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Nonlinear Differential Equations and Applications NoDEA

Strong solutions to stochastic hydrodynamical systems with multiplicative noise of jump type

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Abstract. In this paper we prove the existence and uniqueness of maximal strong (in PDE sense) solution to several stochastic hydrodynamical systems on unbounded and bounded domains of $\mathbb{R}^n$, $n = 2, 3$. This maximal solution turns out to be a global one in the case of 2D stochastic hydrodynamical systems. Our framework is general in the sense that it allows us to solve the Navier–Stokes equations, MHD equations, Magnetic Bénard problems, Boussinesq model of the Bénard convection, Shell models of turbulence and the Leray-α model with jump type perturbation. Our goal is achieved by proving general results about the existence of maximal and global solution to an abstract stochastic partial differential equations with locally Lipschitz continuous coefficients. The method of the proofs are based on some truncation and fixed point methods.

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1. Introduction

Stochastic Partial Differential Equations (SPDEs) are a powerful tool for understanding and investigating mathematically hydrodynamic and turbulence theory. To model turbulent fluids, mathematicians often use stochastic equations obtained from adding a noise term in the dynamical equations of the fluids. This approach is basically motivated by Reynolds' work which stipulates that turbulent flows are composed of slow (deterministic) and fast (stochastic) components. Recently by following the statistical approach of turbulence theory, Flandoli et al. [25], Kupiainen [35] confirm the importance of studying the stochastic version of fluids dynamics. Indeed, the authors of [25] pointed

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out that some rigorous information on questions of turbulence theory might be obtained from these stochastic versions. It is worth emphasizing that the presence of the stochastic term (noise) in these models often leads to qualitatively new types of behavior for the processes. Since the pioneering work of Bensoussan and Temam [4], there has been an extensive literature on stochastic Navier–Stokes equations with Wiener noise and related equations, we refer to [1,2,5,6,17,20,26,40,41,48] amongst other.

In the last 5 years, there has been an extensive effort to tackle SPDEs with Levy noise. There are several examples where the Gaussian noise is not well suited to represent realistically external forces. For example, if the ratio between the time scale of the deterministic part and that of the stochastic noise is large, then the temporal structure of the forcing in the course of each event has no influence on the overall dynamics, and—at the time scale of the deterministic process—the external forcing can be modelled as a sequence of episodic instantaneous impulses. This happens for example in Climatology (see, for instance, [32]). Often the noise observed by time series is typically asymmetric, heavy-tailed and has non trivial kurtosis. These are all features which cannot be captured by a Gaussian noise, but rather by a Lévy noise with appropriate parameters. Lévy randomness requires different techniques from the ones used for Brownian motion and are less amenable to mathematical analysis. We refer to [9,11,21,24,31,39,47] that deal with stochastic hydrodynamical systems driven by Lévy type noise. Most of these articles are about the existence of solution which are weak in the PDEs sense.

In this paper, we are interested in proving the existence and uniqueness of maximal and global strong solution of Lévy driven hydrodynamical systems such as the Navier–Stokes equations (NS), Magnetohydrodynamics equations (MHD), Magnetic Bénard problem (MB), Boussinesq model for Bénard convection (BBC), Shell models of turbulence, and 3-D Leray-α for Navier–Stokes equations. Here, strong solutions should be understood in both the Probability and PDEs senses. Our objectives are achieved by adopting the unified approach initiated and developed in [17] and used later in [9]. This approach is based on rewriting the various equations above into an abstract stochastic evolution equations in a Hilbert space $\mathbf{V}$ of the following form

$$\mathbf{u}(t) = \mathbf{u}_0 - \int_0^t [A\mathbf{u}(s) + F(\mathbf{u}(s))]\, ds + \int_0^t \int_Z G(z, \mathbf{u}(s-))\widetilde{\eta}(dz, ds), \quad (1.1)$$

where $\int_Z G(z, \mathbf{u}(s-))\widetilde{\eta}(dz, ds)$ represents a global Lipschitz continuous multiplicative noise of jump type. In Theorem 3.5 we give sufficient conditions (on A and F) for the existence and uniqueness of a maximal solution to (1.1). Sufficient conditions for non-explosion of the maximal solution in finite time is given in Theorem 3.7. These two theorems are our main results and their assumptions are carefully chosen so that they are verified by the NS, MHD, MB, BBC, Shell models and the Leray-α models. In Sect. 4 we borrow the examples and the notations in [17] and give a detailed account of the applicability of our framework to the fluid models we cited in the previous sentence.

The book [44] contains several results about existence of solution to abstract SPDEs driven by Lévy noise in Hilbert space setting, but the hypotheses in this book do not cover the various hydrodynamical systems that we enumerated above. We also note that while there are several results about the existence of solution which are strong in PDEs sense for stochastic hydrodynamical systems perturbed by Wiener noise (see, for instance, [3,8,29,34,38,41] and references therein), it seems that this is the first paper treating the existence of strong (in PDE sense) solution for stochastic hydrodynamical systems with Lévy noise. However, one should mention the paper [13] in which the existence and uniqueness of a strong solution in PDE sense of a stochastic nonlinear beam equations driven by compensated Poisson random measures was established.

The layout of the present paper is as follows. In Sect. 2, we introduce the abstract stochastic evolution equation that our result will be based on. At the beginning of the section, we give the notations and standing assumptions, and prove some preliminary results that we are using throughout. Section 3 is devoted to the statements and the proofs of our main results. We will mainly show that under the assumptions introduced in Sect. 2 the Eq. (1.1) admits a unique maximal local solution, and with additional conditions on F and G we prove that this maximal local solution turns out to be a global one. The results are obtained by use of cut-off and fixed point methods introduced in [10] (see also [18,19] for similar idea). In Sect. 4 we give a detailed discussion on how our abstract results are used to solve the stochastic NS, MHD, MB, BBC, Shell models and Leray-α models driven by multiplicative noise of jump type. Most of the examples and notations in Sect. 4 are taken from [17]. In appendix we prove the well-posedness of a linear stochastic evolution equations driven by compensated Poisson random measure which is very important for our analysis.

2. Description of an abstract stochastic evolution equation

In this paper we give the necessary notations and standing assumptions used throughout the paper. We also prove some preliminary results that are very important for our analysis.

2.1. Notations and preliminary results

In this section we start with some notations, then introduce the assumptions used throughout the paper and our abstract stochastic equation.

Let $(\mathbf{V}, \|\cdot\|)$, $(\mathbf{H}, |\cdot|)$ and $(\mathbf{E}, \|\cdot\|_{\mathbf{E}})$ be three separable Hilbert spaces. The scalar product in $\mathbf{H}$ is denoted by $\langle u, v\rangle$ for any $u, v \in \mathbf{H}$. The same symbol $\langle \phi, v\rangle$ will also be used to denoted duality pairing of $\phi \in \mathbf{V}^*$ and $v \in \mathbf{V}$. We will identify $\mathbf{H}$ with its dual $\mathbf{H}^*$, and we assume that the embeddings

$$\mathbf{E} \subset \mathbf{V} \subset \mathbf{H} \subset \mathbf{V}^* \subset \mathbf{E}^*$$

are continuous and dense.

We denote by $\mathcal{L}(Y_1, Y_2)$ be the space of bounded linear maps from a Banach space Y_1 into another Banach space Y_2.

For $T_2 > T_1 \geq 0$ we denote by $L^p(T_1, T_2; \mathrm{B})$, $p \in [1, \infty)$, the space of all equivalence classes of functions u defined on $[T_1; T_2]$ and taking values in a separable Banach space B such that u is measurable and

$$\|u(\cdot)\|_{L^p(T_1,T_2;\mathrm{B})} := \left(\int_{T_1}^{T_2} \|u(s)\|_{\mathrm{B}}^p ds \right)^{\frac{1}{p}} < \infty.$$

The space $L^\infty(T_1, T_2; \mathrm{B})$ is the set of all classes of measurable functions $u : [T_1, T_2] \to \mathrm{B}$ such that

$$\|u(\cdot)\|_{L^\infty(T_1,T_2;\mathrm{B})} := \operatorname{esssup}_{s\in[T_1,T_2]} \|u(s)\| < \infty.$$

For $T_2 > T_1 \geq 0$ we set

$$X_{T_1,T_2} = L^\infty(T_1, T_2; \mathbf{V}) \cap L^2(T_1, T_2; \mathbf{E}), \tag{2.1}$$

with the norm $\|u\|_{X_{T_1,T_2}}$ defined by

$$\|u\|_{X_{T_1,T_2}}^2 = \operatorname{esssup}_{s\in[T_1,T_2]} \|u(s)\|^2 + \int_{T_1}^{T_2} \|u(s)\|_{\mathbf{E}}^2 \, ds. \tag{2.2}$$

For $T_1 = 0$ and $T_2 = T > 0$ we simply write $X_T := X_{0,T}$.

Let Y be a separable and complete metric space and $T > 0$. The space $\mathbf{D}([0, T]; Y)$ denotes the space of all right continuous functions $x : [0, 1] \to Y$ with left limits. The space $\mathbf{D}([0, T]; Y)$ equipped with the Skorohod topology J_1, which is the finest of Skorohod's topologies, is both separable and complete. For more information about the Skorohod space and the J_1-topology we refer to Ethier and Kurtz [23, Chapter 3, Section 5].

Let Z be a separable metric space, $\mathcal{B}(Z)$ its Borel σ-algebra and let ν be a σ-finite positive measure on the measure space $(Z, \mathcal{B}(Z))$. For the sake of simplicity the Borel σ-algebra $\mathcal{B}(Z)$ of Z will be denoted by $\mathcal{Z}$ for the remaining part of the paper. We set $\mathbb{R}_+ := [0, \infty)$ and λ the Lebesgue measure on $\mathbb{R}$. Suppose that $\mathfrak{P} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is a right-continuous filtration satisfying the usual condition, and $\eta : \Omega \times \mathcal{B}(\mathbb{R}_+) \times \mathcal{Z} \to \bar{\mathbb{N}}$ is a time homogeneous Poisson random measure with the intensity measure ν defined over the filtered probability space $\mathfrak{P}$. We will denote by $\tilde{\eta} = \eta - \gamma$ the compensated Poisson random measure associated to η where the compensator γ is given by

$$\mathcal{B}(\mathbb{R}_+) \times \mathcal{Z} \ni (I, A) \mapsto \gamma(I, A) = \nu(A)\lambda(I) \in \mathbb{R}_+.$$

For each Banach space B we denote by $M^2(0, T; B)$ the space of all equivalence classes of $\mathbb{F}$-progressively measurable B-valued processes defined on $[0, T]$ such that

$$\|u\|_{M^2(0,T;B)}^2 = \mathbb{E} \int_0^T \|u(s)\|_B^2 ds < \infty.$$

Throughout the paper, let us denote by $M^2(X_T)$, the space of all $\mathbb{F}$-progressively measurable $\mathbf{V} \cap \mathbf{H}$-valued processes whose trajectories belong to X_T almost surely, endowed with a norm

$$\|u\|^2_{M^2(X_T)} = \mathbb{E}\left[\sup_{s\in[0,T]} \|u(s)\|^2 + \int_0^T \|u(s)\|^2_{\mathbf{E}}\, ds\right]. \tag{2.3}$$

Now, let $\mathcal{P}$ be the σ-field on $[0,\infty)\times\Omega$ generated by all real-valued left-continuous and $\mathfrak{F}$-adapted processes. Let $(H,\mathcal{B}(H))$ be a measurable space. We say that an H-valued process $g=(g(t))_{t\geq 0}$ is **predictable** if the mapping $[0,\infty)\times\Omega \ni (t,\omega)\mapsto g(t,\omega)\in H$ is $\mathcal{P}/\mathcal{B}(H)$-measurable. Following the notation of [7], let $\mathcal{M}^2_{\text{loc}}(\mathbb{R}_+, L^2(Z,\nu,H))$ be the class of all predictable measurable processes $\xi:\mathbb{R}_+\times\Omega\to L^2(Z,\nu,H)$ satisfying the condition

$$\mathbb{E}\int_0^T\int_Z |\xi(r,z)|^2_H \nu(dz)\, dr < \infty, \quad \forall T>0. \tag{2.4}$$

If $T>0$, the class of all predictable measurable processes $\xi:\mathbb{R}_+\times\Omega\to L^2(Z,\nu,H)$ satisfying the condition (2.4) just for this one T, will be denoted by $\mathcal{M}^2(0,T,L^2(Z,\nu,H))$. Also, let $\mathcal{M}^2_{\text{step}}(\mathbb{R}_+, L^2(Z,\nu,H))$ be the space of all processes $\xi\in\mathcal{M}^2_{\text{loc}}(\mathbb{R}_+, L^2(Z,\nu,H))$ such that

$$\xi(r) = \sum_{j=1}^n 1_{(t_{j-1},t_j]}(r)\xi_j, \quad 0\leq r,$$

where $\{0=t_0<t_1<\ldots<t_n<\infty\}$ is a partition of $[0,\infty)$, and for all j, ξ_j is an $\mathcal{F}_{t_{j-1}}\otimes\mathcal{Z}$-measurable random variable. For any $\xi\in\mathcal{M}^2_{\text{step}}(\mathbb{R}_+, L^2(Z,\nu,H))$ set

$$\tilde{I}(\xi) = \sum_{j=1}^n \int_Z \xi_j(z)\tilde{\eta}\left(dz,(t_{j-1},t_j]\right). \tag{2.5}$$

This is basically the definition of stochastic integral of a random step process ξ with respect to the compound random Poisson measure $\tilde{\eta}$. The extension of this integral on $\mathcal{M}^2_{\text{loc}}(\mathbb{R}_+, L^2(Z,\nu,H))$ is possible thanks to the following result which is taken from [7, Theorem C.1].

Theorem 2.1. *There exists a unique bounded linear operator*

$$I:\mathcal{M}^2_{\text{loc}}(\mathbb{R}_+, L^2(Z,\nu;H))\to L^2(\Omega,\mathcal{F};H)$$

such that for $\xi\in\mathcal{M}^2_{\text{step}}(\mathbb{R}_+, L^2(Z,\nu;H))$ we have $I(\xi)=\tilde{I}(\xi)$. Moreover, there exists a constant $C=C(H)$ such that for any $\xi\in\mathcal{M}^2(\mathbb{R}_+, L^2(Z,\nu,H))$,

$$\mathbb{E}|I(1_{[0,t]}\xi)|^2 =: \mathbb{E}|\int_0^t\int_Z \xi(r,z)\tilde{\eta}(dz,dr)|^2_H \leq C\,\mathbb{E}\int_0^t\int_Z |\xi(r,z)|^2_H\,\nu(dz)\,dr, \quad t\geq 0. \tag{2.6}$$

Furthermore, for each $\xi\in\mathcal{M}^2_{\text{loc}}(\mathbb{R}_+, L^2(Z,\nu,H))$, the process $I(1_{[0,t]}\xi)$, $t\geq 0$, is an H-valued càdlàg martingale. The process $1_{[0,t]}\xi$ is defined by $[1_{[0,t]}\xi]$ $(r,z,\omega):=1_{[0,t]}(r)\xi(r,z,\omega)$, $t\geq 0$, $r\in\mathbb{R}_+$, $z\in Z$ and $\omega\in\Omega$.

As usual we will write

$$\int_0^t\int_Z \xi(r,z)\tilde{\eta}(dz,dr) := I(\xi)(t), \quad t\geq 0.$$

Now we introduce the following standing assumptions.

Assumption 2.1. Let N be a self-adjoint unbounded operator on $\mathbf{H}$ such that $N \in \mathcal{L}(\mathbf{E},\mathbf{H}) \cap \mathcal{L}(\mathbf{V},\mathbf{V}^*)$. Also let A be a bounded linear map from $\mathbf{E}$ into $\mathbf{H}$. We assume that there exist $C_N, C_A > 0$ such that

$$\langle Au, Nu\rangle \geq C_A\|u\|_{\mathbf{E}}^2 \quad \text{and} \quad \langle Nu, u\rangle \geq C_N\|u\|^2,$$

for any $u \in V$. The norm of $N \in \mathcal{L}(\mathbf{E},\mathbf{H})$ and $N \in \mathcal{L}(\mathbf{V},\mathbf{V}^*)$ will be denoted respectively by $\|N\|_{\mathcal{L}(\mathbf{E},\mathbf{H})}$ and $\|N\|_{\mathcal{L}(\mathbf{V},\mathbf{V}^*)}$ throughout.

Let F and G be two nonlinear mappings satisfying the following sets of conditions.

Assumption 2.2. Suppose that $F : \mathbf{E} \to \mathbf{H}$ is such that $F(0) = 0$ and there exists $p \geq 1$, $\alpha \in [0, 1)$ and $C > 0$ such that

$$|F(y) - F(x)| \leq C \left[\|y - x\|\|y\|^{p-\alpha}\|y\|_{\mathbf{E}}^{\alpha} + \|y - x\|_{\mathbf{E}}^{\alpha}\|y - x\|^{1-\alpha}\|x\|^p\right], \quad (2.7)$$

for any $x, y \in \mathbf{E}$.

Assumption 2.3. (i) Assume that $G : \mathbf{V} \to L^{2q}(Z, \nu, \mathbf{V})$ and there exists a constant $\ell_q > 0$ such that

$$\|G(x) - G(y)\|_{L^{2q}(Z,\nu,\mathbf{V})}^{2q} \leq \ell_q^q\|x - y\|^{2q}, \quad (2.8)$$

for any $x, y \in \mathbf{V}$ and $q = 1, 2$.

Note that this implies in particular that there exists a constant $\tilde{\ell}_q > 0$ such that

$$\|G(x)\|_{L^{2q}(Z,\nu,\mathbf{V})}^{2q} \leq \tilde{\ell}_q^q(1+\|x\|^{2q}), \quad (2.9)$$

for any $x \in \mathbf{V}$ and $p = 1, 2$.

(ii) We also assume that G satisfies the inequality (2.7) with the norm of $\mathbf{V}$ replaced by the norm of $\mathbf{H}$. More precisely, there exists $\ell_p > 0$ such that

$$\|G(x) - G(y)\|_{L^{2p}(Z,\nu,\mathbf{H})}^{2p} \leq \ell_p^p|x - y|^{2p}, \quad (2.10)$$

for any $x, y \in \mathbf{V}$ and $p = 1, 2$.

