# STRONG STABILITY IN A $G/M/1$ QUEUEING SYSTEM

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ABSTRACT. In this paper, we study the strong stability of the stationary distribution of the imbedded Markov chain in the $G/M/1$ queueing system, after perturbation of the service law (see Aissani, 1990, and Kartashov, 1981). We show that under some hypotheses, the characteristics of the $G/G/1$ queueing system can be approximated by the corresponding characteristics of the $G/M/1$ system. After clarifying the approximation conditions, we obtain the stability inequalities by exactly computing the constants.

## INTRODUCTION

Many queueing models have been analyzed since the publishing of the pioneering works by A. K. Erlang. One would be fortunate to have a closed form analytical expression for a desired parameter, but even in such an exceptionally favorable case, evaluation of the expression may be very difficult. For example, the well-known simple and elegant Pollaczek–Khintchine formula requires a numerical inversion of the Laplace transform to compute the waiting time distribution. In addition, in most cases, even the Laplace transform or generating function are not available in closed form (that is the case, for example, in the $G/G/1$ queueing systems). For this reason, there exists, when a practical study is performed in queueing theory, a common technique for substituting the real but complicated elements governing a queueing system by simpler ones in some sense close to the real elements. The queueing model so constructed represents an idealization of the real queueing one, and hence the "stability" problem arises. The stability problem in queueing theory is concerned with the "domain" within which the ideal queueing model may be taken as a good approximation of the real queueing system under consideration.

Elaborated upon at the start of the eighties [2, 5], the strong stability method (also called the "method of operators") can be used to investigate the ergodicity and stability of the stationary and non-stationary characteristics of the imbedded Markov chains, [1, 6]. In contrast to other methods, they suppose that the perturbations of the transition kernel are small with respect to some norms in the operator space. This stringent condition gives better stability estimates and enables us to find precise asymptotic expansions of the characteristics of the "perturbed" system.

The applicability of this method for studying the stability of queueing systems has been known since 1982 [3]. Nevertheless, this applicability is not obvious, particularly for complex queueing systems. The difficulties reside in the identification of the perturbed parameter, writing the transition kernel and especially in the choice of weight

norms. Otherwise, the complexity of some systems forces us to perform some intermediate searches, which have a particular interest (see §3).

In this paper, we study the strong stability of the stationary distribution of the imbedded Markov chain in the $G/M/1$ queueing system, after perturbation of the service law. We show that under some hypotheses, the characteristics of the $G/G/1$ queueing system can be approximated by the corresponding characteristics of the $G/M/1$ system. Note that, unlike [3] where the perturbation concerns the arrival flux, we perturb here the service duration. After clarifying the approximation conditions, we obtain the stability inequalities with exact computation of the constants.

## 1. PRELIMINARIES AND NOTATION

Consider a $G/G/1$ queueing system with a general service times distribution $G$ and a general inter-arrival times probability distribution $F$. The following notation is used: $\theta_n$ (the arrival time of the $n$th demand), $\omega_n$ (the departure time of the $n$th customer), $\gamma_n$ (the time interval from $\theta_n$ to the departure of the next customer) and $V_n = V(\theta_n - 0)$ (the number of customers found in the system immediately prior to $\theta_n$).

Denote by $\nu_{\theta_n} = \min\{m > 0, \omega_m \geq \theta_n\}$. Then, $\gamma_n = \omega_{\nu_{\theta_n}} - \theta_n$. Define recursively the following sequence:

$$
(1) \qquad \begin{cases} T_0 = \omega_{\nu_{\theta_n}} - (\theta_n + \gamma_n) = 0, \\ T_k = T_{k-1} + \xi_{\nu_{\theta_n}+k}, & \text{for all } k > 0. \end{cases}
$$

The sequence $\{T_k\}_{k \in \mathbf{N}}$ describes the departure process after $\theta_n$.

Let's also consider a $G/M/1$ system with exponentially distributed service times with parameter $\mu$ and with the same distribution of the arrival flux than the $G/G/1$ one. We introduce the following corresponding notation: $\bar{\theta}_n$, $\bar{\omega}_n$, $\bar{\gamma}_n$, and $\bar{V}_n = \bar{V}(\bar{\theta}_n - 0)$ defined as above. We also define the process $\{\bar{T}_n\}_{n \in \mathbf{N}}$ as the sequence $\{T_n\}$.

In the sequel, when no domain of integration is indicated, an integral is extended over $\mathbf{R}^+$.

