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STRONG STABILITY OF NEUTRAL EQUATIONS WITH AN ARBITRARY DELAY DEPENDENCY STRUCTURE*

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Abstract. The stability theory for linear neutral equations subjected to delay perturbations is addressed. It is assumed that the delays cannot necessarily vary independently of each other, but depend on a possibly smaller number of independent parameters. As a main result, necessary and sufficient conditions for strong stability are derived along with bounds on the spectrum, which take into account the precise dependency structure of the delays. In the derivation of the stability theory, results from realization theory and determinantal representations of multivariable polynomials play an important role. The observations and results obtained in the paper are first illustrated and validated with a numerical example. Next, the effects of small feedback delays on the stability of a boundary controlled hyperbolic partial differential equation and of a control system involving state derivative feedback are analyzed.

Notation.

$\mathbb{C}$	set of complex numbers
$\mathbb{C}^-$, $\mathbb{C}^+$	open left half plane, open right half plane
i	imaginary identity
$\mathbb{N}$	set of natural numbers, including zero
$\mathbb{R}$	set of real numbers
$\mathbb{R}^+$	$\{r \in \mathbb{R} : \ r \geq 0\}$
$\mathbb{R}_0^+$	$\mathbb{R}^+ \setminus \{0\}$
$e_k \in \mathbb{N}^m$	kth unit vector in $\mathbb{N}^m$
$\Re(\lambda), \Im(\lambda), \lvert\lambda\rvert, \ \lambda \in \mathbb{C}$	real part, imaginary part, and modulus of λ
$\vec{r} \in \mathbb{R}^m, \vec{n} \in \mathbb{N}^m, \ldots$	short notation for $(r_1, \ldots, r_m)$, $(n_1, \ldots, n_m), \ldots$
$r_\sigma(A)$	spectral radius of operator (or matrix) A
$r_e(A)$	radius of the essential spectrum of operator A
$\sigma(A)$	spectrum of operator (or matrix) A
$\sigma_e(A)$	essential spectrum of operator (or matrix) A

$$\text{sign}(x),\ x \in \mathbb{R} \quad \text{sign}(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$$

$\mathbb{Z}$ set of integer numbers

$\alpha(A)$ spectral abscissa of operator (or matrix) A,
$\alpha(A) := \sup\{\Re(\lambda): \ \lambda \in \mathbb{C} \text{ and } \lambda \in \sigma(A)\}$

$\|\vec{a}\|,\ \vec{a} \in \mathbb{R}^m$ Euclidean norm of $\vec{a}$, $\|\vec{a}\| := \sqrt{\sum_{k=1}^m a_k^2}$

$\vec{a} \cdot \vec{b},\ \vec{a}, \vec{b} \in \mathbb{R}^m$ Euclidean inner product of $\vec{a}$ and $\vec{b}$, $\vec{a} \cdot \vec{b} := \sum_{k=1}^m a_k b_k$

1. Introduction. Many engineering systems can be modeled by delay differential equations of neutral type, for instance, lossless transmission lines [17] and partial element equivalent circuits [4] in electrical engineering, and combustion systems [26] and controlled constrained manipulators [27] in mechanical engineering. Equations of neutral type also arise in boundary-controlled hyperbolic partial differential equations subjected to small feedback delays [24, 6] and in implementation schemes of predictive controllers for time-delay systems [7, 25]. In this paper we discuss stability properties of the linear neutral equation

$$(1.1) \qquad \dot{x}(t) + \sum_{k=1}^{p_1} H_k \dot{x}(t - \tau_k) = A_0 x(t) + \sum_{k=1}^{p_2} A_k x(t - v_k),$$

where $x(t) \in \mathbb{R}^n$ is the state variable at time t, $\vec{\tau} := (\tau_1, \ldots, \tau_{p_1}) \in (\mathbb{R}_0^+)^{p_1}$ and $\vec{v} := (v_1, \ldots, v_{p_2}) \in (\mathbb{R}_0^+)^{p_2}$ are time-delays, and H_k and A_k are real matrices.

An important aspect in the stability theory of neutral equations is the possible fragility of stability, in the sense that the asymptotic stability of the null solution of (1.1) may be sensitive to *arbitrarily small perturbations* of the delays $\vec{\tau}$; see, e.g., [12, 21, 24, 18] and the references therein. This has led to the introduction of the notion of strong stability in [11, 13, 14], which explicitly takes into account the effect of small delay perturbations. In [13] a necessary and sufficient condition for the strong stability of the null solution of (1.1) is described for the special case where the delays $(\tau_1, \ldots, \tau_m)$ can vary independently of each other (see also [9]), and in [23] some related spectral properties are discussed, though the focus lies on a stabilization procedure for systems with an external input. Note that robustness against delay perturbations is of primary interest in control problems, as parametric uncertainty and feedback delays are inherent features of control systems.

In the existing literature on the stability of neutral equations, subjected to delay perturbations, the delays, τ_k, $1 \leq k \leq p_1$, in (1.1) are almost exclusively assumed to be either mutually independent or commensurate (all multiples of the same parameter); an exception is formed by [28] where a problem with three delays depending on two independent parameters is analyzed. In this paper we study the dependence of the stability properties of (1.1) on the delay parameters, under the assumption that the delays τ_k, $1 \leq k \leq p_1$, are linear functions of $m \geq 1$ "independent" parameters $\vec{r} = (r_1, \ldots, r_m) \in (\mathbb{R}_0^+)^m$, as described by the following relation:

$$(1.2) \qquad \tau_k = \vec{\gamma}_k \cdot \vec{r}, \quad k = 1, \ldots, p_1,$$

with

$$\vec{\gamma}_k := (\gamma_{k,1}, \ldots, \gamma_{k,m}) \in \mathbb{N}^m \setminus \{\vec{0}\}, \quad k = 1, \ldots, p_1.$$

Note that the cases of mutually independent delays, respectively, commensurate delays, appear in this framework as extreme cases ($m = n$ and $\vec{\gamma}_k = \vec{e}_k$, $k = 1, \ldots, p_1$,

respectively, $m = 1$). The problem studied in [28] corresponds to the relation $(\tau_1, \tau_2, \tau_3) = (r_1, r_2, r_1 + r_2)$, which is also of the form (1.2).

There are several main reasons why it is important to develop a stability theory where any delay dependency structure of the form (1.2) can be taken into account explicitly. First, real systems might give rise to a model of the form (1.1) exhibiting a delay dependency caused by physical or other interactions in the system's dynamics. This is explained with a lossless transmission line example in Chapter 9.6 of [11], where it is shown that a parallel transmission line which consists of a current source, two resistors, and a capacitor gives rise to a system of a neutral type with three delays in the difference part, which are integer combinations of two physical parameters. In [6, 19, 24] boundary-controlled partial differential equations are described that lead to a closed-loop system of neutral type, where the delays in the model are particular linear combinations of (physical) feedback delays and delays induced by propagation phenomena. In [31, 32] the robustness against small feedback delays of linear systems controlled with state derivative feedback is addressed, motivated by vibration control applications. There, the closed-loop system can again be written in the form (1.1), where the delays τ_k are combinations of actuator and sensor delays in input and output channels. All these applications give rise to a (nonextreme case of a) delay dependency of the form (1.2). Second, the precise dependency of the delays has a *major* influence on the stability robustness. For instance, we shall illustrate that the asymptotic stability of (1.1) may be destroyed by arbitrarily small perturbations of the delays τ_k, $1 \leq k \leq p_1$, if these perturbations can be chosen independently of each other, but it may be robust against small perturbations if the (perturbed) delays are restricted by a relation like (1.2). Third, the analysis for an arbitrary delay dependence of the form (1.2) is much more complex than the analysis of the special cases available in the literature (e.g., fully independent delays in [23]), where the derivation of the results heavily relies on specific properties induced by the special case. In this discussion it is worthwhile to note that no assumptions need to be made on the interdependency of the delays $\vec{\nu}$, because, as we shall see, this interdependency does not affect the stability robustness with respect to (w.r.t.) small delay perturbations, unlike the interdependency of the delays $\vec{r}$.

While the general aim of the paper is to develop a stability theory for neutral equations with dependent delays subjected to delay perturbations, the emphasis is on the derivation of explicit strong stability criteria and on related spectral properties. As we shall see, only in specific situations, where severe restrictions are put on the dependency structure, can the criteria available in the literature for independent delays be directly generalized, though the derivation is more complicated. To obtain a *general* solution and, in this way, complete the theory, some type of intermediate lifting step may be necessary, where a delay difference equation with dependent delays is transformed into an equation with independent delays with the same spectral properties. The main step will boil down to the representation of a multivariable polynomial as the determinant of a pencil. Such a representation will follow from arguments of realization theory, more precisely, from the construction of lower fractional representations (LFRs). See, for instance, [33] and the manual of the LFR toolbox [20] for an introduction.

Finally, we note that the strong stability criteria developed in this paper are also important in the context of stabilization and control of neutral systems. If the null solution of the associated difference equation is strongly stable, then the unstable manifold is finite-dimensional and remains so in the presence of delay perturbations.

This opens the possibilities of using controllers which act only on that manifold (see, e.g., [29]) or which are based on shifting or assigning a finite number of eigenvalues as [24]. On the contrary, if the difference equation is not strongly stable, then the closed-loop system lacks robustness against small delay perturbations. This may happen even if the application of the control law involves a noncompact perturbation of the solution operator and, thus, directly affects the difference equation; see [13] for an illustration.

The structure of the paper is as follows: in section 2 some basic notions and results on neutral equations are recalled, in support of the subsequent sections. In section 3 the spectral properties of the neutral equation (1.1)–(1.2) and of the associated delay difference equation are addressed, with the emphasis on stability properties and the sensitivity of stability w.r.t. delay perturbations. The main results are presented in section 4, where computational expressions are presented that lead to explicit strong stability conditions. Section 5 is devoted to applications and illustrations. Section 6 contains the conclusions.

