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# Strong Rigidity of $Q$-Rank 1 Lattices

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## Introduction and Statement of Main Result

A discrete subgroup $\Gamma$ of a locally compact topological group $G$ is said to be a *lattice* in $G$ if the homogeneous space $G/\Gamma$ carries a finite $G$-invariant measure. A lattice $\Gamma$ in $G$ is said to be *uniform* (or co-compact) if $G/\Gamma$ is compact; otherwise it is said to be *non-uniform*. A lattice $\Gamma$ in a linear semi-simple group is said to be *irreducible* if no subgroup of $\Gamma$ of finite index is a direct product of two infinite normal subgroups. Let $G$ be a linear analytic semi-simple group which has trivial center and no compact factors, given a lattice $\Lambda$ in $G$ it is known that $G$ decomposes into a direct product $\Pi G_i$, such that for all $i$, $G_i$ is a normal analytic subgroup of $G$; $\Lambda_i = \Lambda \cap G_i$ is an irreducible lattice in $G_i$ and $\Pi \Lambda_i$ is a subgroup of $\Lambda$ of finite index. Furthermore, if $G = \prod_{j \in J} G_j$ is any decomposition of $G$ into a direct product of normal analytic subgroups with $p_j$: $G \to G_j$ denoting the natural projection and if $\Gamma$ is an irreducible lattice in $G$, then the restriction of $p_j$ to $\Gamma$ is an *injective homomorphism* for all $j \in J$ (cf. [12, Cor. 5.23]).

Let $\Gamma$ be an irreducible lattice in a linear analytic semi-simple group $G$ which has trivial center and no compact factors. $\Gamma$ is called *strongly rigid* if given a lattice $\Gamma'$ in a semi-simple analytic group $G'$, $G'$ having trivial center and no compact factors, any isomorphism[1] $\theta$: $\Gamma \to \Gamma'$ extends to an analytic isomorphism of $G$ onto $G'$. Strong rigidity has been proved by G.D. Mostow for irreducible *uniform* lattices in semi-simple groups which are not locally isomorphic to $SL(2, \mathbf{R})$ and has been announced by G.A. Margolis and M.S. Raghunathan for non-uniform lattices in semi-simple groups which have no rank 1 factors. The purpose of this paper is to complete their results so as to apply to arbitrary irreducible lattices in semi-simple groups.

In order to state our main theorem we need to introduce a definition.

Let $G$ be a linear analytic semi-simple group with trivial center and no compact factors. We call an irreducible non-uniform lattice $\Gamma$ (in $G$) a $\mathbf{Q}$-*rank* 1 *lattice* if it has the following two properties

[1] Note that according to [10, Prop. 3.6], $\Gamma'$ is uniform if and only if $\Gamma$ is uniform.

$(R1)$ Given a nontrivial unipotent element of $\Gamma$, there exists a unique maximal unipotent subgroup of $\Gamma$ which contains it.

$(S2)$ For any maximal unipotent subgroup $\Phi$ of $\Gamma$, the commutator subgroup $[\Phi, \Phi]$ of $\Phi$ is central in $\Phi$.

(In §1 we show that if $G$ has a $\mathbf{R}$-rank 1 factor then every irreducible non-uniform lattice in $G$ is a $\mathbf{Q}$-rank 1 lattice (cf. Lemma 1.1). Also (cf. Lemma 1.4) if $\mathbf{G}$ is a connected semi-simple linear algebraic group defined over the field $\mathbf{Q}$ of rational numbers and of $\mathbf{Q}$-rank 1 and $G=\mathbf{G}_{\mathbf{R}}^0$, the identity component of the real points of $\mathbf{G}$, and if $\Gamma \subset \mathbf{G}_{\mathbf{Q}}$ is an irreducible non-uniform lattice in $G$, then $\pi(\Gamma)$ is a $\mathbf{Q}$-rank 1 lattice in $\bar{G}$ where $\pi: G \to G/G_c = \bar{G}$ is the natural projection and $G_c$ is the maximum compact normal subgroup of $G$.)

We can now state the main theorem of this paper.

**Theorem A.** *Let $G$ be a linear analytic semi-simple group with no compact factors and trivial center. Assume that $G$ is not isomorphic to $SL(2, \mathbf{R})/\pm 1_2$. Then any $\mathbf{Q}$-rank 1 lattice in $G$ is strongly rigid.*

In view of Lemmas 1.1 and 1.4 this theorem supplements the recent results[2] of G.A. Margolis and M.S. Raghunathan on strong rigidity of non-uniform lattices in semi-simple analytic groups. In fact the results of Mostow, Margolis and Raghunathan together with Theorem A provide the following

**Theorem B.** *Let $G$ (resp. $G'$) be a semi-simple analytic group and $\Gamma$ (resp. $\Gamma'$) be an irreducible lattice in $G$ (resp. $G'$). Assume that $G$, $G'$ have trivial centers and no compact factors and $G$ is not locally isomorphic to $SL(2, \mathbf{R})$. Then any isomorphism $\theta: \Gamma \to \Gamma'$ extends to an analytic isomorphism of $G$ onto $G'$.*

Our proof of Theorem A reduces to verifying that the hypothesis of the main theorem (stated below) of Mostow [9] are satisfied for $\mathbf{Q}$-rank 1 lattices. One of the central notions underlying Mostow's procedure is what he calls a *pseudo-isometry* $\varphi: X \to X'$ between two metric spaces $X$, $X'$ that is, a map $\varphi$ for which there exist positive real numbers $k$ and $b$ such that

$$d(\varphi(x), \varphi(y)) \leqq k\, d(x, y) \quad \text{for all } x, y \in X$$

and

$$d(\varphi(x), \varphi(y)) \geqq k^{-1}\, d(x, y) \quad \text{whenever } d(x, y) \geqq b.$$

**Mostow's Theorem** (cf. [9]). *Let $\Gamma$, $\Gamma'$ be lattices in $G$, $G'$ respectively ($G$, $G'$ as in the preceding theorem) and let $\theta: \Gamma \to \Gamma'$ be an isomorphism of $\Gamma$ onto $\Gamma'$. Let $X$, $X'$ be the symmetric riemannian spaces associated with $G$, $G'$ respectively. Assume that there exist pseudo-isometries $\varphi: X \to X'$*

[2] To appear.

*and $\varphi'$: $X' \to X$ such that for all $x \in X$, $x' \in X'$ and $\gamma$ (resp. $\gamma'$) in a suitable subgroup of $\Gamma$ (resp. $\Gamma'$) of finite index*

$$\varphi(x\gamma) = \varphi(x)\,\theta(\gamma)$$

*and*

$$\varphi'(x'\gamma') = \varphi'(x')\,\theta^{-1}(\gamma').$$

*Then $\theta$ extends to an analytic isomorphism of $G$ onto $G'$.*

*Thus to prove Theorem* A *it suffices to establish* (cf. § 1.7 and Lemma 1.8) *the following*

**Theorem C.** *Let $G$ and $G'$ be linear analytic semi-simple groups which have trivial centers and no compact factors. Let $K$ (resp. $K'$) be a maximal compact subgroup of $G$ (resp. $G'$) and let $X = K \backslash G$ and $X' = K' \backslash G'$ be the associated symmetric riemannian spaces with the reimannian structure induced from the Killing form on the Lie algebra $\mathfrak{g}$ and $\mathfrak{g}'$ of $G$ and $G'$ respectively. Let $\Gamma$ (resp. $\Gamma'$) be an irreducible non-uniform lattice in $G$ (resp. $G'$). Assume that both $\Gamma$ and $\Gamma'$ are net*[3] *(and so in particular torsion free), $G$ is not locally isomorphic to $SL(2, \mathbf{R})$ and further that $\Gamma$ is a* **Q***-rank 1 lattice. Let $\theta$: $\Gamma \to \Gamma'$ be an isomorphism. Then there exists a pseudo-isometry*

$$\varphi: X \to X'$$

*such that*

$$\varphi(x\gamma) = \varphi(x)\,\theta(\gamma) \qquad \text{for all } x \in X \text{ and } \gamma \in \Gamma.$$

*We shall achieve the demonstration of Theorem* C *in* § 4.

It is a pleasure to thank Professor Mostow who suggested that I look into strong rigidity of non-uniform lattices and with whom I had useful conversations related to the problem.

## § 0. Preliminaris

In the sequel we let **Z** denote the ring of rational integers. **Q** (resp. **R**, resp. **C**) will denote the field of rational (resp. real, resp. complex) numbers. Let $G$ be a linear semi-simple real analytic group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. It is well known that maximal abelian subspaces of $\mathfrak{p}$ are conjugate under $K$ (where $K$ is the analytic subgroup of $G$ corresponding to the subalgebra $\mathfrak{k}$ of $\mathfrak{g}$) and that every element of $\mathfrak{p}$ is semi-simple and has all the eigenvalues real. By definition **R**-rank of $G$ is the dimension of a maximal abelian subspace of $\mathfrak{p}$.

If **G** is a linear semi-simple algebraic group defined over a field $k$, then the dimension of a maximal $k$-split torus **T** is by definition the $k$-rank of **G**. It is known that when $k = \mathbf{R}$ then the **R**-rank of an algebraic group **G** defined over **R** is the same as the **R**-rank (defined above in terms of Lie

[3] See § 1.7 for the definition of net subgroups.

algebra) of the identity component of the group $\mathbf{G}_{\mathbf{R}}$ of $\mathbf{R}$-rational points in $\mathbf{G}$.

If $G$ is a real analytic semi-simple group with trivial center, then $G$ can be realized as the topological identity component of the real points of a suitable connected algebraic group $\mathbf{G}$ defined over $\mathbf{R}$. We call a subgroup $P$ of $G$ parabolic if $P=\mathbf{P}\cap G$ where $\mathbf{P}$ is a parabolic subgroup of $\mathbf{G}$ defined over $\mathbf{R}$. It is known that any parabolic subgroup of $\mathbf{G}$ is *connected* in the Zariski topology, also since the identity component $\mathbf{H}_{\mathbf{R}}^0$ of real points $\mathbf{H}_{\mathbf{R}}$ of an algebraic group $\mathbf{H}$ defined over $\mathbf{R}$ is of finite index in $\mathbf{H}_{\mathbf{R}}$ [8] it follows that if $P$ is a parabolic subgroup of $G$ then $P/P^0$ is finite.

In the following Ad (resp. ad) denotes the adjoint representation of a Lie group on its Lie algebra (resp. of a Lie algebra on itself). Let $G$ be a real analytic semi-simple subgroup of $SL(n,\mathbf{R})$. We assume that $G$ is self adjoint i.e., if $x\in G$ then ${}^t x\in G$. Here for a matrix $x$, ${}^t x$ denotes its transpose. It is well known that any connected linear semi-simple group can be realized in this form (cf. Mostow [7]). The isomorphism $x\to {}^t x^{-1}$ is a Cartan involution of $G$. Let $\sigma$ be the corresponding Cartan involution of the Lie algebra $\mathfrak{g}$ of $G$ and let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition determined by $\sigma$, $\mathfrak{k}$ being the compact subalgebra. The subgroup $K=\{x\in G\,|\,x={}^t x^{-1}\}$ is a maximal compact subgroup of $G$, $G=K\cdot(P(n,\mathbf{R})\cap G)$ where $P(n,\mathbf{R})$ is the set of positive definite symmetric matrices in $SL(n,\mathbf{R})$ and $P(n,\mathbf{R})\cap G=\exp\mathfrak{p}$. Let $\mathfrak{a}\subset\mathfrak{p}$ be a Cartan subspace (i.e., a maximal abelian subspace of $\mathfrak{p}$). Let $\mathfrak{a}^*$ be the dual of $\mathfrak{a}$. For $\lambda\in\mathfrak{a}^*$ let

$$\mathfrak{g}^\lambda=\{X\in\mathfrak{g}\,|\,[H,X]=\lambda(H)\,X \text{ for } H\in\mathfrak{a}\}$$

and

$$\Phi=\{\lambda\,|\,\lambda\in\mathfrak{a}^*,\,\lambda\neq 0 \text{ and } \mathfrak{g}^\lambda\neq 0\}.$$

Then

$$\mathfrak{g}=\sum_{\lambda\in\Phi}\mathfrak{g}^\lambda+\mathfrak{g}^0$$

and

$$\mathfrak{g}^0=\mathfrak{g}^0\cap\mathfrak{k}+\mathfrak{a}.$$

We fix an (open) Weyl chamber in $\mathfrak{a}$. This gives rise to an ordering on the set $\Phi$ of roots. Let $\Phi^+$ (resp. $\Phi^-$) be the set of positive roots (resp. negative roots) in this ordering and let $\Delta\subset\Phi^+$ be the set of simple roots. For a subset $\Psi\subset\Delta$ we define a parabolic subgroup $B_\Psi$ of $G$ called the standard parabolic group associated to $\Psi$ as follows. Let $\mathfrak{u}_\Psi$ be the subspace generated by $\{\mathfrak{g}^\varphi\,|\,\varphi\in\Phi^+,\ \varphi$ has a component in $\Delta-\Psi\}$; $\mathfrak{u}_\Psi$ is a subalgebra. Let

$$\mathfrak{a}_\Psi=\{H\in\mathfrak{a}\,|\,\alpha(H)=0 \text{ for } \alpha\in\Psi\}$$

and let

$$Z(\mathfrak{a}_\Psi)=\{g\in G\,|\,\mathrm{Ad}\,g\,H=H \text{ for all } H\in\mathfrak{a}_\Psi\}.$$

The Lie algebra $\mathfrak{z}_\Psi$ of $Z(\mathfrak{a}_\Psi)$ is

$$\mathfrak{z}_\Psi = \mathfrak{g}^0 + \sum_{\varphi|\mathfrak{a}_\Psi = 0} \mathfrak{g}^\varphi = \mathfrak{g}^0 + \sum_{\substack{\varphi = \Sigma m_\alpha \alpha \\ \alpha \in \Psi}} \mathfrak{g}^\varphi.$$

Let $U_\Psi$ be the connected (unipotent) subgroup of $G$ corresponding to the subalgebra $\mathfrak{u}_\Psi$. Then it can be seen easily that $Z(\mathfrak{a}_\Psi)$ normalizes $U_\Psi$. Set $B_\Psi = Z(\mathfrak{a}_\Psi) \cdot U_\Psi$. Then $B_\Psi$ is a parabolic subgroup of $G$, $U_\Psi$ is its unipotent radical (i.e., it is the maximal normal unipotent subgroup of $B_\Psi$) and $Z(\mathfrak{a}_\Psi)$ is a reductive Levi supplement.

Since $\mathfrak{a}_\Psi (\subset \mathfrak{p})$ is stable under $\sigma$, it follows that $Z(\mathfrak{a}_\Psi)$ is stable under the Cartan involution and hence $Z(\mathfrak{a}_\Psi) = (Z(\mathfrak{a}_\Psi) \cap K) \cdot (Z(\mathfrak{a}_\Psi) \cap \exp \mathfrak{p})$; $Z(\mathfrak{a}_\Psi) \cap \exp \mathfrak{p}$ is diffeomorphic to a Euclidean space and $Z(\mathfrak{a}_\Psi) \cap K$ is a maximal compact subgroup of $Z(\mathfrak{a}_\Psi)$. Since $B_\Psi = Z(\mathfrak{a}_\Psi) \cdot U_\Psi$ (a semi-direct product) and $U_\Psi$ is unipotent it follows that $K \cap B_\Psi = K \cap Z(\mathfrak{a}_\Psi)$ and $K \cap B_\Psi$ is a maximal compact subgroup of $B_\Psi$.

It is well known that any parabolic subgroup $P$ of $G$ is conjugate by an element of $K$ to a unique $B_\Psi$, $\Psi \subset \Delta$.

Let $Y$ be a nonzero element of $\mathfrak{p}$. By $\mathfrak{u}(Y)$ we denote the subspace of $\mathfrak{g}$ spanned by the eigenspaces corresponding to the positive eigenvalues of $\operatorname{ad} Y$; $\mathfrak{u}(Y)$ is a nilpotent subalgebra. Let $U(Y)$ be the analytic subgroup of $G$ corresponding to the subalgebra $\mathfrak{u}(Y)$. Then $U(Y)$ is a unipotent subgroup. Let $B(Y)$ be the normalizer of $U(Y)$ in $G$, then $B(Y)$ is a parabolic subgroup of $G$. From the above description of parabolic subgroups it can be deduced that

1. $B(Y) = M(Y) \cdot U(Y)$ (a semi-direct product) where $M(Y)$ is the centralizer of the one parameter group $\exp \mathbf{R}\, Y$ in $B(Y)$.
2. $K \cap B(Y)$ is a maximal compact subgroup of $B(Y)$ and the one parameter group $\exp \mathbf{R}\, Y$ centralizes it. I.e., $K \cap B(Y) \subset M(Y)$.

For convenience we collect below some known results which will be used in the sequel. Proofs of all these can be found in Raghunathan [12].