Consider the $\sigma$-algebra $\mathcal{E}$ that represents the product $\mathcal{E}_1 \otimes \mathcal{E}_2$ ($\mathcal{E}_1$ is the $\sigma$-algebra generated by the countable partition of $\mathbf{N}$, and $\mathcal{E}_2$ is the Borel $\sigma$-algebra of $\mathbf{R}^+$).

We introduce in the space $m\mathcal{E}$ of finite measures on $\mathcal{E}$ the special family of norms

$$
(2) \qquad \|m\|_v = \sum_{j \geq 0} \int v(j, y) |m_j|(dy) \quad \text{for all } m \in m\mathcal{E},
$$

where $v$ is a measurable function on $\mathbf{N} \times \mathbf{R}^+$, bounded below away from zero (not necessarily finite).

This norm induces a corresponding norm in the space $f\mathcal{E}$ of bounded measurable functions on $\mathbf{N} \times \mathbf{R}^+$, namely,

$$
(3) \qquad \|f\|_v = \sup_{k \geq 0} \sup_{x \geq 0} [v(k, x)]^{-1} |f(k, x)| \quad \text{for all } f \in f\mathcal{E}
$$

as well as a norm in the space of linear operators, namely,

$$
(4) \qquad \|P\|_v = \sup_{k \geq 0} \sup_{x \geq 0} [v(k, x)]^{-1} \sum_{j \geq 0} \int v(j, y) |P_{kj}(x, dy)|.
$$

We associate to each transition kernel $P$ the linear mapping $P \colon f\mathcal{E} \to f\mathcal{E}$ acting on $f \in f\mathcal{E}$ as follows:

$$
(5) \qquad (Pf)(k, x) = \sum_{j \geq 0} \int P_{kj}(x, dy) f(j, y).
$$

For $m \in m\mathcal{E}$ and $f \in f\mathcal{E}$ the symbol $mf$ denotes the integral

$$(6) \qquad mf = \sum_{j \geq 0} \int m_j(dx) f(j, x)$$

and $f \circ m$ denotes the transition kernel having the form

$$(7) \qquad (f \circ m)_{ij}(x, A) = f(i, x) m_j(A).$$

**Fundamental results.** We shall use in the sequel (§2 and §4), the following results (see [2] and [5]).

**Theorem 1.1.** *The Markov chain $X$ having a transition kernel $P$, such that*

$$\|P\|_v < +\infty,$$

*is strongly $v$-stable if the following conditions are satisfied.*

    a) *There exist a nonnegative measure $\alpha \in m\mathcal{E}$ ($\|\alpha\|_v < +\infty$) and a nonnegative measurable function $h \in f\mathcal{E}$ such that $\pi h > 0$, $\alpha \mathbb{1} = 1$, $\alpha h > 0$.*

    b) *The kernel $T = P - h \circ \alpha$ is nonnegative.*

    c) *There exists $\rho < 1$ such that $Tv(k, x) \leq \rho v(k, x)$ for all $(k, x) \in \mathbf{N} \times \mathbf{R}^+$.*

    Here $\mathbb{1} \in f\mathcal{E}$ is the function identically equal to 1.

**Theorem 1.2** ([5]). *Let $X$ be a Markov chain having a transition kernel $P$ and an invariant probability measure $\pi$. Suppose that $X$ is strongly stable and that the conditions of Theorem 1.1 hold. Then, for a stochastic kernel $Q$ with an invariant measure $\nu$ such that $\|Q - P\|_v$ is sufficiently small, the following equality is true:*

$$\nu = \pi[I - \Delta R_0(I - \Pi)]^{-1} = \pi + \sum_{i \geq 1} \pi[\Delta R_0(I - \Pi)]^i,$$

*where $\Delta = Q - P$, $R_0 = (I - T)^{-1}$, $\Pi = \pi \circ \mathbb{1}$ and $I$ is the identity operator on $m\mathcal{E}$. The operator $T$ and the function $\mathbb{1}$ are defined above.*

**Corollary 1.1.** *Given the conditions of Theorem 1.2 and for $\|\Delta\|_v < C^{-1}(1 - \rho)$, we have the following estimation:*

$$(8) \qquad \|\nu - \pi\|_v \leq \|\Delta\|_v \|\pi\|_v C(1 - \rho - C\|\Delta\|_v)^{-1},$$

*where $C = 1 + \|\mathbb{1}\|_v \|\pi\|_v$.*

## 2. Strong stability in the $G/M/1$ queueing system

We first write the transition kernels of the considered Markov chains. Then we apply Theorem 1.1 to establish the strong stability conditions in the $G/M/1$ queueing system.