2. Preliminaries. The initial condition for the neutral system (1.1)–(1.2) is a function segment $\varphi \in \mathcal{C}([-\bar{\tau},\ 0], \mathbb{R}^n)$, where $\bar{\tau} = \max_{k \in \{1,\dots,p_1\}} \tau_k$ and $\mathcal{C}([-\bar{\tau},\ 0],\ \mathbb{R}^n)$ is the Banach space of continuous functions mapping the interval $[-\bar{\tau},\ 0]$ into $\mathbb{R}^n$ and equipped with the supremum-norm. The fact that the map $\mathcal{D} : \mathcal{C}([-\bar{\tau},\ 0], \mathbb{R}^n) \to \mathbb{R}^n$, defined by

$$\mathcal{D}(\varphi) = \varphi(0) + \sum_{k=1}^{p_1} H_k \varphi(-\tau_k),$$

is atomic at zero guarantees existence and uniqueness of solutions of (1.1). Let $x(\varphi) :$ $t \in [-\bar{\tau},\ \infty) \to x(\varphi)(t) \in \mathbb{R}^n$ be the unique forward solution with initial condition $\varphi \in \mathcal{C}([-\bar{\tau},\ 0], \mathbb{R}^n)$, i.e., $x(\varphi)(\theta) = \varphi(\theta)$ for all $\theta \in [-\bar{\tau},\ 0]$. Then the state at time t is given by the function segment $x_t(\varphi) \in \mathcal{C}([-\bar{\tau},\ 0],\ \mathbb{R}^n)$ defined as $x_t(\varphi)(\theta) = x(\varphi)(t+\theta)$, $\theta \in [-\bar{\tau},\ 0]$. Denote by $\mathcal{T}(t; \vec{r}, \vec{v})$ the solution operator, mapping initial data onto the state at time t, i.e.,

$$(2.1) \qquad (\mathcal{T}(t;\ \vec{r}, \vec{v})\varphi)(\theta) = x_t(\varphi)(\theta) = x(\varphi)(t+\theta), \quad \theta \in [-\bar{\tau},\ 0].$$

This is a strongly continuous semigroup. The associated delay difference equation of (1.1) is given by

$$(2.2) \qquad z(t) + \sum_{k=1}^{p_1} H_k z(t - \vec{\gamma}_k \cdot \vec{r}) = 0.$$

For any initial condition $\varphi \in \mathcal{C}_D([-\bar{\tau},\ 0], \mathbb{R}^n)$, where

$$\mathcal{C}_D([-\bar{\tau},\ 0], \mathbb{R}^n) = \left\{ \varphi \in \mathcal{C}([-\bar{\tau},\ 0], \mathbb{R}^n) :\ \mathcal{D}(\varphi) = 0 \right\},$$

a solution $z(\varphi)(t)$ of (2.2) is uniquely defined and satisfies $z_t(\phi) \in \mathcal{C}_D([-\bar{\tau},\ 0], \mathbb{R}^n)$ for all $t \geq 0$. Let $\mathcal{T}_D(t;\ \vec{r})$ be the corresponding solution operator.

The asymptotic behavior of the solutions and, thus, the stability of the null solution of the neutral equation (1.1) is determined by the spectral radius $r_\sigma(\mathcal{T}(t;\ \vec{r}, \vec{v}))$, satisfying

$$r_\sigma(\mathcal{T}(1;\ \vec{r}, \vec{v})) = \epsilon^{c_N(\vec{r}, \vec{v})},$$

$$(2.3) \qquad c_N(\vec{r}, \vec{v}) = \sup\left\{ \Re(\lambda) :\ \det\left(\Delta_N(\lambda;\ \vec{r}, \vec{v})\right) = 0 \right\},$$

where the characteristic matrix Δ_N is given by

$$(2.4) \qquad \Delta_N(\lambda; \ \vec{r}, \vec{v}) = \left(\lambda \Delta_D(\lambda; \ \vec{r}) - A_0 - \sum_{k=1}^{p_2} A_k e^{-\lambda v_k} \right)$$

and

$$\Delta_D(\lambda; \ \vec{r}) = \left(I + \sum_{k=1}^{p_1} H_k e^{-\lambda \vec{\gamma}_k \cdot \vec{r}} \right).$$

For instance, the null solution is exponentially stable if and only if $r_\sigma(\mathcal{T}(1; \ \vec{r}, \vec{v})) < 1$ or equivalently $c_N(\vec{r}, \vec{v}) < 0$ [13, 12] (see [11] for an overview of stability definitions and their relation to spectral properties). In a similar way, the stability of the delay difference equation (2.2) is determined by the spectral radius

$$(2.5) \qquad r_\sigma(\mathcal{T}_D(1; \ \vec{r})) = e^{c_D(\vec{r})},$$

where

$$(2.6) \qquad c_D(\vec{r}) = \begin{cases} -\infty, & \det(\Delta_D(\lambda; \ \vec{r})) \neq 0 \ \forall \lambda \in \mathbb{C}, \\ \sup\{\Re(\lambda) : \ \det(\Delta_D(\lambda; \ \vec{r})) = 0\}, & \text{otherwise}. \end{cases}$$

An important property in the stability analysis of neutral equations is the relation

$$(2.7) \qquad r_e(\mathcal{T}(1; \ \vec{r}, \vec{v})) = r_\sigma(\mathcal{T}_D(1; \ \vec{r}));$$

see, e.g., [11, 10]. From this follows the well-known result that a *necessary condition* for the exponential stability of the null solution of (1.1)–(1.2) is given by the exponential stability of the null solution of the delay difference equation (2.2).

In the remainder of the paper we will call the solutions of $\det(\Delta_N(\lambda; \ \vec{r}, \vec{v})) = 0$ the characteristic roots of the neutral system (1.1). Analogously we will call the solutions of $\det(\Delta_D(\lambda; \ \vec{r})) = 0$ the characteristic roots of the delay difference equation (2.2).

3. Spectral properties. We discuss some spectral properties of the neutral equation (1.1) which are important for the rest of the paper. In section 3.1–3.2 we make the implicit assumption that

$$\exists \lambda \in \mathbb{C} : \ \det \Delta_D(\lambda; \ \vec{r}) \not\equiv 1.$$

The degenerate case where this condition is not met will be treated separately in section 3.3.

3.1. Difference equation. It is well known that the spectral radius (2.5), although continuous in the system matrices H_k, is *not* continuous in the *delays* $\vec{r}$ (see, e.g., [11, 13, 16, 23]), which carries over to (2.6). As a consequence, we are from a practical point of view led to the smallest upper bound on the real parts of the characteristic roots, which is "*insensitive*" to small delay changes.

DEFINITION 3.1. *For $\vec{r} \in (\mathbb{R}_0^+)^m$, let $\bar{C}_D(\vec{r}) \in \mathbb{R}$ be defined as*

$$\bar{C}_D(\vec{r}) = \lim_{\epsilon \to 0+} c_\epsilon(\vec{r}),$$

where

$$c_\epsilon(\vec{r}) = \sup\{c_D(\vec{r} + \delta\vec{r}) : \ \delta\vec{r} \in \mathbb{R}^m \ and \ \|\delta\vec{r}\| \leq \epsilon\}.$$

Clearly we have $\bar{C}_D(\vec{r}) \geq c_D(\vec{r})$, and the inequality can be *strict*, as shown in [23] and illustrated later on. We have the following results.

PROPOSITION 3.2. *The following assertions hold:*
1. *the function*

$$\vec{r} \in (\mathbb{R}_0^+)^m \mapsto \bar{C}_D(\vec{r})$$

   is continuous;
2. *for every $\vec{r} \in (\mathbb{R}_0^+)^m$, we have*[1]

(3.1)
$$\bar{C}_D(\vec{r}) = \max \left\{ c \in \mathbb{R} : \det \left(I + \sum_{k=1}^{p_1} H_k e^{-c\vec{\gamma}_k \cdot \vec{r}} e^{-i\vec{\gamma}_k \cdot \vec{\theta}} \right) = 0 \right.$$
$$\left. \text{for some } \vec{\theta} \in [0, \ 2\pi]^m \right\};$$

3. $\bar{C}_D(\vec{r}) = c_D(\vec{r})$ *for rationally independent*[2] $\vec{r}$;
4. *for all $\vec{r}_1, \vec{r}_2 \in (\mathbb{R}_0^+)^m$, we have*

(3.2)
$$\text{sign}\left(\bar{C}_D(\vec{r}_1)\right) = \text{sign}\left(\bar{C}_D(\vec{r}_2)\right).$$

Proof. Assertions 1 and 3 are direct corollaries of Lemma 2.5 and Theorem 2.2 of [3]. Combining assertion 3 with Theorem 3.1 of [3] yields assertion 2. The proof of assertion 4 is by contradiction. If (3.2) is not satisfied, then by assertion 1 there exists a vector $\vec{s} \in (\mathbb{R}_0^+)^m$ for which $\bar{C}_D(\vec{s}) = 0$. This implies by (3.1) that $\bar{C}_D(\vec{r}) \geq 0$ for all $\vec{r} \in (\mathbb{R}_0^+)^m$ and we arrive at a contradiction. ☐

The property (3.2) leads us to the following definition.

DEFINITION 3.3. *Let $\Xi := \text{sign}\left(\bar{C}_D(\vec{r})\right), \vec{r} \in (\mathbb{R}_0^+)^m$.*

A consequence of the *non*continuity of c_D w.r.t. $\vec{r}$ is that arbitrarily small perturbations on the delays may destroy stability of the delay difference equation. This phenomenon, which was illustrated in [24], has lead to the introduction of the concept of *strong stability* in [13]: we say that the null solution of (2.2) is strongly exponentially stable if it is exponentially stable and remains so when subjected to small variations in the delays $\vec{r}$. We state this more precisely in the following definition.

DEFINITION 3.4. *The null solution of the delay difference equation (2.2) is strongly exponentially stable if there exists a number $\hat{r} > 0$ such that the null solution of*

$$z(t) + \sum_{k=1}^{p_1} H_k z(t - \vec{\gamma}_k \cdot (\vec{r} + \delta\vec{r})) = 0$$

is exponentially stable for all $\delta\vec{r} \in (\mathbb{R}^+)^m$ satisfying $\|\delta\vec{r}\| < \hat{r}$ and $r_k + \delta r_k > 0$, $1 \leq k \leq m$.

The following condition follows from Proposition 3.2.

PROPOSITION 3.5. *The null solution of (2.2) is strongly exponentially stable if and only if $\Xi < 0$.*

Proof. By definition the null solution of (2.2) is strongly exponentially stable if and only if $\bar{C}_D(\vec{r}) < 0$, which is equivalent to $\Xi < 0$. ☐

Remark 3.6. The condition of Proposition 3.5 does not depend on the particular value of $\vec{r} \in (\mathbb{R}_0^+)^m$, that is, strong exponential stability for one value of $\vec{r}$ implies strong exponential stability for all values of $\vec{r}$.

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[1]The maximum in (3.1) is well defined because $\vec{\theta}$ belongs to a compact set.
[2]The m components of $\vec{r} = (r_1, \ldots, r_m)$ are rationally independent if and only if the conditions $\sum_{k=1}^m n_k r_k = 0$ and $n_k \in \mathbb{Z}$ imply $n_k = 0$ for all $k = 1, \ldots, m$. For instance, two delays r_1 and r_2 are rationally independent if their ratio is an irrational number.

3.2. Neutral equation. Following from (2.7), not only the delay difference equation (2.2) but also the neutral equation (1.1)–(1.2) have characteristic roots with real part *arbitrarily close* to $\bar{C}_D(\vec{r})$ for certain (arbitrarily small) perturbations on $\vec{r}$.