**0.1. Lemma** (Selberg) ([12, Lemma 1.15]). *Let $S$ be a Lie group and $\Lambda$ a lattice in $S$. Let $H$ be a closed subgroup of $S$. If there exists a neighborhood $\Omega$ of the identity in $S$ such that $H \Omega H \cap \Lambda \subset H$, then $H \cap \Lambda$ is a lattice in $H$.*

**0.2. Lemma** (Malcev) ([12, Chap. II]). *Let $U$ be a connected simply connected nilpotent Lie group. Let $Z(U)$ be the center and $\mathfrak{u}$ the Lie algebra of $U$. Then a discrete subgroup of $U$ is a lattice in $U$ if and only if $U$ is the minimal analytic subgroup of $U$ containing the discrete subgroup. Let $\Lambda$ be a lattice in $U$ then $\Lambda$ is uniform and is finitely generated, $Z(U) \cap \Lambda$ is a lattice in $Z(U)$ and the $\mathbf{Z}$ span of* $\exp^{-1}(\Lambda)$ *is a lattice in $\mathfrak{u}$. Let $\Lambda'$ be a lattice in a connected simply connected nilpotent Lie group $U'$ and $\alpha\colon \Lambda \to \Lambda'$*

*be an isomorphism. Then there exists a unique isomorphism* $\boldsymbol{\alpha}: U \to U'$ *such that* $\boldsymbol{\alpha}|_\Lambda = \alpha$.

**0.3. Lemma** (Zassenhaus, Kazdan-Margolis) ([12, Th. 8.16]). *Let* $H$ *be a Lie group. There exists a neighborhood* $\Omega$ *of the identity in* $H$ *such that if* $\Lambda$ *is any discrete subgroup of* $H$, *then* $\Lambda \cap \Omega$ *is contained in a connected nilpotent Lie subgroup of* $H$.

In the sequel a neighborhood $\Omega$ of the identity as above will be called a *Zassenhaus neighborhood.*

**0.4. Lemma** (Garland-Raghunathan [2]). *Let* $\mathbf{G}$ *be a semi-simple algebraic group defined and of rank* 1 *over a field* $\mathbf{k} \subset \mathbf{C}$. *Then any nontrivial unipotent element* $\theta \in \mathbf{G}_\mathbf{k}$ *is contained in a unique maximal unipotent* $\mathbf{k}$*-subgroup of* $\mathbf{G}$. *In particular, if* $G$ *is a linear semi-simple real analytic group of* $\mathbf{R}$*-rank* 1 *then any nontrivial unipotent element* $\theta$ *of* $G$ *is contained in a unique maximal unipotent subgroup of* $G$.

**0.5. Lemma** (Raghunathan [12, Th. 13.1]). *Let* $G$ *be a connected linear semi-simple analytic group with trivial center and no compact factors. Let* $\Gamma$ *be an irreducible non-uniform lattice in* $G$. *Let* $\Phi$ *be a maximal unipotent subgroup of* $\Gamma$ *and let* $U$ *be the minimal analytic subgroup of* $G$ *containing* $\Phi$. *Then the centralizer* $Z(U)$ *of* $U$ *in* $G$ *is contained in* $U$.

## § 1. $Q$-Rank 1 Lattices

**1.1. Lemma.** *Let* $G$ *be a semi-simple linear analytic group which has a* $\mathbf{R}$*-rank* 1 *factor (i.e.,* $G$ *has a normal analytic subgroup* $G_1$ *such that* $\mathbf{R}$ *rank* $G_1 = 1$*). Assume that* $G$ *has trivial center and no compact factors. Let* $\Gamma \subset G$ *be an irreducible non-uniform lattice. Then* $\Gamma$ *is a* $\mathbf{Q}$*-rank* 1 *lattice.*

*Proof.* Let $G = G_1 \times G_2$ where $G_1$ is a $\mathbf{R}$-rank 1 factor and $G_2$ is its normal analytic supplement. Let $p: G \to G_1$ be the canonical projection. Let $\theta \in \Gamma$ be a nontrivial unipotent element. Let $\Phi$ be a maximal unipotent subgroup of $\Gamma$ containing $\theta$ and let $\Theta \subset \Gamma$ be a unipotent group containing $\theta$. To check that $\Gamma$ has property $(R\,1)$ it clearly suffices to show that $\Theta \subset \Phi$. Since $\Gamma$ is an irreducible lattice, $p|_\Gamma$ is an isomorphism. According to Lemma 0.4 the unipotent subgroups $p(\Phi)$ and $p(\Theta)$ which have a nontrivial unipotent element $p(\theta)$ in common are contained in the same maximal unipotent subgroup of $G_1$ and in particular the subgroup of $p(\Gamma)$ generated by $p(\Phi) \cup p(\Theta)$ is nilpotent. Since $p|_\Gamma$ is an isomorphism this implies that the unipotent subgroups $\Phi$ and $\Theta$ together generate a nilpotent and hence a unipotent subgroup of $\Gamma$, since $\Phi$ is a maximal unipotent subgroup this group coincides with $\Phi$ and hence $\Theta \subset \Phi$.

To show that $\Gamma$ has property $(S\,2)$ too we argue as follows. Let $\Phi$ be a maximal unipotent subgroup of $\Gamma$. Again since $p|_\Gamma$ is an isomorphism it suffices to prove that $[p(\Phi), p(\Phi)]$ is central in $p(\Phi)$. Let $\mathfrak{g}_1 = \mathfrak{k}_1 + \mathfrak{p}_1$ be a Cartan decomposition of the Lie algebra $\mathfrak{g}_1$ of $G_1$ and let $\mathfrak{a}_1 \subset \mathfrak{p}_1$ be a Cartan subspace (i.e., a maximal abelian subspace of $\mathfrak{p}_1$). Since $G_1$ has $\mathbf{R}$-rank 1, $\mathfrak{a}_1$ is one dimensional. Let $\mathfrak{g}_1 = \mathfrak{g}_1^{-2\alpha} + \mathfrak{g}_1^{-\alpha} + \mathfrak{g}_1^0 + \mathfrak{g}_1^{\alpha} + \mathfrak{g}_1^{2\alpha}$ be the root space decomposition of $\mathfrak{g}_1$ with respect to $\mathfrak{a}_1$. Then as is well known, $\mathfrak{g}_1^{\alpha} + \mathfrak{g}_1^{2\alpha}$ is the Lie algebra of a maximal unipotent subgroup of $G_1$ and any two maximal unipotent subgroups of $G_1$ are conjugate to each other. Since $\mathfrak{g}_1^{2\alpha}$ is central in $\mathfrak{g}_1^{\alpha} + \mathfrak{g}_1^{2\alpha}$ and $[\mathfrak{g}_1^{\alpha} + \mathfrak{g}_1^{2\alpha}, \mathfrak{g}_1^{\alpha} + \mathfrak{g}_1^{2\alpha}] \subset \mathfrak{g}_1^{2\alpha}$ it follows that if $N_1$ is a maximal unipotent subgroup of $G_1$ then $[N_1, N_1]$ is central in $N_1$ and so a-fortiori $[p_1(\Phi), p_1(\Phi)]$ is central in $p_1(\Phi)$. Thus $\Gamma$ has property $(S2)$. This completes the proof of Lemma 1.1.

**1.2. Lemma.** *Let $G$ be a linear analytic semi-simple group with trivial center and no compact factors. Let $\Gamma$ be an irreducible non-uniform lattice in $G$ which has property $(R1)$. Then $\Gamma$ has the following property*

*$(R2)$ Let $\Phi$ be a maximal unipotent subgroup of $\Gamma$ and $U$ be the minimal analytic subgroup of $G$ containing $\Phi$. Let $N(U)$ be the normalizer of $U$ in $G$ and $N^0(U) = \{g \in N(U) \mid \operatorname{Int} g|_U$ preserves a Haar measure on $U\}$. Then $N^0(U) \cap \Gamma$ is a uniform lattice in $N^0(U)$.*

The proof given below of this lemma is essentially due to M.S. Raghunathan.

*Proof of Lemma* 1.2. Let $\Theta$ be a nontrivial subgroup of $\Phi$ which is normalized by every $\gamma (\in \Gamma)$ which normalizes $\Phi$. Let $V$ be the minimal analytic subgroup of $G$ containing $\Theta$ and let $N^0(V) = \{g \in N(V) \mid \operatorname{Int} g|_V$ preserves a Haar measure on $V\}$, where $N(V)$ is the normalizer of $V$ in $G$. We shall in fact prove that $N^0(V) \cap \Gamma$ is a uniform lattice in $N^0(V)$. First we shall show, using Lemma 0.1, that $N^0(V) \cap \Gamma$ is a lattice in $N^0(V)$.

Let $\Omega_0$ be an open Zassenhaus neighborhood of the identity in $G$ (cf. Lemma 0.3) and let $\Omega$ be an open symmetric neighborhood of the identity such that $\Omega^4 \subset \Omega_0$. We assume (as we may) that $V \subset N$ where $G = K \cdot A \cdot N$ is an Iwasawa decomposition of $G$ ($K$ is a maximal compact subgroup of $G$, $N$ is a maximal unipotent subgroup and $A$ is an analytic diagonalizable subgroup which normalizes $N$). Since given any compact subset $E$ of $N$ and any neighborhood $\omega$ of the identity in $N$ we can find an $a \in A$ such that $aEa^{-1} \subset \omega$ and since by Lemma 0.2, $\Theta$ and hence $V \cap \Gamma$ is a uniform lattice in $V$, after replacing $\Gamma$ by a suitable conjugate we can assume that there is a compact subset $E$ of $V \cap \Omega$ such that $E \cdot (V \cap \Gamma) = V$ and thus in the measure on $V/V \cap \Gamma$ induced by a Haar measure $\mu$ on $V$, $\operatorname{Vol.}(V/V \cap \Gamma) < \mu(V \cap \Omega)$.

Let $\gamma = t_1 \omega t_2$ be an element of $N^0(V)\, \Omega N^0(V) \cap \Gamma$ with $t_1, t_2 \in N^0(V)$ and $\omega \in \Omega$. Since $\mu(t_1 (V \cap \Omega)\, t_1^{-1}) = \mu(V \cap \Omega) = \mu(t_2^{-1} (V \cap \Omega)\, t_2)$ and $\operatorname{Vol}(V/V \cap \Gamma) < \mu(V \cap \Omega)$ it follows that if $\pi\colon V \to V/V \cap \Gamma$ is the natural projection, then the maps $\pi|_{t_1 (V \cap \Omega) t_1^{-1}}$ and $\pi|_{t_2^{-1} (V \cap \Omega) t_2}$ can not be injective.

From this it follows that we can find $(e\neq)\,\gamma_1\in V\cap\Gamma$ (resp. $(e\neq)\,\gamma_2\in V\cap\Gamma$) such that $t_1^{-1}\gamma_1 t_1\in\Omega^2$ (resp. $t_2\gamma_2 t_2^{-1}\in\Omega^2$).) Consider now the subgroup of $\Gamma$ generated by $\gamma_1$ and $\gamma\gamma_2\gamma^{-1}$. Since

$$\begin{aligned} t_1^{-1}\gamma\gamma_2\gamma^{-1}t_1 &= t_1^{-1}(t_1\,\omega t_2\,\gamma_2\,t_2^{-1}\,\omega^{-1}\,t_1^{-1})\,t_1 \\ &= \omega(t_2\,\gamma_2\,t_2^{-1})\,\omega^{-1}\in\Omega^4\subset\Omega_0 \end{aligned}$$

and $t_1^{-1}\gamma_1 t_1\in\Omega^2\subset\Omega_0$, the group generated by $t_1^{-1}\gamma\gamma_2\gamma^{-1}t_1$ and $t_1^{-1}\gamma_1 t_1$ is nilpotent. Thus the unipotent elements $\gamma\gamma_2\gamma^{-1}$ and $\gamma_1$ generate a nilpotent and hence a unipotent subgroup of $\Gamma$. Since (cf. Property $(R1)$) $\Phi$ is the unique maximal unipotent subgroup (of $\Gamma$) containing any nontrivial element of itself, it follows that $\gamma\gamma_2\gamma^{-1}\in\Phi$ i.e., $\gamma_2\in\gamma^{-1}\Phi\gamma$ but since $\gamma_2\in V\cap\Gamma\subset\Phi$, $\gamma^{-1}\Phi\gamma=\Phi$ and hence $\gamma$ normalizes $\Phi$ and therefore it normalizes $\Theta$. Thus $\gamma\in N(V)$. Since $\gamma$ normalizes the lattice $\Theta$ in $V$, we conclude that $\gamma$ preserves a Haar measure on $V$ i.e., $\gamma\in N^0(V)$. Thus we have proved that $N^0(V)\,\Omega N^0(V)\cap\Gamma\subset N^0(V)$. According to Lemma 0.1, $N^0(V)\cap\Gamma$ is a lattice in $N^0(V)$.

Now we claim that $N^0(V)\cap\Gamma=N^0(U)\cap\Gamma$. Since $\Phi$ is a maximal unipotent subgroup of $\Gamma$ and $\Phi\subset U$, $U\cap\Gamma=\Phi$. Thus every element of $N^0(U)\cap\Gamma$ normalizes $\Phi$ and hence also $\Theta$. This clearly implies that $N^0(U)\cap\Gamma\subset N^0(V)\cap\Gamma$. On the other hand if $\gamma\in N^0(V)\cap\Gamma$, $\gamma$ normalizes the nontrivial unipotent subgroup $V\cap\Gamma$ of $\Phi$ and in view of property $(R1)$ it normalizes $\Phi$ and thus $\gamma\in N^0(U)\cap\Gamma$. This shows that $N^0(V)\cap\Gamma\subset N^0(U)\cap\Gamma$ and hence $N^0(U)\cap\Gamma=N^0(V)\cap\Gamma$. In view of this to prove that $N^0(V)\cap\Gamma$ is uniform, it suffices to show that $N^0(U)\cap\Gamma$ is uniform in $N^0(U)$.

We shall first show that the unipotent radical of $N(U)$ is $U$. Let $W$ be the unipotent radical of $N^0(U)$. Since $N^0(U)\cap\Gamma$ is a lattice in $N^0(U)$ by a standard argument using a result of Auslander ([12, Th. 8.24 and Cor. 8.28]), Borel's density theorem and the fact that the centralizer $Z(U)$ of $U$ is contained in $U$ (Lemma 0.5) one can prove that $W\cap\Gamma$ is a lattice in $W$. Since $W\supseteq U$ and since $U\cap\Gamma=\Phi$ is a maximal unipotent subgroup of $\Gamma$ it follows that $W\cap\Gamma=\Phi$ and hence in view of Lemma 0.2, $W=U$. Now since the unipotent radical of $N(U)$ contains $U$ and is evidently contained in the unipotent radical of $N^0(U)$, it follows that $U$ is the unipotent radical of $N(U)$ (this also implies that $N(U)$ is a parabolic subgroup of $G$ cf. [12, Prop. 12.8 (b)].)

By Lemma 0.5 the centralizer $Z(U)$ of $U$ is contained in $U$. Also since $U$ is the minimal analytic subgroup containing $\Phi$, according to Lemma 0.2, $\Phi$ is a uniform lattice in $U$ and the **Z**-span $\mathfrak{u}_{\mathbf{Z}}$ of $\exp^{-1}(\Phi)$ is a lattice in the Lie algebra $\mathfrak{u}$ of $U$. Since the centralizer $Z(U)$ of $U$ in $G$ is contained in $U$, it is the center of $U$ and hence (again by Lemma 0.2) $Z(U)\cap\Phi$ is a uniform lattice in $Z(U)$. Consider now the natural representation of $N^0(U)$ on $\mathfrak{u}=\mathfrak{u}\otimes_{\mathbf{R}}\mathbf{C}$. It induces an algebraic morphism $\rho$: $N^0(U)/Z(U)\to GL(\mathfrak{u})$ which clearly is a monomorphism and the image $H$ of $\rho$ is up to commensurability the real rational points of an algebraic group $\mathbf{H}$ defined over $\mathbf{R}$. Let $\alpha$ be the projection $N^0(U)\to N^0(U)/Z(U)$, then since $Z(U)\cap\Gamma=Z(U)\cap\Phi$ is a lattice in $Z(U)$ and $\Lambda=N^0(U)\cap\Gamma$ is a lattice in $N^0(U)$; $\alpha(\Lambda)$ is a lattice in $\alpha(N^0(U))$ and $\rho\alpha(\Lambda)$ is a lattice in $H$.

Evidently $\rho\alpha(\Lambda)\subset GL(\mathfrak{u}_{\mathbf{Z}})$, hence the Zariski closure ${}^0\mathbf{H}$ of $\rho\alpha(\Lambda)$ is a group defined over $\mathbf{Q}$. Clearly ${}^0\mathbf{H}\subset\mathbf{H}$ and hence ${}^0H={}^0\mathbf{H}\cap GL(\mathfrak{u})$ is commensurable with ${}^0H\cap H$. Thus since $\rho\alpha(\Lambda)(\subset{}^0H)$ is a lattice in $H$, it is a lattice in ${}^0H$. Since the unipotent radical of $N^0(U)$ is $U$ and $Z(U)\subset U$, it follows that the unipotent radical (i.e. the maximal normal unipotent subgroup) of $N^0(U)/Z(U)$ is $U/Z(U)$ and hence the unipotent radical of $H$ is precisely $\rho(U/Z(U))$. Moreover since $\alpha(\Phi)$ is a lattice in $U/Z(U)$ it follows that $\rho\alpha(\Phi)$ is Zariski dense in the unipotent radical $\rho(U/Z(U))$ of $H$. This shows that ${}^0H$ contains the unipotent radical of $H$. According to Borel's density theorem (see e.g. [12, Chap. V]), ${}^0H$ contains also all the non-compact simple, semi-simple analytic subgroups of $H$. Since $H\supset H\cap{}^0H\supset\rho\alpha(\Lambda)$, $H/{}^0H\cap H$ carries a finite invariant measure, from this it readily follows that $H/{}^0H\cap H$ is compact. Thus to prove that $N^0(U)/N^0(U)\cap\Gamma$ is compact it suffices to show that ${}^0H/\rho\alpha(\Lambda)$ is compact. Since $\rho\alpha(\Lambda)$ is a lattice in ${}^0H$ and is contained in ${}^0\mathbf{H}_{\mathbf{Z}}={}^0\mathbf{H}\cap GL(\mathfrak{u}_{\mathbf{Z}})$ it follows that $\rho\alpha(\Lambda)$ is a subgroup of finite index in ${}^0\mathbf{H}_{\mathbf{Z}}$ i.e., $\rho\alpha(\Lambda)$ is an arithmetic lattice. Since $Z(U)$, the kernel of $\alpha$, is a unipotent group and $\Phi$ is a maximal unipotent subgroup of $\Gamma$ it follows that every unipotent element of $\rho\alpha(\Lambda)$ is contained in $\rho\alpha(\Phi)\subset\rho(U/Z(U))$ and hence by Godement's criterion ${}^0H/\rho\alpha(\Lambda)$ is compact. This completes the proof of Lemma 1.2.

