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Strong stationarity for optimal control of the obstacle problem with control constraints

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We consider the distributed optimal control of the obstacle problem with control constraints. Since Mignot proved in 1976 the necessity of a system which is equivalent to strong stationarity, it has been an open problem whether such a system is still necessary in the presence of control constraints. Using moderate regularity of the optimal control and an assumption on the control bounds (which is implied by $u_a < 0 \le u_b$ quasi-everywhere (q.e.) in Ω in the case of an upper obstacle $y \le \psi$), we can answer this question in the affirmative. We also present counterexamples showing that strong stationarity may not hold if $u_a < 0$ or $0 \le u_b$ are violated.

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1. Introduction

We consider the distributed optimal control of the obstacle problem with control constraints

$$
\begin{aligned}
\text{Minimize} \quad & j(y) + \frac{\alpha}{2}\,\|u\|^2_{L^2(\Omega)}, \\
\text{with respect to} \quad & (y, u, \xi) \in H_0^1(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega), \\
\text{such that} \quad & \mathcal{A}\, y = u - \xi + f, \\
& 0 \ge y - \psi \;\perp\; \xi \ge 0, \\
\text{and} \quad & u_a \le u \le u_b \quad \text{a.e. in } \Omega.
\end{aligned}
\tag{$\mathbf{P}$}
$$

Here, the set $\Omega \subset \mathbb{R}^n$, $n \ge 1$, is open and bounded. The objective consists of an Fréchet-differentiable observation term $j : H_0^1(\Omega) \to \mathbb{R}$ of the state y and of an $L^2(\Omega)$-regularization term with $\alpha > 0$. The bounded linear operator $\mathcal{A} : H_0^1(\Omega) \to H^{-1}(\Omega)$ is assumed to be coercive. The right-hand side f belongs to $H^{-1}(\Omega)$. The control bounds satisfy $u_a, u_b \in H^1(\Omega)$. The obstacle $\psi \in H^1(\Omega)$ satisfies $\psi \ge 0$ on Γ in the sense that $\min\{\psi, 0\} \in H_0^1(\Omega)$. The complementarity condition

$$0 \ge y - \psi \;\perp\; \xi \ge 0$$

is to be understood in the dual pairing of $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, that is

$$y - \psi \le 0 \quad \text{a.e. in } \Omega, \tag{1.1a}$$

$$\langle \xi,\, v - y \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \le 0 \quad \text{for all } v \in H_0^1(\Omega) : v \le \psi \text{ a.e. in } \Omega. \tag{1.1b}$$

Note that both statements $\xi \ge 0$ and $y - \psi \perp \xi$ are contained in (1.1b), see Proposition 2.5. Since this complementarity condition is a constraint in ($\mathbf{P}$), all constraint qualifications (CQs) of a certain strength, e.g., the CQ of Robinson-Zowe-Kurcyusz, see Robinson [1976], Zowe and Kurcyusz [1979], are violated by ($\mathbf{P}$). Hence, proving necessary optimality conditions is difficult and, as in the case of finite-dimensional mathematical programs with complementarity constraints (MPCCs), there exists a wide variety of stationarity concepts, see [Scheel and Scholtes, 2000, Sec. 2] for stationarity concepts in finite dimensions and, e.g., Mignot [1976], Mignot and Puel [1984], Barbu [1984], Hintermüller and Kopacka [2009], Outrata et al. [2011], Hintermüller et al. [2013], Herzog et al. [2012, 2013], Schiela and Wachsmuth [2013] for stationarity conditions for MPCCs in function space.

Among these conditions, the so called *strong stationarity conditions* are the strongest. In finite dimensions, strong stationarity conditions are necessary for optimality if the MPCC satisfies the Guignard-CQ. The Guignard-CQ is in turn implied by MPCC-LICQ (and by even weaker CQs) and hence usually satisfied for MPCCs, see Flegel and Kanzow [2005].

For optimal control problems involving infinite dimensional complementarity systems, however, strong stationarity has been proved so far only under rather restrictive conditions, namely that the set of admissible controls U_{ad} (or, strictly speaking, its tangent cone) has to be dense in the set of right-hand sides of the state equation (here: $H^{-1}(\Omega)$). This condition excludes the case where control constraints are present (and active). For the derivation of strong stationarity we refer to [Mignot, 1976, Thm. 5.2], see also Mignot and Puel [1984], for the control of the obstacle problem with control in $L^2(\Omega)$, [Outrata et al., 2011, Thm. 6] for the case with controls in $H^{-1}(\Omega)$, and to [Herzog et al., 2013, Thm. 4.5] for an optimal-control problem arising in elasto-plasticity. Moreover, Mignot [1976] was able to derive a strongly stationary system in the cases where the control-to-state map is Fréchet differentiable, see [Mignot, 1976, p.161], or if the desired state y_d satisfies a certain condition rendering the objective convex, see [Mignot, 1976, p.166].