From the fact that the operator $\mathcal{T}(1; \vec{r}, \vec{v})$, defined in (2.1), has only a *point spectrum* in the set

$$\{\lambda \in \mathbb{C} : |\lambda| > r_e(\mathcal{T}(1; \vec{r}, \vec{v})) = r_\sigma(\mathcal{T}_D(1; \vec{r}))\}$$

(see [13]), it follows that all the characteristic roots of (1.1) in the half plane

$$\left\{\lambda \in \mathbb{C} : \Re(\lambda) \geq \bar{C}_D(\vec{r}) + \epsilon\right\},$$

where $\epsilon > 0$, lie in a *compact set* and that the number of these roots (multiplicity taken into account) is *finite*. Bounds on these roots can be obtained from the following lemma, whose proof can be found in Appendix A.

LEMMA 3.7. *If* $\Delta_N(\lambda; \vec{r}, \vec{v}) = 0$ *and* $\Re(\lambda) > \bar{C}_D(\vec{r})$, *then*

$$|\lambda| \leq \max_{\vec{\theta} \in [0, \, 2\pi]^m} \left\| \left(I + \sum_{k=1}^{p_1} H_k e^{-\Re(\lambda)(\vec{\gamma}_k \cdot \vec{r})} e^{-i\vec{\gamma}_k \cdot \vec{\theta}}\right)^{-1} \right\|$$

$$\left(\|A_0\| + \sum_{i=1}^{p_1} \|A_k\| e^{-\Re(\lambda)v_k}\right).$$

By combining the above results we arrive at the following result.

PROPOSITION 3.8. *The function*

$$(\vec{r}, \vec{v}) \in (\mathbb{R}_0^+)^m \times (\mathbb{R}^+)^{p_2} \mapsto \max(\bar{C}_D(\vec{r}), c_N(\vec{r}, \vec{v}))$$

is continuous.

We refer to Appendix B for a detailed proof.

Proposition 3.8 is an important result, given that the function $(\vec{r}, \vec{v}) \in (\mathbb{R}_0^+)^m \times (\mathbb{R}^+)^{p_2} \mapsto c_N(\vec{r}, \vec{v})$ is not continuous, with discontinuities occurring at delay values where $c_N(\vec{r}, \vec{v}) < \bar{C}_D(\vec{r})$. Such situations do occur and will be illustrated in the first example of section 5.

Furthermore, if we define strong exponential stability for the neutral equation (1.1)–(1.2) analogously as for the associated delay difference equation, then we have the following definition.

DEFINITION 3.9. *The null solution of the neutral equation* (1.1)–(1.2) *is strongly exponentially stable if there exists a number* $\hat{r} > 0$ *such that the null solution of*

$$\dot{x}(t) + \sum_{k=1}^{p_1} H_k \dot{x}(t - \vec{\gamma}_k \cdot (\vec{r} + \delta\vec{r})) = A_0 + \sum_{k=1}^{p_2} A_k x(t - (v_k + \delta v_k))$$

is exponentially stable for all $\delta\vec{r} \in (\mathbb{R}.^+)^m$ *and* $\delta\vec{v} \in (\mathbb{R}^+)^{p_2}$ *satisfying* $\|\delta\vec{r}\| < \hat{r}$, $\|\delta\vec{v}\| < \hat{r}$ *and* $r_k + \delta r_k > 0$, $v_l + \delta v_l > 0$, $1 \leq k \leq m$, $1 \leq l \leq p_2$.

Then we get the following result.

PROPOSITION 3.10. *The null solution of the neutral equation* (1.1) *is strongly exponentially stable if and only if* $c_D(\vec{r}, \vec{v}) < 0$ *and* $\Xi < 0$.

Remark 3.11. Proposition 3.10 implies that the interdependency of the delays $\vec{v}$, if any, does not affect the strong stability of the neutral equation (3.10), unlike the interdependence of the delays $\vec{\tau}$.

3.3. Degenerate case. If $\det \Delta_D(\lambda; \ \vec{r}) \equiv 1$, which occurs, for instance, if all matrices H_k are lower triangular and have zero diagonal, then the zeros of $\det \Delta_N$ $(\lambda; \ \vec{r}, \vec{v})$ are equal to the zeros of

$$(3.3) \qquad Q(\lambda; \ \vec{r}, \vec{v}) := \det \left(\lambda I - \mathrm{adj}(\Delta_D(\lambda; \ \vec{r})) \left(A_0 + \sum_{k=1}^{p_2} A_k e^{-\lambda v_k} \right) \right).$$

Equation (3.3) can also be interpreted as the characteristic function of a linear time-delay system of retarded type, of which the spectral properties carry over (see, e.g., [11, 8, 22] for spectral properties of retarded-type systems).

4. Main results, computational expressions for determining strong stability. The aim of this section is to derive computationally tractable characterizations of the quantities $\bar{C}_D(\vec{r})$ and Ξ, which, by Propositions 3.5 and 3.10, directly result in strong stability conditions. First, we consider special cases where particular conditions are put on the interdependence of the delays. In this way expressions are obtained which directly extend the expressions for the case of independent delays presented in [23], but the derivation is more involved. Next, we show how an *arbitrary* delay dependency of the form (1.2) can be dealt with. The main results will be presented in Theorems 4.3 and 4.7.

4.1. Results for special dependencies in the delays. We start by stating a technical lemma.

LEMMA 4.1. *Assume that there is a vector $\vec{\beta} \in (\mathbb{R}_0)^m$ such that*

$$(4.1) \qquad \vec{\gamma}_k \cdot \vec{\beta} = \vec{\gamma}_l \cdot \vec{\beta} \neq 0 \quad \forall k, l \in \{1, \dots, p_1\}.$$

Let $\vec{r} \in (\mathbb{R}_0^+)^m$ and $c \in \mathbb{R}$. If the function

$$\vec{\theta} \in [0, \ 2\pi]^m \mapsto \alpha \left(-\sum_{k=1}^{p_1} H_k e^{-c\vec{\gamma}_k \cdot \vec{r}} e^{-i\vec{\gamma}_k \cdot \vec{\theta}} \right)$$

has a global maximum, α_0, for $\vec{\theta} = \vec{\theta}_0$, then

$$\alpha_0 \in \sigma \left(-\sum_{k=1}^{p_1} H_k e^{-c\vec{\gamma}_k \cdot \vec{r}} e^{-i\gamma_k \cdot \vec{\theta}_0} \right).$$

Proof. Let $\lambda(\vec{\theta}_0)$ be an *active* eigenvalue of $\left(-\sum_{k=1}^{p_1} H_k e^{-c\vec{\gamma}_k \cdot \vec{r}} e^{-i\gamma_k \cdot \vec{\theta}_0} \right)$, that is,

$$\Re(\lambda) = \alpha \left(-\sum_{k=1}^{p_1} H_k e^{-c\vec{\gamma}_k \cdot \vec{r}} e^{-i\gamma_k \cdot \vec{\theta}_0} \right).$$

Because the spectral abscissa of a matrix which smoothly depends on parameters is a continuously differentiable function of these parameters in the neighborhood of a global maximum (see [5]), the eigenvalue $\lambda(\vec{\theta}_0)$ is either simple or semisimple. Hence, it defines a continuously differentiable function

$$(4.2) \qquad \vec{\theta} \in \mathcal{B}(\vec{\theta}_0) \mapsto \lambda(\vec{\theta}),$$

where $\mathcal{B}(\vec{\theta}_0)$ is some open set of $\mathbb{R}^m$ containing $\vec{\theta}_0$. Let the continuously differentiable functions $\vec{\theta} \mapsto w_0^*(\vec{\theta})$ and $\vec{\theta} \mapsto v_0(\vec{\theta})$ correspond to (normalized) left and

right eigenvectors:

$$(4.3) \qquad \left(\lambda(\vec{\theta})I + \sum_{k=1}^{p_1} H_k e^{-c\vec{\gamma}_k \cdot \vec{r}} e^{-i\vec{\gamma}_k \cdot \vec{\theta}} \right) v_0(\vec{\theta}) = 0,$$

$$(4.4) \qquad w_0^*(\vec{\theta}) \left(\lambda(\vec{\theta})I + \sum_{k=1}^{p_1} H_k e^{-c\vec{\gamma}_k \cdot \vec{r}} e^{-i\vec{\gamma}_k \cdot \vec{\theta}} \right) = 0, \quad \vec{\theta} \in \mathcal{B}(\vec{\theta}_0).$$

Because the spectral abscissa has a maximum at $\vec{\theta}_0$, we have

$$\left. \frac{\partial \Re \lambda(\vec{\theta})}{\partial \theta_j} \right|_{\vec{\theta}=\vec{\theta}_0} = 0, \ j = 1, \ldots, m.$$

Note that

$$\left. \frac{\partial \Re \lambda(\vec{\theta})}{\partial \theta_j} \right|_{\vec{\theta}=\vec{\theta}_0} = \Re \left. \frac{\partial \lambda(\vec{\theta})}{\partial \theta_j} \right|_{\vec{\theta}=\vec{\theta}_0},$$

where $\left. \frac{\partial \lambda(\vec{\theta})}{\partial \theta_j} \right|_{\vec{\theta}=\vec{\theta}_0}$ can be computed by differentiating (4.3) at $\vec{\theta}_0$, premultiplying the result with $w_0^*(\vec{\theta}_0)$ and using (4.4). In this way we arrive at

(4.5)

$$\left. \frac{\partial \Re(\lambda(\vec{\theta}))}{\partial \theta_j} \right|_{\vec{\theta}=\vec{\theta}_0} = \Re \frac{w_0^*(\vec{\theta}_0) \left(\sum_{k=1}^{p_1} \gamma_{k,j} i H_k e^{-c\vec{\gamma}_k \cdot \vec{r}} e^{-i\vec{\gamma}_k \cdot \vec{\theta}_0} \right) v_0(\vec{\theta}_0)}{w_0^*(\vec{\theta}_0) v_0(\vec{\theta}_0)} = 0, \ j = 1, \ldots, m.$$