1.3. *Remark.* Let $G$ be a linear analytic semi-simple group with trivial center and no compact factors. Following Raghunathan we call an irreducible non-uniform lattice in $G$ a *rank* 1 lattice if it has properties $(R1)$ and $(R2)$. In [12] Raghunathan has constructed a *nice* fundamental domain for such lattices, a short description of this fundamental domain will be given in the next section. We should note here that according to Lemma 1.2 if a lattice has property $(R1)$ then it necessarily has property $(R2)$ and hence is a rank 1 lattice.

**1.4. Lemma.** *Let* $\mathbf{G}$ *be a connected linear semi-simple algebraic group defined over* $\mathbf{Q}$ *and let* $G=\mathbf{G}_{\mathbf{R}}^0$ *be the identity component of the* $\mathbf{R}$*-rational points of* $\mathbf{G}$ *and let* $\Gamma\subset\mathbf{G}_{\mathbf{Q}}$ *be an irreducible non-uniform lattice in* $G$. *Let* $G_c$ *be the maximal compact normal subgroup of* $G$ *and let* $\pi$: $G\to G/G_c=\bar{G}$ *be the natural projection. If* $\mathbf{Q}$*-rank* $\mathbf{G}=1$, *then* $\bar{\Gamma}=\pi(\Gamma)$ *is a* $\mathbf{Q}$*-rank* 1 *lattice in* $\bar{G}$.

($\bar{G}$ is a semi-simple analytic group with trivial center and hence can be thought of as the adjoint group of its Lie algebra.)

*Proof.* Let $G_0$ be the normal analytic subgroup of $G$ such that $G = G_c \times G_0$ (almost direct product) and let $\pi_0: G_0 \to \bar{G}$ be the restriction of $\pi$ to $G_0$. Evidently $\pi_0$ is a surjective map whose kernel is precisely the finite center of $G_0$. Also since $G_c$ is compact, every unipotent element of $G$ is contained in $G_0$. Now let $\bar{\varphi} \in \bar{\Gamma}$ be a unipotent element and let $\varphi \in \Gamma \subset \mathbf{G_Q}$ be an element which under $\pi$ maps onto $\bar{\varphi}$ and let $\varphi = \varphi_s \cdot \varphi_u$ be the Jordan decomposition of $\varphi$ with $\varphi_s$ (resp. $\varphi_u$) semi-simple (resp. unipotent). Then both $\varphi_s$ and $\varphi_u$ are contained in $\mathbf{G_Q}$ moreover $\varphi_u$ being unipotent is actually contained in $G_0 \cap \mathbf{G_Q}$. Since $\pi(\varphi) = \bar{\varphi}$ is unipotent, $\pi(\varphi_u) = \bar{\varphi}$ and thus we have shown that given a unipotent element $\bar{\varphi}$ in $\bar{\Gamma}$ there is a (in fact unique, since the kernel of $\pi_0$ is central and hence has only semi-simple elements) unipotent element in $G_0 \cap \mathbf{G_Q}$ which is mapped under $\pi_0$ onto $\bar{\varphi}$. Now let $\bar{\theta}$ be a nontrivial unipotent element of $\bar{\Gamma}$ and $\bar{\Theta}$ be a unipotent subgroup of $\bar{\Gamma}$ containing $\bar{\theta}$. Let $\theta \in G_0 \cap \mathbf{G_Q}$ be the unique unipotent element such that $\pi_0(\theta) = \bar{\theta}$. Let $\mathbf{U}$ be the unique maximal unipotent $\mathbf{Q}$-subgroup of $G$ containing $\theta$ (cf. Lemma 0.4) and $\Phi = \mathbf{U} \cap \mathbf{G_Q}$, clearly $\Phi$ is contained in $G_0$ and it is the unique maximal unipotent subgroup of $\mathbf{G_Q}$ containing $\theta$. Since kernel $\pi_0$ is central, $\pi_0^{-1}(\bar{\Theta})$ is a nilpotent group. Let $\Theta$ be the subgroup of $\pi_0^{-1}(\bar{\Theta})$ generated by the unipotent elements in $\pi_0^{-1}(\bar{\Theta})$ then, $\Theta (\subset \mathbf{G_Q})$ is a unipotent group which contains $\theta$ and hence $\Theta \subset \Phi$. It is clear that $\pi_0(\Theta) = \bar{\Theta}$. Thus we have proved that any unipotent subgroup of $\bar{\Gamma}$ which contains $\bar{\theta}$ is contained in $\pi_0(\Phi)$ and hence $\bar{\Phi} = \pi_0(\Phi) \cap \bar{\Gamma}$ is the unique maximal unipotent subgroup of $\bar{\Gamma}$ containing $\bar{\theta}$. This establishes property ($R1$) for $\bar{\Gamma}$.

To show that $[\bar{\Phi}, \bar{\Phi}]$ is central in $\bar{\Phi}$, it clearly suffices to prove that $[\Phi, \Phi]$ is central in $\Phi$. Since $\Phi$ is contained in the unipotent $\mathbf{Q}$-subgroup $\mathbf{U}$ of $\mathbf{G}$ we will be through if we show that $[\mathbf{U}, \mathbf{U}]$ is central in $\mathbf{U}$. But the latter can be proved for example by considering the root space decomposition of the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$ with respect to a maximal $\mathbf{Q}$-split torus $\mathbf{T}$ (see the proof of Lemma 1.1 above and note that since $\mathbf{Q}$-rank $\mathbf{G} = 1$, $\mathbf{T}$ is one dimensional).

1.5. *Remark.* One can use Lemma 1.2 and certain observations made in the proofs of Lemmas 1.2 and 1.4 to prove a converse of Lemma 1.4. More precisely one can prove that (in the notations of the preceding lemma) if the lattice $\bar{\Gamma}$ has property ($R1$), then $\mathbf{Q}$-rank $\mathbf{G} = 1$.

1.6. *Remark.* It follows from certain results announced by Margolis and proved independently by Raghunathan that property ($S2$) is a consequence of property ($R1$) (thus a rank 1 lattice is a $\mathbf{Q}$-rank 1 lattice and vice versa). In fact let $G$ be a linear analytic semi-simple group which has trivial center and no compact factors and let $\Gamma$ be an irreducible non-uniform lattice in $G$, if $G$ has a $\mathbf{R}$-rank 1 factor then according to Lemma 1.1, $\Gamma$ has property $(S2)$ so we can assume that $G$ has no $\mathbf{R}$-rank 1 factors. Then according to the results of Margolis and Raghunathan there exists a connected semi-simple algebraic group $\mathbf{G}$ defined over $\mathbf{Q}$ such that $G$ is isomorphic to the identity component $\mathbf{G}^0_{\mathbf{R}}$ of the group of $\mathbf{R}$ rational points $\mathbf{G}_{\mathbf{R}}$ of $\mathbf{G}$ and $\Gamma \subset \mathbf{G}_{\mathbf{Q}}$. Now if $\Gamma$ has property $(R1)$, by the previous remark $\mathbf{Q}$-rank $\mathbf{G}=1$ and then according to Lemma 1.4, $\Gamma$ has property $(S2)$ too. Since the results of Margolis and Raghunathan have not yet appeared in print we have preferred stating explicitly property $(S2)$ in the definition of $\mathbf{Q}$-rank 1 lattices.

1.7 *Definition.* An element $g \in GL(n, \mathbf{C})$ is said to be *net* if the subgroup of $\mathbf{C}^*$ generated by the eigenvalues of $g$ is torsion free. A subgroup of $GL(n, \mathbf{C})$ is *net* if its every element is net.

*It is known that lattices in analytic groups are finitely generated* ([12, § 6.18]), *thus according to* [12, Th. 6.11] *any lattice in a linear analytic group admits a subgroup of finite index which is net.*

**1.8. Lemma.** *Let $G$ (resp. $G'$) be a linear semi-simple group with trivial center and no compact factors. Let $\Gamma$ be an irreducible non-uniform* $\mathbf{Q}$*-rank* 1 *lattice in $G$. Let $\Gamma'$ be a lattice in $G'$ and let $\theta\colon \Gamma \to \Gamma'$ be an isomorphism. Assume that both $\Gamma$, $\Gamma'$ are net. Then $\Gamma'$ is also an irreducible non-uniform* $\mathbf{Q}$*-rank* 1 *lattice.*

*Proof.* Since irreducibility of a lattice has been defined above in terms of the group structure of the lattice it follows that $\Gamma'$ which is isomorphic to the irreducible lattice $\Gamma$, is also irreducible. Proposition 3.6 of [10] implies that $\Gamma'$ is a non-uniform lattice. It remains to show that $\Gamma'$ has properties $(R1)$ and $(S2)$. Let us first consider the case when $G$ is locally isomorphic to $SL(2, \mathbf{R})$. In this case according to [11, §3] $\mathbf{R}$-rank $G' =$ $\mathbf{R}$-rank $G = 1$ and then in view of Lemma 1.1, $\Gamma'$ is a $\mathbf{Q}$-rank 1 lattice.

Now we assume that $G$ is not locally isomorphic to $SL(2, \mathbf{R})$ then by [2, Th. 0.12] $G'$ is not locally isomorphic to $SL(2, \mathbf{R})$ and hence according to [10, Th. 3.1] $\theta$ takes unipotent elements into unipotent elements and vice versa. This immediately implies that $\Gamma'$ has properties $(R1)$ and $(S2)$. This completes the proof of Lemma 1.8.

## § 2. Fundamental Domains for Rank 1 Lattices

In the proof of Theorem C we shall use the results connected with Raghunathan's construction (see [12, Chapter XIII]) of fundamental domains for irreducible non-uniform lattices which have property $(R1)$ (and hence also $(R2)$ in view of Lemma 1.2). In the following proposition we collect some of his results which will be used in this paper.

We introduce some notations first.

**Notations.** Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ (resp. $\mathfrak{g}'=\mathfrak{k}'+\mathfrak{p}'$) be the Cartan decomposition of $\mathfrak{g}$ (resp. $\mathfrak{g}'$) determined by the maximal compact subgroup $K$ (resp. $K'$). For $c\in\mathbf{R}$ and a nonzero element $Y\in\mathfrak{p}$ (resp. $Y'\in\mathfrak{p}'$) let

$$A_c(Y)=\{\exp t\,Y\,|\,t\leqq c\} \quad (\text{resp. } A_c(Y')=\{\exp t\,Y'\,|\,t\leqq c\})$$

and

$$A_c^0(Y)=\{\exp t\,Y\,|\,t<c\} \quad (\text{resp. } A_c^0(Y')=\{\exp t\,Y'\,|\,t<c\})$$

and we let $\mathfrak{u}(Y)$ (resp. $\mathfrak{u}(Y')$) denote the subalgebra of $\mathfrak{g}$ (resp. $\mathfrak{g}'$) spanned by the eigenspaces corresponding to the positive eigenvalues of $\operatorname{ad} Y$ (resp. $\operatorname{ad} Y'$). Both $\mathfrak{u}(Y)$ and $\mathfrak{u}(Y')$ are nilpotent subalgebras. Let $U(Y)$ (resp. $U(Y')$) be the analytic subgroup of $G$ (resp. $G'$) corresponding to the subalgebra $\mathfrak{u}(Y)$ (resp. $\mathfrak{u}(Y')$). Then $U(Y)$ and $U(Y')$ are unipotent subgroups. Let $B(Y)$ be the normalizer of $U(Y)$ in $G$ and let

$$D_1(Y)=\{g\in B(Y)\,|\,\operatorname{Int} g|_{U(Y)} \text{ preserves a Haar measure on } U(Y)\}.$$

$B(Y)$ is a parabolic subgroup of $G$ (cf. Preliminaries), $D_1(Y)$ is a normal subgroup of $B(Y)$ and it evidently contains $K\cap B(Y)$ which is a maximal compact subgroup of $B(Y)$. Thus $K_Y=K\cap D_1(Y)=K\cap B(Y)$ and $K_Y$ is a maximal compact subgroup of $D_1(Y)$. We define $B(Y')$ and $D_1(Y')$ analogously.

**2.1. Proposition.** *Let $G$ be a linear semi-simple group which has no compact factors and let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition of the Lie algebra $\mathfrak{g}$ of $G$. Let $\Gamma$ be an irreducible non-uniform lattice which has property* $(R1)$. *Then the set of conjugacy classes of maximal unipotent subgroups of $\Gamma$ is finite and given a maximal unipotent subgroup $\Phi$ of $\Gamma$ there exists a $Y\in\mathfrak{p}$ such that $\Phi=U(Y)\cap\Gamma$ and $\Phi$ is a lattice in the unipotent group $U(Y)$. Let $\mathscr{R}\subset\mathfrak{p}$ be a finite subset such that*

(a) *$U(Y)\cap\Gamma$ is a maximal unipotent subgroup of $\Gamma$ and it is a lattice in $U(Y)$.*

(b) *Any maximal unipotent subgroup of $\Gamma$ is conjugate (in $\Gamma$) to $U(Y)\cap\Gamma$ for a unique $Y\in\mathscr{R}$.*

*Then we can find a constant $c\in\mathbf{R}$ such that*

(1) *If $Y_1, Y_2\in\mathscr{R}$ and $Y_1\neq Y_2$, then*

$$KA_c(Y_1)\,D_1(Y_1)\,\Gamma\cap KA_c(Y_2)\,D_1(Y_2)\,\Gamma=\emptyset.$$

(2) *For* $Y\in\mathscr{R}, \gamma\in\Gamma; t_1, t_2 \leqq c; k_1, k_2\in K$ *and* $d_1, d_2\in D_1(Y)$ *if*

$$k_1 \exp t_1 Y d_1 = k_2 \exp t_2 Y d_2 \gamma,$$

then $t_1 = t_2$ and $\gamma\in D_1(Y)\cap\Gamma$.

*Also* (3) *If* $\pi\colon G\to K\diagdown G = X$ *and* $p\colon X\to X/\Gamma$ *denote the canonical projections, then* $\forall t\in\mathbf{R}$ *the set* $\Omega_t = X - \pi\big(\bigcup_{Y\in\mathscr{R}} KA_t^0(Y)\, D_1(Y)\Gamma\big)$ *is compact modulo* $\Gamma$ i.e., $p(\Omega_t)$ *is compact.*

2.2. *Remark.* A few words on the proof of the above proposition are in order. It has been noted in Preliminaries that $B(Y) = M(Y)\cdot U(Y)$ (semi-direct product) and the reductive subgroup $M(Y)$ is centralized by the one parameter group $\exp\mathbf{R}Y$. Hence, if $\Omega\subset B(Y)$ (and in particular if $\Omega\subset D_1(Y)$) is a relatively compact subset then for any $t_0\in\mathbf{R}$ the set $\{\exp tYg\exp - tY \mid g\in\Omega, t\leqq t_0\}$ is relatively compact. Note that the exponential map restricted to $\mathfrak{u}(Y)$ is a diffeomorphism and since the Lie algebra $\mathfrak{u}(Y)$ of $U(Y)$ is the sum of the eigenspaces of $\operatorname{ad} Y$ corresponding to the positive eigenvalues, for any relatively compact subset $\omega$ of $\mathfrak{u}(Y)$ the set $\bigcup_{t\leqq t_0} \operatorname{Ad}\exp tY\omega$ is a relatively compact subset. Also since $\Gamma$ has property $(R1)$ it has property $(R2)$ (cf. Lemma 1.2) and hence for any $Y\in\mathscr{R}$, $D_1(Y)/D_1(Y)\cap\Gamma$ is compact. Thus we can choose a compact subset $\eta(Y)$ of $D_1(Y)$ such that $\eta(Y)\cdot(D_1(Y)\cap\Gamma) = D_1(Y)$. We can use these observations to modify the proofs in Raghunathan [12, Chapter XIII] to get a proof of Proposition 2.1.

**2.3. Lemma.** *Let* $G, \Gamma, \mathscr{R}$ *be as in Proposition* 2.1. *Then for any* $t\in\mathbf{R}$ *and* $Y\in\mathscr{R}$, $KA_t(Y)\, D_1(Y)\Gamma$ *is a closed subset of* $G$.