Throughout this work we fix a positive number T. One of our objectives is to prove the existence and uniqueness of maximal/local solution of the following stochastic evolution equation

$$\mathbf{u}(t) = \mathbf{u}_0 - \int_0^t [A\mathbf{u}(s) + F(\mathbf{u}(s))]\, ds + \int_0^t \int_Z G(z, \mathbf{u}(s-))\widetilde{\eta}(dz, ds). \quad (2.11)$$

The above identity is the shorthand of the following identity

$$\langle \mathbf{u}(t), v\rangle = \langle \mathbf{u}_0, v\rangle - \int_0^t \langle [A\mathbf{u}(s) + F(\mathbf{u}(s))], v\rangle ds + \int_0^t \int_Z \langle G(\mathbf{u}(s-)), v\rangle\widetilde{\eta}(dz, ds), \quad (2.12)$$

almost surely (a.s.) for any $t \in [0, T]$ and $v \in \mathbf{H}$.

Now, let us introduce the concept of local and maximal local solution.

Definition 2.2. (*Local solution*) By a local solution of (2.11) we mean a pair $(\mathbf{u}, \tau_\infty)$ such that

(1) τ_∞ is a stopping time such that $\tau_\infty \leq T$ a.s. and there exists a nondecreasing sequence $\{\tau_n, n \geq 1\}$ stopping times with $\tau_n \uparrow \tau_\infty$ a.s. as $n \uparrow \infty$,
(2) $\mathbf{u}$ is a progressively measurable stochastic process with càdlàg paths in $\mathbf{V}$, with probability 1 $\mathbf{u} \in X_t$ for any $t \in [0, \tau_\infty)$ and

$$\mathbf{u}(t \wedge \tau_n) = \mathbf{u}_0 - \int_0^{t\wedge\tau_n} [A\mathbf{u}(s) + F(\mathbf{u}(s))]\, ds + \int_0^{t\wedge\tau_n} \int_Z G(z, \mathbf{u}(s-))\widetilde{\eta}(dz, ds), \tag{2.13}$$

holds for any $t \in [0, T]$ and $n \geq 1$ with probability 1.

The identity (2.13) is the shorthand of the following

$$\langle \mathbf{u}(t \wedge \tau_n), v\rangle = \langle \mathbf{u}_0, v\rangle - \int_0^{t\wedge\tau_n} \langle [A\mathbf{u}(s) + F(\mathbf{u}(s))], v\rangle ds + \int_0^{t\wedge\tau_n} \int_Z \langle G(\mathbf{u}(s-)), v\rangle \widetilde{\eta}(dz, ds), \tag{2.14}$$

holds for any $t \in [0, T]$, and $n \geq 1$ with probability 1, and for all $v \in \mathbf{H}$.

Remark 2.3. Note that since $\mathbf{u}$ is càdlàg and progressively measurable, the left limit stochastic process $\{\mathbf{u}(t-);\ t \in [0, \tau_\infty]\}$ is continuous and adapted, hence predictable. Therefore, all the terms, in particular the stochastic integral, in (2.13) (also (2.14)) are well-defined.

We also define the maximal local solution to (2.11).

Definition 2.4. (*Maximal local solution*)

(1) Let $(\mathbf{u}, \tau_\infty)$ be a local solution to (2.11). If $\lim_{t\nearrow\tau_\infty} \|\mathbf{u}\|_{X_t} = \infty$ on $\{\omega, \tau_\infty < T\}$ a.s., then the local process $(\mathbf{u}, \tau_\infty)$ is called a maximal local solution.
(2) A maximal local solution $(\mathbf{u}, \tau_\infty)$ is said to be unique if for any other maximal local solution $(\mathbf{v}, \sigma_\infty)$ we have $\sigma_\infty = \tau_\infty$ and $\mathbf{u}(t) = \mathbf{v}(t)$ for any $0 \leq t < \tau_\infty$ with probability one.
(3) If $(\mathbf{u}, \tau_\infty)$ is a local solution to (2.11) and $\tau_\infty = T$ with probability 1, then the stochastic process $\{\mathbf{u}(t), t \in [0, T)\}$ is called a global solution.

As in [8], we let $\theta : \mathbb{R}_+ \to [0, 1]$ be a $\mathcal{C}_0^\infty$ non increasing function such that

$$\inf_{x\in\mathbb{R}_+} \theta'(x) \geq -1, \quad \theta(x) = 1 \text{ iff } x \in [0, 1] \quad \text{and } \theta(x) = 0 \text{ iff } x \in [2, \infty). \tag{2.15}$$

and for $n \geq 1$ set $\theta_n(\cdot) = \theta(\frac{\cdot}{n})$. Note that if $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a non decreasing function, then for every $x, y \in \mathbb{R}$,

$$\theta_n(x)h(x) \leq h(2n), \quad |\theta_n(x) - \theta_n(y)| \leq \frac{1}{n}|x - y|. \tag{2.16}$$

Proposition 2.5. *Let F be a nonlinear mapping satisfying Assumption 2.2. Let us consider a map $B_n^T : X_T \to L^2(0, T; \mathbf{H})$ defined by*

$$(B_n^T u)(t) := \theta_n(\|u\|_{X_t})F(u(t)), \quad u \in X_T, \quad t \in [0, T].$$

Then B_n^T is globally Lipschitz and moreover, for any $u_1, u_2 \in X_T$,

$$\|(B_n^T u_1)(\cdot) - (B_n^T u_2)(\cdot)\|_{L^2(0,T;\mathbf{H})} \leq C(2n)^p \left[(2n)C + 1\right] T^{\frac{1-\alpha}{2}} \|u_1 - u_2\|_{X_T}. \tag{2.17}$$

Proof. The proof is the same as in [8], but for the sake of completeness we repeat it here. Note that by Assumption 2.2 $(B_n^T\ 0)(\cdot) = 0$. Assume that $u_1, u_2 \in X_T$. Denote, for $i = 1, 2$,

$$\tau_i = \inf\{t \in [0, T] : \|u_i\|_{X_t} \geq 2n\}.$$

Note that by definition, if the set on the RHS above is empty, then $\tau_i = T$. Without loss of generality we may assume that $\tau_1 \leq \tau_2$.

Since, for $i = 1, 2$, $\theta_n(\|u_i\|_{X_t}) = 0$ for $t \geq \tau_2$, we have

$$\begin{aligned}
&\|(B_n^T u_1)(\cdot) - (B_n^T u_2)(\cdot)\|_{L^2(0,T;\mathbf{H})} \\
&\quad = \left[\int_0^T |\theta_n(\|u_1\|_{X_t})F(u_1(t)) - \theta_n(\|u_2\|_{X_t})F(u_2(t))|^2\, dt\right]^{1/2} \\
&\quad = \left[\int_0^{\tau_2} |\theta_n(\|u_1\|_{X_t})F(u_1(t)) - \theta_n(\|u_2\|_{X_t})F(u_2(t))|^2\, dt\right]^{1/2} \\
&\quad = \left[\int_0^{\tau_2} |\left[\theta_n(\|u_1\|_{X_t}) - \theta_n(\|u_2\|_{X_t})\right] F(u_2(t)) \right. \\
&\qquad\qquad \left. + \theta_n(\|u_1\|_{X_t}) \left[F(u_1(t)) - F(u_2(t))\right] |^2\, dt\right]^{1/2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\|(B_n^T u_1)(\cdot) - (B_n^T u_2)(\cdot)\|_{L^2(0,T;\mathbf{H})} \\
&\quad \leq \left[\int_0^{\tau_2} |\left[\theta_n(\|u_1\|_{X_t}) - \theta_n(\|u_2\|_{X_t})\right] F(u_2(t))|^2\, dt\right]^{1/2} \\
&\qquad + \left[\int_0^{\tau_2} |\theta_n(\|u_1\|_{X_t}) \left[F(u_1(t)) - F(u_2(t))\right] |^2\, dt\right]^{1/2} =: I_1 + I_2.
\end{aligned}$$

Next, since θ_n is Lipschitz with Lipschitz constant n^{-1} we have

$$\begin{aligned}
I_1^2 &= \int_0^{\tau_2} |\left[\theta_n(\|u_1\|_{X_t}) - \theta_n(\|u_2\|_{X_t})\right] F(u_2(t))|^2\, dt \\
&\leq n^{-2} C^2 \int_0^{\tau_2} \left[|\|u_1\|_{X_t} - \|u_2\|_{X_t}|\right]^2 |F(u_2(t))|^2\, dt
\end{aligned}$$

from which along with the Minkowski inequality we deduce that

$$\begin{aligned}
I_1^2 &\leq n^{-2} C^2 \int_0^{\tau_2} \|u_1 - u_2\|_{X_t}^2 |F(u_2(t))|^2\, dt \leq 4n^2 C^2 \int_0^{\tau_2} \|u_1 - u_2\|_{X_T}^2 |F(u_2(t))|^2\, dt \\
&\leq n^{-2} C^2 \|u_1 - u_2\|_{X_T}^2 \int_0^{\tau_2} |F(u_2(t))|^2\, dt.
\end{aligned}$$

Next, by assumptions

$$\begin{aligned}
\int_0^{\tau_2} |F(u_2(t))|^2\, dt &\leq C^2 \int_0^{\tau_2} \|u_2(t)\|^{2p+2-2\alpha} \|u_2(t)\|_{\mathbf{E}}^{2\alpha}\, dt \\
&\leq C^2 \sup_{t \in [0,\tau_2]} \|u_2(t)\|^{2p+2-2\alpha} \left(\int_0^{\tau_2} \|u_2(t)\|_{\mathbf{E}}^2\, dt\right)^{\alpha} \tau_2^{1-\alpha} \\
&\leq C^2 \tau_2^{1-\alpha} \|u_2\|_{X_{\tau_2}}^{2p+2} \leq C^2 \tau_2^{1-\alpha} (2n)^{2p+2}.
\end{aligned}$$

Therefore,

$$I_1 \leq C^2 \tau_2^{(1-\alpha)/2} (2n)^p \|u_1 - u_2\|_{X_T}.$$

Also, because $\theta_n(\|u_1\|_{X_t}) = 0$ for $t \geq \tau_1$, and $\tau_1 \leq \tau_2$, we have the following chain of equalities/inequalities

$$\begin{aligned}
I_2 &= \left[\int_0^{\tau_2} |\theta_n(\|u_1\|_{X_t}) \left[F(u_1(t)) - F(u_2(t))\right]|^2 \, dt\right]^{1/2} \\
&= \left[\int_0^{\tau_1} |\theta_n(\|u_1\|_{X_t}) \left[F(u_1(t)) - F(u_2(t))\right]|^2 \, dt\right]^{1/2} \\
&\quad \text{because } \theta_n(\|u_1\|_{X_t}) \leq 1 \text{ for } t \in [0, \tau_1) \\
&\leq \left[\int_0^{\tau_1} |F(u_1(t)) - F(u_2(t))|^2 \, dt\right]^{1/2} \\
&\leq C \left[\int_0^{\tau_1} \|u_1(t) - u_2(t)\|^2 \|u_1(t)\|^{2p-2\alpha} \|u_1(t)\|_{\mathbf{E}}^{2\alpha} \, dt\right]^{1/2} \\
&\quad + C \left[\int_0^{\tau_1} \|u_1(t) - u_2(t)\|_{\mathbf{E}}^{2\alpha} \|u_1(t) - u_2(t)\|^{2-2\alpha} \|u_2(t)\|^{2p} \, dt\right]^{1/2} \\
&\leq C \sup_{t \in [0,\tau_1]} \|u_1(t) - u_2(t)\|^{1-\alpha} \|u_2(t)\|^p \left[\int_0^{\tau_1} \|u_1(t) - u_2(t)\|_{\mathbf{E}}^{2\alpha} \, dt\right]^{1/2} \\
&\quad + C \sup_{t \in [0,\tau_1]} \|u_1(t) - u_2(t)\| \|u_1(t)\|^{p-\alpha} \left[\int_0^{\tau_1} \|u_1(t)\|_{\mathbf{E}}^{2\alpha} \, dt\right]^{1/2} \\
&\leq C \sup_{t \in [0,T]} \|u_1(t) - u_2(t)\| \sup_{t \in [0,\tau_1]} \|u_1(t)\|^{p-\alpha} \left[\int_0^{\tau_1} \|u_1(t)\|_{\mathbf{E}}^2 \, dt\right]^{\alpha/2} \tau_1^{(1-\alpha)/2} \\
&\quad + C \sup_{t \in [0,T]} \|u_1(t) - u_2(t)\|^{1-\alpha} \sup_{t \in [0,\tau_1]} \|u_2(t)\|^p \left[\int_0^{\tau_1} \|u_1(t) - u_2(t)\|_{\mathbf{E}}^2 \, dt\right]^{\alpha/2} \tau_1^{(1-\alpha)/2} \\
&\leq C \|u_1 - u_2\|_{X_T} \|u_1\|_{X_{\tau_1}}^p \tau_1^{(1-\alpha)/2} + C \|u_1 - u_2\|_{X_T} \|u_2\|_{X_{\tau_1}}^p \tau_1^{(1-\alpha)/2} \\
&\quad \text{because} \|u_1\|_{X_{\tau_1}} \leq 2n \text{ and } \|u_2\|_{X_{\tau_1}} \leq \|u_2\|_{X_{\tau_2}} \leq 2n \\
&\leq C \tau_1^{(1-\alpha)/2} \|u_1 - u_2\|_{X_T} \left[\|u_1\|_{X_{\tau_1}}^p + \|u_2\|_{X_{\tau_1}}^p\right] \leq C (2n)^{p+1} \tau_1^{(1-\alpha)/2} \|u_1 - u_2\|_{X_T}
\end{aligned}$$

Summing up, we proved

$$\begin{aligned}
&\|(B_n^T u_1)(\cdot) - (B_n^T u_2)(\cdot)\|_{L^2(0,T;\mathbf{H})} \\
&\quad \leq \left[C^2 \tau_2^{(1-\alpha)/2} (2n)^p + C (2n)^{p+1} \tau_1^{(1-\alpha)/2}\right] \|u_1 - u_2\|_{X_T} \\
&\quad = C (2n)^p \left[2nC + 1\right] \tau_2^{(1-\alpha)/2} \|u_1 - u_2\|_{X_T}
\end{aligned}$$

The proof is complete. □

3. Existence of maximal local and global solution of Eq. (2.11)

This section is devoted to the solvability of (2.11). We will mainly show that under Assumptions 2.1–2.3, Eq. (2.11) admits a unique maximal local solution. Under additional conditions on F and G we prove that this maximal local solution turns out to be a global solution. The results are obtained by use of cut-off and fixed point arguments.

3.1. Global solution of a truncated equation

For simplicity we set $B_n^T(u)(s) = B_n^T(u(s))$ for any $u \in X_T$ and $s \geq 0$. Let

$$\begin{aligned} \mathbf{u}_n(t) + \int_0^t [A\mathbf{u}_n(s) + (B_n^T\mathbf{u}_n)(s)]ds = \mathbf{u}_0 \\ + \int_0^t \int_Z G(z, \mathbf{u}_n(s-))\widetilde{\eta}(dz, ds), \quad t \in [0, T], \end{aligned} \tag{3.1}$$

which is understood as

$$\begin{aligned} \langle \mathbf{u}_n(t), v\rangle + \int_0^t \langle A\mathbf{u}_n(s) + (B_n^T\mathbf{u}_n)(s), v\rangle ds = \langle \mathbf{u}_0, v\rangle \\ + \int_0^t \int_Z \langle G(z, \mathbf{u}_n(s-)), v\rangle \widetilde{\eta}(dz, ds), \quad t \in [0, T], \end{aligned} \tag{3.2}$$

for any $v \in \mathbf{H}$. Here, as in the previous section we set

$$(B_n^T\mathbf{u}_n)(t) = \theta_n(\|u\|_{X_t})F(u(t)), \ \ t \in [0, T]$$

for any $u \in X_T$. For $n \in \mathbb{N}$ we also set

$$\phi(n) = C^2(2n)^{2p}\left[2nC + 1\right]^2. \tag{3.3}$$

Now, let $\mathbf{v} \in M^2(X_T)$, $n > 0$ and let us consider the linear stochastic evolution equation

$$\begin{cases} d\mathbf{u}_n(t) + A\mathbf{u}_n(t)dt = -(B_n^T\mathbf{u}_n)(t)dt + \int_Z G(z, \mathbf{v}(t-))\widetilde{\eta}(dz, dt), \\ \mathbf{u}_n(0) = u_0. \end{cases} \tag{3.4}$$

Thanks to Theorem A.1 for each $\mathbf{v} \in M^2(X_T)$ and $n \geq 1$, there exists a unique $\mathbf{V}$-valued progressively measurable process $\mathbf{u}_n$ solving (3.4). Moreover, $\mathbf{u}^n \in \mathbf{D}([0, T]; \mathbf{V}) \cap L^2(0, T; \mathbf{E})$ with probability 1.

Lemma 3.1. *For each $n \geq 1$ let Λ_n be the mapping defined by*

$$\Lambda_n : M^2(X_T) \ni \mathbf{v} \mapsto \mathbf{u}_n = \Lambda_n(\mathbf{v}),$$

where $\mathbf{u}_n$ is the unique solution to (3.4). For any $\mathbf{v} \in M^2(X_T)$, the stochastic process $\mathbf{u}_n$ belongs to $M^2(X_T)$.

Proof. Let $\Psi : \mathbf{H} \to \mathbb{R}$ be the mapping defined by

$$\Psi(u) = \langle u, Nu\rangle,$$

for any $u \in \mathbf{H}$. This mapping is Fréchet differentiable with first derivative defined by

$$\Psi'(u)[h] = \langle h, Nu\rangle + \langle u, Nh\rangle.$$

Since N is self-adjoint we have

$$\Psi'(u)[h] = 2\langle h, Nu\rangle.$$

Applying Itô's formula (see, for instance, [44, Appendix D]) to $\Psi(\mathbf{u})$ with (3.4) we obtain

$$\begin{aligned}\Psi(\mathbf{u}_n(t)) - \Psi(\mathbf{u}_0) + 2\int_0^t \langle A\mathbf{u}_n(s) + (B_n^T\mathbf{u}_n)(s), N\mathbf{u}_n(s)\rangle ds \\ = \int_0^t\int_Z \Big[\Psi(\mathbf{u}_n(s) + G(z,\mathbf{v}(s))) - \Psi(\mathbf{u}_n(s)) - \Psi'(\mathbf{u}_n(s))[G(z,\mathbf{v}(s))]\Big]\nu(dz)ds \\ + \int_0^t\int_Z \Big[\Psi(\mathbf{u}_n(s-) + G(z,\mathbf{v}(s-))) - \Psi(\mathbf{u}_n(s-))\Big]\widetilde{\eta}(dz,ds). \qquad (3.5)\end{aligned}$$

From the Cauchy-Schwarz inequality we derive that

$$\begin{aligned}\left|\int_0^t \langle (B_n^T\mathbf{u}_n)(s), N\mathbf{u}_n(s)\rangle ds\right| &\le \int_0^t |(B_n^T\mathbf{u}_n)(s)||N\mathbf{u}_n(s)|ds, \\ &\le \|N\|_{\mathcal{L}(\mathbf{E},\mathbf{H})}\int_0^t |B_n^T(\mathbf{v}(s))|\|\mathbf{u}_n(s)\|_{\mathbf{E}}ds.\end{aligned}$$

From the last line along with Cauchy's inequality with ε we deduce that

$$\mathbb{E}\left|\int_0^t \langle (B_n^T\mathbf{u}_n)(s), N\mathbf{u}_n(s)\rangle ds\right| \le \varepsilon\mathbb{E}\int_0^t \|\mathbf{u}_n(s)\|^2_{\mathbf{E}}ds + \frac{\|N\|^2_{\mathcal{L}(\mathbf{E},\mathbf{H})}}{4\varepsilon}\mathbb{E}\int_0^t |B_n^T(\mathbf{v}(s))|^2ds.$$

Now invoking Eq. (2.17) from Proposition 2.5 we infer that

$$\mathbb{E}\left|\int_0^t \langle (B_n^T\mathbf{u}_n)(s), N\mathbf{u}_n(s)\rangle ds\right| \le \varepsilon\mathbb{E}\int_0^t \|\mathbf{u}_n(s)\|^2_{\mathbf{E}}ds + \frac{\|N\|^2_{\mathcal{L}(\mathbf{E},\mathbf{H})}\phi(n)}{4\varepsilon}t^{\alpha-1}\|\mathbf{v}\|^2_{M^2(X_T)}, \qquad (3.6)$$

where $\phi(n)$ is defined in (3.3).