Let $\vec{\beta} \in (\mathbb{R}_0)^m$ be such that condition (4.1) holds. From (4.5) it follows that

$$
\begin{aligned}
0 &= \sum_{j=1}^{m} \beta_j \ \Re \left(\frac{w_0^* \left(\sum_{k=1}^{p_1} \gamma_{k,j} i H_k e^{-c\vec{\gamma}_k \cdot \vec{r}} e^{-i\vec{\gamma}_k \cdot \vec{\theta}_0} \right) v_0}{w_0^* v_0} \right) \\
&= \Re \left(\sum_{j=1}^{m} \beta_j \frac{w_0^* \left(\sum_{k=1}^{p_1} \gamma_{k,j} i H_k e^{-c\vec{\gamma}_k \cdot \vec{r}} e^{-i\vec{\gamma}_k \cdot \vec{\theta}_0} \right) v_0}{w_0^* v_0} \right) \\
&= \Re \left(\frac{w_0^* \left(\sum_{k=1}^{p_1} (\vec{\gamma}_k \cdot \vec{\beta}) \ i H_k e^{-c\vec{\gamma}_k \cdot \vec{r}} e^{-i\vec{\gamma}_k \cdot \vec{\theta}_0} \right) v_0}{w_0^* v_0} \right) \\
&= \Re \left(\frac{w_0^* \left(\sum_{k=1}^{p_1} (\vec{\gamma}_1 \cdot \vec{\beta}) \ i H_k e^{-c\vec{\gamma}_k \cdot \vec{r}} e^{-i\vec{\gamma}_k \cdot \vec{\theta}_0} \right) v_0}{w_0^* v_0} \right) \\
&= (\vec{\gamma}_1 \cdot \vec{\beta}) \ \Re \left(i \frac{w_0^* \left(\sum_{k=1}^{p_1} H_k e^{-c\vec{\gamma} \cdot \vec{r}} e^{-i\vec{\gamma}_k \cdot \vec{\theta}_0} \right) v_0}{w_0^* v_0} \right) \\
&= (\vec{\gamma}_1 \cdot \vec{\beta}) \ \Re \left(i \frac{w_0^* \lambda(\vec{\theta}_0) v_0}{w_0^* v_0} \right) \\
&= -(\vec{\gamma}_1 \cdot \vec{\beta}) \ \Im(\lambda(\vec{\theta}_0)).
\end{aligned}
$$

We conclude that $\Im(\lambda(\vec{\theta}_0)) = 0$ and $\Re(\lambda(\vec{\theta}_0)) = \alpha_0$. $\square$

The next result states that under condition (4.1), the quantity $\bar{C}_D(\vec{r})$ can be computed from the zeros of a scalar function.

PROPOSITION 4.2. *If* $\det \Delta_D(\lambda; \ \vec{r}) \not\equiv 0$ *and there is a vector* $\vec{\beta} \in (\mathbb{R}_0)^m$ *such that*

$$\vec{\gamma}_k \cdot \vec{\beta} = \vec{\gamma}_l \cdot \vec{\beta} \neq 0 \quad \forall k, l \in \{1, \ldots, p_1\},$$

then for every $\vec{r} \in (\mathbb{R}_0^+)^m$, $\bar{C}_D(\vec{r})$ *is the largest zero of the function*

$$c \in \mathbb{R} \to f(c; \ \vec{r}) - 1,$$

where

(4.6)
$$f(c; \ \vec{r}) = \max_{\vec{\theta} \in [0, \ 2\pi]^m} \alpha \left(-\sum_{k=1}^{p_1} H_k e^{-c \vec{\gamma}_k \cdot \vec{r}} e^{-i \vec{\gamma}_k \cdot \vec{\theta}} \right).$$

Proof. From

(4.7)
$$\bar{C}_D(\vec{r}) = \max \left\{ c \in \mathbb{R} : \ \det \left(I + \sum_{k=1}^{p_1} H_k e^{-c \vec{\gamma}_k \cdot \vec{r}} e^{-i \vec{\gamma}_k \cdot \vec{\theta}} \right) = 0 \right.$$
$$\left. \text{for some } \vec{\theta} \in [0, \ 2\pi]^m \right\}$$

(see Proposition 3.2), it follows that there exists at least one value of c such that $f(c; \ \vec{r}) \geq 1$. As $\lim_{c \to +\infty} f(c; \ \vec{r}) = 0$, the following number is well defined:

$$\hat{c}(\vec{r}) := \max\{c : \ f(c; \ \vec{r}) = 1\}.$$

It is clear that $f(c; \ \vec{r}) \leq 1$ if $c \geq \hat{c}(\vec{r})$. By (4.7) this implies that

(4.8)
$$\hat{c}(\vec{r}) \geq \bar{C}_D(\vec{r}).$$

Next, from Lemma 4.1 and the fact that $f(\hat{c}(\vec{r}); \ \vec{r}) = 1$ it follows that there exists a $\vec{\theta}_0(\vec{r}) \in [0, \ 2\pi]^{p_1}$ such that

$$1 \in \sigma \left(-\sum_{k=1}^{p_1} H_k e^{-\hat{c}(\vec{r}) \vec{\gamma}_k \cdot \vec{r}} e^{-i \vec{\gamma}_k \cdot \vec{\theta}_0(\vec{r})} \right).$$

By (4.7) one concludes that

(4.9)
$$\bar{C}_D(\vec{r}) \geq \hat{c}(\vec{r}).$$

From (4.8) and (4.9) we get $\bar{C}_D(\vec{r}) = \hat{c}(\vec{r})$, which is equivalent to the assertion of the proposition. $\square$

By further imposing that the vector $\vec{\beta}$, appearing in Proposition 4.2, has *positive* components only—among others—an explicit expression for Ξ, and thus an explicit strong stability condition, is obtained.

THEOREM 4.3. *Define*

(4.10)
$$\delta_0 := \max_{\vec{\theta} \in [0, \ 2\pi]^m} \alpha \left(-\sum_{k=1}^{p_1} H_k e^{-i \vec{\gamma}_k \cdot \vec{\theta}} \right).$$

If $\det \Delta_D(\lambda; \ \vec{r}) \not\equiv 0$ *and there is a vector* $\vec{\beta} \in (\mathbb{R}_0^+)^m$ *such that*

$$\vec{\gamma}_k \cdot \vec{\beta} = \vec{\gamma}_l \cdot \vec{\beta} \quad \forall k, l \in \{1, \ldots, p_1\},$$

then the assertion of Proposition 4.2, can be strengthened as follows:

1. *for all $\vec{r} \in (\mathbb{R}_0^+)^m$, $\bar{C}_D(\vec{r})$ is the unique zero of the strictly decreasing function $c \in \mathbb{R} \mapsto f(c;\ \vec{\tau}) - 1$, with f given by (4.6);*

2. *we have*

$$\Xi = \operatorname{sign} \log(\delta_0);$$

3. *if $\delta_0 > 1$, then there exists a vector $\vec{r}_0 \in (\mathbb{R}_0^+)^m$ for which $\bar{C}_D(\vec{r}_0) > 0$.*

Proof. We first prove the second and third statements. According to its definition we evaluate Ξ as

$$(4.11) \qquad \Xi = \operatorname{sign}\left(\bar{C}_D(\vec{\beta})\right).$$

From Proposition 4.2 $\bar{C}_D(\vec{\beta})$ is the largest zero of the function

$$c \in \mathbb{R} \mapsto e^{-c\vec{\gamma}_1 \cdot \vec{\beta}} \max_{\vec{\theta} \in [0,\ 2\pi]^m} \alpha\left(-\sum_{k=1}^{p_1} H_k e^{-i\vec{\gamma}_k \cdot \vec{\theta}}\right),$$

thus

$$(4.12) \qquad \bar{C}_D(\vec{\beta}) = \frac{1}{\vec{\gamma}_1 \cdot \vec{\beta}} \log(\delta_0).$$

The second and third assertions of the proposition follow from (4.11) and (4.12).

The proof of the first assumption is analogous to the proof of Theorem 6 of [23] and relies on the second assertion, combined with an approximation and continuation argument. □

Remark 4.4. If $p_1 = m$ and $\tau_k = r_k$, $1 \leq k \leq m$, then Proposition 4.3 reduces to Theorem 6 and Proposition 1 of [23] and δ_0 is an equivalent quantity with γ_0 of [13].

4.2. Results for general case: Lifting procedure. Recall that the characteristic function of (2.2) is given by

$$(4.13) \qquad \Delta_D(\lambda;\ \vec{r}) = \det\left(I + \sum_{k=1}^{p_1} H_k e^{-\lambda\ \vec{\gamma}_k \cdot \vec{r}}\right).$$

By formally setting

$$x_i = e^{-\lambda\ r_i},\ \ i = 1, \ldots, m,$$

the function (4.13) can be interpreted as a *multivariable* polynomial

$$(4.14) \qquad p(x_1, \ldots, x_m) := \det\left(I + \sum_{k=1}^{p_1} H_k \left(\Pi_{l=1}^m x_l^{\gamma_{k,l}}\right)\right),$$

with some constraints on the variables.

Using results from realization theory, one can show that the polynomial (4.14) can be "lifted" and expressed as the determinant of a (linear) pencil. To do so, we write the polynomial matrix

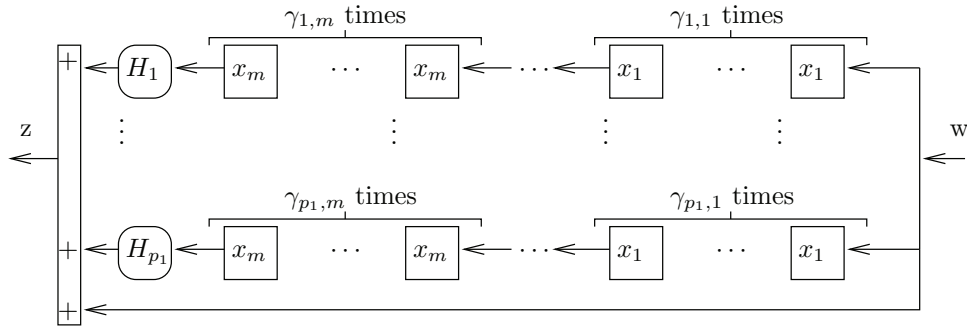$$I + \sum_{k=1}^{p_1} H_k \left(\Pi_{l=1}^m x_l^{\gamma_{k,l}}\right)$$

FIG. 4.1. *Block diagram of the relation* (4.15).

as a so-called lower linear fractional representation (see [33]). Let "input" $w \in \mathbb{R}^n$ and "output" $z \in \mathbb{R}^n$ be such that

$$(4.15) \qquad z = \left(I + \sum_{k=1}^{p_1} H_k \left(\Pi_{l=1}^m x_l^{\gamma_{k,l}} \right) \right) w.$$

This relation can be represented by the block diagram shown in Figure 4.1. By "pulling out" the square blocks, corresponding to the variables, and collecting them in a diagonal matrix, it follows that (4.15) is equivalent to

$$(4.16) \qquad \left[\begin{array}{c} z \\ y \end{array} \right] = M \left[\begin{array}{c} w \\ u \end{array} \right], \quad u = \Delta(x_1, \ldots, x_m)\, y,$$

where

$$(4.17) \qquad M = \left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array} \right] :=$$

and
(4.18)

$$\Delta(x_1, \ldots, x_m) =$$

with $s_k = \sum_{l=1}^{m} \gamma_{k,l}, \quad 1 \le k \le p_1$, and $I_u, \ u \in \mathbb{N}$, denoting the u-by-u unity matrix.
From (4.16) we obtain

$$z = \mathcal{F}_l(M, \Delta(x_1, \ldots, x_m)) \, y$$
$$:= \left(I + M_{12}\Delta(x_1, \ldots, x_m)(I - M_{22}\Delta(x_1, \ldots, x_m))^{-1} M_{21} \right) y.$$

It follows that

$$p(x_1, \ldots, x_m) = \det \left(I + M_{12}\Delta(x_1, \ldots, x_m)(I - M_{22}\Delta(x_1, \ldots, x_m))^{-1} M_{21} \right)$$
$$= \det \left(I + (I - M_{22}\Delta(x_1, \ldots, x_m))^{-1} M_{21} M_{12}\Delta(x_1, \ldots, x_m) \right)$$
$$= \det \left(I + (M_{21}M_{12} - M_{22})\Delta(x_1, \ldots, x_m) \right)$$
$$= \det \left(I + \sum_{k=1}^{m} \tilde{H}_k x_k \right),$$

where

$$(4.19) \qquad \tilde{H}_k = (M_{21}M_{12} - M_{22})\Delta(\vec{e}_k), \quad k = 1, \ldots, m,$$

and $\vec{e}_k$ is the kth unit vector in $\mathbb{R}^m$. In this way, we arrive at the following result.