**Lemma 2.1.** *The sequence $X_n = (V_n, \gamma_n)$ forms a homogeneous Markov chain with state space $\mathbf{N} \times \mathbf{R}^+$ and transition operator $Q = \|Q_{ij}\|_{i,j \geq 0}$, defined by*

$$Q_{ij}(x, dy) = \mathsf{P}\left(V_{n+1} = j, \gamma_{n+1} \in dy \ / \ V_n = i, \gamma_n = x\right)$$

$$= \begin{cases} q_{i-j}(x, dy), & \text{for } 1 \leq j \leq i, \ i \geq 1, \\ \sum_{k \geq i} q_k(x, dy), & \text{for } j = 0, \ i \geq 0, \\ p(x, dy), & \text{for } j = i + 1, \ i \geq 0, \\ 0, & \text{for } j > i + 1, \ i \geq 0, \end{cases}$$

*where*

$$(9) \qquad \begin{cases} q_k(x, dy) = \int_x^\infty \mathsf{P}(T_k \leq u - x < T_{k+1}, T_{k+1} - (u - x) \in dy) \, dF(u), \\ p(x, dy) = \int_0^x \mathsf{P}(x - u \in dy) \, dF(u). \end{cases}$$

**Lemma 2.2.** *The sequence $\bar{X}_n = (\bar{V}_n, \bar{\gamma}_n)$ forms a homogeneous Markov chain with state space $\mathbf{N} \times \mathbf{R}^+$ and transition operator $\bar{Q} = \|\bar{Q}_{ij}\|_{i,j \geq 0}$, having the same form as $Q$ (Lemma 2.1), where*

$$(10) \qquad \bar{q}_k(x) = \int_x^\infty e^{-\mu(u-x)} \frac{[\mu(u-x)]^k}{k!} \, dF(u).$$

*Remark* 2.1. The assumption $\bar{\tau}\mu > 1$, where $\bar{\tau}$ is a mean time between arrivals in the $G/M/1$ queueing system, implies the existence of a stationary distribution $\bar{\pi}$ for the imbedded Markov chain $\bar{X}_n$ in the $G/M/1$ system. This distribution has the following form:

$$(11) \qquad \bar{\pi}(\{k\}, A) = \bar{\pi}_k(A) = p_k \mathsf{E}(A) \quad \text{for all } \{k\} \subset \mathbf{N} \text{ and } A \subset \mathbf{R}^+,$$

where $p_k = \lim_{n \to \infty} \mathsf{P}(\bar{V}_n = k)$ is given by the following relation:

$$(12) \qquad p_k = (1 - \sigma)\sigma^k, \qquad k = 0, 1, 2, \ldots,$$

$\sigma$ is the unique solution of the equation

$$(13) \qquad \sigma = F^*(\mu - \mu\sigma) = \int_0^\infty e^{-(\mu - \mu\sigma)x} \, dF(x),$$

and $F^*$ is the Laplace transform of the probability density function of the demands' inter-arrival times. We can show that $0 < \sigma < 1$ [7]. Otherwise, note that

$$(14) \qquad \begin{aligned} \lim_{t \to \infty} \mathsf{P}(X(t) = k) &= \frac{1}{\bar{\tau}\mu} p_{k-1}, \qquad k = 1, 2, \ldots, \\ \lim_{t \to \infty} \mathsf{P}(X(t) = 0) &= 1 - \frac{1}{\bar{\tau}\mu}, \end{aligned}$$

where $X(t)$ represents the size of the $G/M/1$ system at time $t$.

Formulas (12) and (14) permit us to compute the stationary distribution of the queue length in a $G/M/1$ system. Unfortunately, for the $G/G/1$ system, these exact formulas are not known. So, if we suppose that the $G/G/1$ system is close to the $G/M/1$ system, then we can use formulas (12) and (14) to approximate the $G/G/1$ system characteristics with prior estimation of the corresponding approximation error.

