, $S_n = y$ with $t-a\leqq y\leqq t$ and then $Z_t = S_{n+1}-t = X_{n+1}+y-t>b$. By independence of S_n and X_{n+1}, the conditional probability of the second event is simply $P\{X_{n+1}>t+b-y\} = 1-F(t+b-y)$. Multiplying this by $F^{n*}\{dy\}$, the distribution of S_n, and summing over all $t-a\leqq y\leqq t$ we get

$$P\{Y_t\leqq a, Z_t>b, N_t=n\} = \int_{t-a}^{t}[1-F(t+b-y)]F^{n*}\{dy\}.$$

Summing over all $n \geqq 0$ gives (9.7) since $\sum F^{n^*} = U$.

LEMMA 10. (i) *Let a_t be defined by* (9.5) *with* $0<a<1$. *Then*

$$a_t/t \to 0 \quad but \quad a_t \to \infty \quad as \quad t \to \infty. \tag{9.8}$$

(ii) *Let ε, $\delta > 0$. Then there is a $T>0$ such that for all $t \geqq T$ and all $\frac{1}{2}t \leqq y \leqq 2t$ we have*

$$\frac{1-\varepsilon}{m(t)}\delta < U(y+\delta)-U(y) < \frac{1+\varepsilon}{m(t)}\delta. \tag{9.9}$$

(We prove Lemma 10 later.)

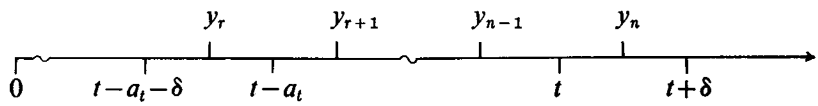
Let ε, $\delta > 0$ with $0<\varepsilon<1$ be fixed but arbitrary. By Lemma 10, $a_t \to \infty$ and $(t-a_t)/t \to 1$ as $t \to \infty$. Hence by choosing T_1 sufficiently large we may assume that both (9.9) and the inequalities

$$\tfrac{1}{2}t+10\delta < t-a_t < t < 2t-10\delta, \quad a_t > 100\delta, \tag{9.10}$$

hold simultaneously for all $t \geqq T_1$. Let $t \geqq T_1$ and consider the partition $0=y_0 < y_1 < y_2 < \cdots$ of $[0, \infty)$ where $y_k = k\delta$. Write

$$\Delta U_k = U(y_{k+1})-U(y_k) = U(y_k+\delta)-U(y_k)$$

and let y_r and y_n be chosen as in the following diagram

(9.11) 

$(y_r \leqq t-a_t,\ y_{n-1} \leqq t)$. Since $y_r > t-a_t-\delta$ and $y_n < t+\delta$ it follows from (9.9) and (9.10) that

$$\frac{1-\varepsilon}{m(t)}\delta < \Delta U_k < \frac{1+\varepsilon}{m(t)}\delta, \qquad k = r, r+1, \dots, n-1, n. \tag{9.12}$$

Now let $f(y) = 1-F(t+b_t-y)$, $0 \leqq y \leqq t+b_t$. Then f is nonnegative, nondecreasing and bounded by 1. Consequently by (9.7), (9.11) and (9.12)

$$\begin{aligned} P\{Y_t \leqq a_t, Z_t > b_t\} &= \int_{t-a_t}^{t} f(y)U\{dy\} \leqq \sum_{k=r}^{n-1} f(y_{k+1})\Delta U_k < \frac{1+\varepsilon}{m(t)} \sum_{k=r}^{n-1} f(y_{k+1})\delta \\ &= \frac{1+\varepsilon}{m(t)} \sum_{k=r+1}^{n} f(y_k)\delta \leqq \frac{1+\varepsilon}{m(t)} \int_{y_{r+1}}^{y_{n+1}} f(y)\,dy \\ &\leqq \frac{1+\varepsilon}{m(t)} \int_{t-a_t}^{t+2\delta} f(y)dy \leqq \frac{1+\varepsilon}{m(t)} \int_{t-a_t}^{t} f(y)dy + \frac{4\delta}{m(t)}. \end{aligned}$$

A similar calculation gives

$$P\{Y_t \leqq a_t, Z_t > b_t\} > \frac{1-\varepsilon}{m(t)} \int_{t-a_t}^{t} f(y)\,dy - \frac{4\delta}{m(t)}.$$

But

$$\int_{t-a_t}^{t} f(y)\,dy = \int_{t-a_t}^{t} [1-F(t+b_t-y)]\,dy = m(a_t+b_t)-m(b_t)$$
$$= m(a_t+b_t)-bm(t).$$