PROPOSITION 4.5. *There always exist real square matrices $\tilde{H}_1, \ldots, \tilde{H}_m$ of equal dimensions such that*

$$(4.20) \qquad p(x_1, \ldots, x_m) = \det \left(I + \sum_{k=1}^{m} \tilde{H}_k x_k \right),$$

or, equivalently,

$$(4.21) \qquad \det \Delta_D(\lambda; \, \vec{r}) = \det \left(I + \sum_{k=1}^{m} \tilde{H}_k e^{-\lambda r_k} \right).$$

A solution is given by (4.19), where M and Δ are defined in (4.17) and (4.18).

Remark 4.6. The lifting of (4.14) to an expression of the form (4.20) is *not* unique. Furthermore, the presented solution (4.17)–(4.19) does not necessarily correspond to a solution where the matrices $\tilde{H}_k$ have minimal dimensions. In fact, a minimal realization can be obtained from a block diagram representation of (4.15) (possibly different from the one shown in Figure 4.1), where the number of square blocks (thus, the dimension of $\Delta(x_1, \ldots, x_m)$) is minimal. As we shall illustrate with two examples, the construction of such minimal realization strongly depends on the specific properties of the polynomial under consideration and is hard to automate. Notice here that finding an algorithm for the automatic construction of a *minimal* realization is still an *open problem* in realization theory. Note also that the lifting procedure presented above is systematic and generally applicable. For more results on linear fractional representations (LFRs) of multivariable polynomials we refer to the specialized literature; see, e.g., Chapter 10 in [33] for representations coming from state-space realizations in control theory, and Chapter 14 in [15] for many references and extensions to symmetric representations and polynomials with noncommutative variables. See also [20] for an excellent user-friendly publicly available MATLAB toolbox which contains—among other things—routines to compute LFRs and numerical heuristics to reduce the order of LFRs.

We now return to the original problem. From the expression (4.21) it follows that $\Delta_D(\lambda; \, \vec{r})$ can be interpreted as the characteristic function of the "lifted"

difference equation

$$\chi(t) + \sum_{k=1}^{m} \tilde{H}_k \chi(t - r_k) = 0.$$

As this equation satisfies the condition assumed in the propositions of section 4.1, the following result directly follows.

THEOREM 4.7. *For the delay difference equation* (2.2) *we have*

$$\Xi = \text{sign} \log(\delta_0),$$

where

$$\delta_0 := \max_{\vec{\theta} \in [0,\, 2\pi]^m} \alpha \left(\sum_{k=1}^{m} -\tilde{H}_k e^{-i\theta_k} \right)$$

and the matrices $\tilde{H}_k$ are such that (4.21) *in Proposition* 4.5 *holds.*

Furthermore, for all $\vec{r} \in (\mathbb{R}_0^+)^m$, $\bar{C}_D(\vec{r})$ is the unique *zero of the* strictly decreasing *function*

$$c \in \mathbb{R} \rightarrow f(c;\ \vec{r}) = \max_{\vec{\theta} \in [0,\, 2\pi]^m} \alpha \left(\sum_{k=1}^{m} -\tilde{H}_k e^{-cr_k} e^{-i\theta_k} \right).$$

With two examples we illustrate the lifting procedure for the computation of the matrices $\tilde{H}_k$, $1 \leq k \leq m$, because this is the main step in the application of Theorem 4.7.

Example 4.8. If $p_1 = 3$, $m = 2$, and

$$\vec{\gamma}_1 = (1,0),\ \vec{\gamma}_2 = (0,1),\ \vec{\gamma}_3 = (1,1),$$

then the delay difference equation (2.2) becomes

(4.22) $$z(t) + H_1 z(t - r_1) + H_2 z(t - r_2) + H_3 z(t - (r_1 + r_2)) = 0.$$

This case is not directly covered in section 4.1 since there does not exist a vector $\vec{\beta} \in (\mathbb{R}_0^+)^M$ such that

$$\vec{\gamma}_k \cdot \vec{\beta} = \vec{\gamma}_l \cdot \vec{\beta} \neq 0 \quad \forall k, l \in \{1, 2, 3\}.$$

The characteristic equation of (4.22) is given by

$$\det \left(I + H_1 e^{-\lambda r_1} + H_2 e^{-\lambda r_2} + H_3 e^{-\lambda(r_1 + r_2)} \right) = 0.$$

An application of Proposition 4.5 leads to the equivalent expression

$$\det \left(I + \begin{bmatrix} H_1 & 0 & 0 & 0 \\ H_1 & 0 & 0 & 0 \\ H_1 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 \end{bmatrix} e^{-\lambda r_1} + \begin{bmatrix} 0 & H_2 & 0 & H_3 \\ 0 & H_2 & 0 & H_3 \\ 0 & H_2 & 0 & H_3 \\ 0 & 0 & 0 & 0 \end{bmatrix} e^{-\lambda r_2} \right) = 0.$$

In Figure 4.2 (top) we show a block diagram of the relation
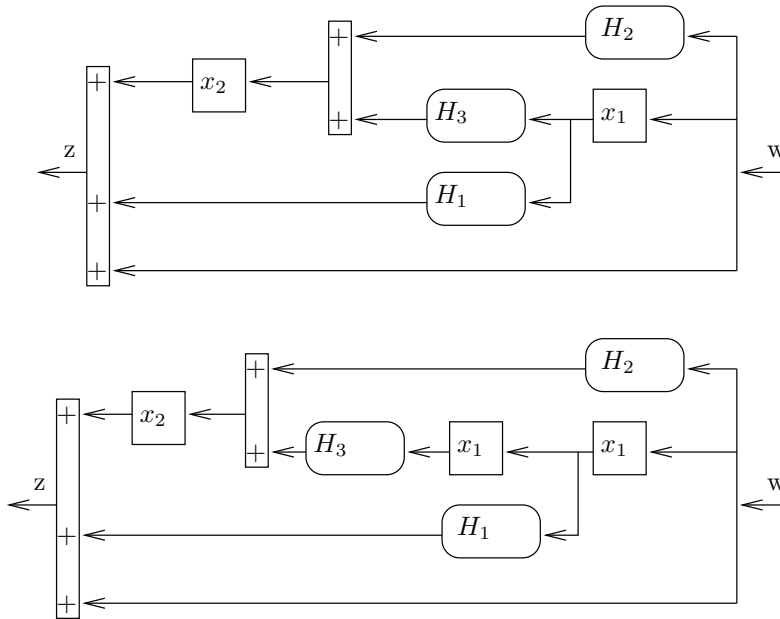
(4.23) $$z = (I + H_1 x_1 + H_2 x_2 + H_3 x_1 x_2)\ w,$$

FIG. 4.2. *Block diagram representation of* (4.23) *(above) and* (4.25) *(below), using a minimum number of square blocks.*

where we have minimized the number of square blocks (corresponding to a variable), that is, we have minimized the dimension of $\Delta(x_1, x_2)$. It leads to the minimal order[3] lifting, given by

$$(4.24) \qquad \det \left(I + \underbrace{\begin{bmatrix} H_1 & 0 \\ H_2 H_1 - H_3 & 0 \end{bmatrix}}_{\tilde{H}_1} e^{-\lambda r_1} + \underbrace{\begin{bmatrix} 0 & I \\ 0 & H_2 \end{bmatrix}}_{\tilde{H}_2} e^{-\lambda r_2} \right) = 0.$$

Example 4.9. If $p_1 = 3$, $m = 2$, and

$$\vec{\gamma}_1 = (1, 0), \ \vec{\gamma}_2 = (0, 1), \ \vec{\gamma}_3 = (2, 1),$$

then the characteristic equation of (2.2) becomes

$$\det \left(I + H_1 e^{-\lambda r_1} + H_2 e^{-\lambda r_2} + H_3 e^{-\lambda(2r_1 + r_2)} \right) = 0.$$

The systematic lifting procedure proposed in Proposition 4.7 leads us to the equivalent expression

$$\det \left(I + \begin{bmatrix} H_1 & 0 & 0 & 0 & 0 \\ H_1 & 0 & 0 & 0 & 0 \\ H_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & -I & 0 \end{bmatrix} e^{-\lambda r_1} + \begin{bmatrix} 0 & H_2 & 0 & 0 & H_3 \\ 0 & H_2 & 0 & 0 & H_3 \\ 0 & H_2 & 0 & 0 & H_3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} e^{-\lambda r_2} \right) = 0.$$

---

[3]without assumptions on the matrices H_k. A further reduction may be possible when the matrices H_k are specified or information about their structure is present.

A minimal order lifting follows from the block diagram representation of

$$(4.25) \qquad\qquad z = (I + H_1 x_1 + H_2 x_2 + H_3 x_1 x_2)\, w,$$

shown in Figure 4.2 (bottom), and it is given by

$$(4.26) \qquad \det \left(I + \underbrace{\begin{bmatrix} H_1 & 0 & 0 \\ -I & 0 & 0 \\ H_2 H_1 & -H_3 & 0 \end{bmatrix}}_{\tilde{H}_1} e^{-\lambda r_1} + \underbrace{\begin{bmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & H_2 \end{bmatrix}}_{\tilde{H}_2} e^{-\lambda r_2} \right) = 0.$$

Finally, we illustrate that the lifting step is necessary if the assumption on the interdependency of the delays of Proposition 4.3 is not satisfied.

Example 4.10. When applying Theorem 4.7 to the delay difference equation

$$z(t) + \frac{20}{101} z(t - r_1) - \frac{40}{101} z(t - r_2) - \frac{80}{101} z(t - (r_1 + r_2)) = 0,$$

for which the lifting (4.24) can be used, we get $\delta_0 = 0.9945 < 1$, thus $\Xi < 0$, and we can conclude strong stability. On the other hand, formula (4.10) would result in $\delta_0 = 1.0066 > 1$. This demonstrates that lifting may be necessary if the assumption of Proposition 4.3 is not satisfied, and that the assertions of Proposition 4.3 are not condensed formulations of the assertions of Theorem 4.7.