Now, note that

$$\Psi(u+h) - \Psi(u) - \Psi'(u)[h] = \langle Nh, h\rangle.$$

Hence

$$\begin{aligned}I_1 := \mathbb{E}\left|\int_0^t\int_Z \Big[\Psi(\mathbf{u}_n(s) + G(z,\mathbf{v}(s))) - \Psi(\mathbf{u}_n(s)) - \Psi'(\mathbf{u}_n(s))[G(z,\mathbf{v}(s))]\Big]\nu(dz)ds\right| \\ \le \|N\|_{\mathcal{L}(\mathbf{V},\mathbf{V}^*)}\mathbb{E}\int_0^t\int_Z \|G(z,\mathbf{v}(s))\|^2\nu(dz)ds. \qquad (3.7)\end{aligned}$$

By making use of (2.9) we easily derive from the last inequality that

$$I_1 \le t\|N\|_{\mathcal{L}(\mathbf{V},\mathbf{V}^*)}\tilde{\ell}_1\left(1 + \|\mathbf{v}\|^2_{M^2(X_T)}\right). \qquad (3.8)$$

Notice also that

$$\Psi(u+h) - \Psi(u) = 2\langle Nu, h\rangle + \langle Nh, h\rangle,$$

thus

$$
\begin{aligned}
I_2 := & \mathbb{E} \sup_{s\in[0,t]} \left| \int_0^s \int_Z \Big[\Psi(\mathbf{u}_n(r-) + G(z, \mathbf{v}(r-)) - \Psi(\mathbf{u}_n(r-)) \Big] \widetilde{\eta}(dz, dr) \right| \\
& \le \mathbb{E} \sup_{s\in[0,t]} \left| \int_0^s \int_Z \langle N\mathbf{u}_n(s-), G(z, \mathbf{v}(s-)) \rangle \widetilde{\eta}(dz, ds) \right| \\
& \quad + \mathbb{E} \sup_{s\in[0,t]} \left| \int_0^s \int_Z \langle NG(z, \mathbf{v}(s-)), G(z, \mathbf{v}(s-)) \rangle \widetilde{\eta}(dz, ds) \right| \\
& \le I_{2,1} + I_{2,2}.
\end{aligned}
$$

Owing to the BDG inequality (see, for instance, [45, Theorem 48]) we infer that

$$
\begin{aligned}
I_{2,1} := & \mathbb{E} \sup_{s\in[0,t]} \left| \int_0^s \int_Z \langle N\mathbf{u}_n(r-), G(z, \mathbf{v}(r-)) \rangle \widetilde{\eta}(dz, dr) \right| \\
& \le C\mathbb{E} \left[\int_0^t \int_Z \langle NG(z, \mathbf{v}(s)), \mathbf{u}_n(s) \rangle^2 ds \right]^{\frac{1}{2}}, \\
& \le C\|N\|_{\mathcal{L}(\mathbf{V},\mathbf{V}^*)} \mathbb{E} \left[\sup_{s\in[0,t]} \|\mathbf{u}_n(s)\| \left(\int_0^t \int_Z \|G(z, \mathbf{v}(s))\|^2 \nu(dz) ds \right)^{\frac{1}{2}} \right] \\
& \quad \text{(by the Young inequality with } \delta > 0 \text{ arbitrary)} \\
& \le \delta \mathbb{E} \left[\sup_{s\in[0,t]} \|\mathbf{u}_n(s)\|^2 \right] + \frac{C^2 \|N\|^2_{\mathcal{L}(\mathbf{V},\mathbf{V}^*)}}{4\delta} \mathbb{E} \int_0^t \int_Z \|G(z, \mathbf{v}(s))\|^2 \nu(dz) ds \\
& \quad \text{(by the inequality (2.9))} \\
I_{2,1} & \le \delta \mathbb{E} \left[\sup_{s\in[0,t]} \|\mathbf{u}_n(s)\|^2 \right] + \frac{C^2 \|N\|^2_{\mathcal{L}(\mathbf{V},\mathbf{V}^*)} \tilde{\ell}_1 t}{4\delta} (1 + \|\mathbf{v}\|^2_{M^2(X_T)}).
\end{aligned} \tag{3.9}
$$

Using again the BDG inequality yields

$$
\begin{aligned}
I_{2,2} := & \mathbb{E} \sup_{s\in[0,t]} \left| \int_0^s \int_Z \langle NG(z, \mathbf{v}(s-)), G(z, \mathbf{v}(s-)) \rangle \widetilde{\eta}(dz, ds) \right| \\
& \le C\mathbb{E} \left[\int_0^t \int_Z [\langle NG(z, \mathbf{v}(s)), G(z, \mathbf{v}(s)) \rangle]^2 \nu(dz) ds \right]^{\frac{1}{2}} \\
& \le C\|N\|_{\mathcal{L}(\mathbf{V},\mathbf{V}^*)} \| \mathbb{E} \left[\int_0^t \int_Z \|G(z, \mathbf{v}(s))\|^4 \nu(dz) ds \right]^{\frac{1}{2}} \\
& \text{(by the inequality (2.9))} \\
& \le C\|N\|_{\mathcal{L}(\mathbf{V},\mathbf{V}^*)} \tilde{\ell}_2 t (1 + \mathbb{E} \sup_{s\in[0,t]} \|\mathbf{v}(s)\|^2), \\
I_{2,2} & \le C\|N\|_{\mathcal{L}(\mathbf{V},\mathbf{V}^*)} \tilde{\ell}_2 t (1 + \|\mathbf{v}\|^2_{M^2(X_T)}).
\end{aligned} \tag{3.10}
$$

Now it follows from Eqs. (3.5), (3.6), (3.8), (3.9) and (3.10) that

$$\begin{aligned}&\mathbb{E}\sup_{s\in[0,t]}\Psi(\mathbf{u}_n(s))-\Psi(\mathbf{u}_0)+2\mathbb{E}\int_0^t\langle A\mathbf{u}_n(s),N\mathbf{u}_n(s)\rangle ds\\&\quad\le 2C(\|N\|,\delta,\varepsilon,n,t)(1+\|\mathbf{v}\|^2_{M^2(X_T)})\\&\qquad+\varepsilon\mathbb{E}\int_0^t\|\mathbf{u}_n(s)\|^2_{\mathbf{E}}ds+\delta\mathbb{E}\Big[\sup_{s\in[0,t]}\|\mathbf{u}_n(s)\|^2\Big],\end{aligned}$$

where

$$\|N\|:=\max\left(\|N\|_{\mathcal{L}(\mathbf{E},\mathbf{H})},\|N\|_{\mathcal{L}(\mathbf{V},\mathbf{V}^*)}\right),$$

$$C(\|N\|,\varepsilon,\delta,n,t):=\left(\frac{\|N\|\phi(n)t^{\alpha-1}}{4\varepsilon}+t\left[\frac{C^2\|N\|\tilde{\ell}_1}{4\delta}+\tilde{\ell}_1+C\tilde{\ell}_2\right]\right)\|N\|.$$

Since $\langle u,Nu\rangle\ge C_N\|u\|^2$ and $\langle Au,Nu\rangle\ge C_A\|u\|^2_{\mathbf{E}}$, it follows that

$$\begin{aligned}&(C_N-\delta)\mathbb{E}\Big[\sup_{s\in[0,t]}\|\mathbf{u}_n(s)\|^2\Big]+(2C_A-\varepsilon)\mathbb{E}\\&\quad\times\int_0^t\|\mathbf{u}_n(s)\|^2_{\mathbf{E}}ds\le C(\|N\|,\varepsilon,\delta,n,t)(1+\|\mathbf{v}\|^2_{M^2(X_T)})+\Psi(\mathbf{u}_0).\end{aligned}$$

Choosing $\varepsilon=C_A$ and $\delta=C_N/2$, we derive from the last inequality that

$$\begin{aligned}&\mathbb{E}\Big[\sup_{s\in[0,t]}\|\mathbf{u}_n(s)\|^2\Big]+\mathbb{E}\int_0^t\|\mathbf{u}_n(s)\|^2_{\mathbf{E}}ds\le\frac{\Psi(\mathbf{u}_0)}{\min(C_N/2,C_A)}\\&\quad+\frac{C(\|N\|,C_A,C_N,n,t)}{\min(C_N/2,C_A)}(1+\|\mathbf{v}\|^2_{M^2(X_T)}).\end{aligned}$$

With this last inequality we easily conclude the proof of the claim. □

Lemma 3.2. *Let Λ_n be the mapping defined in Lemma 3.1 and*

$$\|N\|:=\max\left(\|N\|_{\mathcal{L}(\mathbf{E},\mathbf{H})},\|N\|_{\mathcal{L}(\mathbf{V},\mathbf{V}^*)}\right).$$

Then, there exists a constant $\kappa>0$ depending only on $\|N\|$, n and the constants in Assumptions 2.1–2.3 such that

$$\|\Lambda_n(\mathbf{v}_1)-\Lambda_n(\mathbf{v}_2)\|^2_{M^2(X_T)}\le\kappa\left[T^{\alpha-1}\vee T\right]\|\mathbf{v}_1-\mathbf{v}_2\|^2_{M^2(X_T)},$$

for any $\mathbf{v}_1,\mathbf{v}_2\in M^2(X_T)$.

Proof. Let $\mathbf{v}_i$, $i=1,2$, be two elements of $M^2(X_T)$. To each $\mathbf{v}_i$ one can associate a unique element $\mathbf{u}_i\in M^2(X_T)$ which is a solution to Eq. (3.4) with the stochastic perturbation

$$(B_n^T\mathbf{v}_i)(t)dt+\int_Z G(z,\mathbf{v}_i(t-))\widetilde{\eta}(dz,dt),$$

and initial condition $\mathbf{u}_0$. In this proof we suppress the dependence on n of the solution to (3.4). The difference $\mathbf{u}=\mathbf{u}_1-\mathbf{u}_2$ solves the linear equation

$$\begin{cases}d\mathbf{u}(t)+A\mathbf{u}(t)dt=[(B_n^T\mathbf{v}_1)(t)-(B_n^T\mathbf{v}_2)(t)]dt+\int_Z[G(z,\mathbf{v}_1(t-))-G(z,\mathbf{v}_2(t-))]\widetilde{\eta}(dz,dt),\\ \mathbf{u}(0)=0.\end{cases}\tag{3.11}$$

To simplify our notation we also set $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$.

As before we apply Itô's formula (see, for instance, [44, Appendix D]) to $\Psi(u) = \langle Nu, u\rangle$ with (3.11). We then obtain

$$\Psi(\mathbf{u}(t)) + 2\int_0^t \langle A\mathbf{u}(s), N\mathbf{u}(s)\rangle ds \leq 2\int_0^t |(B_n^T\mathbf{v}_1)(s) - (B_n^T\mathbf{v}_2)(s)||N\mathbf{u}(s)|ds$$
$$+ \int_0^t\int_Z f(z,s,\mathbf{v}_1,\mathbf{v}_2)\nu(dz)ds + \int_0^t\int_Z g(z,s,\mathbf{v}_1,\mathbf{v}_2)\widetilde{\eta}(dz,ds), \qquad (3.12)$$

with

$$g(z,s,\mathbf{v}_1,\mathbf{v}_2) := \langle N[G(z,\mathbf{v}_1(s-)) - G(z,\mathbf{v}_2(s-))], G(z,\mathbf{v}_1(s-)) - G(z,\mathbf{v}_2(s-))\rangle$$
$$+2\langle [G(z,\mathbf{v}_1(s-)) - G(z,\mathbf{v}_2(s-))], N\mathbf{u}(s-)\rangle,$$

and

$$f(z,s,\mathbf{v}_1,\mathbf{v}_2) := \langle N[G(z,\mathbf{v}_1(s)) - G(z,\mathbf{v}_2(s))], G(z,\mathbf{v}_1(s)) - G(z,\mathbf{v}_2(s))\rangle.$$

Arguing as in the proofs of Eqs. (3.6), (3.8), (3.9) and (3.10), respectively, we obtain the following inequalities

$$\mathbb{E}\int_0^t |(B_n^T\mathbf{v}_1)(s) - (B_n^T\mathbf{v}_2)(s)||N\mathbf{u}(s)|ds \leq \frac{\|N\|^2\phi(n)}{4\varepsilon}t^{\alpha-1}\|\mathbf{v}\|^2_{M^2(X_T)}$$
$$+ \varepsilon\mathbb{E}\int_0^t \|\mathbf{u}(s)\|^2_{\mathbf{E}}ds,$$

$$\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^t\int_Z g(z,s,\mathbf{v}_1,\mathbf{v}_2)\widetilde{\eta}(dz,ds)\right| \leq \left[\frac{C^2\|N\|^2\ell_1}{4\delta} + \|N\|\ell_2\right] t\|\mathbf{v}\|^2_{M^2(X_T)}$$
$$+ \delta\mathbb{E}\left[\sup_{s\in[0,t]}\|\mathbf{u}(s)\|^2\right],$$

$$\mathbb{E}\int_0^t\int_Z f(z,s,\mathbf{v}_1,\mathbf{v}_2)\nu(dz)ds \leq \|N\|\ell_1 t\|\mathbf{v}\|^2_{M^2(X_T)},$$

where ε, δ are arbitrary positive numbers. By setting $T^* = T \vee T^{\alpha-1}$ and

$$\tilde{\kappa} := \left(\|N\|\left[\frac{\phi(n)}{4\varepsilon} + \frac{C^2\ell_1}{4\delta}\right] + \ell_1 + C\ell_2\right)\|N\|,$$

it follows from these inequalities and Eq. (3.12) that

$$(C_N - \delta)\mathbb{E}\left[\sup_{s\in[0,t]}\|\mathbf{u}(s)\|^2\right] + (2C_A - \varepsilon)\mathbb{E}\int_0^t \|\mathbf{u}(s)\|^2_{\mathbf{E}}ds \leq \tilde{\kappa}T^*\|\mathbf{v}\|^2_{M^2(X_T)},$$

where we have used the fact that $\langle u, Nu\rangle \geq C_N\|u\|^2$ and $\langle Au, Nu\rangle \geq C_A\|u\|^2_{\mathbf{E}}$. By choosing $\delta = C_N/2$ and $\varepsilon = C_A$ we get from the last estimate that

$$\mathbb{E}\left[\sup_{s\in[0,t]}\|\mathbf{u}(s)\|^2\right] + \mathbb{E}\int_0^t \|\mathbf{u}(s)\|^2_{\mathbf{E}}ds \leq \kappa T^*\|v\|^2_{M^2(X_T)},$$

where $\kappa := \tilde{\kappa}/\min(C_N/2, C_A)$. The last estimate means that

$$\|\Lambda_n(\mathbf{v}_1) - \Lambda_n(\mathbf{v}_2)\|^2_{M^2(X_T)} \leq \kappa T^*\|\mathbf{v}_1 - \mathbf{v}_2\|^2_{M^2(X_T)}.$$

This completes the proof of our lemma. $\square$

Let n be a fixed positive integer. It follows from Lemma 3.1 that $\Lambda^n_{T,\mathbf{u}_0} := \Lambda_n$ maps $M^2(X_T)$ into itself. From the proof of Lemma 3.2 we deduce that $\Lambda^n_{T,\mathbf{u}_0}$ is globally Lipschitz. Moreover it is a strict contraction for small T. Therefore we can find a time $\delta_n > 0$ such that for any initial condition $\mathbf{u}_0$ the map $\Lambda^n_{\delta_n,\mathbf{u}_0}$ is $\frac{1}{2}$-contraction. Hence it admits a unique fixed point $\mathbf{u}_{n,\delta_n} \in M^2(X_{\delta_n})$ which solves on the small interval $[0, \delta_n]$ the nonlinear stochastic evolution equation

$$\mathbf{u}(t)+\int_0^t [A\mathbf{u}(s)+(B^T_n\mathbf{u})(s)]ds = \mathbf{u}_0+\int_0^t \int_Z G(z, \mathbf{u}(s-))\widetilde{\eta}(dz, ds), \quad t \in [0, \delta_n). \tag{3.13}$$

Lemma 3.3. *Let $\mathbf{u}_{n,\delta_n}$ be a solution of (3.13). Then $\mathbb{P}$-almost surely $\mathbf{u}_{n,\delta_n} : [0, \delta_n] \to \mathbf{V}$ is càdlàg .*

Proof. For the sake of simplicity we just write $\delta := \delta_n$. Since the solution $\mathbf{u}_{n,\delta}$ to the truncated Eq. (3.1) belongs to $M^2(X_\delta)$, from Proposition 2.5 and the fact that $A \in \mathcal{L}(\mathbf{E}, \mathbf{H})$ we infer that $A\mathbf{u}_{n,\delta}(\cdot) + B^T_n(\mathbf{u}_{n,\delta}(\cdot))$ is an element of $M^2(0, \delta; \mathbf{H})$. From Theorem 2.1 we derive that the process $\int_0^\cdot \int_Z G(z, \mathbf{u}_{n,\delta}(-s))$ $\widetilde{\eta}(dz, s)$ belongs to $L^2(\Omega, \mathbf{D}([0, \delta]; \mathbf{V}))$ and define an $\mathbb{F}$-martingale. Since $\mathbb{P}$-a.s

$$\mathbf{u}_{n,\delta}(t) + \int_0^t [A\mathbf{u}_{n,\delta}(s) + (B^T_n\mathbf{u}_{n,\delta})(s)]ds = u_0 + \int_0^t \int_Z G(z, \mathbf{u}_{n,\delta}(s-))\widetilde{\eta}(dz, ds),$$

$t \in (0, \delta]$, it follows from the above remarks and [30, Theorem 2] that $\mathbb{P}$-a.s. $\mathbf{u}_{n,\delta} \in \mathbf{D}([0, \delta]; \mathbf{V})$. ☐

Now, we are able to formulate the result about the global existence of solution to the truncated Eq. (3.1).