*Proof.* Since $K$ is compact and in view of property $(R2)$ $D_1(Y)/D_1(Y)\cap\Gamma$ is compact, to prove that $KA_t(Y)\, D_1(Y)\Gamma$ is closed it suffices to show that if $\{\exp t_i Y d\gamma_i\}$ with $t_i\leqq t$, $d\in D_1(Y)$ and $\gamma_i\in\Gamma$ is a convergent sequence, then it converges to a limit in $A_t(Y)\, D_1(Y)\Gamma$. If $t_i$'s are bounded then the sequence $\{\exp t_i Y\}$ is contained in a compact subset of $A_t(Y)$ and hence if necessary by passing to a subsequence we can assume that $\{\exp t_i Y d\gamma_i\}$ as well as $\{\exp t_i Y\}$ are convergent and hence $\{d\gamma_i\}$ and therefore also the sequence $\{\gamma_i\}$ is convergent. Clearly in this case $\{\exp t_i Y d\gamma_i\}$ converges to a point in $A_t(Y)\, D_1(Y)\Gamma$. So if possible, let us assume that $t_i\to-\infty$ and $\{\exp t_i Y d\gamma_i\}$ converges. As noted in Remark 2.2, $\{\exp t_i Y d\exp - t_i Y\}$ is contained in a relatively compact subset of $D_1(Y)$ and hence has a convergent subsequence, so we can assume (after passing to a subsequence) that $\{\exp t_i Y\gamma_i\}$ converges to $\lambda\in G$. Let $\varphi\,(\neq e)$ be an element of $U(Y)\cap\Gamma$. Since $U(Y)$ is the subgroup corresponding to the Lie algebra $\mathfrak{u}(Y)$ which is the sum of eigenspaces of $\operatorname{ad} Y$ corresponding to the positive eigenvalues, if $t_i\to-\infty$, $\{\operatorname{Ad}(\exp t_i Y)Z\}$ for a fixed $Z\in\mathfrak{u}(Y)$ converges to 0 and hence $\exp t_i Y\varphi\exp - t_i Y\to e$.

Thus the sequence

$$\{(\exp t_i Y\gamma_i)(\gamma_i^{-1}\varphi\gamma_i)(\exp t_i Y\gamma_i)^{-1}\}\,(=\{\exp t_i Y\varphi\exp - t_i Y\})$$

converges to the identity. Hence $\{\gamma_i^{-1}\varphi\gamma_i\}$ is a convergent sequence converging to the identity. Since $\Gamma$ is discrete this implies that for all large $i$, $\gamma_i^{-1}\varphi\gamma_i = e$ and hence $\varphi = e$, a contradiction. This completes the proof of the lemma.

2.4. *Remark.* Let $\mathfrak{R}$ and $c\in\mathbf{R}$ be as in Proposition 2.1. For $b\in\mathbf{R}$, the canonical map

$$K\times A_b^0(Y)\times D_1(Y)\to G$$

$$(k,a,d)\mapsto k\cdot a\cdot d \quad \text{for } k\in K, a\in A_b^0(Y) \text{ and } d\in D_1(Y)$$

is an analytic map of maximal rank and hence by rank theorem it is an open map. In view of Proposition 2.1 and the observation that the group $\exp\mathbf{R}\,Y$ centralizes $K\cap B(Y)(=K\cap D_1(Y))$ made in the Preliminaries, it follows that if $b\leqq c$ this map gives rise to a diffeomorphism,

$$\bigcup_{Y\in\mathfrak{R}} A_b^0(Y)\times\big((K\cap D_1(Y))\backslash D_1(Y)/D_1(Y)\cap\Gamma\big)\to X/\Gamma$$

such that the image is an open co-compact subset of $X/\Gamma$ (i.e., it contains complement of a compact subset of $X/\Gamma$). It is also evident now that for any $b\leqq c$ if

$$\Omega_b = X - \bigcup_{Y\in\mathfrak{R}} \pi\big(KA_b^0(Y)\,D_1(Y)\Gamma\big),$$

then $p(\Omega_b)$ is a strong deformation retract of $X/\Gamma$.

In the sequel $\mathfrak{S}_b$ (resp. $\mathfrak{S}_b^0$) will denote the closed set

$$\bigcup_{Y\in\mathfrak{R}} \pi\big(KA_b(Y)\,D_1(Y)\Gamma\big) \quad \text{(resp. the open set } \bigcup_{Y\in\mathfrak{R}} \pi\big(KA_b^0(Y)\,D_1(Y)\Gamma\big)\text{)}.$$

## § 3. Two Lemmas

In this section we shall prove two results of technical nature.

**3.1. Lemma.** *Let $G$ be a real analytic semi-simple group with trivial center and let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition of the Lie algebra $\mathfrak{g}$ of $G$. Let $B$ be a parabolic subgroup of $G$ and $U$ the unipotent radical of $B$. Let $\mathfrak{u}$ (resp. $\mathfrak{b}$) be the Lie algebra of $U$ (resp. $B$). If $[U,U]$ is central in $U$, then we can find a* (unique) $Y\in\mathfrak{p}$ *such that*

(1) $\mathfrak{u}=\mathfrak{g}_Y^1+\mathfrak{g}_Y^2$, $\quad\mathfrak{b}=\mathfrak{g}_Y^0+\mathfrak{g}_Y^1+\mathfrak{g}_Y^2$ *and*
$\mathfrak{g}=\mathfrak{g}_Y^{-2}+\mathfrak{g}_Y^{-1}+\mathfrak{g}_Y^0+\mathfrak{g}_Y^1+\mathfrak{g}_Y^2$,

*where for* $\alpha=0, \pm 1, \pm 2$, $\mathfrak{g}_Y^\alpha$ *is the eigenspace of* $\operatorname{ad} Y$ *corresponding to the eigenvalue* $\alpha$.

(2) $\mathfrak{g}_Y^2=[\mathfrak{u},\mathfrak{u}]$.

*Proof.* Let $\mathfrak{a}\subset\mathfrak{p}$ be a Cartan subspace. Let $\mathfrak{g}=\sum_{\varphi\in\Phi}\mathfrak{g}^\varphi+\mathfrak{g}^0$ be the root space decomposition of the Lie algebra $\mathfrak{g}$ with respect to $\mathfrak{a}$. Let us fix a Weyl chamber in $\mathfrak{a}$, this gives rise to an ordering on the set $\Phi$ of roots. Let $\Phi^+$ be the set of positive roots and $\Delta\subset\Phi^+$ be the simple roots. Let $\mathfrak{n}^+$ be the nilpotent subalgebra $\sum_{\varphi\in\Phi^+}\mathfrak{g}^\varphi$.

Let $K$ be the analytic subgroup of $G$ with Lie algebra $\mathfrak{k}$. Since any parabolic subgroup of $G$ is conjugate by an element of $K$ to a standard parabolic subgroup (cf. Preliminaries) and since $\mathfrak{p}$ is stable under $\operatorname{Ad}K$ it suffices to prove the lemma in the case when $B=B_\Psi$ for a $\Psi\subset\Delta$.

Let $Y\in\mathfrak{a}$ be the unique element such that $\alpha(Y)=0$ for $\alpha\in\Psi$ and $\alpha(Y)=1$ if $\alpha\in\Delta-\Psi$. Let $\mathbf{N}$ be the set of natural numbers and for $m\in\mathbf{Z}$, let $\mathfrak{g}_Y^m$ denote the eigenspace of $\operatorname{ad} Y$ corresponding to the eigenvalue $m$. Then clearly

$$\mathfrak{g}=\sum_{n\in\mathbf{N}}\mathfrak{g}_Y^{-n}+\mathfrak{g}_Y^0+\sum_{n\in\mathbf{N}}\mathfrak{g}_Y^n$$

$$\mathfrak{u}=\mathfrak{u}_\Psi=\sum_{n\in\mathbf{N}}\mathfrak{g}_Y^n$$

$$\mathfrak{b}=\mathfrak{b}_\Psi=\mathfrak{g}_Y^0+\sum_{n\in\mathbf{N}}\mathfrak{g}_Y^n \quad\text{and}\quad \mathfrak{n}^+\supset\mathfrak{u}=\sum_{n\in\mathbf{N}}\mathfrak{g}_Y^n.$$

As is well known $\mathfrak{n}^+$ is generated (as a Lie algebra) by the space $\sum_{\alpha\in\Delta}\mathfrak{g}^\alpha$ and clearly $\sum_{\alpha\in\Delta}\mathfrak{g}^\alpha\subset\mathfrak{g}_Y^0+\mathfrak{g}_Y^1$, it follows that $\mathfrak{u}=\sum_{n\in\mathbf{N}}\mathfrak{g}_Y^n$ is contained in the Lie algebra generated by $\mathfrak{g}_Y^0+\mathfrak{g}_Y^1$. Now since by the hypothesis $[\mathfrak{u},\mathfrak{u}]$ is central in $\mathfrak{u}$ and since $\mathfrak{g}_Y^1\subset\mathfrak{u}$ it follows (using Jacobi's identity) that $\mathfrak{g}_Y^0+\mathfrak{g}_Y^1+[\mathfrak{g}_Y^1,\mathfrak{g}_Y^1]$ is a subalgebra. This implies that

$$\sum_{n\in\mathbf{N}}\mathfrak{g}_Y^n=\mathfrak{g}_Y^1+[\mathfrak{g}_Y^1,\mathfrak{g}_Y^1]$$

so

$$[\mathfrak{g}_Y^1,\mathfrak{g}_Y^1]=\mathfrak{g}_Y^2 \quad\text{and}\quad \mathfrak{g}_Y^n=0 \quad \text{for } n>2.$$

Since $\mathfrak{g}_Y^{-n}$ is dual to $\mathfrak{g}_Y^n$ under the Killing form, it follows that $\mathfrak{g}_Y^{-n}=0$ if $n>2$. Hence

$$\mathfrak{g}=\mathfrak{g}_Y^{-2}+\mathfrak{g}_Y^{-1}+\mathfrak{g}_Y^0+\mathfrak{g}_Y^1+\mathfrak{g}_Y^2$$

$$\mathfrak{b}=\mathfrak{g}_Y^0+\mathfrak{g}_Y^1+\mathfrak{g}_Y^2, \qquad \mathfrak{u}=\mathfrak{g}_2^1+\mathfrak{g}_Y^2$$

and

$$[\mathfrak{u},\mathfrak{u}]=[\mathfrak{g}_Y^1,\mathfrak{g}_Y^1]=\mathfrak{g}_Y^2.$$

The uniqueness assertion of the lemma[4] follows from the conjugacy of maximal **R**-split tori in $B$ and the fact that $\mathfrak{a}$ is the Lie algebra of a maximal **R**-split torus in $B$.

In the following the real rational points of a connected reductive algebraic group defined over **R** will be called a reductive real algebraic group. Note that a reductive real algebraic group need not be connected but in any case it has only finitely many connected components.

The proof of our next lemma depends upon the strong rigidity of uniform lattices (proved in Mostow [9]) and the classification of two dimensional closed differentiable manifolds.

**3.2. Lemma.** *Let $\Lambda$ (resp. $\Lambda'$) be a uniform lattice in a reductive real algebraic group $M$ (resp. $M'$). We assume that $\Lambda$, $\Lambda'$ are net. Let $L$ (resp. $L'$) be a maximal compact subgroup of $M$ (resp. $M'$) and $Y = L\backslash M$, $Y' = L'\backslash M'$ be the associated "symmetric spaces". Let $\theta\colon \Lambda \to \Lambda'$ be an isomorphism. Then there is a $C^\infty$ diffeomorphism $\varphi\colon Y \to Y'$ such that $\varphi(y\lambda) = \varphi(y)\,\theta(\lambda)$ for $y \in Y$, $\lambda \in \Lambda$.*

*Proof.* Let $\mathbf{M} \subset GL(\mathbf{V})$ (resp. $\mathbf{M}' \subset GL(\mathbf{V}')$) be a connected reductive algebraic group defined over **R** such that $M = \mathbf{M}_{\mathbf{R}}$ (resp. $M' = \mathbf{M}'_{\mathbf{R}}$) and let $\mathbf{M} = \mathbf{S}\cdot\mathbf{T}$, $\mathbf{M}' = \mathbf{S}'\cdot\mathbf{T}'$ (almost direct products) where $\mathbf{T}$ (resp. $\mathbf{T}'$) is the maximal central torus in $\mathbf{M}$ (resp. $\mathbf{M}'$) and $\mathbf{S} = [\mathbf{M}, \mathbf{M}]$, $\mathbf{S}' = [\mathbf{M}', \mathbf{M}']$ are connected normal semi-simple algebraic subgroups defined over **R**. If $\mathbf{H}$ and $\bar{\mathbf{H}}$ are algebraic groups defined over **R** and if $\pi\colon \mathbf{H} \to \bar{\mathbf{H}}$ is a surjective morphism defined over **R**, then $\pi(\mathbf{H}_{\mathbf{R}})$ is a subgroup of $\bar{\mathbf{H}}_{\mathbf{R}}$ of finite index. This implies that if $\mathbf{K}$ is a maximal compact subgroup of $\bar{\mathbf{H}}_{\mathbf{R}}$ then $\mathbf{K} \cap \pi(\mathbf{H}_{\mathbf{R}})$ is a maximal compact subgroup $\pi(\mathbf{H}_{\mathbf{R}})$ and the natural inclusion $\pi(\mathbf{H}_{\mathbf{R}}) \cap \mathbf{K} \backslash \pi(\mathbf{H}_{\mathbf{R}}) \to \mathbf{K} \backslash \bar{\mathbf{H}}_{\mathbf{R}}$ is a diffeomorphism. Thus since $L$ (resp. $L'$) obviously contains the maximal compact normal subgroup of $M$ (resp. $M'$) we can, after dividing out the center of $\mathbf{S}$, the maximal connected normal **R**-anisotropic subgroup[5] of $\mathbf{S}$ and also the **R**-anisotropic component of $\mathbf{T}$, assume that $\mathbf{T}$ is split over **R**, $\mathbf{S}$ has no **R**-anisotropic factors and further $\mathbf{S}$ has trivial center. Thus $\mathbf{T} \cap \mathbf{S} = \{e\}$ which implies that $\mathbf{M}$ is actually direct product of $\mathbf{T}$ and $\mathbf{S}$ and $\mathbf{M}_{\mathbf{R}} = M = \mathbf{S}_{\mathbf{R}} \times \mathbf{T}_{\mathbf{R}}$. Similarly we can assume that $\mathbf{T}'$ is split over **R**, $\mathbf{S}'$ has trivial center and no nontrivial **R**-anisotropic factors and then $\mathbf{M}' = \mathbf{S}' \times \mathbf{T}'$ (direct product), $\mathbf{M}'_{\mathbf{R}} = M' = \mathbf{S}'_{\mathbf{R}} \times \mathbf{T}'_{\mathbf{R}}$. Let $\alpha\colon \mathbf{S}_{\mathbf{R}} \times \mathbf{T}_{\mathbf{R}} \to \mathbf{S}_{\mathbf{R}}$ (resp. $\alpha'\colon \mathbf{S}'_{\mathbf{R}} \times \mathbf{T}'_{\mathbf{R}} \to \mathbf{S}'_{\mathbf{R}}$) be the natural projection. Then since an algebraic morphism takes net subgroups into net subgroups, both $\alpha(\Lambda)$ and $\alpha'(\Lambda')$ are net.

$\mathbf{T}_{\mathbf{R}}$ is direct product of its maximal finite subgroup $\mathbf{F}$ and the identity component $\mathbf{T}^0_{\mathbf{R}}$, similarly $\mathbf{T}'_{\mathbf{R}}$ is direct product of its maximal finite sub-

[4] In the sequel we shall not use uniqueness.

[5] An algebraic group defined over **R** is said to be anisotropic over **R** if it has no nontrivial **R**-split torus. A reductive algebraic group defined over **R** is anisotropic over **R** if and only if the group of its **R**-rational points is compact in the Hausdorff topology.

group $\mathbf{F}'$ and the identity component $\mathbf{T}_{\mathbf{R}}'^0$. Clearly the finite central subgroups $\mathbf{F}$ and $\mathbf{F}'$ are contained respectively in $L$ and $L'$. Thus dividing out $\mathbf{M}$ and $\mathbf{M}'$ by $\mathbf{F}$ and $\mathbf{F}'$ respectively we can assume that we are in the following situation. $M=\mathbf{S}_{\mathbf{R}}\times T$ and $M'=\mathbf{S}'_{\mathbf{R}}\times T'$ (direct products) where $T$ and $T'$ are (connected) vector groups; $\mathbf{S}$, $\mathbf{S}'$ are *connected* semi-simple algebraic groups defined over $\mathbf{R}$ which have no nontrivial $\mathbf{R}$-anisotropic normal subgroups and in particular they have trivial center. (Of course we no longer assume that $M$ and $M'$ are real reductive algebraic groups.) $\Lambda$ (resp. $\Lambda'$) is a uniform lattice in $M$ (resp. $M'$) such that if we denote the natural projection $\mathbf{S}_{\mathbf{R}}\times T\to\mathbf{S}_{\mathbf{R}}$ (resp. $\mathbf{S}'_{\mathbf{R}}\times T'\to\mathbf{S}'_{\mathbf{R}}$) again by $\alpha$ (resp. $\alpha'$), then $\alpha(\Lambda)$ (resp. $\alpha'(\Lambda')$) is a net subgroup of $\mathbf{S}_{\mathbf{R}}$ (resp. $\mathbf{S}'_{\mathbf{R}}$); $L$ (resp. $L'$) is a maximal compact subgroup of $M$ (resp. $M'$). Since $T$ and $T'$ are vector groups, $L\subset\mathbf{S}_{\mathbf{R}}$ and $L'\subset\mathbf{S}'_{\mathbf{R}}$.