5. Illustrations and applications.

5.1. Numerical example.
We apply the theoretical results derived above to the system

$$(5.1) \qquad \frac{d}{dt}\left(x(t) + \sum_{k=1}^{3} H_k x(t - \tau_k) \right) = A_0 x(t) + A_1 x(t - \nu_1),$$

where the system matrices are given by

$$(5.2) \qquad \begin{aligned} H_1 &= \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{8} & \frac{1}{2} \end{bmatrix}, \quad H_2 = \begin{bmatrix} -\frac{1}{8} & 1 \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix}, \quad H_3 = \begin{bmatrix} -\frac{1}{8} & -\frac{1}{4} \\ 0 & \frac{1}{8} \end{bmatrix}, \\ A_0 &= \begin{bmatrix} 0 & -\frac{11}{40} \\ \frac{11}{80} & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -\frac{1}{64} & -\frac{1}{8} \\ -\frac{1}{8} & -\frac{1}{32} \end{bmatrix} \end{aligned}$$

and the dependency of the delays is described by

$$(5.3) \qquad\qquad \tau_1 = r_1, \ \ \tau_2 = r_2, \ \ \tau_3 = 2r_1 + r_2,$$

with r_1 and r_2 independent.

In Figure 5.1 we show the rightmost characteristic roots of (5.1)–(5.3) for $(r_1, r_2) = (1, 2)$ and $\nu_1 = 1$, computed with the quasi-polynomial mapping–based rootfinder (QPMR) [30]. Note that the exponentially transformed characteristic roots correspond to the eigenvalues of the operator $\mathcal{T}(1;\ (r_1, r_2), \nu_1)$. We have

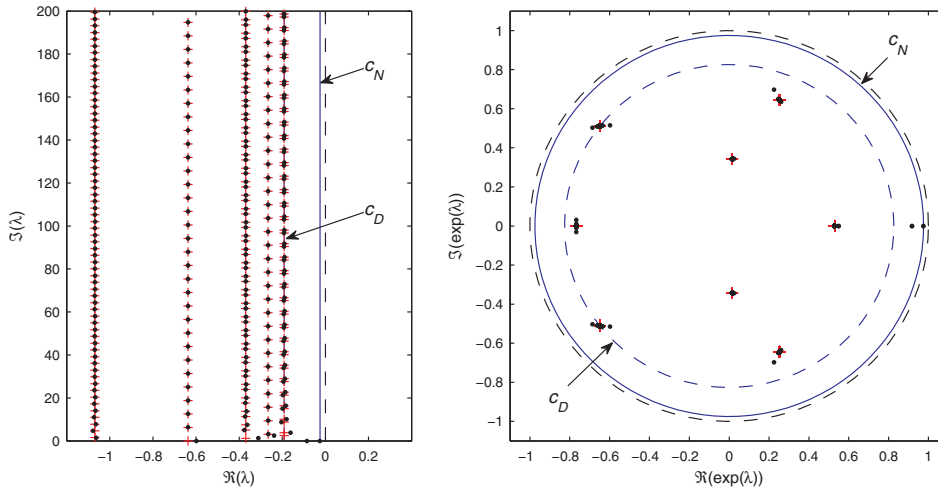$$c_N((1,2),1) = -0.025 \ \ \text{and} \ \ c_D((1,2)) = -0.191.$$

FIG. 5.1. *Rightmost characteristic roots of the system* (5.1)–(5.3) *with* $(r_1, r_2) = (1, 2)$ *and* $\nu_1 = 1$. *Dots—characteristic roots of the neutral system; crosses—characteristic roots of the associated difference equation.*

Let us remark that the latter quantity can be calculated from the zeros of a polynomial, because

$$\Delta_D(\lambda; \ (1, 2)) = \det(I + H_1 \chi + H_2 \chi^2 + H_3 \chi^4),$$

provided $\chi = e^{-\lambda}$. Thus, if the characteristic roots of the delay difference equation with the commensurate delays are exponentially transformed, they are mapped to a finite number of points. Due to the relation

$$\sigma_e(\mathcal{T}(t; \ (r_1, r_2), \nu_1)) = \sigma(\mathcal{T}_D(t; \ (r_1, r_2))),$$

the transformed roots of the neutral system accumulate to these points. This can be seen in the right frame of Figure 5.1.

In order to show the effect of small delay perturbations, we depict in Figure 5.2 the characteristic roots of (5.1)–(5.3) for $(r_1, r_2) = (1, 2 + \pi/100)$ and $\nu_1 = 1$. We also indicate the quantity

$$\bar{C}_D((1, 2), \nu_1) = -0.066,$$

which can be computed by applying Theorem 4.7, starting from the representation (4.26). The fact that $\bar{C}_D((1, 2)) > c_D((1, 2))$ illustrates the noncontinuity of the function $\vec{r} \mapsto c_D(\vec{r})$. Notice from Figures 5.1–5.2 that in any right half plane $\{\lambda \in \mathbb{C} : \Re(\lambda) > \bar{C}_D + \epsilon\}$, $\epsilon > 0$, the neutral equation has only a finite number of characteristic roots.

Because $\bar{C}_D((1, 2)) < 0$, which implies $\Xi < 0$, and $c_D((1, 2), 1) < 0$, the null solution of (5.1)–(5.3) is strongly exponentially stable.

If one is only interested in checking strong stability of the delay difference equation, then according to Theorem 4.7 it is sufficient to check whether $\delta_0 < 1$, where

$$(5.4) \qquad \delta_0 = \max_{\vec{\theta} \in [0, \ 2\pi]^2} \alpha \left(-\tilde{H}_1 e^{-i\theta_1} - \tilde{H}_2 e^{-i\theta_2} \right),$$
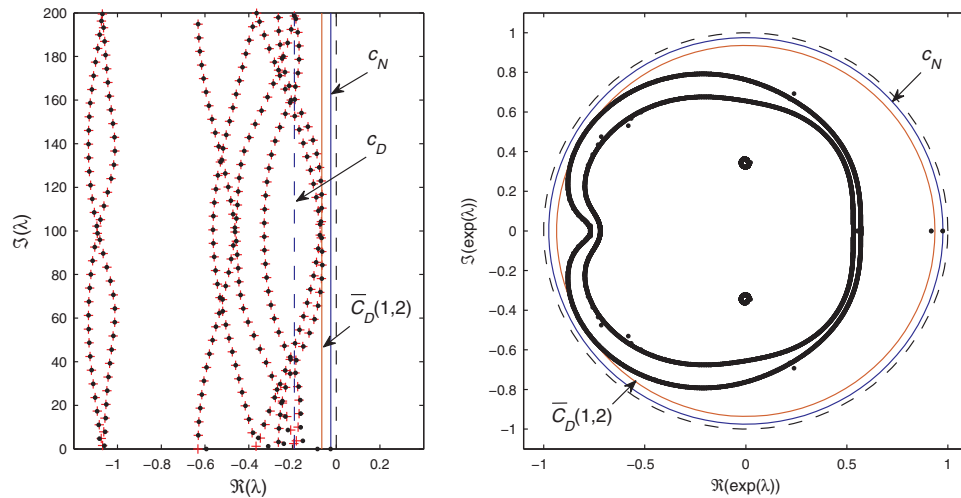
FIG. 5.2. *Rightmost characteristic roots of the system* (5.1)–(5.3) *with* $(r_1, r_2) = (1, 2 + \frac{\pi}{100})$ *and* $\nu_1 = 1$.

with $\tilde{H}_1, \tilde{H}_2$ defined in (4.26). From (5.4) we get

$$\delta_0 = \max_{\theta \in [0, 2\pi]} \alpha(-\tilde{H}_1 - \tilde{H}_2 e^{-i\theta}) = 0.901.$$

In Figure 5.3 we show contour lines of the spectral abscissa function

$$(5.5) \qquad\qquad (\theta_1, \theta_2) \mapsto \alpha(-\tilde{H}_1 e^{-i\theta_1} - \tilde{H}_2 e^{-i\theta_2}),$$

as well as curves corresponding to the values of θ_1 and θ_2 for which a rightmost eigenvalue of

$$(5.6) \qquad\qquad -\tilde{H}_1 e^{-i\theta_1} - \tilde{H}_2 e^{-i\theta_2}$$

is real. As can be seen from the figure, the matrix (5.6) has a real rightmost eigenvalue if (θ_1, θ_2) is a global maximizer of (5.5). This is in accordance with the statement of Lemma 4.1.

Finally, let us illustrate that the effect of delay perturbations strongly depends on the interdependence of the delays. If, instead of the relation (5.3), we assume that the delays τ_k, $1 \le k \le 3$, in (5.1) can vary independently of each other independent, that is,

$$\tau_k = r_k, \ k = 1, \ldots, 3,$$

then we get

$$\bar{C}_D((1, 2, 4)) = 0.055,$$

which shows that strong stability is lost. Note for comparison that with the previously considered dependency structure (5.3) the nominal values $\vec{r} = (1, 2)$ also corresponded to $\vec{\tau} = (1, 2, 4)$.
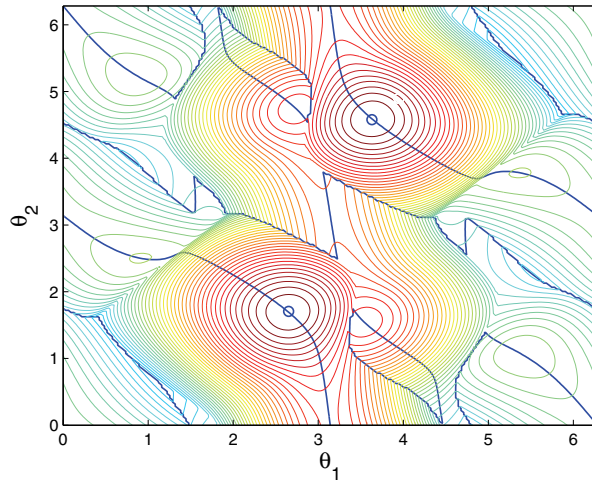
FIG. 5.3. *Contour lines of the function* (5.5). *The global maxima are indicated with* "∘". *The dark curves correspond to values of* θ_1 *and* θ_2 *for which the rightmost eigenvalue of* (5.6) *is real.*

5.2. Boundary-controlled partial differential equation. The following model from [19] (see also [6, 24] for a simplified version) describes movement of a string fixed at one side and controlled by changing the direction of the external force at the other side:

$$(5.7) \qquad w_{tt}(x,t) - w_{xx}(x,t) + 2aw_t(x,t) + a^2 w(x,t) = 0, \ t \geq 0, x \in [0, \ 1],$$

$$(5.8) \qquad\qquad\qquad w(0,t) = 0, \quad w_x(1,t) = -kw_t(1,t-h).$$

The variable $w(x,t)$ describes the movement at position x at time t. The parameter $h \geq 0$ represents a small delay in the velocity feedback, $k \geq 0$ is the controller gain, and $a \geq 0$ represents a damping constant.