Theorem 3.4. *Let Assumptions 2.1–2.3 hold. Then, for each $n \geq 1$ the truncated Eq. (3.1) admits a unique global solution $\mathbf{u}^n \in M^2(X_T)$ for any $T \in (0,\infty)$. Moreover, $\mathbf{u}^n \in \mathbf{D}([0, \delta]; \mathbf{V})$ with probability one.*

Proof. Let n be a positive integer and $\delta_n > 0$ such that $\Lambda^n_{\delta_n,\mathbf{u}_0}$ is a $\frac{1}{2}$-contraction. To keep the notation simple we just write $\delta := \delta_n$. For $k \in \mathbb{N}$ let $(t_k)_{k\in\mathbb{N}}$ be a sequence of times defined by $t_k = k\delta$. By the $\frac{1}{2}$-contraction property of Λ^n_{δ,u_0} we can find a càdlàg process $\mathbf{u}^{[n,1]} \in M^2(X_\delta)$ such that $\mathbf{u}^{[n,1]} = \Lambda^n_{\delta,u_0}(\mathbf{u}^{[n,1]})$. Since $\mathbf{u}^{[n,1]} \in M^2(X_\delta)$ it follows from Lemma 3.3 that $\mathbf{u}^{[n,1]}$ is $\mathcal{F}_t$-measurable and $\mathbf{u}^{[n,1]}(t) \in L^2(\Omega, \mathbb{P}; \mathbf{V})$ for any $t \in [0, \delta]$. Thus replacing $\mathbf{u}_0$ with $\mathbf{u}^{[n,1]}(\delta) = \mathbf{u}^{[n,1]}(\delta-)$ and using the same argument as above we can find a càdlàg process $\mathbf{u}^{[n,2]} \in M^2(X_{t_1,t_2})$ such that $\mathbf{u}^{[n,2]} = \Lambda^n_{\delta,\mathbf{u}^{[n,1]}(\delta)}(\mathbf{u}^{[n,2]})$. By induction we can construct a sequence of càdlàg processes $\mathbf{u}^{[n,k]} \subset M^2(X_{t_{k-1},t_k})$ such that $\mathbf{u}^{[n,k]} = \Lambda_{\delta,\mathbf{u}^{[n,k-1]}}(\mathbf{u}^{[n,k]})$. Now let $\mathbf{u}^n$ be the process defined by $\mathbf{u}^n(t) = \mathbf{u}^{[n,1]}(t)$, $t \in [0, \delta)$, and for $k = [\frac{T}{\delta}] + 1$ and $0 \leq t < \delta$, let $\mathbf{u}^n(t + k\delta) = \mathbf{u}^{[n,k]}(t)$. By construction $\mathbf{u}^n \in M^2(X_T)$ and $\mathbf{u}^n = \Lambda^n_{T,u_0}(\mathbf{u}^n)$, consequently $\mathbf{u}^n$ is a global solution to the truncated Eq. (3.1). The fact that $\mathbf{u}^n \in \mathbf{D}([0, \delta]; \mathbf{V})$ with probability one follows from Lemma 3.3 and the construction of $\mathbf{u}^n$.

Now let $(\mathbf{v}, \tau)$ be a another local solution of Eq. (3.1), we shall show that $\mathbf{u}^n(t) = \mathbf{v}(t)$, for all $t \in [0, \tau)$ almost surely. For this purpose let $t_1 = \tau \wedge \delta$ and $t_k = \tau \wedge (k\delta)$ where k and δ are as above; note that as $k \to \lfloor \frac{T}{\delta} \rfloor$ we have $t_k \uparrow \tau$ almost surely. With the same contraction principle used above we infer that $1_{[0,\tau\wedge\delta)}\mathbf{u}^n(.) = 1_{[0,\tau\wedge\delta)}\mathbf{v}(.)$ and $1_{[0,t_k)}\mathbf{u}^n(.) = 1_{[0,t_k)}\mathbf{v}(.)$ almost surely. By letting $k \to \infty$ we infer that $\mathbf{u}^n(t) = \mathbf{v}(t)$, for all $t \in [0, \tau)$ almost surely. $\square$

3.2. Existence and uniqueness of maximal/global solution to Eq. (2.11)

In this subsection we will use what we have learnt from the solvability of the truncated Eq. (3.1) to construct a unique maximal local and global solution to the original problem (2.11).

We start with the existence and uniqueness of a maximal local solution.

Theorem 3.5. *If all the assumptions of Theorem 3.4 hold, then there exists a unique pair $(\mathbf{u}, \tau_\infty)$ which is a maximal local solution to (2.11).*

Proof. We have seen that for each $n \in \mathbb{N}$ Eq. (3.1) has an unique global strong solution $\mathbf{u}^n$. Let us construct a sequence of stopping times $\{\tau_n, n \in \mathbb{N}\}$ as follows

$$\tau_n = \inf\{t \geq 0, \|\mathbf{u}^n\|_{X_t} \geq n\} \wedge T, n \in \mathbb{N}.$$

Now let $k > n$ and $\tau_{n,k} = \inf\{t \geq 0, \|\mathbf{u}^k\|_{X_t} \geq n\} \wedge T$. Since $\tau_{n,k} \leq \tau_k$ a.s., $(\mathbf{u}^k, \tau_{n,k})$ is a local solution to Eq. (3.1) and $(\mathbf{u}^n, \tau_n)$ is also a local solution to Eq. (3.1). Hence by the uniqueness we proved in Theorem 3.4 we infer that $\mathbf{u}^n(t) = \mathbf{u}^k(t)$ a.s for all $t \in [0, \tau_n \wedge \tau_{n,k}]$ which implies that

$$\mathbf{u}^n(t) = \mathbf{u}^k(t) \text{ a.s. for } t \in [0, \tau_n]. \tag{3.14}$$

This also proves that $\tau_n < \tau_k$ a.s. for all $n < k$, and the sequence $\{\tau_n, n \in \mathbb{N}\}$ has a limit $\tau_\infty := \lim_{n\uparrow\infty} \tau_n$ a.s..

Now let $\{\mathbf{u}(t), 0 \leq t < \tau_\infty\}$ be the stochastic process defined by

$$\mathbf{u}(t) = \mathbf{u}^n(t),\ t \in [\tau_{n-1}, \tau_n],\ n \geq 1, \tag{3.15}$$

where $\tau_0 = 0$. Since by definition $\theta_n(\|\mathbf{u}^n(s)\|_{X_s}) = 1$ for any $s \in [0, t \wedge \tau_n]$, it follows that $B_n^T(\mathbf{u}^n(s)) = F(\mathbf{u}^n(s))$ for any $s \in [0, t \wedge \tau_n)$. By (3.14) we have $\mathbf{u}(t \wedge \tau_n) = \mathbf{u}^n(t \wedge \tau_n]$, thus we can derive that $\mathbb{P}$-a.s.

$$\begin{aligned}
\mathbf{u}(t \wedge \tau_n) &= \mathbf{u}_0 - \int_0^{t\wedge\tau_n} \left[A\mathbf{u}^n(s) + (B_n^T\mathbf{u}^n)(s)\right] ds + \int_0^{t\wedge\tau_n} \int_Z G(z, \mathbf{u}^n(s-))\widetilde{\eta}(dz, ds), \\
&= \mathbf{u}_0 - \int_0^{t\wedge\tau_n} [A\mathbf{u}(s) + F(\mathbf{u}(s))]\, ds + \int_0^{t\wedge\tau_n} \int_Z G(z, \mathbf{u}(s-))\widetilde{\eta}(dz, ds),
\end{aligned}$$

for any $t \in [0, T]$. This proves that $(\mathbf{u}, \tau_n)$ is a local solution to (2.11). If $\{\tau_\infty(\omega) < T\}$, then

$$\begin{aligned}
\lim_{t\uparrow\tau_\infty} \|\mathbf{u}\|_{X_t} &\geq \lim_{n\uparrow\infty} \|\mathbf{u}\|_{X_{\tau_n}}, \\
&\geq \lim_{n\uparrow\infty} \|\mathbf{u}^n\|_{X_{\tau_n}} = \infty,
\end{aligned}$$

because $\|\mathbf{u}^n\|_{X_{\tau_n}} \geq n$. Therefore $(\mathbf{u}, \tau_\infty)$ is a maximal local solution to Eq. (2.11).

We will prove that this maximal solution is unique. For this let $(\mathbf{v}, \sigma_\infty)$ be another maximal local solution and $\{\sigma_n, n \geq 0\}$ a sequence of stopping times converging to σ_∞ defined by

$$\sigma_n = \inf\{t \geq 0, \|\mathbf{v}\|_{X_t} \geq n\} \wedge \sigma_\infty \wedge T.$$

Arguing as above we can prove that $\mathbf{u}(t) = \mathbf{v}(t)$ for all $t \in [0, \tau_n \wedge \sigma_n]$ a.s. which, upon letting $n \uparrow \infty$, implies that

$$\mathbf{u}(t) = \mathbf{v}(t) \text{ for all } t \in [0, \tau_\infty \wedge \sigma_\infty] \text{ a.s..}$$

From this last identity we can conclude that $\tau_\infty = \sigma_\infty$ almost surely. Indeed if the last conclusion were not true then we either have

$$\begin{aligned}\lim_{t\uparrow\sigma_\infty} \|1_{\{\sigma_\infty > \tau_\infty\}}\mathbf{v}\|_{X_t} &= \lim_{n\uparrow\infty} \|1_{\{\sigma_\infty > \tau_\infty\}}\mathbf{v}\|_{X_{\sigma_n}}\\ &= \lim_{n\uparrow\infty} \|1_{\{\sigma_\infty > \tau_\infty\}}\mathbf{u}\|_{X_{\tau_n}} = \infty,\end{aligned} \tag{3.16}$$

or

$$\begin{aligned}\lim_{t\uparrow\tau_\infty} \|1_{\{\sigma_\infty < \tau_\infty\}}\mathbf{u}\|_{X_t} &= \lim_{n\uparrow} \|1_{\{\sigma_\infty < \tau_\infty\}}\mathbf{u}\|_{X_{\tau_n}}\\ &= \lim_{n\uparrow} \|1_{\{\sigma_\infty < \tau_\infty\}}\mathbf{v}\|_{X_{\sigma_n}} = \infty.\end{aligned} \tag{3.17}$$

The identity (3.16) (resp. Eq. (3.17)) contradicts the fact that $\mathbf{v}$ (resp. $\mathbf{u}$) does not explode before time σ_∞ (resp. τ_∞). Therefore one must have $\tau_\infty = \sigma_\infty$ almost surely, which yields the uniqueness of the maximal local solution $(\mathbf{u}, \tau_\infty)$. ☐

Proposition 3.6. *In addition to the assumptions of Theorem 3.5 we assume that* $\mathbb{E}|\mathbf{u}_0|^4 < \infty$ *and there exists* $\tilde{C}_A > 0$ *such that* $\langle Au, u\rangle \geq \tilde{C}_A \|u\|^2$ *for any* $u \in \mathbf{E}$. *We also suppose that* F

$$\langle F(u), u\rangle = 0, \tag{3.18}$$

for all $u \in \mathbf{E}$. *Let* $\mathbf{u} \in \mathbf{E}$ *be the stochastic process we constructed in Theorem 3.5. Let* $(\tau_n)_{n\geq 1}$ *be a sequence of stopping times defined by*

$$\tau_n = \inf\{t \geq 0 : \|\mathbf{u}\|^2_{X_t} \geq n^2\} \wedge T.$$

Then for $r = 1, 2$, *for any* $t \geq 0$ *there exists a constant* $\tilde{C} > 0$ *such that the local solution* $(\mathbf{u}, \tau_n)$ *to* (2.11) *satisfies*

$$\mathbb{E} \sup_{s\in[0,t\wedge\tau_n]} |\mathbf{u}(s)|^{2r} + \mathbb{E}\int_0^{t\wedge\tau_n} |\mathbf{u}(s)|^{2r-2}\|\mathbf{u}(s)\|^2 ds \leq \tilde{C}, \tag{3.19}$$

and

$$\mathbb{E}\left[\int_0^{t\wedge\tau_n} \|\mathbf{u}(s)\|^2 ds\right]^2 \leq \tilde{C}, \tag{3.20}$$

for any $n \geq 1$.

Proof. Note that $(\mathbf{u},\tau_\infty)$, where $\tau_\infty=\lim_{n\uparrow\infty}\tau_n$ a.s., is the unique maximal solution to (2.11). Throughout let n be a fixed positive integer. To shorten notation we define $t_n=t\wedge\tau_n$ for any $t\in[0,T]$. Let $\Psi(\mathbf{u}):=|\mathbf{u}|^2$, $\mathbf{u}\in\mathbf{H}$, and

$$g(s,z):=\langle G(z,\mathbf{u}(s)),G(z,\mathbf{u}(s))\rangle,$$
$$f(s,z):=g(s,z)+2\langle G(z,\mathbf{u}(s)),\mathbf{u}(s)\rangle.$$

Now, for $t\geq 0$ let $y(t):=\langle \mathbf{u}(t),\mathbf{u}(t)\rangle$ and $\Psi'(\mathbf{u}(t))[h]=2\langle\mathbf{u}(t),h\rangle$ for any $h\in\mathbf{H}$. The estimate (3.19) can be proved by using the Itô' formula to $[\Psi(\mathbf{u}(t_n))]^r$, $r=1,2$, with $t_n=t\wedge\tau_n$ for every $t\in[0,T]$.

First, for $r=1$ we should notice that by Itô's formula and the assumption about F in Proposition 3.6 we have

$$\begin{aligned} y(t_n)+\int_0^{t_n}\Psi'(\mathbf{u}(s))[A\mathbf{u}(s)]ds &= y(0)+\int_0^{t_n}\int_Z g(s,z)\nu(dz)ds\\ &\quad+\int_0^{t_n}\int_Z f(s-,z)\widetilde{\eta}(dz,ds). \end{aligned}\tag{3.21}$$

The same calculations with $N=\mathrm{Id}$ as in proof of Lemma 3.2 yields

$$\begin{aligned} \mathbb{E}\sup_{s\in[0,t_n)}\Psi(\mathbf{u}(s))+2\mathbb{E}\int_0^{t_n}\langle A\mathbf{u}(s),\mathbf{u}(s)\rangle ds &\leq \mathbb{E}\Psi(\mathbf{u}_0)+\bar\ell_1\mathbb{E}\int_0^{t_n}|\mathbf{u}(s)|^2ds\\ &\quad+\bar\ell_1T+\mathbb{E}\sup_{s\in[0,t_n)}\left|\int_0^s\int_Z f(s-,z)\widetilde{\eta}(dz,ds)\right|, \end{aligned}\tag{3.22}$$

for any $t\in[0,T]$ and $n\geq 1$. Arguing as in the proofs of Eqs. (3.9) and (3.10) we obtain the following inequality

$$\begin{aligned} \mathbb{E}\sup_{s\in[0,t_n]}\left|\int_0^s\int_Z f(s-,z)\widetilde{\eta}(dz,ds)\right| &\leq \left[\frac{C^2\bar\ell_1}{4\varepsilon}+\bar\ell_2\right]\mathbb{E}\int_0^{t_n}|\mathbf{u}(s)|^2ds\\ &\quad+\varepsilon\mathbb{E}\left[\sup_{s\in[0,t_n]}|\mathbf{u}(s)|^2\right], \end{aligned}$$

which along with (3.22) implies that

$$\begin{aligned} \mathbb{E}\sup_{s\in[0,t_n)}\Psi(\mathbf{u}(s))+2\mathbb{E}\int_0^{t_n}\langle A\mathbf{u}(s),\mathbf{u}(s)\rangle ds &\leq \mathbb{E}\Psi(\mathbf{u}_0)+\varepsilon\mathbb{E}\left[\sup_{s\in[0,t_n]}|\mathbf{u}(s)|^2\right]\\ &\quad+\left[\bar\ell_1\left[\frac{C^2\bar\ell_1}{4\varepsilon}+1\right]+\bar\ell_2\right]\mathbb{E}\int_0^{t_n}|\mathbf{u}(s)|^2ds. \end{aligned}$$

Using Assumption (2.1), choosing $\varepsilon=1/2$ and invoking the Gronwall lemma yield

$$\mathbb{E}\sup_{s\in[0,t_n)}|\mathbf{u}(s)|^2+\mathbb{E}\int_0^{t_n}\|\mathbf{u}(s)\|^2ds\leq\frac{\mathbb{E}\Psi(\mathbf{u}_0)}{\min(\frac{1}{2},2\tilde C_A)}[e^{\ell T}+1],\tag{3.23}$$

where

$$\ell=\frac{}{\min(\frac{1}{2},2\tilde C_A)}\left[\bar\ell_1\left[\frac{C^2\bar\ell_1}{21}+1\right]+\bar\ell_2\right].$$

This completes the proof of the theorem for $r = 1$.