Suppose that the service law of the $G/G/1$ system is close to the exponential one with parameter $\mu$. This proximity is characterized by the distance of variation,

$$(15) \qquad W^* = W^*(G, E) = \int e^{\delta t} |G - E|(dt), \quad \text{where } \delta > 0.$$

Let us also consider the following deviation:

$$(16) \qquad W_0 = W_0(G, E) = \int |G - E|(dt).$$

We apply Theorem 1.1 to the imbedded Markov chain $\bar{X}_n$ (defined in Lemma 2.2). Consider the test function

$$(17) \qquad \begin{aligned} v \colon \mathbf{N} \times \mathbf{R}^+ &\to \mathbf{R}^+, \\ (k, x) &\mapsto v(k, x) = \beta^k e^{\delta x}, \end{aligned}$$

where $1 < \beta < 1/\sigma$ and $0 < \delta = \mu - \mu/\beta < \mu$ ($\sigma$ is given by relation (13)).

Let $\alpha$ be a measure defined as follows: for $\{j\} \times dy \in \mathcal{E}$, we have

$$(18) \qquad \alpha(\{j\} \times dy) = \alpha_j(dy) = \begin{cases} E(dy), & \text{for } j = 0, \\ 0, & \text{for } j \neq 0, \end{cases}$$

and the measurable function

$$h \colon \mathbf{N} \times \mathbf{R}^+ \to \mathbf{R},$$

(19)
$$(i, x) \quad \mapsto h_i(x) = h(i, x) = \sum_{k \geq i} \bar{q}_k(x),$$

where $\bar{q}_k(x)$ is defined by relation (10).

**Lemma 2.3.** *Let $\bar{X}_n$ be the Markov chain defined in Lemma 2.2. Then the operator $T = \|T_{ij}(x, dy)\|_{i,j \geq 0}$ is nonnegative.*

*Proof.* In fact, it is easily seen that

$$T_{ij}(x, dy) = \begin{cases} 0, & \text{for } j = 0, \\ \bar{Q}_{ij}(x, dy), & \text{for } j > 0; \end{cases}$$

hence the result. $\qquad \square$

**Lemma 2.4.** *For all $x > 0$, for all $\beta > 1$, and for $\delta = \mu - \mu/\beta > 0$ the inequality*

(20)
$$\int_0^x dF(u) \int_0^x e^{\delta y} \mathsf{P}(x - u \in dy) \leq \int_0^x e^{\delta(u-x)} \, dF(u)$$

*holds.*

*Proof.* It is sufficient to notice that

$$\int_0^x dF(u) \int_0^x e^{\delta y} \mathsf{P}(x - u \in dy) = \int_0^x dF(u) \int_0^x e^{\delta y} \mathsf{P}(y < x - u \leq y + dy)$$
$$\leq \int_0^x e^{\delta(x-u)} \, dF(u). \qquad \square$$

**Lemma 2.5.** *Suppose that in the $G/M/1$ system the following geometric ergodicity condition*

(21)
$$\mu \bar{\tau} > 1$$

*holds. Then for all $\beta \in \mathbf{R}$ such that $1 < \beta < 1/\sigma$, the inequality*

(22)
$$\beta F^* \left( \mu - \frac{\mu}{\beta} \right) < 1$$

*is true, where $\sigma$ and $F^*$ have been defined in relation (13).*

*Proof.* Let us consider the function

$$\psi \colon [1, 1/\sigma] \to \mathbf{R}^+,$$

(23)
$$\beta \quad \mapsto \beta F^* \left( \mu - \frac{\mu}{\beta} \right).$$

From the convexity of $\psi$ and from relation (13), we have the result. $\qquad \square$

**Lemma 2.6.** *For a function $v$ such that $v(k, x) = \beta^k e^{\delta x}$, with $1 < \beta < 1/\sigma$ and $\delta = \mu(1 - 1/\beta) > 0$, the inequality*

(24)
$$(Tv)(k, x) \leq \rho v(k, x)$$

*holds, where $\rho = \beta F^*(\mu - \mu/\beta) < 1$.*

*Proof.* From relation (5) and Lemmas 2.1, 2.2, and 2.3, we have

$$(Tv)(k, x) = \sum_{j \geq 0} \int T_{kj}(x, dy) v(j, y) = \sum_{j > 0} \int \bar{Q}_{kj}(x, dy) v(j, y)$$

$$\leq \beta^{k+1} \left[ \int_x^\infty e^{-\mu(u-x)} e^{(\mu/\beta)(u-x)} \, dF(u) + \int_0^x e^{\delta(x-u)} \, dF(u) \right]$$

$$\leq \beta^k e^{\delta x} \beta \int_0^\infty e^{-\delta u} \, dF(u),$$

and from Lemma 2.5, we obtain the result.                                    □

**Lemma 2.7.** *Let $\bar{Q}$ be the transition kernel of the imbedded Markov chain $\bar{X}_n$ in the $G/M/1$ system. Then $\|\bar{Q}\|_v < \infty$.*

*Proof.* From relations (5) and (4), the proof can be easily established by Lemmas 2.3 and 2.6.                                    □

All the conditions of Theorem 1.1 are satisfied; hence we can state the following result.