When substituting a solution of the form $w(x,t) = e^{\lambda t}z(x)$ in (5.7)–(5.8) the following characteristic equation is obtained:

$$(5.9) \qquad\qquad 1 + e^{-2a}e^{-\lambda 2} + ke^{-\lambda h} - ke^{-2a}e^{-\lambda(2+h)} = 0.$$

Note that this equation can be interpreted as the characteristic equation of a delay difference equation of the form (2.2), exhibiting three delays $(\tau_1, \tau_2, \tau_3) = (2, h, 2+h)$ that depend on two independent delays $(r_1, r_2) = (2, h)$.

If $h = 0$, the characteristic roots are

$$\lambda = -\frac{1}{2}\log\left|\frac{1+k}{1-k}\right| - a + i\left(\pi l + \frac{\pi}{4}(1+\text{sign}(k-1))\right), \quad l \in \mathbb{Z}.$$

As for all $k \neq 1$,

$$(5.10) \qquad\qquad c(k) := -\frac{1}{2}\log\left|\frac{1+k}{1-k}\right| - a < 0,$$

the system with $h = 0$ is stable for all $k \neq 1$. As k approaches 1, the real parts of the characteristic roots move off to $-\infty$, which indicates superstability at $k = 1$ (meaning

that perturbations disappear in a finite time). This is indeed the case and can be explained as follows: the general solution of (5.7) can be written as a combination of two traveling waves: a solution $\phi(x-t)e^{-at}$ moving to the right and a solution $\psi(x+t)e^{-at}$ moving to the left. If $k = 1$, then $\phi(x-t)e^{-at}$ satisfies the second boundary condition, and thus the reflection coefficient at $x = 1$ is zero; at $x = 0$ the wave $\phi(x+t)$ is reflected completely. Consequently all perturbations of the zero solution disappear in a finite time (at most 2 time units).

Next, we look at the effect of a small feedback delay h in the application of the boundary control. If the delays $(r_1, r_2) = (2, h)$ are rationally independent, which occurs if h is an irrational number, then we have $c_D(\vec{r}) = \bar{C}_D(\vec{r})$ (Proposition 3.2), and the stability condition is given by $\Xi < 0$ (which also guarantees stability for all $h > 0$). To compute Ξ, we apply Theorem 4.7, based on the lifting (4.24). This yields

$$\delta_0 = \max_{(\theta_1, \theta_2) \in [0, \ 2\pi]^2} \ \alpha \left(- \begin{bmatrix} e^{-2a} & 0 \\ 2ke^{-2a} & 0 \end{bmatrix} e^{-i\theta_1} - \begin{bmatrix} 0 & 1 \\ 0 & k \end{bmatrix} e^{-i\theta_2} \right)$$

$$= \max_{\theta \in [0, \ 2\pi]} \ r_\sigma \left(\begin{bmatrix} e^{-2a} & 0 \\ 2ke^{-2a} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & k \end{bmatrix} e^{-i\theta} \right)$$

$$= \max \left\{ |\lambda| : \ 1 - \frac{k(\lambda + e^{-2a})}{\lambda^2 - e^{-2a}\lambda} e^{i\theta} = 0, \ \theta \in [0, \ 2\pi], \lambda \in \mathbb{C} \right\}$$

$$= \max \left\{ |\lambda| : \ \left| \frac{k(\lambda + e^{-2a})}{\lambda^2 - e^{-2a}\lambda} \right| = 1, \ \lambda \in \mathbb{C} \right\}$$

$$= \max \left\{ |\lambda| : \ \frac{k\left(1 + \frac{e^{-2a}}{|\lambda|}\right)}{|\lambda - e^{-2a}|} = 1, \ \lambda \in \mathbb{C} \right\}$$

$$= \frac{1}{2} \left(e^{-2a} + k + \sqrt{(e^{-2a} + k)^2 + 4ke^{-2a}} \right).$$

It follows that

$$\Xi = \text{sign} \log(\delta_0) < 0 \Leftrightarrow k < \tanh(a),$$

where $<$ can be replaced with $>, =$. We conclude with the following:

1. If $k < \tanh(a)$, then the system (5.7)–(5.8) is exponentially stable for all $h \geq 0$.
2. If $k > \tanh(a)$, then the system (5.7)–(5.8) is exponentially unstable for all irrational values of h. Consequently, there exist *arbitrarily small* values of h that destroy the exponential stability of the system without delay in the boundary control.

5.3. Delay robustness of state derivative feedback control. In [1, 2] the problem of stabilization and control of the linear system

(5.11) $$\dot{x}(t) = Ax(t) + Bu(t),$$

where $x(t) \in \mathbb{R}^n$ is the vector of state variables, $u \in \mathbb{R}^{n_u}(t)$ is the vector of inputs, and A, B are constant coefficient matrices of compatible dimension, has been solved by the state derivative feedback controller

(5.12) $$u(t) = -K_d \dot{x}(t).$$

The use of state derivative control law is motivated by its easy implementation in applications where accelerometers are used for measuring the system motion, e.g.,

applications in vibration control, where the state variables typically correspond to positions and velocities. In [1, 2], it is shown that if the system (5.11) is controllable, and $\det(A) \neq 0$, then all the characteristic roots of the closed-loop system can be assigned at arbitrary positions in $\mathbb{C} \setminus \{0\}$. However, results described in [31] indicate that stability of the state derivative feedback control may not be robust against small feedback delays. This issue is investigated in what follows.

If we assume that there is a delay τ_{u_k} on the kth component of input u, $1 \leq k \leq n_u$, and a delay τ_{x_l} in the measurement of the lth component of $\dot{x}$, $1 \leq l \leq n$, then the closed-loop system (5.11)–(5.12) becomes

$$(5.13) \qquad \dot{x}(t) + \sum_{k=1}^{n_u} BE_k \sum_{l=1}^{n} K_d F_l \dot{x}(t - \tau_{u_k} - \tau_{x_l}) = Ax(t),$$

where $E_k = [e_{i,j}^k] \in \mathbb{R}^{n_u \times n_u}$ and $F_l = [f_{i,j}^l] \in \mathbb{R}^{n \times n}$ satisfy

$$e_{i,j}^k = \begin{cases} 1, & i = j = k, \\ 0, & \text{otherwise,} \end{cases} \qquad f_{i,j}^l = \begin{cases} 1, & i = j = l, \\ 0, & \text{otherwise} \end{cases}$$

for $k = 1, \ldots, n_u$ and $l = 1, \ldots, n$. Equation (5.13) is of the general form (1.1), provided that we set

$$
\begin{aligned}
& p_1 = n_u n, \quad p_2 = 0, \quad m = n_u + n, \\
(5.14) \quad & (\tau_1, \ldots, \tau_{p_1}) = (\tau_{u_1} + \tau_{x_1}, \ldots, \tau_{u_1} + \tau_{x_n}, \ldots, \tau_{u_{n_u}} + \tau_{x_1}, \tau_{u_{n_u}} + \tau_{x_n}), \\
& (r_1, \ldots, r_m) = (\tau_{u_1}, \ldots, \tau_{u_{n_u}}, \tau_{x_1}, \ldots, \tau_{x_n}),
\end{aligned}
$$

and we define vectors $\vec{\gamma}_k$, $1 \leq k \leq p_1$, and matrices A_0, H_k, $1 \leq k \leq p_1$, accordingly. We have the following result.

PROPOSITION 5.1. *Assume the system (5.11) is stabilized with the control law (5.12).*

If the feedback gain K_d is such that

$$\gamma_0(K_d) := \max \left\{ \alpha \left(- \sum_{k=1}^{n_u} BE_k \sum_{l=1}^{n} K_d F_l e^{i(\mu_k + \nu_l)} \right) : \vec{\mu} \in [0, \ 2\pi]^{n_u}, \ \vec{\nu} \in [0, \ 2\pi]^n \right\} < 1,$$

then the exponential stability of the closed-loop system is robust against small feedback delays.

If $\gamma_0(K_d) \geq 1$, then the exponential stability of the closed-loop stability is not robust against small delay perturbations.

Proof. The interdependence between the delays of the neutral system (5.13) satisfies the condition of Proposition 4.3. Furthermore, for this system the quantity δ_0, defined in Proposition 4.3, reduces to $\gamma_0(K_d)$. Consequently, if $\gamma_0(K_d) < 1$, then $\Xi < 0$. By the bounds on the characteristic roots given in Lemma 3.7, the continuity of the individual characteristic roots w.r.t. the delay parameters and the exponential stability of the delay-free system, we conclude that $c_D(\vec{r}, \vec{v}) < 0$ for sufficiently small values of $\vec{r}$ and $\vec{v}$. Robustness of stability follows. If $\gamma_0(K_d) > 1$, then the null solution of (5.13) is not strongly exponentially stable, which implies that infinitesimal perturbations on the (arbitrarily small) delays destroy exponential stability. $\square$

6. Conclusions. The stability theory for neutral equations and delay difference equation subjected to delay perturbations has been developed for the case where

the delays have an arbitrary dependency structure, with the emphasis on spectral properties and computational expressions for $\bar{C}_D$ and Ξ that, among others, lead to explicit strong stability conditions.

Instrumental to this, it has been shown that a general delay difference equation with dependent delays can always be transformed, without changing the characteristic equation, into a delay difference equation with possibly larger dimension but with independent delays, such that the stability theory for systems with independent delays can be applied to complete the theory. An essential step of the constructive procedure consists of representing a multivariate polynomial as the determinant of a pencil. In this sense it is remarkable how the realization theory, commonly used in robust control and optimization, has proven its usefulness to the problems considered in the paper, which are of a different nature. In addition special cases have been addressed for which the lifting step, which may increase the computational complexity, can be omitted.

More specifically the main results are presented in Theorem 4.3, holding for a special dependency of the delays, and Theorem 4.7, holding for the general case. Theorem 4.7 depends on a lifting of the characteristic function for which Proposition 4.5 guarantees the existence and provides a *constructive* solution.

The results derived in the paper have been applied to various problems, including the study of the effects of unmodeled delays on the stability of a boundary-controlled hyperbolic partial differential equation and of a control scheme involving state derivative feedback, being of importance in vibration control applications. These examples illustrate the importance of taking into account small delays or delay perturbations, as well as the dependency structure of the delays.