By applying Itô's formula to $[y(t)]^2 =: z(t)$ we obtain

$$
\begin{aligned}
&\mathbb{E} \sup_{r\in[0,t_n)} \left[z(r) + 2\int_0^r y(s)\Psi'(\mathbf{u}(s))[A\mathbf{u}(s)]ds - 2\int_0^r \int_Z y(s)g(s,z)\nu(dz)ds \right] \\
&\quad = \mathbb{E} \sup_{r\in[0,t_n)} \left[z(0) + \int_0^r \int_Z [f(s,z)]^2 \nu(dz)ds \right] \\
&\qquad + \mathbb{E} \sup_{r\in[0,t_n)} \left[\int_0^r \int_Z \left([f(s-,z)]^2 + 2y(s-)f(s-,z) \right) \widetilde{\eta}(dz,ds) \right].
\end{aligned} \tag{3.24}
$$

By performing elementary calculation and using part (ii) of Assumption 2.3 one can show that

$$
\begin{aligned}
\mathbb{E}\int_0^{t_n} \int_Z [f(s,z)]^2 \nu(dz)ds &\leq 2\mathbb{E}\int_0^{t_n} \int_Z [\langle G(z,\mathbf{u}(s)), G(z,\mathbf{u}(s))\rangle]^2 \nu(dz)ds \\
&\quad + 2\mathbb{E}\int_0^{t_n} \int_Z [\langle \mathbf{u}(s), G(z,\mathbf{u}(s))\rangle]^2 \nu(dz)ds, \\
&\leq 2[\bar{\ell}_1 + \bar{\ell}_2^2] \left(T + \mathbb{E}\int_0^{t_n} |\mathbf{u}(s)|^4 ds \right).
\end{aligned} \tag{3.25}
$$

Similarly,

$$
2\mathbb{E}\int_0^{t_n} \int_Z y(s)g(s,z)\nu(dz)ds \leq \bar{\ell}_1 \left(T + \mathbb{E}\int_0^{t_n} |\mathbf{u}(s)|^4 ds \right) \tag{3.26}
$$

Since $[f(s,z)]^2 > 0$, by using [46, Theorem 3.10, Eq. (3.10)] we derive that

$$
\mathbb{E} \sup_{r\in[0,t_n)} \left| \int_0^r \int_Z [f(s-,z)]^2 \widetilde{\eta}(dz,ds) \right| \leq \mathbb{E}\int_0^{t_n} \int_Z [f(s,z)]^2 \nu(dz)ds
$$

and by arguing as above we infer that

$$
\mathbb{E}\int_0^{t_n} \int_Z [f(s-,z)]^2 \widetilde{\eta}(dz,ds) \leq 2^2[\bar{\ell}_1 + \bar{\ell}_2^2] \left(T + \mathbb{E}\int_0^{t_n} |\mathbf{u}(s)|^4 ds \right). \tag{3.27}
$$

By using the BDG inequality and Cauchy inequality with epsilon we obtain

$$
\begin{aligned}
\mathbb{E} \sup_{r\in[0,t_n)} \left| \int_0^r \int_Z 2y(s-)f(s-,z)\widetilde{\eta}(dz,ds) \right| &\leq 4K\mathbb{E}\left[\int_0^{t_n} \int_Z [y(s)]^2 |f(s,z)|^2 \Big) \nu(dz)ds \right]^{\frac{1}{2}}, \\
&\leq \frac{16K^2}{4\varepsilon} \mathbb{E}\int_0^{t_n} \int_Z |f(s,z)|^2 \Big) \nu(dz)ds \\
&\quad + \varepsilon\mathbb{E} \sup_{s\in[0,t_n)} |\mathbf{u}(s)|^4.
\end{aligned} \tag{3.28}
$$

And arguing as in (3.27) we derive that

$$
\begin{aligned}
&\mathbb{E} \sup_{r\in[0,t_n)} \left| \int_0^r \int_Z 2y(s-)f(s-,z)\widetilde{\eta}(dz,ds) \right| - \varepsilon\mathbb{E} \sup_{s\in[0,t_n)} |\mathbf{u}(s)|^4 \\
&\qquad \leq \frac{32K^2(1 + \bar{\ell}_1 + \bar{\ell}_2^2)}{4\varepsilon} \left(T + \mathbb{E}\int_0^{t_n} |\mathbf{u}(s)|^4 ds \right).
\end{aligned} \tag{3.29}
$$

Plugging (3.25), (3.26), (3.27) and (3.29) in (3.24), using Assumption 2.1 and choosing $\varepsilon = 1/2$ yield the existence of positive constants $\tilde{L}$, $\bar{\ell}$ such that

$$\mathbb{E} \sup_{s\in[0,t_n)} |\mathbf{u}(s)|^4 + \int_0^{t_n} |\mathbf{u}(s)|^2\|\mathbf{u}(s)\|^2 ds \leq \tilde{L}T + \tilde{L}\mathbb{E}\int_0^{t_n} |\mathbf{u}(s)|^4 ds + \bar{\ell}\mathbb{E}[\Psi(\mathbf{u}_0)]^2.$$

Thanks to the Gronwall lemma we infer that

$$\mathbb{E} \sup_{s\in[0,t_n)} |\mathbf{u}(s)|^4 + \int_0^{t_n} |\mathbf{u}(s)|^2\|\mathbf{u}(s)\|^2 ds \leq \left(\tilde{L}T + \bar{\ell}\mathbb{E}[\Psi(\mathbf{u}_0]^2)\right)\left[e^{\tilde{L}T} + 1\right]. \tag{3.30}$$

The above inequality completes the proof of (3.19) for $r = 2$, and hence the first part of our theorem.

To prove the second part we will use (3.21). In fact, from (3.21) we derive that

$$\mathbb{E}\left[\int_0^{t_n} \Psi'(\mathbf{u}(s))[A\mathbf{u}(s)]ds\right]^2 \leq C\mathbb{E}[y(0)]^2 + C\mathbb{E}\left[\int_0^{t_n}\int_Z g(s,z)\nu(dz)ds\right]^2$$
$$+ C\mathbb{E}\left[\int_0^{t_n}\int_Z f(s-,z)\tilde{\eta}(dz,ds)\right]^2. \tag{3.31}$$

Note that the stochastic integral in the last term of the RHS of the above estimate is real-valued, so from Itô's isometry we infer that

$$\mathbb{E}\left[\int_0^{t_n}\int_Z f(s-,z)\tilde{\eta}(dz,ds)\right]^2 = \mathbb{E}\int_0^{t_n}\int_Z [f(s,z)]^2\nu(dz)ds,$$

from which altogether with (3.27) and (3.19) we derive that for any $t \geq 0$ there exists a constant $\tilde{C} > 0$ such that

$$\mathbb{E}\left[\int_0^{t_n}\int_Z f(s-,z)\tilde{\eta}(dz,ds)\right]^2 \leq \tilde{C} \tag{3.32}$$

for any $n \geq 1$. By imitating the proof of (3.25) we infer that there exists $C > 0$ such that

$$\mathbb{E}\left[\int_0^{t_n}\int_Z g(s,z)\nu(dz)ds\right]^2 \leq C\left(t + \mathbb{E}\int_0^{t_n} |\mathbf{u}(s)|^4 ds\right), \tag{3.33}$$

from which and (3.19) we deduce that for any $t \geq 0$ there exists $\tilde{C} > 0$ such that

$$\mathbb{E}\left[\int_0^{t_n}\int_Z g(s,z)\nu(dz)ds\right]^2 \leq \tilde{C}, \tag{3.34}$$

for any $n \geq 1$. Taking (3.32) and (3.34) into (3.31) implies that

$$\mathbb{E}\left[\int_0^{t_n} \Psi'(\mathbf{u}(s))[A\mathbf{u}(s)]ds\right]^2 \leq C\mathbb{E}[y(0)]^2 + 2\tilde{C}. \tag{3.35}$$

Thanks to this last estimate and the fact that $\langle A\mathbf{u}, \mathbf{u}\rangle \geq \tilde{C}_A\|\mathbf{u}\|^2$ we easily derive that for any $t \geq 0$ there exists $\tilde{C} > 0$ such that for any $n \geq 1$

$$\mathbb{E}\left[\int_0^{t_n} \|\mathbf{u}(s)\|^2 ds\right] \leq \tilde{C}.$$

This completes the proof of (3.20), and hence the whole Proposition. □

Now we turn our attention to the existence and uniqueness of global solution.

Theorem 3.7. *Assume that F satisfies the assumptions of Proposition 3.6 with $p = 1$ and $\alpha \in [0, \frac{1}{2}]$. Moreover, we suppose that there exists $\tilde{c} > 0$ such that*

$$|F(u) - F(v)| \leq \tilde{c}\Big[|u|^{1-\alpha}\|u\|^{\alpha}\|u-v\|^{1-\alpha}\|u-v\|_{\mathbf{E}}^{\alpha} + |u-v|^{1-\alpha}\|u-v\|^{\alpha}\|v\|^{1-\alpha}\|v\|_{\mathbf{E}}^{\alpha}\Big] \tag{3.36}$$

for any $u, v \in \mathbf{E}$. Then Problem (2.11) has a unique global solution.

Proof. Let $\mathbf{u}$ be the stochastic process we constructed in Theorem 3.5 and

$$\|N\| := \max\left(\|N\|_{\mathcal{L}(\mathbf{E},\mathbf{H})}, \|N\|_{\mathcal{L}(\mathbf{V},\mathbf{V}^*)}\right).$$

Let $(\tau_n)_{n\geq 1}$ be a sequence of stopping times defined by

$$\tau_n = \inf\left\{t \in [0,T] : \sup_{s\in[0,t]} \|\mathbf{u}(s)\|^2 + \int_0^t \|\mathbf{u}(s)\|_{\mathbf{E}}^2 ds \geq n^2\right\}.$$

Note that $(\mathbf{u}, \tau_\infty)$, where $\tau_\infty = \lim_{n\uparrow\infty} \tau_n$ a.s., is the unique maximal solution to (2.11). To deal with the structure of the nonlinearity F (see Eq. (3.36)) we introduce another sequence of stopping times $(\sigma_m)_{m\geq 1}$ defined by

$$\sigma_m = \inf\left\{t \in [0,T] : \int_0^t |\mathbf{u}(s)|^2 \|\mathbf{u}(s)\|^{\frac{2\alpha}{1-\alpha}} ds \geq m\right\}, \quad \text{for any } m \geq 1.$$

To shorten notation we define $t_{m,n} = t \wedge (\sigma_m \wedge \tau_n)$ for any $t \in [0,T]$, $n \geq 1$ and $m \geq 1$. Let

$$\begin{aligned} f(s,z) &:= \langle NG(z,\mathbf{u}(s)), G(z,\mathbf{u}(s))\rangle \\ &\quad + 2\langle G(z,\mathbf{u}(s)), N\mathbf{u}(s)\rangle, \end{aligned}$$

and

$$g(s,z) := \langle NG(z,\mathbf{u}(s)), G(z,\mathbf{u}(s))\rangle.$$

Applying Itô's formula to $\Psi(u) = \langle u, Nu\rangle$ we obtain

$$\begin{aligned} \Psi(\mathbf{u}(t_{m,n})) &= \Psi(\mathbf{u}_0) - 2\int_0^{t_{m,n}} [\langle A\mathbf{u}(s) + F(\mathbf{u}(s)), N\mathbf{u}(s)\rangle]\, ds \\ &\quad + \int_0^{t_{m,n}} \int_Z g(s,z)\nu(dz)ds + \int_0^{t_{m,n}} \int_Z f(s-,z)\widetilde{\eta}(dz,ds), \end{aligned}$$

for any $t \in [0,T]$. For any $\delta > 0$ and $p, q \geq 1$ with $p^{-1} + q^{-1} = 1$ let $C(\delta, p, q)$ be the constant from the Young inequality

$$ab \leq C(\delta,p,q)a^p + \delta b^q.$$

From Eq. (3.36) and the above Young inequality with $p = \frac{2}{1+\alpha}$, $q = \frac{2}{1-\alpha}$, and $\delta = C_A$ we obtain

$$|2\langle F(\mathbf{u}(s)), N\mathbf{u}(s)\rangle| \leq C(C_A,p,q)[2\tilde{c}\|N\|]^q |\mathbf{u}(s)|^2 \|\mathbf{u}(s)\|^{\frac{2\alpha}{1-\alpha}} + C_A\|\mathbf{u}(s)\|_{\mathbf{E}}^2.$$

By making use of the definition of σ_m we get that

$$2\left|\int_0^{t_{m,n}} \langle F(\mathbf{u}(s)), N\mathbf{u}(s)\rangle ds\right| \le C(C_A,p,q)[2\tilde{c}\|N\|]^q mT + C_A\int_0^{t_{m,n}} \|\mathbf{u}(s)\|_{\mathbf{E}}^2. \tag{3.37}$$

From the assumption on G we derive that

$$\int_0^{t_{m,n}}\int_Z g(s,z)\nu(dz)ds \le \|N\|\tilde{\ell}_1 T + \|N\|\tilde{\ell}_1\int_0^{t_{m,n}} \|\mathbf{u}(s)\|^2 ds. \tag{3.38}$$

By taking the mathematical expectation to both sides of this estimate and by using Assumption 2.1 altogther with Eqs. (3.37), (3.38) we infer that

$$\begin{aligned}\mathbb{E}[\|\mathbf{u}(t_{m,n})\|^2] + \mathbb{E}\int_0^{t_{m,n}} \|\mathbf{u}(s)\|_{\mathbf{E}}^2 ds &\le \tilde{L}^{-1}\|N\|\tilde{\ell}_1\mathbb{E}\int_0^{t_{m,n}}\|\mathbf{u}(s)\|^2 ds\\ &+ \tilde{L}^{-1}[\mathbb{E}\Psi(\mathbf{u}_0) + \|N\|\tilde{\ell}_1 T + C_{mA}T],\end{aligned}$$

where $\tilde{L} = \min(C_N, C_A)$ and $C_{mA} := C(C_A,p,q)[2\tilde{c}\|N\|]^q m$. From the Gronwall's lemma we infer that

$$\begin{aligned}\mathbb{E}[\|\mathbf{u}(t_{m,n})\|^2] + \mathbb{E}\int_0^{t_{m,n}} \|\mathbf{u}(s)\|_{\mathbf{E}}^2 ds &\le \tilde{L}^{-1}[\mathbb{E}\Psi(\mathbf{u}_0) + \|N\|\tilde{\ell}_1 T\\ &+ C_{mA}T]e^{\tilde{L}^{-1}\|N\|\tilde{\ell}_1 t_{m,n}}[1+\|N\|\tilde{\ell}_1 T].\end{aligned} \tag{3.39}$$

Next, note that

$$\begin{aligned}\mathbb{P}(\tau_n < t) &= \mathbb{P}(\{\tau_n < t\}\cap(\Omega_m\cup\Omega_m^c)),\\ &= \mathbb{P}(\{\tau_n < t\}\cap\Omega_m) + \mathbb{P}(\{\tau_n < t\}\cap\Omega_m^c),\end{aligned}$$

where $\Omega_m = \{\sigma_m \ge T\}$, $m \ge 1$. Now, by arguing as in [12, pp. 123] we have

$$\begin{aligned}\mathbb{P}(\tau_n < t) &\le \frac{1}{n^2}\mathbb{E}\left(1_{\{\tau_n<t\}\cap\Omega_m}\left[\|\mathbf{u}(t_{m,n})\|^2 + \int_0^{t_{m,n}}\|\mathbf{u}(s)\|_{\mathbf{E}}^2 ds\right]\right) + \mathbb{P}[\Omega_m^c],\\ &\le \frac{1}{n^2}\mathbb{E}\left[\|\mathbf{u}(t_{m,n})\|^2 + \int_0^{t_{m,n}}\|\mathbf{u}(s)\|_{\mathbf{E}}^2 ds\right] + \frac{1}{m}\mathbb{E}\int_0^{t_{m,n}}|\mathbf{u}(s)|^2\|\mathbf{u}(s)\|^{\frac{2\alpha}{1-\alpha}} ds.\end{aligned}$$

Thanks to Eq. (3.39)

$$\begin{aligned}\mathbb{P}(\tau_n < t) &\le \frac{1}{n^2}\tilde{L}^{-1}[\mathbb{E}\Psi(\mathbf{u}_0) + \|N\|\tilde{\ell}_1 T + C_{mA}T]e^{\tilde{L}^{-1}\|N\|\tilde{\ell}_1 T}\\ &+ \frac{1}{m}\mathbb{E}\int_0^{t_{m,n}}|\mathbf{u}(s)|^2\|\mathbf{u}(s)\|^{\frac{2\alpha}{1-\alpha}} ds,\end{aligned}$$

from which we derive that

$$\lim_{n\nearrow\infty}\mathbb{P}(\tau_n < t) \le \frac{1}{m}\left\{\mathbb{E}\left[\sup_{s\in[0,t_{m,n}]}|\mathbf{u}(s)|^4\right] + \left(\mathbb{E}\left[\int_0^{t_{m,n}}\|\mathbf{u}(s)\|^2 ds\right]^2\right)^{\frac{2\alpha}{1-\alpha}}\right\}.$$

Since $\alpha \in [0,\frac{1}{2}]$ it follows from Proposition 3.6 (see (3.19)–(3.20)) that the solution $\mathbf{u}$ satisfies

$$\mathbb{E}[\sup_{s\in[0,t_{m,n}]}\|\mathbf{u}(s)\|^4] + \left(\mathbb{E}\left[\int_0^{t_{m,n}}\|\mathbf{u}(s)\|^2 ds\right]^2\right)^{\frac{2\alpha}{1-\alpha}} \le \tilde{C}.$$

Hence, combining this latter equation with the former one yields that

$$\lim_{n\to\infty} \mathbb{P}\left(\tau_n < t\right) = 0,$$

from which we derive that $\mathbb{P}\left(\tau_\infty < T\right) = 0$ for any $T > 0$. This implies that $\mathbf{u}$ is a global solution. $\square$

Remark 3.8. All of our results in this section remain true if we replace $F(u)$ by $B(u) + R(u)$ with $R \in \mathcal{L}(H, H)$ and B satisfying the assumptions of Theorems 3.5 and 3.7.

4. Examples

The examples, notations and references used in this section are taken from [17].

4.1. Notations

Let $n \in \{2, 3\}$ and assume that $\mathcal{O} \subset \mathbb{R}^n$ is a Poincaré's domain (its definition is given below) with boundary $\partial\mathcal{O}$ of class $\mathcal{C}^\infty$. For any $p \in [1, \infty)$ and $k \in \mathbb{N}$, $\mathbb{L}^p(\mathcal{O})$ and $\mathbb{W}^{k,p}(\mathcal{O})$ are the well-known Lebesgue and Sobolev spaces, respectively, of $\mathbb{R}^n$-valued functions. The corresponding spaces of scalar functions we will denote by standard letter, e.g. $W^{k,p}(\mathcal{O})$.

A domain $\mathcal{O} \subset \mathbb{R}^d$ is called a Poincaré's domains if following Poincaré's inequality holds

$$|\mathbf{u}| \leq c|\nabla \mathbf{u}|, \quad \text{for all } \mathbf{u} \in H^1(\mathcal{O}). \tag{4.1}$$

For $p = 2$ we denote $\mathbb{W}^{k,2}(\mathcal{O}) = \mathbb{H}^k$ and its norm are denoted by $\|\mathbf{u}\|_k$. By $\mathbb{H}_0^1$ we mean the space of functions in $\mathbb{H}^1$ that vanish on the boundary on $\mathcal{O}$; $\mathbb{H}_0^1$ is a Hilbert space when endowed with the scalar product induced by that of $\mathbb{H}^1$. The usual scalar product on $\mathbb{L}^2$ is denoted by $\langle u, v\rangle$ for $u, v \in \mathbb{L}^2$. Its associated norm is $|u|$, $u \in \mathbb{L}^2$. We also introduce the following spaces

$$\begin{aligned}
\mathcal{V}_1 &= \{\mathbf{u} \in [\mathcal{C}_c^\infty(\mathcal{O}, \mathbb{R}^n)] \text{ such that } \nabla \cdot \mathbf{u} = 0\} \\
\mathbf{V}_1 &= \text{ closure of } \mathcal{V} \text{ in } \ \mathbb{H}_0^1(\mathcal{O}) \\
\mathbf{H}_1 &= \text{ closure of } \mathcal{V} \text{ in } \ \mathbb{L}^2(\mathcal{O}).
\end{aligned}$$

We also consider the Hilbert spaces $\mathbf{H}_2 = \mathbf{H}_1$ and $\mathbf{V}_2 = \mathbb{H}^1 \cap \mathbf{H}_2$.