**Theorem 2.1.** *Suppose that in the $G/M/1$ system, the geometric ergodicity condition (21) holds. Then for all $\beta \in \mathbf{R}^+$ such that $1 < \beta < \sigma^{-1}$ the Markov chain $\bar{X}_n$ is strongly v-stable for a function $v(k, x) = \beta^k e^{\delta x}$, where $0 < \delta = \mu - \mu/\beta < \mu$ and $\rho = \beta F^*(\mu - \mu/\beta) < 1$.*

## 3. Quantitative estimation of the deviation norm of the transition operator in the $G/M/1$ system

Introduce the following notation:

$$(25) \qquad \psi_k(x, A) = \mathsf{P}\big(T_k < x \leq T_{k+1}, T_{k+1} - x \in A\big),$$

$$(26) \qquad \bar{\psi}_k(x, A) = \mathsf{P}\big(\bar{T}_k < x \leq \bar{T}_{k+1}, \bar{T}_{k+1} - x \in A\big)$$

and

$$(27) \qquad \Delta_k(x, A) = \psi_k(x, A) - \bar{\psi}_k(x, A).$$

Note that the functions introduced in relations (9) and (10) can be written

$$(28) \qquad \begin{cases} q_k(x, dy) = \int_x^\infty \psi_k(u - x, dy) \, dF(u), \\ \bar{q}_k(x, dy) = \int_x^\infty \bar{\psi}_k(u - x, dy) \, dF(u). \end{cases}$$

For a nonnegative and bounded measurable function $f$ taking values in $\mathbf{R}^+$ and for a measure $m$ on $\mathcal{E}_2$ ($\sigma$-algebra of Borel sets of $\mathbf{R}^+$), we shall denote by $f \star m$ the convolution product of $f$ and $m$. We shall use the same notation for the convolution product of two measures on $\mathcal{E}$.

**Lemma 3.1.** *Consider in the $G/G/1$ queueing system the sequence of random variables $\{\xi_n\}_{n \geq 1}$, where $\xi_n$ represents the service duration of the $n$th customer. Then the following equality*

$$\psi_k(x, A) = \int_0^x \psi_{k-1}(x - s, A) \, dG(s) = \psi_{k-1} \star G(x, A)$$

*holds for all $k \in \mathbf{N}^*$, for all $x \in \mathbf{R}^*$, and for all $A \subset \mathbf{R}^+$.*

*Proof.* The result can be easily obtained by using relation (1) and the convolution product properties.                                    □

**Lemma 3.2.** *Let $Q$ and $\bar{Q}$ be the transition kernels of the imbedded Markov chains in the $G/G/1$ and $G/M/1$ systems, respectively. Then for all $j \in \mathbf{N}^*$, for all $x > 0$, and for all $A \subset \mathbf{R}^+$ we have that*

$$\Delta_j(x, A) = \sum_{i=0}^{j-1} \psi_0 \star G^{\star i} \star (G - E) \star E^{\star(j-i-1)}(x, A) + \Delta_0 \star E^{\star j}(x, A).$$

*Proof.* It is sufficient to note that

$$\Delta_j(x, A) = \sum_{i \leq j-1} \psi_i \star (G - E) \star E^{\star(j-i-1)}(x, A) + \Delta_0 \star E^{\star j}(x, A).$$

The result is easily deduced by using Lemma 3.1. $\qquad\square$

Introduce now the measure

$$K(ds) = \sum_{j \geq 0} E^{\star j}(ds).$$

**Lemma 3.3.** *Let $\bar{Q}$ be the transition kernel of the imbedded Markov chain in the $G/M/1$ system. Then*

$$K(ds) = \delta_0(ds) + \mu \, ds,$$

*where $\delta_0(ds) = E^{\star 0}(ds)$ (Dirac's measure in $0$).*

*Proof.* It is sufficient to remark that $E^{\star j}(ds)$ is the probability density function of a sum of $j$ independent and identically exponentially distributed random variables with the same parameter $\mu$. Then, it is the Erlang probability density function with parameters $\mu$ and $j$. $\qquad\square$

Let us make the following notation:

$$A_2 = \sup_{x \geq 0} \left\{ e^{-\delta x} \sum_{j \geq 0} \int e^{\delta y} \left| \int_x^\infty dF(u) \Delta_0 \star E^{\star j}(u - x, dy) \right| \right\},$$

where $\delta$ is given in Lemma 2.4.