It has been a long-standing issue to prove or disprove the necessity of strong stationarity also in the case of control constraints. Besides this, we remark that strong stationarity is part of second order sufficient conditions (SSC), see [Kunisch and Wachsmuth, 2012, Thm. 2.2]. These SSCs provide a quadratic growth condition, which is in turn essential to prove discretization error estimates, see [Meyer and Thoma, 2013, Ass. 5.3, Thm. 5.7].

The main goal of this paper is to provide the necessity of strong stationarity under less restrictive assumptions than previously known. We only require some moderate regularity of the optimizer $\bar{u} \in H_0^1(\Omega)$ and a technical assumption (5.1) on the control constraints. This assumption is satisfied, e.g., if $u_a < 0 \le u_b$ holds q.e. on Ω, see Lemma 5.3. We refer to Section 2 for the notion of quasi-everywhere (q.e.). In particular, we do *not* require regularity of the domain Ω or of any of the active sets. For the problem under consideration, strong stationarity is defined in (1.3) after the introduction of some notation.

Let us mention that the discussion of the state-constrained problem (5.8), which appears as an auxiliary problem, is interesting in its own right. In dependence on the active sets, there may be no interior point of the feasible set w.r.t. the topology of $C(\bar{\Omega})$. We prove the existence of multipliers which belong to $H^{-1}(\Omega)$. This space is different from the measure space $\mathcal{M}(\Omega) = C_0(\Omega)'$, which is typically expected for state-constrained problems, see, e.g., Casas [1986]. A similar phenomenon was observed in Schiela [2009], where the existence of an interior point, however, was assumed in a space more regular than the state space.

Let us give a brief outline of the paper. In the remainder of the introduction, we fix some notation and introduce the system of strong stationarity. Some basic results on capacity theory are recalled in Section 2. In Section 3 we consider a linearization of (**P**), which is used in Sections 4 and 5 to prove additional properties of a local minimizer $\bar{u}$ and to show that strong stationarity is a necessary condition, respectively. We present two counterexamples in Section 6 demonstrating that strong stationarity may not hold when the assumption $u_a < 0 \le u_b$ is violated. In Appendix A, we give an explicit characterization of the strictly active set $\tilde{A}_s$, which differs from the usual definition of A_s in the literature. Our definition of $\tilde{A}_s$, see also Lemma 3.1, is more suited for our analysis, since it allows for a quasi-every formulation (see Section 2 for the definition of quasi-everywhere) of the cone $\mathcal{K}(\bar{u})$, which occurs in the linearized state equation, see (3.1).

Notation

We define the set of admissible controls

$$U_{\mathrm{ad}} := \{u \in L^2(\Omega) : u_a \le u \le u_b \text{ a.e. on } \Omega\},$$

and the closed convex set

$$K := \{y \in H_0^1(\Omega) : y \le \psi \text{ a.e. on } \Omega\}.$$

For a convex set $M \subset Y$ in a normed space Y and $y \in M$ we denote by $\mathcal{T}_M(y)$ the tangent cone of M at y, which is the closed conic hull of $M - y$. We use this notation for the sets $K \subset H_0^1(\Omega)$ and $U_{\mathrm{ad}} \subset L^2(\Omega)$.

For sets $M \subset H_0^1(\Omega)$ and $N \subset H^{-1}(\Omega)$ we define, as usual, the polar cones

$$\begin{aligned} M^\circ &:= \{f \in H^{-1}(\Omega) : \langle f,\, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \le 0 \text{ for all } v \in M\}, \\ N^\circ &:= \{v \in H_0^1(\Omega) : \langle f,\, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \le 0 \text{ for all } f \in N\}. \end{aligned}$$

Using this notation, the complementarity condition (1.1b) is equivalent to

$$\xi \in \mathcal{T}_K(y)^\circ. \tag{1.2}$$

In Section 5 we work with a closed subspace $V \subset H_0^1(\Omega)$. For subsets $M \subset V$ and $N \subset V'$ we define the polars w.r.t. the V-V' duality. We make also use of the polar cone of $\mathcal{T}_{U_{\mathrm{ad}}}(\bar u)$ w.r.t. the $L^2(\Omega)$-inner product which is denoted by

$$\mathcal{N}_{U_{\mathrm{ad}}}(\bar u) = \Big\{v \in L^2(\Omega) : \int_\Omega u\, v \,\mathrm{d}x \le 0 \text{ for all } u \in \mathcal{T}_{U_{\mathrm{ad}}}(\bar u)\Big\}.$$

Finally, for $\xi \in H^{-1}(\Omega)$, we define the annihilator

$$\xi^\perp := \{v \in H_0^1(\Omega) : \langle \xi,\, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = 0\}.$$

Strong stationarity

Using standard arguments, the existence of minimizers $(\bar y, \bar u, \bar \xi)$ of (**P**) can be shown under additional assumptions (in particular, j has to be bounded from below and weakly lower semicontinuous; and $u_a \le u_b$), see, e.g., [Mignot and Puel, 1984, Thm. 2.1].