It easily follows from a result of Auslander (see [12, Th. 8.24]) and Borel's density theorem that $\alpha(\Lambda)$ is discrete and hence is a uniform lattice in $\mathbf{S}_{\mathbf{R}}$ and $\Lambda\cap T$ is a lattice in $T$. From this it is also clear that (since $\mathbf{S}$ has trivial center) $\Lambda\cap T$ is precisely the center of $\Lambda$. Similarly $T'\cap\Lambda'$ is the center of $\Lambda'$, $\alpha'(\Lambda')$ is a uniform lattice in $\mathbf{S}'_{\mathbf{R}}$ and $T'\cap\Lambda'$ is a lattice in $T'$. The isomorphism $\theta\colon\Lambda\to\Lambda'$ thus defines isomorphisms $\theta|_{T\cap\Lambda}\colon T\cap\Lambda\to T'\cap\Lambda'$ and $\bar\theta\colon\Lambda/T\cap\Lambda\to\Lambda'/T'\cap\Lambda'$. Since $L\diagdown M=(L\diagdown\mathbf{S}_{\mathbf{R}})\times T$, $L'\diagdown M'=(L'\diagdown\mathbf{S}'_{\mathbf{R}})\times T'$ and the isomorphism $\theta|_{T\cap\Lambda}\colon T\cap\Lambda\to T'\cap\Lambda'$ extends to a unique analytic isomorphism $T\to T'$ it suffices to prove the result assuming that $T$ and $T'$ are trivial i.e., when $M=\mathbf{S}_{\mathbf{R}}$ and $M'=\mathbf{S}'_{\mathbf{R}}$.

Now since $\Lambda$ is a (uniform) lattice in $\mathbf{S}_{\mathbf{R}}$ by a well known argument using Borel's density theorem (cf. [12, Chap. V]) it follows that $\mathbf{S}=\prod_{i\in I}\mathbf{S}^i$ where $\mathbf{S}^i$'s are connected normal algebraic subgroups of $\mathbf{S}$ defined over $\mathbf{R}$ and $\Lambda_i=\mathbf{S}^i\cap\Lambda$ is an irreducible uniform lattice in $\mathbf{S}^i_{\mathbf{R}}$. As $\Pi\Lambda_i$ is a uniform lattice in $\mathbf{S}_{\mathbf{R}}=\prod_{i\in I}\mathbf{S}^i_{\mathbf{R}}$, it is a subgroup of $\Lambda$ of finite index. Let $\Lambda'_i=\theta(\Lambda_i)$ and let $\mathbf{S}'^i$ be the identity component of the Zariski closure of $\Lambda'_i$ in $\mathbf{S}'$. Clearly $\mathbf{S}'^i$ is an algebraic group defined over $\mathbf{R}$ and $\tilde\Lambda'_i=\Lambda'_i\cap\mathbf{S}'_i$ is a subgroup of finite index in $\Lambda'_i$. Let $\theta^{-1}(\tilde\Lambda'_i)=\tilde\Lambda_i$, then $\tilde\Lambda_i$ is an irreducible uniform lattice in $\mathbf{S}^i_{\mathbf{R}}$. Since $\Lambda'_i$ is normal in the lattice $\theta(\Pi\Lambda_i)$, by Borel's density theorem it follows that $\mathbf{S}'^i$ is a normal subgroup of $\mathbf{S}'$. By density arguments it also follows that $\Pi\mathbf{S}'^i=\mathbf{S}'$ and if $i\neq j$ then $\mathbf{S}'^i$ commutes with $\mathbf{S}'^j$ and so $\mathbf{S}'^i\cap\mathbf{S}'^j$ is trivial. Thus $\mathbf{S}'=\prod_{i\in I}\mathbf{S}'^i$ (direct product). Since $\mathbf{S}'^i_{\mathbf{R}}\cdot\Pi\tilde\Lambda'_j=\mathbf{S}'^i_{\mathbf{R}}\cdot\prod_{j\neq i}\tilde\Lambda'_j$ is a closed subgroup of $\mathbf{S}'_{\mathbf{R}}$, $\mathbf{S}'^i_{\mathbf{R}}\cap(\Pi\tilde\Lambda'_i)=\tilde\Lambda'_i$ is a uniform lattice in $\mathbf{S}'^i_{\mathbf{R}}$.

For $i\in I$ let $p_i\colon\mathbf{S}\to\mathbf{S}^i$ (resp. $p'_i\colon\mathbf{S}'\to\mathbf{S}'^i$) be the natural projection. Let $\Delta=\prod_{i\in I}p_i(\Lambda)$ (resp. $\Delta'=\prod_{i\in I}p'_i(\Lambda')$). Obviously $\Lambda\subset\Delta$ (resp. $\Lambda'\subset\Delta'$). Since for $i\in I$, $p_i(\Lambda)$ (resp. $p'_i(\Lambda')$) is a uniform lattice in $\mathbf{S}^i_{\mathbf{R}}$ (resp. $\mathbf{S}'^i_{\mathbf{R}}$), $\Delta$ (resp. $\Delta'$)

is a uniform lattice in $\mathbf{S_R}$ (resp. $\mathbf{S'_R}$). It can be seen (using Borel's density theorem) that there exists a canonical isomorphism $\boldsymbol{\theta}$: $\Delta \to \Delta'$ such that $\boldsymbol{\theta}|_\Lambda = \theta$. Thus it suffices to prove the lemma assuming that $\Lambda = \Delta$ and $\Lambda' = \Delta'$. Since corresponding to the decomposition $\prod_{i\in I} \mathbf{S}^i_\mathbf{R}$ (resp. $\prod_{i\in I} \mathbf{S}'^i_\mathbf{R}$) of $\mathbf{S_R}$ (resp. $\mathbf{S'_R}$) there is a decomposition of the maximal compact subgroup $L$ (resp. $L'$) and hence also of the associated symmetric spaces it follows that we can further assume that $\Lambda$ and hence $\Lambda'$ are irreducible.

Now we shall consider the two possible cases separately. First we consider the case when $\mathbf{S_R}$ is locally isomorphic to $SL(2, \mathbf{R})$. In this case according to [11, §3] $\mathbf{S'_R}$ has $\mathbf{R}$-rank 1 and since a uniform lattice determines the dimension of the symmetric space associated to the ambient semi-simple Lie group (this follows from a simple cohomology argument), $\dim L' \backslash \mathbf{S'_R} = \dim L \backslash \mathbf{S_R} = 2$ and hence $\mathbf{S'_R}$ is also locally isomorphic to $SL(2, \mathbf{R})$. Now $\xi = (L \backslash \mathbf{S_R}, \beta, L \backslash \mathbf{S_R}/\Lambda)$ (resp. $\xi' = (L' \backslash \mathbf{S'_R}, \beta', L' \backslash \mathbf{S'_R}/\Lambda')$) is a locally trivial $\Lambda$ (resp. $\Lambda'$) bundle where $\beta$: $L \backslash \mathbf{S_R} \to L \backslash \mathbf{S_R}/\Lambda$ (resp. $\beta'$: $L' \backslash \mathbf{S'_R} \to L' \backslash \mathbf{S'_R}/\Lambda'$) is the canonical projection. Since $L \backslash \mathbf{S_R}$ and $L' \backslash \mathbf{S'_R}$ are contractible ($L$, $L'$ being maximal compact subgroups of respectively $\mathbf{S_R}$ and $\mathbf{S'_R}$) it follows that these bundles are classifying principal bundles. Now if we identify $\Lambda'$ with $\Lambda$ with the help of the isomorphism $\theta$ then from the properties of classifying bundles (see for example [4]) it follows that there is a homotopy equivalence $\bar{\varphi}_0$: $L \backslash \mathbf{S_R}/\Lambda \to L' \backslash \mathbf{S'_R}/\Lambda'$ such that the bundle induced by $\bar{\varphi}_0$ from $\xi'$ is isomorphic to the bundle $\xi$ (note that we have identified $\Lambda'$ with $\Lambda$). Now since $L \backslash \mathbf{S_R}/\Lambda$ and $L' \backslash \mathbf{S'_R}/\Lambda'$ are two dimensional closed manifolds, it follows from the classification of such manifolds (see [1, §7] and [6]) that there is a diffeomorphism $\bar{\varphi}$: $L \backslash \mathbf{S_R}/\Lambda \to L' \backslash \mathbf{S'_R}/\Lambda'$ which is homotopic to $\bar{\varphi}_0$. Since homotopic maps induce isomorphic bundles it follows that $\bar{\varphi}^*(\xi') \approx \xi$ and hence there is a map $\varphi$: $L \backslash \mathbf{S_R} \to L' \backslash \mathbf{S'_R}$ such that: $\varphi(x\lambda) = \varphi(x)\,\theta(\lambda)$ and the induced map $L \backslash \mathbf{S_R}/\Lambda \to L' \backslash \mathbf{S'_R}/\Lambda'$ is the diffeomorphism $\bar{\varphi}$. Clearly then $\varphi$ is a diffeomorphism and in this case the proof is complete.

Next we consider the case when $\mathbf{S_R}$ (and therefore $\mathbf{S'_R}$) is not locally isomorphic to $SL(2, \mathbf{R})$. Since[6] $\mathbf{S_R}/\mathbf{S}^0_\mathbf{R}$ and $\mathbf{S'_R}/\mathbf{S}'^0_\mathbf{R}$ are finite, there exists a subgroup $\Lambda_1$ of $\Lambda$ of finite index such that $\Lambda_1$ is contained in $\mathbf{S}^0_\mathbf{R}$ and $\theta(\Lambda_1)$ is contained in $\mathbf{S}'^0_\mathbf{R}$. Clearly $\Lambda_1$ (resp. $\theta(\Lambda_1)$) is an irreducible uniform lattice in $\mathbf{S}^0_\mathbf{R}$ (resp. in $\mathbf{S}'^0_\mathbf{R}$). According to Mostow [9] there exists an isomorphism $\boldsymbol{\theta}$: $\mathbf{S}^0_\mathbf{R} \to \mathbf{S}'^0_\mathbf{R}$ such that $\boldsymbol{\theta}|\Lambda_1 = \theta|\Lambda_1$. Let us consider now the groups $\mathbf{S}^0_\mathbf{R}\Lambda$ and $\mathbf{S}'^0_\mathbf{R}\Lambda'$, clearly these are subgroups of finite index in $\mathbf{S_R}$ and $\mathbf{S'_R}$ respectively. We define a map $\psi$: $\mathbf{S}^0_\mathbf{R}\Lambda \to \mathbf{S}'^0_\mathbf{R}\Lambda'$ by setting $\psi(x\lambda) = \boldsymbol{\theta}(x)\,\theta(\lambda)$ for $x \in \mathbf{S}^0_\mathbf{R}$ and $\lambda \in \Lambda$. It can be checked that $\psi$ is a well defined isomorphism.

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[6] For a group $H$ we let $H^0$ denote the connected component of the identity in $H$.

Since the natural inclusions $(L \cap \mathbf{S}^0_{\mathbf{R}} \Lambda) \backslash \mathbf{S}^0_{\mathbf{R}} \Lambda \to L \backslash \mathbf{S}_{\mathbf{R}}$ and

$$(L' \cap \mathbf{S}'^0_{\mathbf{R}} \Lambda') \backslash \mathbf{S}'^0_{\mathbf{R}} \Lambda' \to L' \backslash \mathbf{S}'_{\mathbf{R}}$$

are diffeomorphisms it suffices to show that there is a $C^\infty$ diffeomorphism

$$\varphi\colon (L \cap \mathbf{S}^0_{\mathbf{R}} \Lambda) \backslash \mathbf{S}^0_{\mathbf{R}} \Lambda \to (L' \cap \mathbf{S}'^0_{\mathbf{R}} \Lambda') \backslash \mathbf{S}'^0_{\mathbf{R}} \Lambda'$$

such that

$$\varphi(x\lambda) = \varphi(x)\,\theta(\lambda) \quad \text{for } x \in (L \cap \mathbf{S}^0_{\mathbf{R}} \Lambda) \backslash \mathbf{S}^0_{\mathbf{R}} \Lambda \quad \text{and} \quad \lambda \in \Lambda.$$

Now if $L' \cap \mathbf{S}'^0_{\mathbf{R}} \Lambda' = \psi(L \cap \mathbf{S}^0_{\mathbf{R}} \Lambda)$ then $\psi$ induces a diffeomorphism of required type between the symmetric spaces, but in general $L' \cap \mathbf{S}'^0_{\mathbf{R}} \Lambda'$ may not be equal to $\psi(L \cap \mathbf{S}^0_{\mathbf{R}} \Lambda)$. In any case the groups $L' \cap \mathbf{S}'^0_{\mathbf{R}} \Lambda'$ and $\psi(L \cap \mathbf{S}^0_{\mathbf{R}} \Lambda)$ are maximal compact subgroups of the group $\mathbf{S}'^0_{\mathbf{R}} \Lambda'$ which has finitely many connected components, hence there is an element $g'$ in the identity component $\mathbf{S}'^0_{\mathbf{R}}$ of $\mathbf{S}'^0_{\mathbf{R}} \Lambda'$ such that

$$\psi(L \cap \mathbf{S}^0_{\mathbf{R}} \Lambda) = g'(L' \cap \mathbf{S}'^0_{\mathbf{R}} \Lambda')\, g'^{-1}$$

Let us consider the inner automorphism $y' \mapsto g' y' g'^{-1}$ of the group $\mathbf{S}'^0_{\mathbf{R}} \Lambda'$. This induces a diffeomorphism

$$\varphi'_0\colon (L' \cap \mathbf{S}'^0_{\mathbf{R}} \Lambda') \backslash \mathbf{S}'^0_{\mathbf{R}} \Lambda' \to \psi(L \cap \mathbf{S}^0_{\mathbf{R}} \Lambda) \backslash \mathbf{S}'^0_{\mathbf{R}} \Lambda'$$

such that,

$$\varphi'_0(y' \cdot a') = \varphi_0(y') \cdot g' a' g'^{-1} \quad \text{for } y' \in (L' \cap \mathbf{S}'^0_{\mathbf{R}} \Lambda') \backslash \mathbf{S}'^0_{\mathbf{R}} \Lambda'$$

and $a' \in \mathbf{S}'^0_{\mathbf{R}} \Lambda'$.

Thus to complete the proof of the lemma it suffices to show that there is a diffeomorphism

$$\varphi_0\colon (L \cap \mathbf{S}^0_{\mathbf{R}} \Lambda) \backslash \mathbf{S}^0_{\mathbf{R}} \Lambda \to \psi(L \cap \mathbf{S}^0_{\mathbf{R}} \Lambda) \backslash \mathbf{S}'^0_{\mathbf{R}} \Lambda'$$

such that

$$\varphi_0(x\lambda) = \varphi_0(x)\, g' \theta(\lambda)\, g'^{-1} \quad \text{for } x \in (L \cap \mathbf{S}^0_{\mathbf{R}} \Lambda) \backslash \mathbf{S}^0_{\mathbf{R}} \Lambda \quad \text{and} \quad \lambda \in \Lambda.$$

For convenience we shall denote the symmetric space $(L \cap \mathbf{S}^0_{\mathbf{R}} \Lambda) \backslash \mathbf{S}^0_{\mathbf{R}} \Lambda$ by $Y$ and shall identify the group $\mathbf{S}'^0_{\mathbf{R}} \Lambda'$ with $\mathbf{S}^0_{\mathbf{R}} \Lambda$ and the symmetric space $(L \cap \mathbf{S}^0_{\mathbf{R}} \Lambda) \backslash \mathbf{S}'^0_{\mathbf{R}} \Lambda'$ with $Y$ with the help of the isomorphism $\psi$. Let $\gamma\colon [0, 1] \to \mathbf{S}'^0_{\mathbf{R}}$ be a differentiable curve such that $\gamma(0)$ is the identity and $\gamma(1) = g'$. We have a differentiable map

$$I \times \Lambda \times Y \to Y$$

$$(t, \lambda, y) \mapsto y \cdot \gamma(t) \lambda \gamma(t^{-1}) \quad \text{for } t \in I, \lambda \in \Lambda \quad \text{and} \quad y \in Y.$$

We can now use Theorem 4 of Koszul [5, §3 (p. 59)], in a suitably modified form, and compactness of [0, 1] to produce a diffeomorphism $\varphi_0\colon Y \to Y$ such that

$$\varphi_0(y\lambda) = \varphi_0(y) \cdot \gamma(1) \lambda \gamma(1)^{-1} = \varphi_0(y) \cdot g' \lambda g'^{-1} \quad \text{for } y \in Y \quad \text{and} \quad \lambda \in \Lambda$$

(note that we have identified $\mathbf{S}'^0_{\mathbf{R}} A'$ with $\mathbf{S}^0_{\mathbf{R}} A$ and hence $A'$ with $A$). This completes the proof of Lemma 3.2.

## § 4. Proof of Theorem C

In this section we shall use the notations introduced in §2. Let $\mathscr{R}(\subset \mathfrak{p})$ and $c \in \mathbf{R}$ be as in Proposition 2.1. For $Y \in \mathscr{R}$ since $\Phi_Y = U(Y) \cap \Gamma$ is a maximal unipotent subgroup of $\Gamma$ and since $\Gamma$ is a $\mathbf{Q}$-rank 1 lattice, $[\Phi_Y, \Phi_Y]$ is central in $\Phi_Y$ and hence $[U(Y), U(Y)]$ is central in $U(Y)$. Also since $B(Y)$ is a parabolic subgroup with unipotent radical $U(Y)$ we can assume, in view of Lemma 3.1, that $\mathscr{R}$ is so chosen that for every $Y \in \mathscr{R}$ (in the notations of Lemma 3.1)

$$\mathfrak{u}(Y) = \mathfrak{g}^1_Y + \mathfrak{g}^2_Y, \qquad \mathfrak{b}(Y) = \mathfrak{g}^0_Y + \mathfrak{g}^1_Y + \mathfrak{g}^2_Y$$

and $[\mathfrak{u}(Y), \mathfrak{u}(Y)] = \mathfrak{g}^2_Y$.