Let (e_1, e_2) be the standard basis in $\mathbb{R}^2$ and $x = (x^1, x^2)$ an element of $\mathbb{R}^2$. When $\mathcal{O} = (0, l) \times (0, 1)$ is a rectangular domain in the vertical plane we consider the following spaces

$$\mathbf{H}_3 = \left\{u \in \mathbb{L}^2,\ \mathrm{div} u = 0,\ u^2|_{x^2=0} = u^2|_{x^2=1} = 0,\ u^1|_{x^1=0} = u^1|_{x^1=l}\right\}$$

and $\mathbf{H}_4 = L^2(\mathcal{O})$. We also denote

$$\begin{aligned}
\mathbf{V}_3 &= \left\{u \in \mathbf{H}_3 \cap \mathbb{H}^1,\ u|_{x^2=0} = u|_{x^2=1} = 0,\ u \text{ is } l\text{-periodic in } x^1\right\},\\
\mathbf{V}_4 &= \left\{\theta \in H^1(\mathcal{O}),\ \ \theta|_{x^2=0} = \theta|_{x^2=1} = 0,\ \theta \text{ is } l\text{-periodic in } x^1\right\},\\
\mathbf{H}_5 &= \mathbf{H}_3,\\
\mathbf{V}_5 &= \mathbf{H}_5 \cap \mathbb{H}^1.
\end{aligned}$$

Let $\Pi_i : \mathbb{L}^2 \to \mathbf{H}_i$ be the projection from $\mathbb{L}^2$ onto $\mathbf{H}_i$, $i = 1, 2, 3, 4, 5$. We denote by A_i the Stokes operator defined by

$$\begin{cases} D(\mathrm{A}_i) = \{u \in \mathbf{H}_i,\ \Delta u \in \mathbf{H}_i\},\\ \mathrm{A}_i u = -\Pi_i \Delta u,\ u \in D(\mathrm{A}_i), \end{cases} \tag{4.2}$$

$i = 1, \ldots, 5$. In all cases the A_i-s are self-adjoint, positive linear operators on $\mathbf{H}_i$. Finally we set $\mathbf{E}_i = D(\mathrm{A}_i)$, $i \in \{1, 2, 3, 4, 5\}$. Note that $\mathbf{E}_i \subset \mathbb{H}^2 \cap \mathbf{V}_i$, $i = 1, 2, 3, 5$ and $\mathbf{E}_4 \subset H^2 \cap \mathbf{V}_4$.

We endow the spaces $\mathbf{H}_i$, $i \in \{1, 2, 3, 4, 5\}$, with the scalar product and norm of $\mathbb{L}^2$. We equip the space $\mathbf{V}_i$, $i \in \{1, 2, 3, 4, 5\}$, with the scalar product $\langle \mathrm{A}_i^{\frac{1}{2}} \mathbf{u}, \mathrm{A}_i^{\frac{1}{2}} \mathbf{v}\rangle$ which is equivalent to the $\mathbb{H}^1(\mathcal{O})$-scalar product on $\mathbf{V}_i$. The spaces $\mathbf{E}_i$, $i \in \{1, 2, 3, 4, 5\}$ are equipped with the norm $|\mathrm{A}_i \mathbf{u}|$ which is equivalent to the $\mathbb{H}^2$-norm on $\mathbf{E}_i$.

Remark 4.1. In the case of an general unbounded domain we equip the space $\mathbf{V}_i$, $i \in \{1, 2, 3, 4, 5\}$, with the scalar product $\langle (\mathrm{Id} + \mathrm{A}_i)^{\frac{1}{2}} \mathbf{u}, (\mathrm{Id} + \mathrm{A}_i)^{\frac{1}{2}} \mathbf{v}\rangle$. The spaces $\mathbf{E}_i$, $i \in \{1, 2, 3, 4, 5\}$ are equipped with the norm $|(\mathrm{Id} + \mathrm{A}_i)\mathbf{u}|$ which is equivalent to the $\mathbb{H}^2$-norm on $\mathbf{E}_i$.

Next we define two trilinear forms $b_1(\cdot, \cdot, \cdot)$ and $b_2(\cdot, \cdot, \cdot)$ by setting

$$b_1(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \sum_{i,j=1}^{n} \int_O \mathbf{u}^i(x) \frac{\partial}{\partial x_i} \mathbf{v}^j(x) \mathbf{w}^j(x) dx, \text{ for any } (\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbb{L}^4 \times \mathbb{W}^{1,4} \times \mathbb{L}^2, \tag{4.3}$$

$$b_2(\mathbf{u}, \theta_2, \theta_3) = \sum_{i=1}^{n} \int_O \mathbf{u}^i(x) \frac{\partial}{\partial x_i} \theta_2(x) \theta_3(x) dx, \text{ for any } (\mathbf{u}, \theta_2, \theta_3) \in \mathbb{L}^4 \times W^{1,4} \times L^2. \tag{4.4}$$

Recall that for $\alpha = \frac{n}{4}$, the following estimate, valid for all $\mathbf{u} \in \mathbb{H}^1$ (or $\mathbf{u} \in H^1$), is a special case of Gagliardo–Nirenberg's inequalities:

$$\|\mathbf{u}\|_{\mathbb{L}^4} \leq |\mathbf{u}|^{1-\alpha} |\nabla \mathbf{u}|^{\alpha}. \tag{4.5}$$

The inequality (4.5) can be written in the spirit of the continuous embedding

$$\mathbb{H}^1 \subset \mathbb{L}^4. \tag{4.6}$$

Using Cauchy-Schwarz inequality, (4.5) and (4.6) in (4.3), (4.4) we derive that for any $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbb{H}^1 \times \mathbb{H}^2 \times \mathbb{L}^2$

$$|b_1(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq c \|\mathbf{u}\|_{\mathbb{H}^1}\ |\nabla \mathbf{v}|^{1-\alpha}\ |D^2 \mathbf{v}|^{\alpha}\ |\mathbf{w}| \quad \text{for any } (\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbb{H}^1 \times \mathbb{H}^2 \times \mathbb{L}^2, \tag{4.7}$$

$$|b_2(\mathbf{u}, \theta_2, \theta_3)| \leq c \|\mathbf{u}\|_{\mathbb{H}^1}\ |\nabla \theta_2|^{1-\alpha}\ |D^2 \theta_2|^{\alpha}\ |\theta_3| \quad \text{for any } (\mathbf{u}, \theta_2, \theta_3) \in \mathbb{H}^1 \times H^2 \times L^2. \tag{4.8}$$

From Eq. (4.7) (resp., Eq. (4.8)) we infer that there exists a bilinear map $B_1(\cdot,\cdot)$ (resp., $B_2(\cdot,\cdot)$) defined on $\mathbf{V}_i\times\mathbf{E}_i$ and taking values in $\mathbf{H}_i$, for appropriate values of i. Moreover, there exist $c>0$ such that

$$|B_1(\mathbf{u},\mathbf{v})|\leq c\|\mathbf{u}\|_{\mathbb{H}^1}\| \ \|\mathbf{v}\|_{\mathbb{H}^1}^{1-\alpha}\|\mathbf{v}\|_{\mathbb{H}^2}^{\alpha}, \quad \text{for any } (\mathbf{u},\mathbf{v})\in\mathbf{V}_i\times\mathbf{E}_i, \tag{4.9}$$

$$|B_2(\mathbf{u},\theta_2)|\leq c\|\mathbf{u}\|_{\mathbb{H}^1}\| \ \|\theta_2\|_{\mathbb{H}^1}^{1-\alpha}\|\theta_2\|_{\mathbb{H}^2}^{\alpha}, \quad \text{for any } (\mathbf{u},\mathbf{v})\in\mathbf{V}_i\times\mathbf{E}_i, \tag{4.10}$$

for appropriate values of i. Note that using Cauchy–Schwarz inequality, (4.5) and (4.6) in (4.3), (4.4) we also derive that

$$|B_1(\mathbf{u},\mathbf{v})|\leq c|\mathbf{u}|^{1-\alpha}\|\mathbf{u}\|_{\mathbb{H}^1}^{\alpha}\| \ \|\mathbf{v}\|_{\mathbb{H}^1}^{1-\alpha}\|\mathbf{v}\|_{\mathbb{H}^2}^{\alpha}, \quad \text{for any } (\mathbf{u},\mathbf{v})\in\mathbf{V}_i\times\mathbf{E}_i, \tag{4.11}$$

$$|B_2(\mathbf{u},\theta_2)|\leq c|\mathbf{u}|^{1-\alpha}\|\mathbf{u}\|_{\mathbb{H}^1}^{\alpha} \ \|\theta_2\|_{\mathbb{H}^1}^{1-\alpha}\|\theta_2\|_{\mathbb{H}^2}^{\alpha}, \quad \text{for any } (\mathbf{u},\mathbf{v})\in\mathbf{V}_i\times\mathbf{E}_i, \tag{4.12}$$

for appropriate values of i.

4.2. Stochastic hydrodynamical systems

In this subsection we use exactly the same notations as used in [17].

4.2.1. Stochastic Navier–Stokes equations. Let $\mathcal{O}$ be a bounded, open and simply connected domain of $\mathbb{R}^n$, $n=2,3$. The boundary $\partial\mathcal{O}$ of $\mathcal{O}$ is assumed to be smooth. Let $(Z,\mathcal{Z},\nu)$ be a measure space where the ν is a σ-finite, positive measure and $\tilde{\eta}$ be a compensated Poisson random measure having intensity measure ν defined on filtered complete probability space $\mathfrak{P}=(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$, where the filtration $\mathbb{F}=(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions. We consider the Navier–Stokes equation with the Dirichlet (no-slip) boundary conditions:

$$\begin{aligned} &du+[-\kappa\Delta u+u\nabla u+\nabla p]\,dt\\ &\quad=\int_Z \tilde{G}(t,\mathbf{u}(t-),z)\tilde{\eta}(dz,dt), \quad \operatorname{div} u=0 \ \text{ in } \ D, \qquad u=0 \quad \text{on} \quad \partial\mathcal{O}, \end{aligned} \tag{4.13}$$

where $u=(u^1(x,t),u^2(x,t))$ is the velocity of a fluid, $p(x,t)$ is the pressure, κ the kinematic viscosity. Here $\int_Z \tilde{G}(t,u(t),z)\tilde{\eta}(dz,dt)$ represents a state-dependent random external forcing of jump type.

Let $\mathbf{H}=\mathbf{H}_1$, $\mathbf{V}=\mathbf{V}_1$ and $\mathbf{E}=\mathbf{E}_1$ where the hilbert spaces $\mathbf{H}_i$, $\mathbf{V}_i$ and $\mathbf{E}_i$ are defined as in Eq. (4.2) of Sect. 4.1. The norms of $\mathbf{H}$, $\mathbf{V}$ and $\mathbf{E}$ are denoted by $|\cdot|$, $\|\cdot\|$, $\|\cdot\|_{\mathbf{E}}$, respectively.

Let $A=\mathrm{A}_1$ and $B=B_1$ be the linear and bilinear maps defined in Sect. 4.1. We also set $N=A$. Note that in this case N is self-adjoint and $N\in\mathcal{L}(\mathbf{E},\mathbf{H})\cap\mathcal{L}(\mathbf{V},\mathbf{V}^*)$.

We suppose that $\tilde{G}$ satisfies the following sets of conditions.

Assumption 4.1. We assume that $\tilde{G}$ maps $\mathbf{V}$ into $L^{2p}(Z,\nu,\mathbf{V})$ and there exists a constant $\ell_p>0$ such that

$$\|\tilde{G}(x)-\tilde{G}(y)\|_{L^{2p}(Z,\nu,\mathbf{V})}^{2p}\leq \ell_p^p\|x-y\|^{2p}, \tag{4.14}$$

for any $x,y\in\mathbf{V}$ and $p=1,2$.

Note that this implies in particular that there exists a constant $\tilde{\ell}_p > 0$ such that

$$\|\tilde{G}(x)\|^{2p}_{L^{2p}(Z,\nu,\mathbf{H})} \leq \tilde{\ell}^p_p(1+\|x\|^{2p}), \tag{4.15}$$

for any $x \in \mathbf{V}$ and $p = 1, 2$.

By setting $R \equiv 0$ and projecting on the space of divergence free vector fields the system (4.13) can be rewritten in the following abstract form

$$\begin{aligned} &d\mathbf{u} + [A\mathbf{u} + B(\mathbf{u},\mathbf{u}) + R(\mathbf{u})]dt = \int_Z \tilde{G}(t, \mathbf{u}(t-), z)\tilde{\eta}(dz, dt), \\ &\mathbf{u}(0) = \xi, \end{aligned} \tag{4.16}$$

Theorem 4.2. *The stochastic Navier–Stokes problem* (4.16) *admits a local maximal strong solution which is global if* $n = 2$.

Remark 4.3. This theorem remains true in the case $\mathcal{O}$ being a general unbounded domain. For the proof it is sufficient to take $A = A_1 + \mathrm{Id}$, $R(\mathbf{u}) := -\mathbf{u}$ and argue as in the case of bounded domain.

Proof. The existence and uniqueness of a maximal local solution will follow from Theorem 3.5 if we are able to prove that $F(\mathbf{u}) = B(\mathbf{u},\mathbf{u})$ satisfies (2.7). But from (4.9) we deduce that there exists $C > 0$ such that for

$$|B(y) - B(x)| \leq C\left[\|y - x\|\|y\|^{1-\frac{n}{4}}\|y\|^{\frac{n}{4}}_{\mathbf{E}} + \|y - x\|^{\frac{n}{4}}_{\mathbf{E}}\|y - x\|^{1-\frac{n}{4}}\|x\|\right],$$

for any $x, y \in \mathbf{E}$. This means that B satisfies (2.7) with $p = 1$ and $\alpha = \frac{n}{4}$. Since $\alpha = \frac{3}{4} \notin [0, \frac{1}{2}]$ for $n = 3$, the solution is only maximal. For $n = 2$ we have $\alpha = \frac{1}{2}$ and $\langle B(\mathbf{u},\mathbf{u}), \mathbf{u}\rangle = 0$. So thanks to Remark 3.8, we only need to check that (3.36) is verified by B. But this will follow from (4.11). $\square$

4.2.2. Magnetohydrodynamic equations. Let $\mathcal{O} \subset \mathbb{R}^n$, $n = 2, 3$ be a simply connected, possibly unbounded domain. As above we assume that $\mathcal{O}$ has a smooth boundary $\partial\mathcal{O}$. Let $(Z_i, \mathcal{Z}_i, \nu_i)$, $i = 1, 2$ be two measure spaces where the measures ν_i are σ-finite and positive. We consider two mutually independent compensated Poisson random measures $\tilde{\eta}_i$ with intensity measure ν_i defined on a complete filtered probability space $\mathfrak{P} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. We consider the magneto-hydrodynamic (MHD) equations

$$\begin{aligned} du + [-\Delta u + u\nabla u]dt = [-\nabla\left(p + \frac{1}{2}|b|^2\right) + b\nabla b]dt \\ + \int_{Z_1} \tilde{f}(t, u(t-), b(t-), z_1)\tilde{\eta}_1(dz_1, dt), \end{aligned} \tag{4.17}$$

$$db + [-\nu_2\Delta b + u\nabla b]dt = [b\nabla u]dt + \int_{Z_2} \tilde{g}(t, u(t-), b(-t), z_2)\tilde{\eta}_2(dz_2, dt), \tag{4.18}$$

$$\operatorname{div} u = 0, \quad \operatorname{div} b = 0 \tag{4.19}$$

where $u = (u^{(1)}(x,t), u^{(2)}(x,t), u^{(3)}(x,t))$ and $b = (b^{(1)}(x,t), b^{(2)}(x,t), b^{(3)}(x,t))$ denote velocity and magnetic fields, $p(x,t)$ is a scalar pressure. We consider the following boundary conditions

$$u = 0, \quad b \cdot n = 0, \quad \operatorname{curl} b \times n = 0 \quad \text{on } \partial\mathcal{O} \tag{4.20}$$

The terms $\int_{Z_1} \tilde{f}(t, u(t), b(t), z_1)\tilde{\eta}_1(dz_1, dt)$ and $\int_{Z_2} \tilde{g}(t, u(t), b(t), z_2)\tilde{\eta}_2(dz_2, dt)$, represent random external volume forces and the curl of random external current applied to the fluid. We refer to [22,36,49] for the mathematical theory for the MHD equations.

Let $\mathbf{H} = \mathbf{H}_1 \times \mathbf{H}_2$, $\mathbf{V} = \mathbf{V}_1 \times \mathbf{V}_2$ and $\mathbf{E} = \mathbf{E}_1 \times \mathbf{E}_2$. We define a bilinear map $B(\cdot, \cdot)$ on $\mathbf{V} \times \mathbf{E}$ by

$$\begin{aligned}\langle B(z_1, z_2), z_3\rangle &= \langle B_1(u_1, u_2), u_3\rangle - \langle B_1(b_1, b_2), u_3\rangle \\ &\quad + \langle B_1(u_1, b_2), b_3\rangle - \langle B_1(b_1, u_2), b_3\rangle,\end{aligned}$$

for $z_1 = (u_1, b_1) \in \mathbf{V}$, $z_2 = (u_2, b_2) \in \mathbf{E}$ and $z_3 = (u_3, b_3) \in \mathbf{H}$. We also set

$$Az = \begin{pmatrix} \mathrm{Id} + A_1 & 0 \\ 0 & \mathrm{Id} + A_2 \end{pmatrix} \begin{pmatrix} u \\ b \end{pmatrix}$$

for $z = (u, b) \in \mathbf{E}$.

We set $\mathbf{u} := (u, b)$ and

$$\int_Z \tilde{G}(t, \mathbf{u}(t-), z)\tilde{\eta}(dz, dt) := \begin{pmatrix} \int_{Z_1} \tilde{f}(t, \mathbf{u}(t-), z_1)\tilde{\eta}_1(dz_1, dt) \\ \int_{Z_2} \tilde{g}(t, \mathbf{u}(t-)), z_2)\tilde{\eta}_2(dz_2, dt) \end{pmatrix}.$$

We assume that $\tilde{f}$, $\tilde{g}$ are chosen in such a way that $\tilde{G}$ maps $\mathbf{V}$ into $L^{2p}(Z, \nu, \mathbf{V})$ and satisfies Assumption 4.1.