Appendix A. Proof of Lemma 3.7. Because $\Delta_D(\lambda;\ \vec{r})$ is invertible, we can write the characteristic equation in the form

$$\det\left(\lambda I - \Delta_D(\lambda;\ \vec{r})^{-1}\left(A_0 + \sum_{k=1}^{p_2} A_k e^{-\lambda v_k}\right)\right) = 0.$$

This equation can be interpreted as

$$\lambda \in \sigma\left(\Delta_D(\lambda;\ \vec{r})^{-1}\left(A_0 + \sum_{k=1}^{p_2} A_k e^{-\lambda v_k}\right)\right),$$

which implies

$$|\lambda| \leq \left\|\Delta_D(\lambda;\ \vec{r})^{-1}\left(A_0 + \sum_{k=1}^{p_2} A_k e^{-\lambda v_k}\right)\right\|.$$

By further working out the estimate, we arrive at the assertion.   □

Appendix B. Proof of Proposition 3.8. We prove continuity at $(\vec{r}, \vec{v}) = (\vec{r}_0, \vec{v}_0)$, where we consider two cases.

Case 1. $\bar{C}_D(\vec{r}_0) \geq c_N(\vec{r}_0, \vec{v}_0)$.

The proof is by contradiction. By item (1) of Proposition 3.2 a violation of the continuity property would imply the existence of sequences $\left\{\vec{r}^{(\varrho)}\right\}_{\varrho \geq 1}, \left\{\vec{v}^{(\varrho)}\right\}_{\varrho \geq 1}$ and the existence of a number $\epsilon > 0$ such that

$$\lim_{\varrho \to \infty} \vec{r}^{(\varrho)} = \vec{r}_0, \quad \lim_{\varrho \to \infty} \vec{v}^{(\varrho)} = \vec{v}_0$$

and

$$c_N(\vec{r}^{(\varrho)}, \vec{v}^{(\varrho)}) \geq \bar{C}_D(\vec{r}_0) + \epsilon \ \ \forall \varrho \geq 1.$$

As a consequence, there exists a sequence of complex numbers $\{\lambda^{(\varrho)}\}_{\varrho \geq 1}$ satisfying

$$\Delta_N(\lambda^{(\varrho)}; \ \vec{r}^{(\varrho)}, \vec{v}^{(\varrho)}) = 0, \ \Re(\lambda^{(\varrho)}) > \bar{C}_D(\vec{r}_0) + \epsilon/2 \ \ \forall \varrho \geq 1.$$

By Lemma 3.7, there is a compact subset of $\mathbb{C}$ which contains all elements of the sequence $\{\lambda^{(\varrho)}\}_{\varrho \geq 1}$. Consequently, this sequence has at least one accumulation point $\hat{\lambda}$. From Rouché's theorem it follows that

$$\Delta_N(\hat{\lambda}; \ \vec{r}_0, \vec{v}_0) = 0.$$

Because $\Re(\hat{\lambda}) > \bar{C}_D(\vec{r}_0)$, we arrive at $c_N(\vec{r}_0, \vec{v}_0) > \bar{C}_D(\vec{r}_0)$ and have a contradiction.

Case 2. $\bar{C}_D(\vec{r}_0) < c_N(\vec{r}_0, \vec{v}_0)$.

Let $\epsilon > 0$ be such that $\bar{C}_D(\vec{r}_0) + \epsilon < c_N(\vec{r}_0, \vec{v}_0)$ and $\Delta_N(\bar{C}_D(\vec{r}_0) + \epsilon + j\omega; \ \vec{r}_0, \vec{v}_0) \neq 0$ for all $\omega \geq 0$. From Lemma 3.7 one concludes that the number of zeros of Δ_N in the right half plane

$$\mathcal{H} := \{\lambda \in \mathbb{C} : \ \Re(\lambda) > \bar{C}_D(\vec{r}_0) + \epsilon\}$$

is finite and invariant for $\|\vec{r} - \vec{r}_0\| < \delta$ and $\|v - v_0\| < \delta$, with δ sufficiently small. The assertion is a consequence of the continuity of the zeros of Δ_N in the half place $\mathcal{H}$ w.r.t. the delay parameters $\vec{r}, \vec{v}$ and the continuity of $\bar{C}_D$ w.r.t. $\vec{r}$. $\quad\Box$

REFERENCES

[1] T. H. S. ABDELAZIZ AND M. VALÁŠEK, *Direct algorithm for pole placement by state-derivative feedback for multi-input linear systems—nonsingular case*, Kybernetika, 41 (2005), pp. 637–660.

[2] T. H. S. ABDELAZIZ AND M. VALÁŠEK, *State derivative feedback by LQR for linear time-invariant systems*, in Proceedings of the 16th IFAC World Congress, Prague, Czech Republic, 2005.

[3] C. E. AVELLAR AND J. K. HALE, *On the zeros of exponential polynomials*, J. Math. Anal. Appl., 73 (1980), pp. 434–452.

[4] A. BELLEN, N. GUGLIELMI, AND A. E. RUEHLI, *Methods for linear systems of circuit delay differential equations of neutral type*, IEEE Trans. Circuits Syst. I, 76 (1999), pp. 212–215.

[5] J. BURKE AND M. OVERTON, *Differential properties of the spectral abscissa and the spectral radius for analytic matrix valued mappings*, Nonlinear Anal., 23 (1994), pp. 467–488.

[6] R. DATKO, *Two examples of ill-posedness with respect to time delays revisited*, IEEE Trans. Automat. Control, 42 (1997), pp. 434–452.

[7] K. ENGELBORGHS, M. DAMBRINE, AND D. ROOSE, *Limitations of a class of stabilization methods for delay equations*, IEEE Trans. Automat. Control, 46 (2001), pp. 336–339.

[8] K. GU, V. L. KHARITONOV, AND J. CHEN, *Stability of Time-Delay Systems*, Birkhäuser, Boston, 2003.

[9] J. K. HALE, *Parametric stability in difference equations*, Boll. Un. Mat. Ital., 11 (1975), pp. 209–214.

[10] J. K. HALE, *Effects of delays on dynamics*, in Topological Methods in Differential Equations and Inclusions, A. Granas, M. Frigon, and G. Sabidussi, eds., Kluwer Academic, Dordrecht, The Netherlands, 1995, pp. 191–238.

[11] J. K. HALE AND S. M. VERDUYN LUNEL, *Introduction to Functional Differential Equations*, Appl. Math. Sci. 99, Springer-Verlag, New York, 1993.

[12] J. K. HALE AND S. M. VERDUYN LUNEL, *Effects of small delays on stability and control*, Oper. Theory Adv. Appl., 122 (2001), pp. 275–301.

[13] J. K. HALE AND S. M. VERDUYN LUNEL, *Strong stabilization of neutral functional differential equations*, IMA J. Math. Control Inform., 19 (2002), pp. 5–23.

[14] J. K. HALE AND S. M. VERDUYN LUNEL, *Stability and control of feedback systems with time delays*, Internat. J. Systems Sci., 34 (2003), pp. 497–504.

[15] J. HELTON, S. A. MCCULLOUGH, AND V. VINNIKOV, *Noncommutative convexity arises from linear matrix inequalities*, J. Funct. Anal., 240 (2006), pp. 105–191.

[16] D. HENRY, *Linear autonomous neutral functional differential equations*, J. Differential Equations, 19 (1974), pp. 106–128.

[17] V. B. KOLMANOVSKII AND A. MYSHKIS, *Introduction to the Theory and Applications of Functional Differential Equations*, Math. Appl. 463, Kluwer Academic, Dordrecht, The Netherlands, 1999.

[18] H. LOGEMANN AND S. TOWNLEY, *The effect of small delays in the feedback loop on the stability of neutral systems*, Systems Control Lett., 27 (1996), pp. 267–274.

[19] H. LOGEMANN, R. REBARBER, AND G. WEISS, *Conditions for robustness and nonrobustness of the stability of feedback control systems with respect to small delays in the feedback loop*, SIAM J. Control Optim., 34 (1996), pp. 572–600.

[20] J.-F. MAGNI, *User Manual of the Linear Fractional Representation Toolbox*, Version 2.0, Technical Report TR 5/10403.01F DCSC, ONERA, Systems, Control and Flight Dynamics Department, Toulouse, France, 2005.

[21] W. MELVIN, *Stability properties of functional difference equations*, J. Math. Anal. Appl., 48 (1974), pp. 749–763.

[22] W. MICHIELS AND S.-I. NICULESCU, *Stability and Stabilization of Time-Delay Systems. An Eigenvalue-Based Approach*, Adv. Des. Control 12, SIAM, Philadelphia, 2007.

[23] W. MICHIELS AND T. VYHLÍDAL, *An eigenvalue based approach for the stabilization of linear time-delay systems of neutral type*, Automatica, 41 (2005), pp. 991–998.

[24] W. MICHIELS, K. ENGELBORGHS, D. ROOSE, AND D. DOCHAIN, *Sensitivity to infinitesimal delays in neutral equations*, SIAM J. Control Optim., 40 (2001), pp. 1134–1158.

[25] S. MONDIÉ AND W. MICHIELS, *Finite spectrum assignment of unstable time-delay systems with a safe implementation*, IEEE Trans. Automat. Control, 48 (2003), pp. 2207–2212.

[26] R. M. MURRAY, C. A. JACOBSON, R. CASAS, A. I. KHIBNIK, C. R. JOHNSON JR., B. R., A. A. PERACCHIO, AND W. M. PROSCIA, *System identification for limit cycling systems: A case study for combustion instabilities*, in Proceedings of the 1998 American Control Conference, Philadelphia, 1998, pp. 2004–2008.

[27] S.-I. NICULESCU AND B. BROGLIATO, *Force measurements time-delays and contact instability phenomenon*, European J. Control, 5 (1999), pp. 279–289.

[28] N. OLGAC, T. VYHLÍDAL, AND R. SIPAHI, *A new perspective in the stability assessment of neutral systems with multiple and cross-talking delays*, SIAM J. Control Optim., 47 (2008), pp. 327–344.

[29] L. PANDOLFI, *Stabilization of neutral functional differential equations*, J. Optim. Theory Appl., 20 (1976), pp. 191–204.

[30] T. VYHLÍDAL AND P. ZÍTEK, *Quasipolynomial mapping based rootfinder for analysis of time delay systems*, in Proceedings of the Fourth IFAC Workshop on Time-Delay Systems, Rocquencourt, France, 2003.

[31] T. VYHLÍDAL, P. MCGAHAN, W. MICHIELS, AND P. ZÍTEK, *Stability impact of the latency in the state derivative feedback*, in Proceedings of the 16th International Conference on Process Control, Strbske Pleso, Slovakia, 2007.

[32] T. VYHLÍDAL, W. MICHIELS, P. ZÍTEK, AND P. MCGAHAN, *Stability impact of small delays in proportional-derivative state feedback*, Control Engineering Practice, to appear.

[33] K. ZHOU, J. C. DOYLE, AND K. GLOVER, *Robust and Optimal Control*, Prentice–Hall, Englewood Cliffs, NJ, 1995.