Throughout the paper, we denote by $(\bar y, \bar u, \bar \xi)$ a local minimum of (**P**). We define (up to sets of zero capacity, see Definition 2.1 for the notion of capacity) the active sets w.r.t. the control constraints

$$\begin{aligned} A_a &:= \{x \in \Omega : \bar u(x) = u_a(x)\}, \\ A_b &:= \{x \in \Omega : \bar u(x) = u_b(x)\}, \end{aligned}$$

as well as the active set w.r.t. the constraint $y - \psi \le 0$ in the obstacle problem

$$A := \{x \in \Omega : 0 = \bar y(x) - \psi(x)\}.$$

By modifying A (or, equivalently, $\bar y$) on a set of zero capacity if necessary, we may assume that A is a Borel set, see Lemma 2.2. Note that the active sets A_a, A_b, A are quasi-closed, see Definition 2.1 for the notion of quasi-closeness.

We say that a feasible point $(\bar y, \bar u, \bar\xi)$ of (**P**) is strongly stationary, if there exist multipliers $(p, \mu, \nu) \in H_0^1(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$, such that the system of strong stationarity

$$\mathcal{A}^\star p + j'(\bar y) + \mu = 0 \quad \text{in } H^{-1}(\Omega), \tag{1.3a}$$
$$\alpha\,\bar u - p + \nu = 0 \quad \text{in } L^2(\Omega), \tag{1.3b}$$
$$-p \in \mathcal{T}_K(\bar y) \cap \bar\xi^\perp, \tag{1.3c}$$
$$\mu \in (\mathcal{T}_K(\bar y) \cap \bar\xi^\perp)^\circ, \tag{1.3d}$$
$$\nu \in \mathcal{N}_{U_{\text{ad}}}(\bar u) \tag{1.3e}$$

is satisfied. Here, $\mathcal{A}^\star : H_0^1(\Omega) \to H^{-1}(\Omega)$ is the adjoint operator of $\mathcal{A}$. Note that (1.3) is a generalization of the necessary conditions in the case without control constraints [Mignot, 1976, Thm. 4.3], [Mignot and Puel, 1984, Thm. 2.2], which can be obtained by setting $U_{\text{ad}} = L^2(\Omega)$ and hence $\nu = 0$ in (1.3).

Using the representation (3.4) of the set $\mathcal{T}_K(\bar y) \cap \bar\xi^\perp$, we can rewrite (1.3c) and (1.3d) equivalently as

$$p \ge 0 \text{ q.e. on } \tilde B \text{ and } p = 0 \text{ q.e. on } \tilde A_s, \tag{1.3c'}$$
$$\langle \mu,\, v\rangle_{H^{-1},H_0^1} \ge 0 \text{ for all } v \in H_0^1(\Omega) : v \ge 0 \text{ q.e. on } \tilde B \text{ and } v = 0 \text{ q.e. on } \tilde A_s. \tag{1.3d'}$$

The strongly active set $\tilde A_s$ and the biactive set $\tilde B = A \setminus \tilde A_s$ are defined in Lemma 3.1, see also Appendix A. We remark that (1.3c') and (1.3d') are falsely stated in many papers as

$$p \ge 0 \text{ a.e. on } A \setminus A_s \text{ and } p = 0 \text{ a.e. on } A_s, \tag{1.3c''}$$
$$\langle \mu,\, v\rangle_{H^{-1},H_0^1} \ge 0 \text{ for all } v \in H_0^1(\Omega) : v \ge 0 \text{ a.e. on } B \text{ and } v = 0 \text{ a.e. on } A_s, \tag{1.3d''}$$

where $A_s := \{x \in \Omega : \bar\xi(x) > 0\}$ for $\bar\xi \in L^2(\Omega)$ and $B := A \setminus A_s$. The first condition is weaker than (1.3c'), but the second one is stronger than (1.3d'). Moreover, it is easy to construct an example with $U_{\text{ad}} = L^2(\Omega)$, where A is a set of measure zero, but non-zero capacity. Then, (1.3d') is satisfied by a solution, whereas (1.3d'') may not hold.

We do not denote the strictly (biactive) set by A_s (B) in order to remind the reader that our definition of it differs from the usual definition in the literature.