Since $G$ and therefore $G'$ is not locally isomorphic to $SL(2, \mathbf{R})$ and since $\Gamma$, $\Gamma'$ are net according to [10, Theorem 3.1] the isomorphism $\theta$ takes unipotent elements into unipotent elements and vice-versa. Thus for any $Y \in \mathscr{R}$, $\theta(\Phi_Y)$ is a maximal unipotent subgroup of $\Gamma'$ and hence in view of Proposition 2.1 and Lemma 3.1, there exists a $Y' \in \mathfrak{p}'$ such that $\theta(\Phi_Y) = U(Y') \cap \Gamma' = \Phi'_{Y'}$; $\Phi'_{Y'}$ is a lattice in $U(Y')$ and $\mathfrak{u}(Y') = \mathfrak{g}'^1_{Y'} + \mathfrak{g}'^2_{Y'}$, $\mathfrak{b}(Y') = \mathfrak{g}'^0_{Y'} + \mathfrak{g}'^1_{Y'} + \mathfrak{g}'^2_{Y'}$. Thus we get a finite subset $\mathscr{R}'$ of $\mathfrak{p}'$ and a bijection $\mathscr{R} \to \mathscr{R}'$. In the sequel image of any $Y \in \mathscr{R}$ under this bijection will be denoted by $Y'$.

Clearly any maximal unipotent subgroup of $\Gamma'$ is conjugate (in $\Gamma'$) to $U(Y') \cap \Gamma'$ for a unique $Y' \in \mathscr{R}'$. Hence according to Proposition 2.1, there exists a constant $c' \in \mathbf{R}$ such that if $Y'_1, Y'_2 \in \mathscr{R}$ and $Y'_1 \neq Y'_2$ then

$$K' A'_{c'}(Y'_1) D_1(Y'_1) \Gamma' \cap K' A'_{c'}(Y'_2) D_1(Y'_2) \Gamma' = \emptyset$$

and for $Y' \in \mathscr{R}'$, $\gamma' \in \Gamma'$; $t_1, t_2 \leqq c'$; $k'_1, k'_2 \in K'$ and $d'_1, d'_2 \in D_1(Y')$ if

$$k'_1 \exp t_1 Y' d'_1 = k'_2 \exp t_2 Y' d'_2 \gamma',$$

then $t_1 = t_2$, $\gamma' \in D_1(Y') \cap \Gamma'$. Also if $\pi'$: $G' \to K' \setminus G'$ and $p'$: $X' \to X'/\Gamma'$ are the natural projections then for any $t \in \mathbf{R}$ the set

$$\Omega'_t = X' - \pi'\Big(\bigcup_{Y' \in \mathscr{R}'} K' A^0_t(Y') D_1(Y') \Gamma'\Big)$$

is compact modulo $\Gamma'$, i.e., $p'(\Omega'_t)$ is compact.

In the sequel we shall denote

$$\bigcup_{Y' \in \mathscr{R}'} \pi'(K' A^0_t(Y') D_1(Y') \Gamma') \qquad \Big(\text{resp.} \bigcup_{Y' \in \mathscr{R}'} \pi'(K' A_t(Y') D_1(Y') \Gamma')\Big)$$

by $'\mathfrak{S}_t^0$ (resp. $'\mathfrak{S}_t$). Also for any $Y \in \mathscr{R}$ (resp. $Y' \in \mathscr{R}'$) $K_Y$ will denote $K \cap D_1(Y)$ (resp. $K'_{Y'}$ will denote $K' \cap D'_1(Y')$). As has been remarked earlier $K_Y$ (resp. $K'_{Y'}$) is a maximal compact subgroup of $D_1(Y)$ (resp. $D_1(Y')$).

For $Y \in \mathscr{R}$, let $\alpha_Y$ be the analytic isomorphism from $U(Y)$ to $U(Y')$ determined by the isomorphism $\theta|_{\Phi_Y}: \Phi_Y \to \Phi'_{Y'}$ (cf. Lemma 0.2, note that a connected unipotent group is simply connected). For a subgroup $H$ of a group $M$, let $N_M(H)$ denote the normalizer of $H$ in $M$. Since $B(Y) \cap \Gamma = N_\Gamma(\Phi_Y)$ (resp. $B(Y') \cap \Gamma' = N_{\Gamma'}(\Phi'_{Y'})$) and since $\Phi_Y$ is a lattice in $U(Y)$ it follows that every element in $N_\Gamma(\Phi_Y)$ preserves a Haar measure on $U(Y)$ thus $N_\Gamma(\Phi_Y) \subset D_1(Y)$ and hence $\Lambda_Y = N_\Gamma(\Phi_Y) = D_1(Y) \cap \Gamma$. By similar considerations $\Lambda'_{Y'} = N_{\Gamma'}(\Phi'_{Y'}) = D_1(Y') \cap \Gamma'$. The subgroups $U(Y)\Lambda_Y$ and $U(Y')\Lambda'_{Y'}$ are closed subgroups of $G$ and $G'$ respectively since

$$U(Y) \cap \Lambda_Y = \Phi_Y \qquad (\text{resp. } U(Y') \cap \Lambda'_{Y'} = \Phi'_{Y'})$$

is a (uniform) lattice in $U(Y)$ (resp. $U(Y')$). We define a map $\beta_Y: U(Y)\Lambda_Y \to U(Y')\Lambda'_{Y'}$ by setting

$$\beta_Y(u \cdot \lambda) = \alpha_Y(u) \cdot \theta(\lambda) \qquad \text{for } u \in U(Y) \quad \text{and} \quad \lambda \in \Lambda_Y$$

It can be easily checked that this is a well defined analytic isomorphism.

Now let us consider the spaces $K_Y \backslash D_1(Y)$ and $K'_{Y'} \backslash D_1(Y')$. They are contractible since $K_Y$ (resp. $K'_{Y'}$) is a maximal compact subgroup of $D_1(Y)$ (resp. $D_1(Y')$). We assert that $(K_Y \backslash D_1(Y), \pi_Y, K_Y \backslash D_1(Y)/U(Y)\Lambda_Y)$ is a locally trivial principal $U(Y)\Lambda_Y$ bundle, where

$$\pi_Y: K_Y \backslash D_1(Y) \to K_Y \backslash D_1(Y)/U(Y)\Lambda_Y$$

is the natural projection.

Since $B(Y) = M(Y) \cdot U(Y)$ (a semi-direct product, cf. Preliminaries) and $D_1(Y) \supset U(Y)$,

$$D_1(Y) = (D_1(Y) \cap M(Y)) \cdot U(Y) \qquad (\text{a semi-direct product}).$$

Also recall that $K_Y \subset D_1(Y) \cap M(Y)$ thus $K_Y \backslash D_1(Y) \to K_Y \backslash D_1(Y)/U(Y)$ is a trivial principal $U(Y)$ bundle. Since $U(Y)$ is a normal subgroup of $D_1(Y)$, there is a natural action of the discrete subgroup $U(Y)\Lambda_Y/U(Y)$ on the space $K_Y \backslash D_1(Y)/U(Y)$ on the right. Since $\Gamma$ and hence $\Lambda_Y$ is net, $U(Y)\Lambda_Y/U(Y)$ has no nontrivial torsion element, for if $\lambda \in \Lambda_Y$ projects onto a torsion element in $U(Y)\Lambda_Y/U(Y)$ then for a suitable positive integer $n$, $\lambda^n$ is contained in the unipotent group $U(Y)$ and hence all the eigenvalues of $\lambda$ are roots of unity, since $\lambda$ is net this implies that $\lambda$ is unipotent and hence it is contained in $U(Y)$ which shows that $U(Y)\Lambda_Y/U(Y)$ has no nontrivial torsion elements. From this one easily concludes that the action of $U(Y)\Lambda_Y/U(Y)$ on $K_Y \backslash D_1(Y)/U(Y)$ is fixed point free and hence $K_Y \backslash D_1(Y)/U(Y) \to K_Y \backslash D_1(Y)/U(Y)\Lambda_Y$ is a locally trivial principal $U(Y)\Lambda_Y/U(Y)$ bundle and thus $(K_Y\setminus D_1(Y), \pi_Y, K_Y D_1(Y)/U(Y)\Lambda_Y)$ is a locally trivial principal $U(Y)\Lambda_Y$ bundle. Similarly for $Y'\in\mathscr{R}'$,

$$(K'_{Y'}\setminus D_1(Y'), \pi_{Y'}, K'_{Y'}\setminus D_1(Y')/U(Y')\Lambda'_{Y'})$$

is a locally trivial principal $U(Y')\Lambda'_{Y'}$ bundle, where $\pi_{Y'}$ is the canonical projection

$$K'_{Y'}\setminus D_1(Y')\to K'_{Y'}\setminus D_1(Y')/U(Y')\Lambda'_{Y'}.$$

Since $K_Y\setminus D_1(Y)$ and $K'_{Y'}\setminus D_1(Y')$ are contractible, these bundles are classifying principal bundles. Now if we identify $U(Y)\Lambda_Y$ and $U(Y')\Lambda'_{Y'}$ with the help of the isomorphism $\beta_Y$, then it follows from the theory of principal bundles (cf. [4]) that there is a map

$$K_Y\setminus D_1(Y)\to K'_{Y'}\setminus D_1(Y')$$

which we denote again by $\alpha_Y$ such that

$$\alpha_Y(\delta\cdot\lambda)=\alpha_Y(\delta)\,\beta_Y(\lambda)\quad \text{for } \delta\in K_Y\setminus D_1(Y) \text{ and } \lambda\in U(Y)\Lambda_Y. \tag{1}$$

Let us consider the commutative diagram

$$\begin{array}{ccc}
K_Y\setminus D_1(Y) & \xrightarrow{\alpha_Y} & K'_{Y'}\setminus D_1(Y') \\
\downarrow & & \downarrow \\
K_Y\setminus D_1(Y)/U(Y) & \xrightarrow{\bar\alpha_Y} & K'_{Y'}\setminus D_1(Y')/U(Y') \\
\downarrow & & \downarrow \\
K_Y\setminus D_1(Y)/U(Y)\Lambda_Y & \xrightarrow{\bar{\bar\alpha}_Y} & K'_{Y'}\setminus D_1(Y')/U(Y')\Lambda'_{Y'}
\end{array}$$

where the spaces

$$(K_Y\setminus D_1(Y)/U(Y))/(U(Y)\Lambda_Y/U(Y))$$

and

$$(K'_{Y'}\setminus D_1(Y')/U(Y'))/(U(Y')\Lambda'_{Y'}/U(Y'))$$

have been identified with respectively $K_Y\setminus D_1(Y)/U(Y)\Lambda_Y$ and $K'_{Y'}\setminus D_1(Y')/U(Y')\Lambda'_{Y'}$ in the canonical way, the vertical arrows are the natural projections and $\bar\alpha_Y$, $\bar{\bar\alpha}_Y$ are the maps induced by $\alpha_Y$. Since $D_1(Y)/U(Y)$ and $D_1(Y')/U(Y')$ are reductive real algebraic groups and $U(Y)\Lambda_Y/U(Y)$ (resp. $U(Y')\Lambda'_{Y'}/U(Y')$) is a uniform lattice in $D_1(Y)/U(Y)$ (resp. $D_1(Y')/U(Y')$) and since $\beta_Y$: $U(Y)\Lambda_Y\to U(Y')\Lambda'_{Y'}$ induces an isomorphism

$$\bar\beta_Y\colon\ U(Y)\Lambda_Y/U(Y)\to U(Y')\Lambda'_{Y'}/U(Y')$$

it follows from Lemma 3.2 that there is a $C^\infty$ diffeomorphism

$$\bar{\varphi}_Y\colon\ K_Y \setminus D_1(Y)/U(Y) \to K'_{Y'} \setminus D_1(Y')/U(Y')$$

such that for $x \in K_Y \setminus D_1(Y)/U(Y)$ and $\bar{\lambda} \in U(Y)\Lambda_Y/U(Y)$

$$\bar{\varphi}_Y(x\bar{\lambda}) = \bar{\varphi}_Y(x)\,\bar{\beta}_Y(\bar{\lambda}).$$

Let $\bar{\bar{\varphi}}_Y$ be the map

$$K_Y \setminus D_1(Y)/U(Y)\Lambda_Y \to K'_{Y'} \setminus D_1(Y')/U(Y')\Lambda'_{Y'}$$

induced by $\bar{\varphi}_Y$. Again since $K_Y \setminus D_1(Y)/U(Y)$ and $K'_{Y'} \setminus D_1(Y')/U(Y')$ are contractible, the bundle

$$K_Y \setminus D_1(Y)/U(Y) \to K_Y \setminus D_1(Y)/U(Y)\Lambda_Y$$

$$\left(\text{resp. } K'_{Y'} \setminus D_1(Y')/U(Y') \to K'_{Y'} \setminus D_1(Y')/U(Y')\Lambda'_{Y'}\right)$$

is a universal $U(Y)\Lambda_Y/U(Y)$ $\left(\text{resp. } U(Y')\Lambda'_{Y'}/U(Y')\right)$ bundle. Since $\bar{\alpha}_Y$ induces a bundle map it follows (from the property of classifying bundles) that $\bar{\alpha}_Y$ is homotopic to the diffeomorphism $\bar{\bar{\varphi}}_Y$. Since homotopic maps induce isomorphic bundles it follows that the bundle

$$\bar{\bar{\varphi}}_Y^*\left(K'_{Y'} \setminus D_1(Y'), \pi_{Y'}, K'_{Y'} \setminus D_1(Y')/U(Y')\Lambda'_{Y'}\right)$$

is isomorphic to the bundle $\left(K_Y \setminus D_1(Y), \pi_Y, K_Y \setminus D_1(Y)/U(Y)\Lambda_Y\right)$ and thus we get a map

$$\varphi_Y\colon\ K_Y \setminus D_1(Y) \to K_{Y'} \setminus D_1(Y')$$

such that

$$\varphi_Y(\delta \cdot \lambda) = \varphi_Y(\delta)\,\beta_Y(\lambda) \qquad \text{for } \delta \in K_Y \setminus D_1(Y) \quad \text{and} \quad \lambda \in U(Y)\Lambda_Y. \quad (2)$$

Further $\varphi_Y$ is a diffeomorphism since all the bundles under consideration are locally trivial, differentiable and $\bar{\bar{\varphi}}_Y$ is a diffeomorphism.

In the sequel $\tilde{\varphi}_Y$ will denote the composite

$$D_1(Y) \to K_Y \setminus D_1(Y) \xrightarrow{\ \varphi_Y\ } K'_{Y'} \setminus D_1(Y')$$

where the first map is the natural projection.

Let $d = \min(c, c') - 1$ and let $\varphi_0$ be a diffeomorphism from $\mathfrak{S}_d$ onto ${}'\mathfrak{S}_d$ defined as follows. For $k \in K$, $t \leq d$, $Y \in \mathscr{R}$ and $u \in D_1(Y)$, $\gamma \in \Gamma$ let

$$\varphi_0\left(\pi(k \cdot \exp tY \cdot u \cdot \gamma)\right) = \pi'\left(\exp tY' \cdot \tilde{\varphi}_Y(u)\,\theta(\gamma)\right)$$

$$\langle \text{with some abuse of notations} \rangle.$$

It is easily seen using Proposition 2.1 and the properties of $\varphi_Y$ that $\varphi_0$ is a well defined diffeomorphism and for $x$ in its domain of definition and $\gamma \in \Gamma$,

$$\varphi_0(x\gamma) = \varphi_0(x)\,\theta(\gamma).$$

Thus $\varphi_0$ induces a map $\bar{\varphi}_0\colon p(\mathfrak{S}_d) \to p'({}'\mathfrak{S}_d)$.

In the following we shall identify $\Gamma'$ with $\Gamma$ with the help of the isomorphism $\theta$ and call a map $\psi$: $X \to X'$ $\Gamma$-equivariant if $\psi(x\gamma) = \psi(x)\gamma$. Since $X$ and $X'$ are contractible spaces, the spaces $X/\Gamma$ and $X'/\Gamma$ are classifying spaces for principal $\Gamma$ bundles, it follows that the spaces $X/\Gamma$ and $X'/\Gamma$ are homotopically equivalent and there exists a homotopy equivalence $\bar{\psi}$: $X/\Gamma \to X'/\Gamma$ such that the $\Gamma$-bundle induced by $\bar{\psi}$ from the bundle $\xi' = (X', p', X'/\Gamma)$ is isomorphic to the bundle $\xi = (X, p, X/\Gamma)$. For $Y \in \mathscr{R}$ let $T_Y = K_Y \setminus D_1(Y)/\Lambda_Y$ and for $t \leqq c$ let

$$\varepsilon_Y^t \colon\ T_Y \to X/\Gamma$$

be the map induced from the map

$$u \mapsto p \circ \pi(\exp t\, Y u) \qquad \text{for } u \in D_1(Y).$$

It is clear that for all $t \leqq c$, $\varepsilon_Y^t$ is a homeomorphism onto its image (cf. Remark 2.4). Since $\xi \approx \bar{\psi}^*(\xi')$, the $\Gamma$ bundle on $T_Y$ induced from $\xi'$ by the map $\bar{\psi} \cdot \varepsilon_Y^t$ is isomorphic to the bundle induced from $\xi$ by $\varepsilon_Y^t$. Also since $\varphi_0$ induces a $\Gamma$-bundle isomorphism from $\xi|_{p(\mathfrak{S}_d)}$ to $\xi'|_{p'('\mathfrak{S}_d)}$ it follows that the bundle induced on $T_Y$ from $\xi'$ by $\varphi_0 \cdot \varepsilon_Y^d$ is isomorphic to the bundle induced from $\xi$ by $\varepsilon_Y^d$. But since $\varepsilon_Y^{d+1}$ and $\varepsilon_Y^d$ are clearly homotopic the $\Gamma$-bundles induced on $T_Y$ from $\xi$ by $\varepsilon_Y^{d+1}$ and $\varepsilon_Y^d$ are isomorphic. This proves that the maps $\bar{\psi} \cdot \varepsilon_Y^{d+1}$ and $\bar{\varphi}_0 \cdot \varepsilon_Y^d$ induce isomorphic bundles on $T_Y$ and as $X'/\Gamma$ is a classifying space for $\Gamma$-bundles it follows that for every $Y \in \mathscr{R}$, $\bar{\varphi}_0 \cdot \varepsilon_Y^d$ is homotopic to $\bar{\psi} \cdot \varepsilon_Y^{d+1}$.