**Lemma 3.4.** *Let $\bar{Q}$ and $Q$ be the transition operators of the imbedded Markov chains in the $G/M/1$ and $G/G/1$ system, respectively. Let $\bar{\tau}$ be the mean inter-arrival time. Then the inequality*

$$A_2 \leq W^*(1 + \mu \bar{\tau})$$

*holds, where $W^*$ is given by relation (15).*

*Proof.* Taking into account that

$$\Delta_0 \star E^{\star j}(x, A) = \int_0^x \Delta_0(x - s, A) E^{\star j}(ds),$$

we easily show that

$$A_2 \leq \sup_{x \geq 0} \left\{ e^{-\delta x} \int |G - E|(dt) \int_x^\infty dF(u) \sum_{j \geq 0} \int_0^{u-x} E^{\star j}(ds) \int e^{\delta y} \mathbb{1}_{\{t+x+s-u \in dy\}} \right\},$$

and from Lemma 3.3 we obtain the result. $\qquad\square$

Introduce the following measure:

$$R(ds) = \sum_{j \geq 0} G^{\star j}(ds).$$

**Lemma 3.5.** *Let $Q$ and $\bar{Q}$ be the transition kernels of the imbedded Markov chains in the $G/G/1$ and $G/M/1$ systems (respectively). Suppose that the common probability distribution function $F$ of the demands' inter-arrival times satisfies the Cramér condition*

$$(29) \qquad \text{there exists } a > 0 \text{ such that} \quad \int e^{at}\, dF(t) = N < +\infty$$

*and*

$$(30) \qquad W_0 = \int |G - E|(dt) < \frac{a}{a+\mu} < 1.$$

*Denote*

$$g(u) = \int_0^u e^{-as} R(ds).$$

*Then the inequality*

$$(31) \qquad g(u) < \frac{1}{1 - C_0}, \quad \text{where } C_0 = W_0 + \frac{\mu}{a+\mu},$$

*holds for all $u \in \mathbf{R}^+$.*

*Proof.* We can show by induction that

$$M^{\star k}(ds) = e^{-as} G^{\star k}(ds) \quad \text{for all } k > 0,$$

where

$$M(ds) = e^{-as}\, G(ds).$$

Taking into account

$$g(u) = \sum_{j \geq 0} \int_0^u M^{\star j}(ds),$$

we conclude that

$$\int g(ds) = \sum_{j \geq 0} \int M^{\star j}(ds).$$

Using the Laplace transform properties and condition (30), we obtain the result. $\qquad \square$

**Lemma 3.6.** *Under the conditions of Lemma 3.5, the inequality*

$$\int dF(u) \int_0^u R(ds) \leq \frac{N}{1 - C_0}$$

*holds, where $N = \int e^{at}\, dF(t) < \infty$.*

*Proof.* The result can be easily deduced from Lemma 3.5. $\qquad \square$

**Lemma 3.7.** *Assuming that the conditions of Lemma 3.5 are fulfilled, we have*

$$\int dF(u) \int_0^u K \star R(ds) \leq \frac{N + \mu M}{1 - C_0},$$

*where*

$$M = \int u e^{au}\, dF(u) < +\infty.$$

*Proof.* Using Lemmas 3.3 and 3.6, we can show that

$$\int dF(u) \int_0^u K \star R(ds) \leq \frac{N}{1 - C_0} + \mu \int u\, dF(u) \int_0^u R(ds)$$

and from Lemma 3.5 we obtain the following inequality:

$$\int u\, dF(u) \int_0^u R(ds) \leq \frac{M}{1 - C_0}.$$

From condition (29), we obtain the desired result. $\qquad \square$

Put

$$A_1 = \sup_{x \geq 0} \left\{ e^{-\delta x} \int_x^\infty dF(u) \int_0^{u-x} |G-E| \star R \star K(dt) \int e^{\delta y} \psi_0(u-x-t, dy) \right\}.$$