By setting $R \equiv -\mathrm{Id}$ and projecting on $\mathbf{H}$ we can see that (4.17), (4.18) can be rewritten in the form (4.16). Now, by choosing $N = A$ we can show by arguing as in Theorem 4.2 that the stochastic Magnetohydrodynamic equations (4.17), (4.18) has a local maximal solution which is global if the dimension $n = 2$.

4.2.3. Magnetic Bénard problem. Let $\mathcal{O} = (0, l) \times (0, 1)$ be a rectangular domain in the vertical plane, (e_1, e_2) the standard basis in $\mathbb{R}^2$. Let $(Z_i, \mathcal{Z}_i, \nu_i)$, $i = 1, 2, 3$ be three measure spaces where the measures ν_i are σ-finite and positive. We consider three mutually independent compensated Poisson random measures $\tilde{\eta}_i$ with intensity measure ν_i defined on a complete filtered probability space $\mathfrak{P} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

We consider the equations

$$\begin{aligned}
&du + [u\nabla u - \kappa_1 \Delta u + \nabla\left(p + \frac{s}{2}|b|^2\right) - sb\nabla b]dt = \theta e_2 dt \\
&\quad + \int_{Z_1} \tilde{f}(t, u(t-), \theta(t-), b(t-), z_1)\tilde{\eta}_1(dz_1, dt), \\
&\operatorname{div} u = 0, \\
&d\theta + [u\nabla\theta - u^{(2)} - \kappa\Delta\theta]dt = \int_{Z_2} \tilde{g}(t, u(t-), \theta(t-), b(t-), z_2)\tilde{\eta}_2(dz_2, dt), \\
&db + [-\kappa_2\Delta b + u\nabla b - b\nabla u]dt = \int_{Z_3} \tilde{h}(t, u(t-), \theta(t-), b(t-), z_3)\tilde{\eta}_2(dz_3, dt), \\
&\operatorname{div} b = 0,
\end{aligned}$$

with boundary conditions

$$\begin{aligned}
&u = 0,\ \ \theta = 0,\ \ b^{(2)} = 0,\ \ \partial_2 b^{(1)} = 0\ \text{ on }\ x^{(2)} = 0 \text{ and } x^{(2)} = 1, \\
&\quad u, p, \theta, b, u_{x^{(1)}}, \theta_{x^{(1)}}, b_{x^{(1)}}\ \text{ are periodic in } x^{(1)} \text{ with period } l.
\end{aligned}$$

This is the Boussinesq model coupled with magnetic field (see [28]) with stochastic perturbations. Throughout we assume that $\kappa_1 = \kappa_2 = s = 1$. Let $\mathbf{H} = \mathbf{H}_3 \times \mathbf{H}_4 \times \mathbf{H}_5$, $\mathbf{V} = \mathbf{V}_3 \times \mathbf{V}_4 \times \mathbf{V}_5$ and $\mathbf{E} = \mathbf{E}_3 \times \mathbf{E}_4 \times \mathbf{E}_5$.

We define a bilinear map $B(\cdot,\cdot)$ on $\mathbf{V} \times \mathbf{E}$ by

$$\begin{aligned}\langle B(z_1, z_2), z_3\rangle = {} & \langle B_1(u_1, u_2), u_3\rangle - \langle B_1(b_1, b_2), u_3\rangle \\ & + \langle B_1(u_1, b_2), b_3\rangle - \langle B_1(b_1, u_2), b_3\rangle + \langle B_2(u_1, \theta_2,), \theta_3\rangle,\end{aligned}$$

for $z_1 = (u_1, \theta_1, b_1) \in \mathbf{V}$, $z_2 = (u_2, \theta_2, b_2) \in \mathbf{E}$ and $z_3 = (u_3, \theta_3, b_3) \in \mathbf{H}$. Using the notations in (4.2), we set

$$Az = \begin{pmatrix} A_3 & 0 & 0 \\ 0 & A_4 & 0 \\ 0 & 0 & A_5 \end{pmatrix} \begin{pmatrix} u \\ \theta \\ b \end{pmatrix}$$

for $z = (u, \theta, b) \in E$.

We also set $R(u, \theta, b) = -(\theta e_2\, , u^{(2)}, 0)$ and $N = A$. Note that in this case $R \in \mathcal{L}(\mathbf{H}, \mathbf{H})$ and $N \in \mathcal{L}(\mathbf{E}, \mathbf{H}) \cap \mathcal{L}(\mathbf{V}, \mathbf{V}^*)$.

We set $\mathbf{u} := (u, \theta, b)$ and

$$\int_Z \tilde{G}(t, \mathbf{u}(t-), z)\tilde{\eta}(dz, dt) := \begin{pmatrix} \int_Z \tilde{f}(t, \mathbf{u}(t-), z_1)\tilde{\eta}_1(dz_1, dt) \\ \int_Z \tilde{g}(t, \mathbf{u}(t-), z_2)\tilde{\eta}_2(dz_2, dt) \\ \int_Z \tilde{h}(t, \mathbf{u}(t-), z_3)\tilde{\eta}_3(dz_3, dt). \end{pmatrix}$$

We assume that $\tilde{f}, \tilde{g}, \tilde{h}$ are chosen such that $\tilde{G}$ verifies Assumption 4.1. With these notations we can put the stochastic Magnetic Bénard problem into the abstract stochastic evolution Eq. (4.16).

Theorem 4.4. *The stochastic Magnetic Bénard problem* (4.16) *admits a unique global strong solution.*

Proof. The maximal local solution will follow from Theorem 3.5 if we are able to prove that $F(\mathbf{u}) = B(\mathbf{u}, \mathbf{u}) + R(\mathbf{u})$ satisfies (2.7). Since R is a bounded linear map, it follows from Remark 3.8 that it is sufficient to check (2.7) for B. But from (4.9) and (4.10) we deduce that there exists $C > 0$ such that for

$$|B(y) - B(x)| \leq C \left[\|y - x\| \|y\|^{1-\frac{n}{4}} \|y\|_{\mathbf{E}}^{\frac{n}{4}} + \|y - x\|_{\mathbf{E}}^{\frac{n}{4}} \|y - x\|^{1-\frac{n}{4}} \|x\| \right],$$

for any $x, y \in \mathbf{E}$. This means that B satisfies (2.7) with $p = 1$ and $\alpha = \frac{n}{4}$. Since $n = 2$ we have $\alpha = \frac{1}{2}$ and $\langle B(\mathbf{u}, \mathbf{u}), \mathbf{u}\rangle = 0$. So thanks to Remark 3.8, we only need to check that (3.36) is verified by B. But this will follow from (4.11) and (4.12). ☐

4.2.4. Boussinesq model for the Bénard convection. Let $\mathcal{O}$ be a (possibly) domain of $\mathbb{R}^n$, $n = 2, 3$, $\{e_i, \ldots, e_n\}$ a standard basis in $\mathbb{R}^n$ and $x=(x^{(1)}, \ldots, x^{(n)})$ an element of $\mathbb{R}^n$. We assume that $\mathcal{O}$ has a smooth boundary $\partial\mathcal{O}$. Let $(Z_i, \mathcal{Z}_i, \nu_i)$, $i = 1, 2$ be two measure spaces where the measures ν_i are σ-finite and positive. We consider two mutually independent compensated Poisson random measures $\tilde{\eta}_i$ with intensity measure ν_i defined on a complete filtered probability space $\mathfrak{P} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

Let us consider the Bénard convection problem (see e.g. [27] and the references therein) given by the following system

$$du + [u\nabla u - \Delta u + \nabla p]dt = \theta e_n dt + \int_{Z_1} \tilde{f}(t, u(t-), b(t-), z_1)\tilde{\eta}_1(dz_1, dt), \quad \operatorname{div} u = 0, \tag{4.21}$$

$$d\theta + [u\nabla\theta - u^{(n)} - \Delta\theta]dt = \int_{Z_2} \tilde{g}(t, u(t-), \theta(t-), z_2)\tilde{\eta}_2(dz_2, dt), \tag{4.22}$$

with boundary conditions

$$u = 0 \;\; \& \;\; \theta = 0 \;\text{ on } \partial\mathcal{O}.$$

Here $p(x,t)$ is the pressure field, $\int_{Z_1} \tilde{f}(t, u(t), b(t), z_1)\tilde{\eta}_1(dz_1, dt)$, $\int_{Z_2} \tilde{g}(t, u(t), b(t), z_2)\tilde{\eta}_2(dz_2, dt)$ represent two random external forces, $u = (u^{(1)}(x,t), \ldots, u^{(n)}(x,t))$ is the velocity field and $\theta = \theta(x,t)$ is the temperature field.

We set $\mathbf{H} = \mathbf{H}_3 \times \mathbf{H}_4$, $\mathbf{V} = \mathbf{V}_3 \times \mathbf{V}_4$, $\mathbf{E} = \mathbf{E}_3 \times \mathbf{E}_4$. Following the notations given in (4.2) we define

$$Az = \begin{pmatrix} A_3 & 0 \\ 0 & A_4 \end{pmatrix} \begin{pmatrix} u \\ \theta \end{pmatrix}$$

for $z = (u, \theta) \in E$. We define a bilinear map $B(\cdot, \cdot)$ on $\mathbf{V} \times \mathbf{E}$ by

$$\langle B(z_1, z_2), z_3\rangle = \langle B_1(u_1, u_2), u_3\rangle + \langle B_2(u_1, \theta_2,), \theta_3\rangle,$$

for $z_1 = (u_1, \theta_1) \in \mathbf{V}$, $z_2 = (u_2, \theta_2) \in \mathbf{E}$ and $z_3 = (u_3, \theta_3) \in \mathbf{H}$. We also put $R(u, \theta, b) = -(\theta e_2\,,\, u^{(n)})$ and $N = A$.

As before we set $\mathbf{u} := (u, \theta)$ and

$$\int_Z \tilde{G}(t, \mathbf{u}(t-), z)\tilde{\eta}(dz, dt) := \begin{pmatrix} \int_{Z_1} \tilde{f}(t, u(t-), b(t-), z_1)\tilde{\eta}_1(dz_1, dt) \\ \int_{Z_2} \tilde{g}(t, u(t-), b(t-), z_2)\tilde{\eta}_2(dz_2, dt) \end{pmatrix}.$$

We assume that $\tilde{f}$, $\tilde{g}$ are chosen in such a way that $\tilde{G}$ maps $\mathbf{V}$ into $L^{2p}(Z, \nu, \mathbf{V})$ and satisfies Assumption 4.1.

By Arguing as in the case of Navier–Stokes equations, Magnetic Bénard problem and MHD equations we can show that if the random external force satisfies Assumption 4.1, then the Boussinesq model for the Bénard convection admits a unique maximal strong solution which is global is $n = 2$.

4.3. Shell models of turbulence

Here, we use again the same notations as used in [17]. Let H be a set of all sequences $u = (u_1, u_2, \ldots)$ of complex numbers such that $\sum_n |u_n|^2 < \infty$. We consider H as a *real* Hilbert space endowed with the inner product $(\cdot, \cdot)$ and the norm $|\cdot|$ of the form

$$(u, v) = \operatorname{Re} \sum_{n=1}^{\infty} u_n v_n^*, \quad |u|^2 = \sum_{n=1}^{\infty} |u_n|^2,$$

where v_n^* denotes the complex conjugate of v_n. In this space H we consider the evolution equation (4.16) with $R = 0$ and with linear operator A and bilinear mapping B defined by the formulas

$$(Au)_n = \nu k_n^2 u_u, \quad n = 1, 2, \ldots, \quad D(A) = \left\{ u \in H \,:\, \sum_{n=1}^{\infty} k_n^4 |u_n|^2 < \infty \right\},$$

where $\nu > 0$, $k_n = k_0 \mu^n$ with $k_0 > 0$ and $\mu > 1$, and

$$[B(u,v)]_n = -i \\ \times \left(a k_{n+1} u_{n+1}^* v_{n+2}^* + b k_n u_{n-1}^* v_{n+1}^* - a k_{n-1} u_{n-1}^* v_{n-2}^* - b k_{n-1} u_{n-2}^* v_{n-1}^*\right)$$

for $n = 1, 2, \ldots$, where a and b are real numbers (here above we also assume that $u_{-1} = u_0 = v_{-1} = v_0 = 0$). This choice of A and B corresponds to the so-called GOY-model (see, e.g., [42]). If we take

$$[B(u,v)]_n = -i \\ \times \left(a k_{n+1} u_{n+1}^* v_{n+2} + b k_n u_{n-1}^* v_{n+1} + a k_{n-1} u_{n-1} v_{n-2} + b k_{n-1} u_{n-2} v_{n-1}\right),$$

then we obtain the Sabra shell model introduced in [37].

One can easily show (see [1] for the GOY model and [16] for the Sabra model) that the trilinear form

$$\langle B(u,v), w\rangle \equiv \mathrm{Re} \sum_{n=1}^{\infty} [B(u,v)]_n \, w_n^*$$

satisfies the inequality

$$|\langle B(u,v), w\rangle| \le C|u||A^{1/2}v||w|, \quad \forall u, w \in H, \quad \forall v \in D(A^{1/2}).$$

Hence taking $\mathbf{H} = H$, $(\mathbf{V}, \|\cdot\|) = (D(A^{\frac{1}{2}}), |A^{\frac{1}{2}} \cdot|)$ and $(\mathbf{E}, \|\cdot\|_{\mathbf{E}}) := (D(A), |A\cdot|)$ we infer that the nonlinear term for the shell models satisfies Assumption 2.2 with $\alpha = 0$ and $p = 1$. By Arguing as before we can show that if the random external force satisfies Assumption 4.1, then stochastic shell models admits a unique global strong solution.

Let us consider the following dyadic model (see, e.g., [33] and the references therein)

$$\partial_t u_n + \nu \lambda^{2\alpha n} u_n - \lambda^n u_{n-1}^2 + \lambda^{n+1} u_n u_{n+1} = f_n, \quad n = 1, 2, \ldots, \tag{4.23}$$

where $\nu, \alpha > 0$, $\lambda > 1$, $u_0 = 0$. By setting $[B(u,v)]_n = -\lambda^n u_{n-1} v_{n-1} + \lambda^{n+1} u_n \, v_{n+1}$ and $(Au)_n = \nu \, \lambda^{2\alpha n} \, u_n$, it is not difficult to show that the system (4.23) falls also in the framework of the shell models of turbulence provided that $\alpha \ge 1/2$.

4.4. 3D Leray α-model for Navier–Stokes equations

As in the previous subsections, we use the same notations as used in [17].

In a bounded 3D domain $\mathcal{O}$ we consider the following equations:

$$\partial_t u - \Delta u + v \nabla u + \nabla p = f, \tag{4.24}$$

$$(1 - \alpha\Delta) v = u, \quad \operatorname{div} u = 0, \quad \operatorname{div} v = 0 \;\text{ in }\; \mathcal{O}, \tag{4.25}$$

$$v = u = 0 \quad \text{on} \quad \partial\mathcal{O}. \tag{4.26}$$

where $u = (u^{(1)}, u^{(2)}, u^{(3)})$ and $v = (v^{(1)}, v^{(2)}, v^{(3)})$ are unknown fields, $p(x,t)$ is the pressure. We refer to [14,15] and references for results related to (4.24)–(4.26).

Let $\mathbf{H} = \mathbf{H}_1$, $\mathbf{V} = \mathbf{V}_1$ and $\mathbf{E} = \mathbf{E}_1$ be the Hilbert spaces defined in Sect. 4.1. Set $A = A_1$, $G_\alpha = (Id + \alpha A)^{-1}$ and define a bilinear mapping $B(\cdot, \cdot)$ on $\mathbf{V} \times \mathbf{E}$ by setting

$$B(u, v) = B_1(G_\alpha u, v),$$

for any $u \in \mathbf{V}$ and $v \in \mathbf{E}$.

Arguing as in [17, Subsubsection 2.1.5] we can show that there exists $C > 0$ such that

$$|B(u, v)| \leq C\|u\|_{L^3}\|\nabla v\|_{L^3}, \tag{4.27}$$

for any $u \in \mathbb{L}^3$ and $v \in \mathbb{W}^{1,3}$. Recall that in three dimensional case we have the following Gagliardo–Nirenberg inequality

$$\|u\|_{L^3} \leq c|u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}_{\mathbb{H}^1}, \quad u \in \mathbb{H}^1. \tag{4.28}$$

Now using this inequality and the continuous embedding $\mathbb{H}^1 \subset \mathbb{L}^3$

we infer from (4.27) that

$$|B(u, v)| \leq C\|u\|_{\mathbf{V}}\|v\|^{\frac{1}{2}}_{\mathbf{V}}\|v\|_{\mathbf{E}}, \tag{4.29}$$

$$|B(u, v)| \leq C|u|^{\frac{1}{2}}_{\mathbf{H}}\|u\|^{\frac{1}{2}}_{\mathbf{V}}\|v\|^{\frac{1}{2}}_{\mathbf{V}}\|v\|_{\mathbf{E}}, \tag{4.30}$$

for any $u \in \mathbf{V}$, $v \in \mathbf{E}$.

Now we set $R \equiv 0$ and $N = A$. Thanks to (4.29)–(4.30) we see that the nonlinear term for the 3D Leray α-model for Navier–Stokes equations satisfies the assumptions of Theorem 3.7 with $\alpha = \frac{1}{2}$ and $p = 1$. Therefore we can argue as in the case of 2D stochastic Navier–Stokes equations and show that the stochastic 3D Leray α-model for Navier–Stokes equations admits a global solution if the random external force satisfies Assumption 4.1.

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Appendix A. Existence of solution to the linear SPDE (3.4)

Throughout this appendix we assume that the separable Hilbert spaces $\mathbf{E}, \mathbf{V}$, and $\mathbf{H}$ are defined as before.