Let $I$ be the unit interval $[0, 1]$. For $Y \in \mathscr{R}$ we fix a homotopy $\delta_Y$: $T_Y \times I \to X'/\Gamma$ between $\bar{\varphi}_0 \cdot \varepsilon_Y^d$ and $\bar{\psi} \cdot \varepsilon_Y^{d+1}$ such that the composite

$$T_Y \to T_Y \times \{0\} \xrightarrow{\delta_Y|} X'/\Gamma \quad \text{is} \quad \bar{\varphi}_0 \cdot \varepsilon_Y^d$$

and the composite

$$T_Y \to T_Y \times \{1\} \xrightarrow{\delta_Y|} X'/\Gamma \quad \text{is} \quad \bar{\psi} \cdot \varepsilon_Y^{d+1}.$$

Let $\bar{\varphi}_1$: $X/\Gamma \to X'/\Gamma$ be the map defined as follows

$$\bar{\varphi}_1|_{p(\Omega_{d+1})} = \bar{\psi}, \qquad \bar{\varphi}_1|_{p(\mathfrak{S}_d)} = \bar{\varphi}_0$$

and for $d \leqq t \leqq d+1$, $Y \in \mathscr{R}$, $u \in D_1(Y)$

$$\bar{\varphi}_1(p \cdot \pi(\exp t\, Y u)) = \delta_Y(v_Y u, t-d)$$

where $v_Y$ is the natural projection $D_1(Y) \to T_Y = K_Y \setminus D_1(Y)/\Lambda_Y$. Using Lemma 2.3 one can check that $\bar{\varphi}_1$ is continuous. Since $p(\Omega_{d+1})$ is a strong deformation retract of $X/\Gamma$ and since $\bar{\varphi}_1|_{p(\Omega_{d+1})} = \bar{\psi}$ it follows that $\bar{\varphi}_1$ is homotopic to $\bar{\psi}$. Also since $\bar{\varphi}_1$ restricted to the open co-compact set $p(\mathfrak{S}_d^0)$ is a $C^\infty$ map, by standard results in differential topology it follows that there exists a $C^\infty$ map

$$\overline{\varphi}\colon X/\Gamma \to X'/\Gamma'$$

which is homotopic to $\overline{\varphi}_1$ and which coincides with $\overline{\varphi}_1$ on the open co-compact set $p(\mathfrak{S}^0_{d-1})$. Clearly $\overline{\varphi}$ is homotopic to $\overline{\psi}$. Thus $\xi \approx \overline{\psi}^*(\xi') \approx \overline{\varphi}^*(\xi')$. Let $\varphi\colon X \to X'$ be a $\Gamma$-equivariant map lying over $\overline{\varphi}$. Then $\varphi$ is a $C^\infty$ map. *We claim that $\varphi$ is a pseudo-isometry.* We shall prove our claim in several steps.

In the sequel for a differentiable map $\psi\colon M \to N$ we denote by $\dot{\psi}$ the map between the total spaces of the tangent bundles on $M$ and $N$ induced by $\psi$.

We will first show that there exists a constant $\alpha_2 > 0$ such that if $Z$ is a tangent vector to $X$ at a point in $\mathfrak{S}^0_{d-1}$ then

$$\|\dot{\varphi} Z\| \leqq \alpha_2 \|Z\|. \tag{3}$$

(Here we use $\|\ \|$ to denote the norm in the riemannian structure on both $X$ as well as $X'$. We shall use $\|\ \|$ to denote also the norms on $\mathfrak{p}$ and $\mathfrak{p}'$ determined by the Killing forms on $\mathfrak{g}$, $\mathfrak{g}'$ respectively.)

In the following we denote $D_1(Y) \cap M(Y)$ by $M_1(Y)$. Recall that $D_1(Y) = M_1(Y) \cdot U(Y)$ (a semi-direct product). For $Y \in \mathscr{R}$, $D_1(Y)/\Lambda_Y$ is compact so there exists a compact subset $\eta(Y)$ of $M_1(Y)$ such that $D_1(Y) = \eta(Y) \cdot U(Y) \Lambda_Y$. Since the riemannian structures on $X$ and $X'$ are respectively $\Gamma$ and $\Gamma'$ invariant (in fact they are respectively $G$ and $G'$ invariant) and since

$$\overline{\varphi}|_{p(\mathfrak{S}^0_{d-1})} = \overline{\varphi}_1|_{p(\mathfrak{S}^0_{d-1})} = \overline{\varphi}_0|_{p(\mathfrak{S}^0_{d-1})}$$

it suffices to show that there is a constant $\alpha_2 > 0$ such that for any $Y \in \mathscr{R}$, if $Z$ is a tangent vector to $X$ at a point in $\pi(A^0_{d-1}(Y)\,\eta(Y)\,U(Y))$ then

$$\|\dot{\varphi}_0(Z)\| \leqq \alpha_2 \|Z\|. \tag{4}$$

Let us consider the composite $\tilde{\varphi}_Y$ of the maps

$$M_1(Y) \cdot U(Y) = D_1(Y) \to K_Y \diagdown D_1(Y) \xrightarrow{\varphi_Y} K'_{Y'} \diagdown D_1(Y')$$
$$\approx (K'_{Y'} \diagdown M_1(Y')) \cdot U(Y').$$

Given $m \in M_1(Y)$, we can use local sections of the locally trivial fibration $M_1(Y') \to K'_{Y'} \diagdown M_1(Y')$ and the fact that, $\tilde{\varphi}_Y(\delta \cdot \lambda) = \tilde{\varphi}_Y(\delta)\,\beta_Y(\lambda)$ for $\lambda \in U(Y)$, to find a relatively compact open neighborhood $\omega_m$ of $m$ in $M_1(Y)$ and a map $\psi_{\omega_m}\colon \omega_m \cdot U(Y) \to D_1(Y')$ such that

1. The composite $\omega_m \cdot U(Y) \xrightarrow{\psi_{\omega_m}} D_1(Y') \to K'_{Y'} \diagdown D_1(Y')$ is the restriction of $\tilde{\varphi}_Y$ to $\omega_m \cdot U(Y)$.

2. $\psi_{\omega_m}(\delta \cdot \lambda) = \psi_{\omega_m}(\delta) \cdot \beta_Y(\lambda) = \psi_{\omega_m}(\delta) \cdot \alpha_Y(\lambda)$ for $\delta \in \omega_m \cdot U(Y), \lambda \in U(Y)$ (recall that $\beta_Y|_{U(Y)} = \alpha_Y$). And

3. $\psi_{\omega_m}(\omega_m)$ is a relatively compact subset of $D_1(Y')$. Let

$$\psi_{Y,\omega_m}\colon A^0_{d-1}(Y)\cdot\omega_m\cdot U(Y)\to A^0_{d-1}(Y')\cdot D_1(Y')$$

be the map

$$\exp tY\cdot\delta\mapsto\exp tY'\,\psi_{\omega_m}(\delta)\quad\text{for }\delta\in\omega_m\cdot U(Y)\quad\text{and}\quad t<d-1.$$

For $m\in M_1(Y)$ let $\omega^0_m$ be an open neighborhood of $m$ such that the closure $\bar\omega^0_m$ of $\omega^0_m$ is contained in $\omega_m$. Since $\mathscr{R}$ is finite and $\eta(Y)$ is compact and therefore can be covered by finitely many $\omega^0_m$'s, to prove (4) it suffices to show that given a $\omega^0_m$ there is a constant $\alpha_3$ such that if $Z$ is a tangent vector to $X$ at a point in $\pi\big(A^0_{d-1}(Y)\cdot\omega^0_m\cdot U(Y)\big)$ then

$$\|\dot\varphi_0(Z)\|\leqq\alpha_3\,\|Z\|. \tag{5}$$

For a fixed $Y\in\mathscr{R}$ consider the eigenspace decomposition $\mathfrak{g}=\mathfrak{g}_Y^{-2}+\mathfrak{g}_Y^{-1}+\mathfrak{g}_Y^0+\mathfrak{g}_Y^1+\mathfrak{g}_Y^2$ of $\mathfrak{g}$ with respect to $\operatorname{ad}Y$. Since $Y\in\mathfrak{p}$, $\mathfrak{g}_Y^0=\mathfrak{g}_Y^0\cap\mathfrak{k}\oplus\mathfrak{g}_Y^0\cap\mathfrak{p}$ and by our choice of $Y$, $\mathfrak{u}(Y)=\mathfrak{g}_Y^1+\mathfrak{g}_Y^2$, $[\mathfrak{u}(Y),\mathfrak{u}(Y)]=\mathfrak{g}_Y^2$ and the Lie algebra of $B(Y)$ (resp. $M(Y)$) is $\mathfrak{g}_Y^0+\mathfrak{g}_Y^1+\mathfrak{g}_Y^2$ (resp. $\mathfrak{g}_Y^0$). Let $\{Z^i\}_{1\leqq i\leqq n}$ be a basis of $\mathfrak{u}(Y)$ such that each $Z^i$ is either in $\mathfrak{g}_Y^1$ or in $\mathfrak{g}_Y^2$ and $\{\tfrac12(Z^i-\sigma Z^i)\}_{1\leqq i\leqq n}$ ($\sigma$ is the Cartan involution) is an ortho-normal set with respect to the Killing form. Clearly every element in $\mathfrak{g}_Y^0$ is orthogonal to $Z^i-\sigma Z^i$ for $i\leqq n$. Let $\{Y^j\}_{j=0}$ be an orthonormal basis of $\mathfrak{g}_Y^0\cap\mathfrak{p}$ such that $Y^0=Y/\|Y\|$.

In the following we shall identify the Lie algebra $\mathfrak{g}$ (resp. $\mathfrak{g}'$) with the tangent space to $G$ (resp. $G'$) at the identity and for an element $Z\in\mathfrak{g}$ (resp. $Z'\in\mathfrak{g}'$) and a point $g\in G$ (resp. $g'\in G'$) we let $Z_g$ (resp. $Z'_{g'}$) denote the value of the right invariant vector field on $G$ (resp. $G'$) determined by $Z$ (resp. $Z'$) at $g$ (resp. $g'$). Now it can be easily seen that for $t<d-1$, $\delta\in\omega_m$ and $u\in U(Y)$

$$\dot\psi_{Y,\omega_m}(Y_{\exp tY\cdot\delta\cdot u})=Y'_{\psi_{Y,\omega_m}(\exp tY\cdot\delta\cdot u)}$$

for $Z$ in the Lie algebra of $M_1(Y)$ we get

$$\dot\psi_{Y,\omega_m}(Z_{\exp tY\cdot\delta\cdot u})=\big(\operatorname{Ad}\exp tY'f(Z,\delta)\big)_{\psi_{Y,\omega_m}(\exp tY\cdot\delta\cdot u)}$$

where $f(Z,\delta)$ is the element in the Lie algebra of $D_1(Y')$ such that

$$\big(f(Z,\delta)\big)_{\psi_{\omega_m}(\delta)}=\dot\psi_{\omega_m}(Z_\delta).$$

Finally for $Z^i\in\mathfrak{u}(Y)$

$$\begin{aligned}&\dot\psi_{Y,\omega_m}(Z^i_{\exp tY\cdot\delta\cdot u})\\&\quad=\big(\operatorname{Ad}(\exp tY'\cdot\psi_{\omega_m}(\delta))\,\dot\alpha_Y(\operatorname{Ad}(\delta^{-1}\exp-tY)Z^i)\big)_{\psi_{Y,\omega_m}(\exp tY\cdot\delta\cdot u)}.\end{aligned}$$

We note that since $[\mathfrak{u}(Y),\mathfrak{u}(Y)]=\mathfrak{g}_Y^2$ and $[\mathfrak{u}(Y'),\mathfrak{u}(Y')]=\mathfrak{g}'^2_{Y'}$, any Lie algebra automorphism of $\mathfrak{u}(Y)$ (resp. $\mathfrak{u}(Y')$) stabilizes $\mathfrak{g}_Y^2$ (resp. $\mathfrak{g}'^2_{Y'}$) and $\dot\alpha_Y$ which is a Lie algebra isomorphism maps $\mathfrak{g}_Y^2$ onto $\mathfrak{g}'^2_{Y'}$. Since $\operatorname{Ad}\exp-tY$ (resp. $\operatorname{Ad}\exp tY'$) restricted to $\mathfrak{g}_Y^2$ (resp. $\mathfrak{g}'^2_{Y'}$) is multiplication by $e^{-2t}$

(resp. $e^{2t}$) it follows that for $Z^i \in \mathfrak{g}_Y^2$

$$\begin{aligned}
&\dot{\psi}_{Y,\omega_m}(Z^i_{\exp tY\cdot\delta\cdot u}) \\
&\quad=\big(\mathrm{Ad}(\exp tY'\cdot\psi_{\omega_m}(\delta))\,\dot{\alpha}_Y(\mathrm{Ad}(\delta^{-1}\exp -tY)Z^i)\big)_{\psi_{Y,\omega_m}(\exp tY\cdot\delta\cdot u)}\\
&\quad=\big(\mathrm{Ad}(\exp tY'\cdot\psi_{\omega_m}(\delta)\cdot\exp -tY')\\
&\qquad\cdot\mathrm{Ad}\exp tY'\cdot\dot{\alpha}_Y(\mathrm{Ad}(\delta^{-1}\exp -tY)Z^i)\big)_{\psi_{Y,\omega_m}(\exp tY\cdot\delta\cdot u)}\\
&\quad=\big(\mathrm{Ad}(\exp tY'\cdot\psi_{\omega_m}(\delta)\cdot\exp -tY')\\
&\qquad\cdot(e^{2t}\dot{\alpha}_Y\cdot\mathrm{Ad}\,\delta^{-1}(e^{-2t}Z^i))\big)_{\psi_{Y,\omega_m}(\exp tY\cdot\delta\cdot u)}\\
&\quad=\big(\mathrm{Ad}(\exp tY'\cdot\psi_{\omega_m}(\delta)\cdot\exp tY')\\
&\qquad\cdot(\dot{\alpha}_Y\cdot\mathrm{Ad}\,\delta^{-1}(Z^i))\big)_{\psi_{Y,\omega_m}(\exp tY\cdot\delta\cdot u)}.
\end{aligned}$$

If $Z^i \in \mathfrak{g}_Y^1$ then

$$\begin{aligned}
&\dot{\psi}_{Y,\omega_m}(Z^i_{\exp tY\cdot\delta\cdot u}) \\
&\quad=\big(\mathrm{Ad}(\exp tY'\cdot\psi_{\omega_m}(\delta)\cdot\exp -tY')\,\mathrm{Ad}\exp tY'\\
&\qquad\cdot\dot{\alpha}_Y\cdot\mathrm{Ad}\,\delta^{-1}\,\mathrm{Ad}\exp -tY(Z^i)\big)_{\psi_{Y,\omega_m}(\exp tY\cdot\delta\cdot u)}\\
&\quad=\big(\mathrm{Ad}(\exp tY'\,\psi_{\omega_m}(\delta)\cdot\exp -tY')\,\mathrm{Ad}\exp tY'\\
&\qquad\cdot\dot{\alpha}_Y(e^{-t}\,\mathrm{Ad}\,\delta^{-1}(Z^i))\big)_{\psi_{Y,\omega_m}(\exp tY\cdot\delta\cdot u)}\\
&\quad=\big(e^{-t}\,\mathrm{Ad}(\exp tY'\cdot\psi_{\omega_m}(\delta)\cdot\exp -tY')\\
&\qquad\cdot\{e^t(\dot{\alpha}_Y\,\mathrm{Ad}\,\delta^{-1}Z^i)_1+e^{2t}(\dot{\alpha}_Y\,\mathrm{Ad}\,\delta^{-1}Z^i)_2\}\big)_{\psi_{Y,\omega_m}(\exp Y\cdot\delta\cdot u)}\\
&\quad=\big(\mathrm{Ad}(\exp tY'\cdot\psi_{\omega_m}(\delta)\cdot\exp -tY')\\
&\qquad\cdot\{(\dot{\alpha}_Y\,\mathrm{Ad}\,\delta^{-1}Z^i)_1+e^t(\dot{\alpha}_Y\,\mathrm{Ad}\,\delta^{-1}Z^i)_2\}\big)_{\psi_{Y,\omega_m}(\exp tY\cdot\delta\cdot u)}
\end{aligned}$$

where

$$\dot{\alpha}_Y\,\mathrm{Ad}\,\delta^{-1}(Z^i)=(\dot{\alpha}_Y\,\mathrm{Ad}\,\delta^{-1}Z^i)_1+(\dot{\alpha}_Y\,\mathrm{Ad}\,\delta^{-1}Z^i)_2$$

with

$$(\dot{\alpha}_y\,\mathrm{Ad}\,\delta^{-1}Z^i)_1\in\mathfrak{g}'^1_{Y'}\quad\text{and}\quad(\alpha_Y\,\mathrm{Ad}\,\delta^{-1}Z^i)_2\in\mathfrak{g}'^2_{Y'}.$$