**Lemma 3.8.** *Under the conditions of Lemma 3.5 and supposing that*

$$G^* = G^*(-\delta) = \int e^{\delta s} \, dG(s) < +\infty,$$

*we have that the inequality*

$$A_1 \leq G^* W_0 \frac{N + \mu M}{1 - C_0}$$

*holds.*

*Proof.* From Lemma 3.4, we obtain

$$\int e^{\delta y} \psi_0(u-x-t, dy) \leq \int e^{\delta s} \, G(ds).$$

Substituting in the expression of $A_1$, we obtain the result. $\qquad \square$

From the previous lemmas, we can state the following result, which gives us the quantitative estimation of the transition operators' deviation norm in the $G/M/1$ system after perturbation of the service duration.

**Theorem 3.1.** *Let $Q$ and $\bar{Q}$ be the transition kernels of the imbedded Markov chains in the $G/G/1$ and $G/M/1$ systems, respectively. Suppose that for each $\beta$, $1 < \beta < 1/\sigma$, the following conditions hold:*

(1) $G^* = \int e^{\delta t} G(dt) < +\infty$;
(2) *there exists $a > 0$ such that*

$$\int e^{au} \, dF(u) = N < +\infty;$$

(3) $W_0 = \int |G - E|(dt) < a/(a+\mu)$;
(4) *the geometric ergodicity condition* (21) *holds.*

*Then the following inequality holds:*

$$\|Q - \bar{Q}\|_v \leq W^*(1 + \mu\bar{\tau}) + W_0 G^* \frac{N + \mu M}{1 - C_0},$$

*where $C_0 = W_0 + \mu/(a+\mu) < 1$ and*

$$M = \int u e^{au} \, dF(u) < +\infty.$$

*Proof.* From relation (4) we have

$$\|Q - \bar{Q}\|_v = \sup_{k \geq 0} \sup_{x \geq 0} \left\{ e^{-\delta x} \beta^{-k} \sum_{j \geq 0} \int \beta^j e^{\delta y} |Q_{kj}(x, dy) - \bar{Q}_{kj}(x, dy)| \right\}.$$

Make the following notation:

$$S_k(x) = \sum_{j \geq 0} \int \beta^j e^{\delta y} |Q_{kj}(x, dy) - \bar{Q}_{kj}(x, dy)|.$$

*Case one:* $k = 0$. Using Lemmas 2.1 and 2.2, we obtain

$$S_0(x) \leq \sum_{j \geq 0} \int e^{\delta y} \left| \int_x^\infty \Delta_j(u - x, dy) \, dF(u) \right|;$$

therefore,

$$(32) \qquad \sup_{x \geq 0} \left\{ e^{-\delta x} S_0(x) \right\} \leq \sup_{x \geq 0} \left\{ e^{-\delta x} \sum_{j \geq 0} \int e^{\delta y} \left| \int_x^\infty \Delta_j(u - x, dy) \, dF(u) \right| \right\}.$$

*Case two:* $k \neq 0$. From Lemmas 2.1 and 2.2, we obtain

$$S_k(x) \leq \beta^k \sum_{j \geq 0} \int e^{\delta y} |(q_j(x, dy) - \bar{q}_j(x, dy))|;$$

therefore,

$$(33) \qquad \sup_{x \geq 0} \left\{ e^{-\delta x} \beta^{-k} S_k(x) \right\} \leq \sup_{x \geq 0} \left\{ e^{-\delta x} \sum_{j \geq 0} \int e^{\delta y} \left| \int_x^\infty \Delta_j(u - x, dy) \, dF(u) \right| \right\}.$$

From equations (32) and (33) we conclude that

$$\|Q - \bar{Q}\|_v \leq \sup_{x \geq 0} \left\{ e^{-\delta x} \sum_{j \geq 0} \int e^{\delta y} \left| \int_x^\infty \Delta_j(u - x, dy) \, dF(u) \right| \right\}.$$

Using Lemma 3.2, we obtain

$$\|Q - \bar{Q}\|_v \leq A_2 + A_1,$$

and using Lemmas 3.4 and 3.8, we easily obtain the result. $\qquad \square$

## 4. Stability inequality in a $G/M/1$ queueing system

Consider the imbedded Markov chains $X_n$ and $\bar{X}_n$ in the $G/G/1$ and $G/M/1$ queueing systems, respectively. Their transition kernels $Q$ and $\bar{Q}$ are given by Lemmas 2.1 and 2.2. Let $\pi$ and $\bar{\pi}$ be their stationary probabilities. Note that $\bar{\pi}$ is given by the relation (11). As was shown in Theorem 2.1, $\bar{X}_n$ is strongly $v$-stable. Now, we apply Corollary 1.1.