Let $(Z, \mathcal{B}(Z))$ be a separable metric space and let ν be a σ-finite positive measure on it. For the sake of simplicity we write $\mathcal{Z} := \mathcal{B}(Z)$. Let $\eta : \Omega \times$

$\mathcal{B}(\mathbb{R}_+) \times \mathcal{Z} \to \bar{\mathbb{N}}$ is a time homogeneous Poisson random measure with the intensity measure ν. We will denote by $\tilde{\eta} = \eta - \gamma$ the compensated Poisson random measure associated to η where the compensator γ is given by

$$\mathcal{B}(\mathbb{R}_+) \times \mathcal{Z} \ni (A, I) \mapsto \gamma(A, I) = \nu(A)\lambda(I) \in \mathbb{R}_+.$$

Let $\phi \in M^2(0, T; \mathbf{H})$ and $\psi \in \mathcal{M}^2(0, T; L^4(Z, \nu; \mathbf{V}))$. We will show in the next theorem that the following linear SPDEs (which is (3.4)) has a unique solution

$$\begin{cases} d\mathbf{u}(t) + [A\mathbf{u}(t) + \phi(t)]dt = \int_Z \psi(t, z)\widetilde{\eta}(dz, dt), \ t \in [0, T], \\ \mathbf{u}(0) = \mathbf{u}_0. \end{cases} \tag{A.1}$$

Theorem A.1. *Let A, N be as in Assumption 2.1, $\phi \in M^2(0, T; \mathbf{H})$, $\psi \in \mathcal{M}^2(0, T; L^2(Z, \nu; \mathbf{V}))$. Let $\mathbf{u}_0$ be a $\mathbf{V}$-valued $\mathcal{F}_0$-measurable random variable satisfying $\mathbb{E}|\mathbf{u}_0|^2 < \infty$. Then there exists a unique progressively measurable process $\mathbf{u}$ such that $\mathbf{u} \in L^2(0, T; \mathbf{E}) \cap \mathbf{D}(0, T; \mathbf{V})$ with probability 1, and almost surely*

$$\langle \mathbf{u}(t), w\rangle + \int_0^t \langle A\mathbf{u}(s) + \phi(s), w\rangle ds = \langle \mathbf{u}_0, w\rangle + \int_0^t \int_Z \langle \psi(s, z), w\rangle \widetilde{\eta}(dz, ds), \tag{A.2}$$

for all $t \in [0, T]$ and $w \in \mathbf{H}$.

Proof. We will use the Picard method as presented in [43, Chapter 3, Section 1]. Throughout this proof we set

$$\|N\| := \max\left(\|N\|_{\mathcal{L}(\mathbf{E},\mathbf{H})}, \|N\|_{\mathcal{L}(\mathbf{V},\mathbf{V}^*)}\right).$$

For positive integer n we define a sequence $\{\mathbf{u}^{[n]}(t), t \in [0, T]\}$ of stochastic processes as follows

$$\begin{cases} \mathbf{u}^{[1]}(t) = \mathbf{u}_0, \\ \mathbf{u}^{[n+1]}(t) = \mathbf{u}_0 - \int_0^t [A\mathbf{u}^{[n]}(s) + \phi(s)]ds + \int_0^t \int_Z \psi(s, z)\widetilde{\eta}(dz, ds), \ t \in [0, T], \ n \geq 2. \end{cases}$$

Thanks to our assumption and [30, Theorem 2] the strochastic processe

$$\mathbf{u}^{[2]}(t) = \mathbf{u}_0 - \int_0^t [A\mathbf{u}^{[1]}(s) + \phi(s)]ds + \int_0^t \int_Z \psi(s, z)\widetilde{\eta}(dz, ds)$$

is a well-defined $\mathbf{V}$-valued adapted and càdlàg process. By iterating this definition we see that for each $n \geq 2$ $\mathbf{u}^{[n]}$ is also a well-defined $\mathbf{V}$-valued adapted and càdlàg process.

Now we will show that the sequence $\mathbf{u}^{[n]}$ is converging in appropriate topology to the solution $\mathbf{u}$ of (A.1). In fact we will show that $\mathbf{u}^{[n]} \in L^2(\Omega; L^\infty(0, T; \mathbf{V}))$ is a Cauchy sequence. For this aim define $\Phi^n(t) = \mathbb{E}\sup_{s\in[0,t]} \|\mathbf{u}^{[n+1]}(s) - \mathbf{u}^{[n]}(s)\|^2$ for all $n \geq 1$. We have

$$\mathbf{u}^{[n+1]}(t) - \mathbf{u}^{[n]}(t) = -\int_0^t A(\mathbf{u}^{[n]}(s) - \mathbf{u}^{[n-1]})ds,$$

for any $t \in [0, T]$ and $n \geq 2$. Multiplying this equation by $N(\mathbf{u}^{[n+1]} - \mathbf{u}^{[n]})$, using Assumption 2.1 and the Cauchy inequality with arbitrary $\varepsilon > 0$ we infer that

$$(C_N-\varepsilon)\sup_{s\in[0,t]}\|\mathbf{u}^{[n+1]}(s)-\mathbf{u}^{[n]}(s)\|^2\le\frac{\|N\|^2\|A\|^2}{4\varepsilon}\int_0^t\|\mathbf{u}^{[n]}(s)-\mathbf{u}^{[n-1]}(s)\|^2ds.$$

Choosing $\varepsilon=C_N/2$ taking the mathematical expectation to both side of the last estimate implies

$$\Phi^n(t)\le\frac{\|N\|^2\|A\|^2}{2C_N^2}\int_0^t\Phi^{n-1}(s)ds. \tag{A.3}$$

As in [43] we iterate (A.3) and obtain

$$\Phi^n(t)\le\left(\frac{\|N\|^2\|A\|^2}{2C_N^2}\right)^n\frac{1}{n!}\Phi^1(t),$$

from which we deduce that $(\mathbf{u}^{[n]};n\ge 1)$ forms a Cauchy sequence in $L^2(\Omega;L^\infty(0,T;\mathbf{V}))$. Therefore, there exists $\mathbf{u}\in L^2(\Omega;L^\infty(0,T;\mathbf{V}))$ such that

$$\mathbf{u}^{[n]}\to\mathbf{u}\text{ strongly in }L^2(\Omega;L^\infty(0,T;\mathbf{V})). \tag{A.4}$$

Now, we prove that $\mathbf{u}^{[n]}$ is bounded in $L^2(\Omega;L^2(0,T;\mathbf{E}))$. For this purpose we apply Itô formula to $\Psi(u)=\langle u,Nu\rangle$ and use Assumption 2.1 to infer that

$$\begin{aligned}&C_N\|\mathbf{u}^{[n]}(t\wedge\tau)\|^2+2C_A\int_0^{t\wedge\tau}\|\mathbf{u}^{[n]}(s)\|^2_{\mathbf{E}}ds\le\int_0^T\left[|\phi(s)|^2+\int_Z\langle N\psi(s,z),\psi(s,z)\rangle\nu(dz)\right]ds\\&+\int_0^{t\wedge\tau}\int_Z\left[\langle\psi(s,z),N\mathbf{u}^{[n]}(s-)\rangle+\langle\psi(s,z),N\psi(s,z)\rangle\right]\widetilde{\eta}(dz,ds)\\&+\Psi(\mathbf{u}_0)+\|N\|^2\int_0^T|\mathbf{u}^{[n]}(s)|^2ds.\end{aligned} \tag{A.5}$$

where τ is an arbitrary stopping time localizing the local martingale

$$\int_0^t\int_Z\left[\langle\psi(s,z),N\mathbf{u}^{[n]}(s-)\rangle+\langle\psi(s,z),N\psi(s,z)\rangle\right]\widetilde{\eta}(dz,ds).$$

We easily derive from (A.5) that

$$\begin{aligned}&C_N\|\mathbf{u}^{[n]}(t\wedge\tau)\|^2+2C_A\int_0^{t\wedge\tau}\|\mathbf{u}^{[n]}(s)\|^2_{\mathbf{E}}ds\le\int_0^T\left[|\phi(s)|^2+\|N\|^2\int_Z\|\psi(s,z)\|^2\nu(dz)\right]ds\\&+\int_0^{t\wedge\tau}\int_Z\left[\langle\psi(s,z),N\mathbf{u}^{[n]}(s-)\rangle+\langle\psi(s,z),N\psi(s,z)\rangle\right]\widetilde{\eta}(dz,ds)\\&+\Psi(\mathbf{u}_0)+\|N\|^2\int_0^T\|\mathbf{u}^{[n]}(s)\|^2ds.\end{aligned}$$

Since, by the first part of our proof, $\int_0^T\mathbb{E}\|\mathbf{u}^{[n]}(s)\|^2ds$ is bounded and τ is arbitrary, by taking mathematical expectation to both sides of the last inequality we derive that there exists $C>0$ such that

$$\mathbb{E}\int_0^T\|\mathbf{u}^{[n]}(s)\|^2_{\mathbf{E}}ds\le C.$$

This implies that one can find a subsequence of $\mathbf{u}^{[n]}$, which will be denoted with the same fashion, such that

$$\mathbf{u}^{[n]}\to\mathbf{u}\text{ weakly in }L^2(\Omega;L^2(0,T;\mathbf{E})). \tag{A.6}$$

Since, by assumption, $A \in \mathcal{L}(\mathbf{E}, \mathbf{H})$ it follows from (A.6) that

$$Au^{[n]} \to A\mathbf{u} \text{ weakly in } L^2(\Omega; L^2(0,T;\mathbf{H})). \tag{A.7}$$

Owing to the convergences (A.4) and (A.7) we easily derive that, with probability 1, $\mathbf{u}$ satisfies (A.2) for all $t \in [0,T]$ and $w \in \mathbf{H}$. This means that (A.1) holds for all $t \in [0,T]$ and all $w \in \mathbf{H}$ with probability 1. Since $\mathbf{u}$ is the limit in $L^2(\Omega; L^\infty(0,T;\mathbf{V}))$ of a sequence of adapted processes, we infer that $\mathbf{u}$ is adapted. Thanks to our assumption and [30, Theorem 2] the process $\mathbf{u}$ is càdlàg . Because $\mathbf{u}$ is adapted and càdlàg it admits a progressively measurable version which is still denoted with the same symbol. The proof of our theorem is complete. □

References

[1] Barbato, D., Barsanti, M., Bessaih, H., Flandoli, F.: Some rigorous results on a stochastic Goy model. J. Stat. Phys. **125**, 677–716 (2006)

[2] Bensoussan, A.: Stochastic Navier–Stokes equations. Acta Appl. Math. **38**, 267–304 (1995)

[3] Bensoussan, A., Freshe, J.: Local solutions for stochastic Navier–Stokes equations. M2AN Math. Model. Numer. Anal. **34**(2), 241–273 (2000)

[4] Bensoussan, A., Temam, R.: Equations Stochastiques du type Navier-Stokes. J. Funct. Anal. **13**, 195–222 (1973)

[5] Bessaih, H., Millet, A.: Large deviation principle and inviscid shell models. Electron. J. Probab. **14**, 2551–2579 (2009)

[6] Bessaih, H., Flandoli, F., Titi, E.S.: Stochastic attractors for shell phenomenological models of turbulence. J. Stat. Phys. **140**, 688–717 (2010)

[7] Brzeźniak, Z., Hausenblas, E.: Maximal regularity for stochastic convolutions driven by Lévy processes. Probab. Theory Rel. Field. **145**(3-4), 615–637 (2009)

[8] Brzeźniak, Z., Hausenblas, E., Razafimandimby, P.: Stochastic nonparabolic dissipative systems modeling the flow of liquid crystals: strong solution. RIMS Symp. Math. Anal. Incompress. Flow RIMS Kôkyûroku **1875**, 41–73 (2013)

[9] Brzeźniak, Z., Hausenblas, E., Zhu, J.: 2D stochastic Navier–Stokes equations driven by jump noise. Nonlinear Anal. **79**, 122–139 (2013)

[10] Brzeźniak, Z., Millet, A.: On the stochastic strichartz estimates and the stochastic nonlinear Schrödinger equation on a compact riemannian manifold. Potential Anal. **41**(2), 269–315 (2014)

[11] Brzeźniak, Z., Motyl, E.: Existence of a martingale solution of the stochastic Navier–Stokes equations in unbounded 2D and 3D domains. J. Differ. Eqs. **254**(4), 1627–1685 (2013)

[12] Brzeźniak, Z., Maslowski, B., Seidler, J.: Stochastic nonlinear beam equations. Probab. Theory Rel. Fields **132**(1), 119–149 (2005)

[13] Brzeźniak, Z., Zhu, J.: Stochastic nonlinear beam equations driven by compensated poisson random measures. arXiv:1011.5377

[14] Chepyzhov, V., Titi, E., Vishik, M.: On the convergence of solutions of the Leray-α model to the trajectory attractor of the 3D Navier–Stokes system. Discrete Contin. Dyn. Syst. **17**, 481–500 (2007)

[15] Cheskidov, A., Holm, D., Olson, E., Titi, E.: On a Leray-α model of turbulence. Proc. R. Soc. Lond. Ser. A **461**, 629–649 (2005)

[16] Constantin, P., Levant, B., Titi, E.S.: Analytic study of the shell model of turbulence. Phys. D **219**, 120–141 (2006)

[17] Chueshov, I., Millet, A.: Stochastic 2D hydrodynamical type systems: well posedness and large deviations. Appl. Math. Optim. **61**(3), 379–420 (2010)

[18] de Bouard, A., Debussche, A.: A stochastic nonlinear Schrödinger equation with multiplicative noise. Commun. Math. Phys. **205**(1), 161–181 (1999)

[19] de Bouard, A., Debussche, A.: The stochastic nonlinear Schrödinger equation in H^1. Stoch. Anal. Appl. **21**(1), 97–126 (2003)

[20] Deugoue, G., Sango, M.: On the strong solution for the 3D stochastic Leray-Alpha model. Bound. Value Probl. vol. 2010, Article ID 723018, p. 31 (2010). doi:10.1155/2010/723018

[21] Dong, Z., Jianliang, Z.: Martingale solutions and Markov selection of stochastic 3D Navier–Stokes equations with jump. J. Differ. Eqs. **250**, 2737–2778 (2011)

[22] Duvaut, G., Lions, J.L.: Inéquations en thermoélasticité et magnéto hydrodynamique. Arch. Ration. Mech. Anal. **46**, 241–279 (1972)

[23] Ethier, S.N., Kurtz, T.G.: Markov Processes, Characterization and Convergence. Wiley Series in Probability and Mathematical Statistics. Wiley, New York (1986)

[24] Fernando, B.P.W., Sritharan, S.S.: Nonlinear filtering of stochastic Navier–Stokes equation with Itô-Lévy noise Stoch. Anal. Appl **31**(3), 381–426 (2013)

[25] Flandoli, F., Gubinelli, M., Hairer, M., Romito, M.: Rigourous remarks about scaling laws in turbulent fluid. Commun. Math. Phys **278**, 1–29 (2008)

[26] Flandoli, F., Gatarek, D.: Martingale and stationary solutions for stochastic Navier–Stokes equations Probab. Theory Rel. Fields **102**(03), 367–391 (1995)

[27] Foias, C., Manley, O., Temam, R.: Attractors for the Bénard problem: existence and physical bounds on their fractal dimension. Nonlinear Anal. **11**, 939–967 (1987)

[28] Galdi, G.P., Padula, M.: A new approach to energy theory in the stability of fluid motion. Arch. Ration. Mech. Anal **110**, 187–286 (1990)

[29] Glatt-Holtz, N., Ziane, M.: Strong pathwise solutions of the stochastic Navier–Stokes system. Adv. Differ. Eqs. **14**(5–6), 567–600 (2009)

[30] Gyöngy, I., Krylov, N.V.: On stochastics equations with respect to semimartingales. II. Itô formula in Banach spaces. Stochastics **6**(3–4), 153–173 (1981)

[31] Hausenblas, E., Razafimandimby, P.A., Sango, M.: Martingale solution to differential type fluids of grade two driven by random force of Lévy type. Potential Anal. **38**(4), 1291–1331 (2013)

[32] Imkeller, P., Pavlyukevich, I.: First exit times of SDEs driven by stable Lévy processes. Stoch. Process. Appl. **116**, 611–642 (2006)

[33] Katz, N.H., Pavlović, N.: Finite time blow-up for a dyadic model of the Euler equations. Trans. Am. Math. Soc. **357**, 695–708 (2005)

[34] Kim, J.U.: Strong solutions of the stochastic Navier–Stokes equations in $\mathbb{R}^3$. Indiana Univ. Math. J. **59**(4), 1417–1450 (2010)

[35] Kupiainen, A.: Statistical theories of turbulence. In: Wehr, J. (ed.) Random Media 2000, p. 29. Wydawnictwa ICM, Warszawa (2004)

[36] Ladyzhenskaya, O., Solonnikov, V.: Solution of some nonstationary magnetohydrodynamical problems for incompressible fluid. Trudy Steklov Math. Inst. **59**, 115–173 (1960)

[37] Lvov, V.S., Podivilov, E., Pomyalov, A., Procaccia, I., Vandembroucq, D.: Improved shell model of turbulence. Phys. Rev. E **58**, 1811–1822 (1998)

[38] Mikulevicius, R.: On strong H^1_2-solutions of stochastic Navier–Stokes equation in a bounded domain. SIAM J. Math. Anal **41**(3), 1206–1230 (2009)

[39] Motyl, E.: Stochastic Navier–Stokes equations driven by Lévy noise in unbounded 3D domains. Potential Anal. **38**(3), 863–912 (2013)

[40] Menaldi, J.-L., Sritharan, S.S.: Stochastic 2-D Navier–Stokes equation. Appl. Math. Optim. **46**(1), 31–53 (2012)

[41] Mikulevicius, R., Rozovskii, B.L.: Stochastic Navier-Stokes Equations and Turbulent Flows SIAM J. Math. Anal. **35**(5), 1250–1310 (2004)

[42] Ohkitani, K., Yamada, M.: Temporal intermittency in the energy cascade process and local Lyapunov analysis in fully developed model of turbulence. Prog. Theor. Phys. **89**, 329–341 (1989)

[43] Pardoux, E.: Equations aux dérivées partielles stochastiques non linéaires monotones; Etude de solutions fortes de type Itô. Thèse (PhD Thesis), Université Paris Sud (1975)

[44] Peszat, S., Zabzcyk, J.: Stochastic partial differential equations with Lévy noise: an evolution equation approach. Encyclopedia of mathematics and its applications vol. 113. Cambridge University Press, Cambridge (2007)

[45] Protter, P.E.: Stochastic integration and differential equations. Applications of Mathematics (New York). Stochastic Modelling and Applied Probability vol. 21, 2nd edn. Springer, Berlin, Heidelberg (2005)

[46] Rüdiger, B., Ziglio, G.: Itô formula for stochastic integrals w.r.t. compensated Poisson random measures on separable Banach spaces. Stochastics **78**(6), 377–410 (2006)

[47] Sakthivel, K., Sritharan, S.S.: Martingale solutions for stochastic Navier–Stokes equations driven by Lévy noise Evol. Equ. Control Theory **1**(2), 355–392 (2012)

[48] Sango, M.: Magnetohydrodynamic turbulent flows: Existence results. Phys. D: Nonlinear Phenom. **239**(12), 912–923 (2010)

[49] Sermange, M., Temam, R.: Some mathematical questions related to MHD equations. Commun. Pure Appl. Math. **36**, 635–664 (1983)

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