If $Z$ is in the Lie algebra of $M_1(Y)$ then,

$$\begin{aligned}
&\dot{\psi}_{Y,\omega_m}(Z_{\exp tY\cdot\delta\cdot u})=\big(\mathrm{Ad}\exp tY'(f(Z,\delta))\big)_{\psi_{Y,\omega_m}(\exp tY\cdot\delta\cdot u)}\\
&\quad=(\mathrm{Ad}\exp tY'\{(f(Z,\delta))_0+(f(Z,\delta))_1+(f(Z,\delta))_2\}_{\psi_{Y,\omega_m}(\exp tY\cdot\delta\cdot u)}
\end{aligned}$$

where

$$f(Z,\delta)=(f(Z,\delta))_0+(f(Z,\delta))_1+(f(Z,\delta))_2$$

with

$$(f(Z,\delta))_i \in \mathfrak{g}'^{\,i}_{Y'} \quad \text{for } i=0,1,2.$$

$$=\big((f(Z,\delta))_0+e^t(f(Z,\delta))_1+e^{2t}(f(Z,\delta))_2\big)_{\psi_{Y,\omega_m}(\exp tY\cdot\delta\cdot u)}.$$

Now let

$$\zeta_{\exp tY\cdot\delta\cdot u}=\Big(\sum_j r_j Y^j+\sum_i s_i Z^i\Big)_{(\exp tY\cdot\delta\cdot u)}$$

be a tangent vector at a point in $A^0_{d-1}(Y)\cdot\omega^0_m\cdot U(Y)$ (thus $t<d-1$, $\delta\in\omega^0_m$ and $u\in U(Y)$) such that

$$\|\dot\pi\zeta_{\exp tY\cdot\delta\cdot u}\|=1$$

then since

$$\begin{aligned}\|\dot\pi\zeta_{\exp tY\cdot\delta\cdot u}\| &= \Big\|\dot\pi\Big(\sum_j r_j Y^j+\sum_i s_i Z^i\Big)\Big\|\\ &=\Big\|\sum_j r_j Y^j+\sum_i \tfrac12 s_i(Z^i-\sigma Z^i)\Big\|\\ &=\Big(\sum_j r_j^2+\sum_i s_i^2\Big)^{\frac12}\end{aligned}$$

it follows that for all $i$ and $j$; $r_j,\ s_i\leqq 1$.

From the commutative diagram

$$\begin{array}{ccc} A^0_{d-1}(Y)\cdot\omega_m\cdot U(Y) & \xrightarrow{\ \psi_{Y,\omega_m}\ } & A^0_{d-1}(Y')\cdot M_1(Y')\cdot U(Y') \\ \Big\downarrow{\scriptstyle\pi|} & & \Big\downarrow{\scriptstyle\pi'|} \\ \pi\big(A^0_{d-1}(Y)\cdot\omega_m\cdot U(Y)\big) & \xrightarrow{\ \varphi_0|\ } & \pi'\big(A^0_{d-1}(Y')\cdot M_1(Y')\cdot U(Y')\big)\end{array}$$

we get:

$$\begin{aligned}\|\dot\varphi_0\dot\pi\zeta_{\exp tY\cdot\delta\cdot u}\| &= \|\dot\pi'\dot\psi_{Y,\omega_m}\{(\Sigma r_j Y^j+\Sigma s_i Z^i)_{\exp Y\cdot\delta\cdot u}\}\|\\ &=\Big\|\dot\pi'\Big\{r_0\frac{Y'}{\|Y\|}+\sum_{j>0} r_j(f(Y^j,\delta))_0\\ &\quad+\sum_{j>0} r_j e^t(f(Y^j,\delta))_1+\sum_{j>0} r_j e^{2t}(f(Y^j,\delta))_2\\ &\quad+\sum_{Z^i\in\mathfrak{g}^2_Y} s_i\operatorname{Ad}(\exp tY'\cdot\psi_{\omega_m}(\delta)\cdot\exp -tY')(\dot\alpha_Y\operatorname{Ad}\delta^{-1}(Z^i))\\ &\quad+\sum_{Z^i\in\mathfrak{g}^1_Y} s_i\operatorname{Ad}(\exp tY'\cdot\psi_{\omega_m}(\delta)\cdot\exp -tY)(\dot\alpha_Y\operatorname{Ad}\delta^{-1}Z^i)_1\\ &\quad+\sum_{Z^i\in\mathfrak{g}^1_Y} s_i e^t\operatorname{Ad}(\exp tY'\cdot\psi_{\omega_m}(\delta)\cdot\exp -tY')(\dot\alpha_Y\operatorname{Ad}\delta^{-1}Z^i)\Big\}\Big\|.\end{aligned}$$

Since $\omega_m$ and $\psi_{\omega_m}(\omega_m)$ are relatively compact subsets, $\bar\omega^0_m(\subset\omega_m)$ is compact and (cf. Remark 2.2) $\bigcup_{t<d-1}\exp tY'\cdot\psi_{\omega_m}(\omega^0_m)\cdot\exp -tY'$ is relatively compact. Also for $t<d-1$ both $e^t$, $e^{2t}$ are bounded. Thus it follows that for a suitable constant $\alpha_3$

$$\|\dot\varphi_0\dot\pi\zeta_{\exp tY\cdot\delta\cdot u}\|\leqq\alpha_3.$$

For any $Z \neq 0$ considering $Z/\|Z\|$ it follows that

$$\|\dot{\varphi}_0 Z\| \leqq \alpha_3 \|Z\|.$$

This establishes (5) and therefore (4) and hence (3) too.

Now since $\varphi|_{\mathfrak{S}^0_{d-1}}: \mathfrak{S}^0_{d-1} \to {'\mathfrak{S}^0_{d-1}}$ is a diffeomorphism, considering its inverse, $\varphi^{-1}$, we can in the same way prove that there is a constant $\alpha_1$ such that for a tangent vector $Z'$ to $X'$ at a point in ${'\mathfrak{S}^0_{d-1}}$

$$\|\dot{\varphi}^{-1} Z'\| \leqq \frac{1}{\alpha_1} \|Z'\|. \tag{6}$$

From this one deduces that

$$\alpha_1 \|Z\| \leqq \|\dot{\varphi} Z\| \tag{7}$$

for a tangent vector $Z$ to $X$ at any point in $\mathfrak{S}^0_{d-1}$.

Now since $\Omega_{d-1} = X - \mathfrak{S}^0_{d-1}$ is compact modulo $\Gamma$ and since $\varphi$ is $\Gamma$-equivariant and the riemannian structures on $X$ and $X'$ are $\Gamma$ invariant, in view of (3) it follows that there is a constant $k_1 \geqq \alpha_2$ such that if $Z$ is a tangent vector to $X$ then

$$\|\dot{\varphi} Z\| \leqq k_1 \|Z\|.$$

This implies that for $x, y \in X$

$$d(\varphi(x), \varphi(y)) \leqq k_1\, d(x, y)$$

and in particular $\varphi$ is uniformly continuous.

To complete the proof of Theorem C we have only to show that there exist positive constants $k$ and $b$ such that

$$d(\varphi(x), \varphi(y)) \geqq k^{-1}\, d(x, y) \quad \text{for } x, y \in X \quad \text{with} \quad d(x, y) \geqq b.$$

We need the following lemma.

**4.1. Lemma.** *Let $G$ and $G'$ be semi-simple linear analytic groups and let $\Gamma$ be a lattice in $G$. Let $\Gamma'$ be a discrete subgroup of $G'$ and $\theta: \Gamma \to \Gamma'$ be an isomorphism. Let $K$ (resp. $K'$) be a maximal compact subgroup of $G$ (resp. $G'$) and let $X = K \diagdown G$ (resp. $X' = K' \diagdown G'$) be the symmetric riemannian space associated with $G$ (resp. $G'$). Let $\varphi: X \to X'$ be a uniformly continuous map such that*

$$\varphi(x\gamma) = \varphi(x)\,\theta(\gamma) \quad \text{for all } x \in X \quad \text{and} \quad \gamma \in \Gamma.$$

*Then $\varphi$ is a proper map.*

*Proof.* Since lattices in analytic groups are finitely generated, according to a result of Selberg, a lattice in a linear analytic group admits a subgroup of finite index which is torsion free ([cf. 12, §6]). Thus it is enough to prove the lemma assuming $\Gamma$ (and hence $\Gamma'$) torsion free.

Let $C'$ be a compact subset of $X'$. If possible let us assume that $\varphi^{-1}(C')$ is non-compact. Then there exists a sequence $\{x_i\} \subset \varphi^{-1}(C')$ which has no convergent subsequence. There are two cases to be considered.

(i) If $\{x_i\}$ is has a convergent subsequence modulo $\Gamma$, then if necessary by passing to a subsequence we can assume that there exist $\gamma_i \in \Gamma$ such that $\{x_i \gamma_i\}$ converges to $x \in X$. Then $\varphi(x_i)\,\theta(\gamma_i)$ converges to $y = \varphi(x)$. Also since $\varphi(x_i) \in C'$ there is a subsequence of $\varphi(x_i)$ which converges. Let $\pi'\colon G' \to K' \setminus G'$ be the natural projection and let $\{g'_i\}$ be a sequence in $G'$ such that $\varphi(x_i) = x'_i = \pi'(g'_i)$. Then since $\pi'$ is proper, we can after passing to a subsequence assume that $\{g'_i\}$ as well as $\{g'_i\,\theta(\gamma_i)\}$ converge. Hence the sequence $\{\theta(\gamma_i)\}$ converges. Since $\Gamma'$ is discrete, $\{\theta(\gamma_i)\}_{i \geqq m\,(\mathrm{say})}$ is a constant sequence. Thus $\{\gamma_i\}_{i \geqq m}$ is a constant sequence. But then since $\{x_i \gamma_i\}$ converges to $x$ this sequence is contained in a compact subset of $X$ and hence $\{x_i\}$ is contained in a compact subset of $X$, a contradiction.

(ii) Now let us assume that $\{x_i\}$ has no convergent subsequence modulo $\Gamma$. Let $\pi\colon G \to K \setminus G$ be the natural projection and let $g_i$ be such that $\pi(g_i) = x_i$. Evidently $\{g_i\}$ has no subsequence convergent modulo $\Gamma$. Hence by [12, Theorem 1.12] there exists a sequence $\{\gamma_i\} \subset \Gamma$ such that $\{g_i \gamma_i g_i^{-1}\}$ converges to the identity and for no $i$, $\gamma_i = e$. Thus $d(\pi(e), \pi(g_i \gamma_i g_i^{-1})) \to 0$ which implies that $d(\pi(g_i), \pi(g_i \gamma_i)) \to 0$ i.e., $d(x_i, x_i \gamma_i) \to 0$. Since $\varphi$ is uniformly continuous this implies that

$$d(\varphi(x_i), \varphi(x_i \gamma_i)) = d(\varphi(x_i), \varphi(x_i)\,\theta(\gamma_i)) \to 0.$$

After passing to a subsequence if necessary, we can assume (since $x_i \in C'$, $C'$ is compact and since $\pi'$ is proper) that $\{g'_i\}$ and $\{g'_i \gamma'_i\}$ are convergent where $\{g'_i\} \subset G'$ is a fixed sequence such that $\pi'(g'_i) = x'_i$ and $\gamma'_i = \theta(\gamma_i)$. From this we can conclude as before that for large $i$, $\gamma'_i = \gamma'$. Then $d(x'_i, x'_i \gamma') = d(x'_i, x'_i \gamma'_i) \to 0$. This implies that if $\{x'_i\}$ converges to $y'$ then $d(y', y'\gamma') = 0$ i.e., $y' = y'\gamma'$. Since isotropy subgroup of any point in $X'$ is compact, this implies that $\gamma'$ is a torsion element and since $\Gamma'$ is torsion free $\gamma' =$ identity. Thus for large $i$, $\gamma'_i$ and therefore $\gamma_i$ are the identities in respective groups. This again is a contradiction.

We now complete the proof of Theorem C. We first observe that there exists a real number $e \leqq d-1$ such that $\varphi^{-1}({}'\mathfrak{S}_e^0) \subset \mathfrak{S}_e^0$. This follows from the fact that $\Omega_{d-1}$ is compact modulo $\Gamma$, so $\varphi(\Omega_{d-1})$ is compact modulo $\Gamma'$ and hence for $e$ sufficiently small, $\varphi(\Omega_{d-1}) \cap {}'\mathfrak{S}_e^0 = \emptyset$. Now since $\bar{\varphi}|_{p(\mathfrak{S}_{d-1}^0)} = \bar{\varphi}_0$ we actually get that $\varphi^{-1}({}'\mathfrak{S}_e^0) \subset \mathfrak{S}_e^0$ and $\varphi$ restricted to $\mathfrak{S}_e^0$ is a diffeomorphism.

Let $a$ be a positive real number fixed once and for all. For a subset $E$ of a metric space $M$ we denote by $T_a(E)$ the set $\{x \in M \mid d(E, x) \leqq a\}$. Since $p'\colon X' \to X'/\Gamma'$ is a distance reducing map it follows that $p'(T_a(\Omega'_e)) \subset$

$T_a(p'(\Omega'_e))$. As $X'/\Gamma'$ is a complete riemannian manifold (cf. [3, Prop. 10.6 and Th. 10.3 in Chapter I]) and $p'(\Omega'_e)$ is compact, $T_a(p'(\Omega'_e))$ and hence $p'(T_a(\Omega'_e))$ are compact. Now since $p'$ is a covering map it can be shown that there exist a positive real number $\alpha \leqq a$ and a compact subset $C'_\alpha$ of $T_a(\Omega'_e)$ such that any closed ball of diameter $\leqq \alpha$ contained in $T_a(\Omega'_e)$ can be brought in $C'_\alpha$ by an element of $\Gamma'$. Let $s = \operatorname{diam} \varphi^{-1}(C'_\alpha)$ then in view of Lemma 4.1, $s$ is finite. Clearly if $B'$ is a ball of diameter $\leqq \alpha$ contained in $T_a(\Omega'_e)$ then $\operatorname{diam} \varphi^{-1}(B') \leqq s$. We now claim that there exists a real number $r$ such that if $B' \subset X'$ is a closed ball of diameter $\leqq \alpha$ then $\operatorname{diam} \varphi^{-1}(B') \leqq r$. To see this we argue as follows. If $B' \cap \Omega'_e \neq \emptyset$ then since $a \geqq \alpha$ it follows that $B' \subset T_a(\Omega'_e)$ and then $\operatorname{diam} \varphi^{-1}(B') \leqq s$. If $B' \cap \Omega'_e = \emptyset$, then since $X' = \Omega'_e \cup {'\mathfrak{S}'^0_e}$, $B' \subset {'\mathfrak{S}^0_e}$. Now since a ball is geodesically convex, $\varphi^{-1}({'\mathfrak{S}^0_e}) \subset \mathfrak{S}^0_e$ and $\varphi$ is a diffeomorphism restricted to $\mathfrak{S}^0_e$; we can easily conclude from the inequality (7) that

$$\operatorname{diam}(\varphi^{-1}(B')) \leqq \alpha/\alpha_1.$$

Thus if $r = \max(\alpha/\alpha_1, s)$ then clearly, for any ball $B' \subset X'$ of diameter $\leqq \alpha$, $\operatorname{diam}(\varphi^{-1}(B')) \leqq r$ which proves our claim.

Let $k_2 = r/\alpha$. For $x, y \in X$ with $d(\varphi(x), \varphi(y)) < m\alpha$ there is a path in $X$ of length less than $mr$ joining $x$ to $y$. Given now $x$ and $y$ in $X$ with $d(x,y) \geqq r$ we can choose $n \geqq 1$ so that

$$n\alpha \leqq d(\varphi(x), \varphi(y)) < (n+1)\alpha,$$

then

$$\begin{aligned} d(x,y) < (n+1)\,r = \frac{n+1}{n} \cdot \frac{r}{\alpha} \cdot n\alpha &\leqq \frac{n+1}{n} \cdot k_2\, d(\varphi(x), \varphi(y)) \\ &\leqq 2k_2\, d(\varphi(x), \varphi(y)). \end{aligned}$$

Set $k = \max(k_1, 2k_2)$ and $b = r$, then

$$d(x,y) \leqq k\, d(\varphi(x), \varphi(y)) \quad \text{for } x, y \in X \quad \text{with} \quad d(x,y) \geqq b$$

and

$$d(\varphi(x), \varphi(y)) \leqq k\, d(x,y)$$

Thus $\varphi$ is a pseudo-isometry. This completes the proof of Theorem C.

4.2. *Remark.* It is well known that lattices in $SL(2,\mathbf{R})/\pm\mathbf{1}_2$ are not strongly rigid in general. In fact since a non-uniform, torsion free lattice in $SL(2,\mathbf{R})/\pm\mathbf{1}_2$ is (non-abelian and) free, its group of outer automorphisms is infinite. This, as can be seen easily, implies that such a lattice can not be strongly rigid. Also since two compact riemann surfaces of the same genus and with isomorphic fundamental groups need not be analytically equivalent it follows that in general even a uniform lattice in $SL(2,\mathbf{R})/\pm\mathbf{1}_2$ may not be strongly rigid (cf. [9, § 1]).

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