*Estimation of* $\|\bar{\pi}\|_v$. From the relation (2) we have

$$\|\bar{\pi}\|_v = \sum_{j \geq 0} \beta^j \int e^{\delta y} |\bar{\pi}_j|(dy)$$

and conclude

$$(34) \qquad \|\bar{\pi}\|_v = \frac{\beta(1 - \sigma)}{1 - \beta \sigma}.$$

*Estimation of* $\|\mathbb{1}\|_v$. From the equation (3) we have

$$(35) \qquad \|\mathbb{1}\|_v = \sup_{x \geq 0} \sup_{k \geq 0} \beta^{-k} e^{-\delta x} = 1$$

because of $\beta > 1$ and $\delta > 0$. Then $C = 1 + \|\bar{\pi}\|_v = 1 + \beta(1 - \sigma)/(1 - \beta \sigma)$. Therefore

$$(36) \qquad C = 1 + \|\bar{\pi}\|_v = \frac{1 + \beta(1 - 2\sigma)}{1 - \beta \sigma}.$$

The results of Corollary 1.1 are obtained under the condition $\|\Delta\|_v < C^{-1}(1 - \rho)$. We impose the following condition:

$$(37) \qquad \|\Delta\|_v < \frac{1 - \rho}{2C} < \frac{1 - \rho}{C}.$$

This inequality is satisfied when the following condition is fulfilled:

$$(38) \qquad W^* < \frac{1-\rho}{2C(1+\mu\bar{\tau}+C_1)} \quad \text{and} \quad W_0 < \frac{a}{a+\mu},$$

where

$$(39) \qquad C_1 = \frac{N+\mu M}{1-C_0}G^*.$$

In fact, from Theorem 3.1 we have

$$\|\Delta\|_v < (1+\mu\bar{\tau})W^* + W_0 C_1 < [(1+\mu\bar{\tau})+C_1]W^*$$
$$< \frac{1-\rho}{2C(1+\mu\bar{\tau}+C_1)}(1+\mu\bar{\tau}+C_1)$$
$$< \frac{1-\rho}{2C}.$$

**Theorem 4.1.** *Let $X_n$ and $\bar{X}_n$ be the imbedded Markov chains in the $G/G/1$ and $G/M/1$ systems, respectively (they are given by Lemmas 2.1 and 2.2). Let $\pi$ and $\bar{\pi}$ be their stationary probabilities. If*

$$W^* = W^*(G,E) < \frac{1-\rho}{2C(1+\mu\bar{\tau}+C_1)}$$

*and if*

$$W_0 < \frac{a}{a+\mu},$$

*then the inequality*

$$\|\pi - \bar{\pi}\|_v \le 2[(1+\mu\bar{\tau})W^* + C_1 W_0]\frac{C(C-1)}{1-\rho}$$

*holds.*

*Proof.* From Corollary 1.1 and Theorem 3.1, we obtain

$$\|\pi - \bar{\pi}\|_v \le [(1+\mu\bar{\tau})W^* + C_1 W_0]C\|\bar{\pi}\|_v(1-\rho-C\|\Delta\|_v)^{-1}.$$

Using the relation (36) we have $\|\bar{\pi}\|_v = C-1$ and from the condition (37) we obtain

$$(1-\rho-C\|\Delta\|_v)^{-1} < \frac{2}{1-\rho}.$$

Therefore

$$\|\pi - \bar{\pi}\|_v \le 2[(1+\mu\bar{\tau})W^* + C_1 W_0]\frac{C(C-1)}{1-\rho}. \quad \square$$

## 5. Conclusion

To "measure" the performances of the strong stability method in a $G/M/1$ queueing system, after disrupting the service duration, we can apply a general approach based on discrete-event simulation [4]. In fact, we note that practically, for a low margin ($W^*$) between the service laws of the $G/G/1$ and $G/M/1$ queueing systems, it is possible to approximate the $G/G/1$ system's characteristics by the corresponding ones of the $G/M/1$ system. In addition, this approximation is very precise when the distance of variation $W^*$ is near to zero (recall that $0 < W_0 < W^*$).

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