Therefore for all $t \geqq T_1$

$$P\{Y_t \leqq a_t, Z_t > b_t\} \lesseqgtr (1\pm\varepsilon)\left(\left(\frac{m(a_t+b_t)}{m(t)}\right)-b\right)\pm\frac{4\delta}{m(t)}. \tag{9.13}$$

Assume for the moment

$$\lim_{t\to\infty}\frac{m(a_t+b_t)}{m(t)} = \max\{a,b\}. \tag{9.14}$$

Then since $m(t)\to\infty$ as $t\to\infty$ we conclude from (9.13) and (9.14):

$$\begin{aligned}(1-\varepsilon)(\max\{a,b\}-b) &\leqq \liminf P\{Y_t \leqq a_t, Z_t > b\}\\ &\leqq \limsup P\{Y_t \leqq a_t, Z_t > b\}\\ &\leqq (1+\varepsilon)(\max\{a,b\}-b)\end{aligned}$$

and (9.6) follows.

It remains to prove (9.14). Let $c=\max\{a,b\}$ and $c_t=\rho(cm(t))$. Then $cm(t) = m(c_t) \leqq m(a_t+b_t) \leqq m(2c_t)$, or

$$c \leqq m(a_t+b_t)/m(t) \leqq m(2c_t)/m(t) = (m(2c_t)/m(c_t))c. \tag{9.15}$$

Now m is slowly varying by Lemma 1 and $c_t\to\infty$ by Lemma 10, hence

$$m(2c_t)/m(c_t)\to 1$$

as $t\to\infty$. Letting $t\to\infty$ in (9.15) gives (9.14). This completes the proof of Theorem 6.

Proof of Lemma 10. (i) Since both $\rho(t)\to\infty$ and $m(t)\to\infty$ it is clear that $a_t=\rho(am(t))\to\infty$ as $t\to\infty$ for any $a>0$. Let $0<a<b$ we show

$$\rho(am(t))/\rho(bm(t)) = a_t/b_t\to 0, \qquad t\to\infty. \tag{9.16}$$

To get (9.8) take $b=1$, $0<a<1$ in (9.16).

Suppose (9.16) fails. Then for some $0<\theta<1$ and some sequence $t_n\to\infty$ we have $\theta\leqq a_{t_n}/b_{t_n}\leqq 1$ for all n. Hence $m(\theta b_{t_n})\leqq m(a_{t_n})<m(b_{t_n})$, or since $m(a_t)=am(t)$, $m(b_t)=bm(t)$,

$$m(\theta b_{t_n})/m(b_{t_n}) \leqq a/b < 1. \tag{9.17}$$

But $m(\theta b_{t_n})/m(b_{t_n})\to 1$ as $t_n\to\infty$, since m is slowly varying and $b_{t_n}\to\infty$, so (9.17) leads to the contradiction $1\leqq a/b<1$. Hence (9.16) must be true.

(ii) Let ε, ε_1, ε_2, δ be positive numbers with $\varepsilon_1, \varepsilon_2<1$. Since m is slowly varying there is a $t_1>0$ such that

$$1-\varepsilon_1 < m(t/2)/m(2t) < 1+\varepsilon_1 \quad \text{for all } t \geqq t_1. \tag{9.18}$$

By Theorem 1, $\alpha=1$, we can find $t_2>0$ so that

$$(9.19)\qquad (1-\varepsilon_2)\cdot\frac{\delta}{m(y)} < U(y+\delta)-U(y) < (1+\varepsilon_2)\cdot\frac{\delta}{m(y)},\quad \text{for } y \geqq t_2.$$

Suppose now that $\frac{1}{2}t \geqq \max\{t_1, t_2\}$ and $\frac{1}{2}t \leqq y \leqq 2t$. Then since m is increasing

$$m(t/2)/m(2t) \leqq m(t)/m(y) \leqq m(2t)/m(t/2).$$

Consequently $1-\varepsilon_1 < m(t)/m(y) < 1/(1-\varepsilon_1)$ by (9.18), and from (9.19) it follows that

$$(1-\varepsilon_1)(1-\varepsilon_2)\frac{\delta}{m(t)} < U(y+\delta)-U(y) < \left(\frac{1+\varepsilon_2}{1-\varepsilon_1}\right)\frac{\delta}{m(t)}.$$

By (pre) choosing ε_1, ε_2 so that $(1-\varepsilon_1)(1-\varepsilon_2) \geqq 1-\varepsilon$ and $(1+\varepsilon_2)/(1-\varepsilon_1) \leqq 1+\varepsilon$ we get (9.9) with $T=\max\{2t_1, 2t_2\}$.

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