2. Basics about capacity theory

In this section, we will recall some basic results in capacity theory. First, we give the definitions, see, e.g., [Attouch et al., 2006, Sec. 5.8.2], [Bonnans and Shapiro, 2000, Def. 6.47], and [Delfour and Zolésio, 2001, Sec. 8.6.1].

Definition 2.1. The *capacity* of a set $A \subset \Omega$ (w.r.t. $H_0^1(\Omega)$) is defined as

$$\operatorname{cap}(A) := \inf\big\{\|\nabla v\|^2_{L^2(\Omega)^n} : v \in H_0^1(\Omega), v \geq 1 \text{ a.e. in a neighbourhood of } A\big\}. \tag{2.1}$$

A function $v : \Omega \to \mathbb{R}$ is called *quasi-continuous* if for all $\varepsilon > 0$, there exists an open set $G_\varepsilon \subset \Omega$, such that $\operatorname{cap}(G_\varepsilon) < \varepsilon$ and v is continuous on $\Omega \setminus G_\varepsilon$.

A set $O \subset \Omega$ is called *quasi-open* if for all $\varepsilon > 0$, there exists an open set $G_\varepsilon \subset \Omega$, such that $\operatorname{cap}(G_\varepsilon) < \varepsilon$ and $O \cup G_\varepsilon$ is open.

Finally, $D \subset \Omega$ is called *quasi-closed* if $\Omega \setminus D$ is quasi-open.

A set of zero capacity has measure zero, but the converse does not hold.

It is known, see [Delfour and Zolésio, 2001, Thm. 6.1], that every $v \in H^1(\Omega)$ possesses a quasi-continuous representative. This representative is uniquely determined up to sets of zero capacity. When we speak about a function $v \in H^1(\Omega)$, we always mean the quasi-continuous representative. For every quasi-continuous function v, the set $\{x \in \Omega : v(x) \leq 0\}$ is quasi-closed, whereas $\{x \in \Omega : v(x) > 0\}$ is quasi-open. Every sequence which converges in $H_0^1(\Omega)$ possesses a pointwise quasi-everywhere convergent subsequence, see [Bonnans and Shapiro, 2000, Lem. 6.52].

The next lemma shows that quasi-open (and, similarly, quasi-closed) sets are "almost" Borel sets. This result is classical, but it is not contained in any of the above references. Hence, for the convenience of the reader, we state its proof.

Lemma 2.2. Let $O \subset \Omega$ be quasi-open. Then there exists a set $M \subset \Omega$, $\operatorname{cap}(M) = 0$, such that $O \cup M$ is a Borel set.

Proof. By definition, for all $\varepsilon > 0$, there exists an open set $G_\varepsilon \subset \Omega$, such that $\operatorname{cap}(G_\varepsilon) \leq \varepsilon$ and $O \cup G_\varepsilon$ is open. Then, we have $O \subset \bigcap_{i\in\mathbb{N}}(O \cup G_{1/i})$ and

$$\operatorname{cap}\Big(\Big[\bigcap_{i\in\mathbb{N}} O \cup G_{1/i}\Big] \setminus O\Big) \leq \operatorname{cap}\Big(\Big[\bigcap_{i\in\mathbb{N}} G_{1/i}\Big]\Big) = 0,$$

by the monotonicity of the capacity, see [Bonnans and Shapiro, 2000, Lem. 6.48]. Hence, O differs from the Borel set $\bigcap_{i\in\mathbb{N}}(O \cup G_{1/i})$ only by a set $M \subset \Omega$ with $\operatorname{cap}(M) = 0$.

We say that $v \geq 0$ holds quasi-everywhere (q.e.) on $O \subset \Omega$ if

$$\operatorname{cap}\big(\{v < 0\} \cap O\big) = 0.$$

The next lemma is essential for converting a.e.-statements into q.e.-statements. It is a slight generalization of [Bonnans and Shapiro, 2000, Lem. 6.49].

Lemma 2.3. Let $O \subset \Omega$ be a quasi-open subset and $v : \Omega \to \mathbb{R}$ a quasi-continuous function. Then, $v \geq 0$ a.e. on O implies $v \geq 0$ q.e. on O.

Proof. Let $\varepsilon > 0$ be given. Since O is quasi-open and v is quasi-continuous, there exist open sets $G_\varepsilon, H_\varepsilon$ such that v is continuous on $\Omega \setminus G_\varepsilon$, $O \cup H_\varepsilon$ is open, and $\operatorname{cap}(G_\varepsilon) \leq \varepsilon$, $\operatorname{cap}(H_\varepsilon) \leq \varepsilon$.

We set $U_\varepsilon = G_\varepsilon \cup H_\varepsilon$. Using the continuity of v on $\Omega \setminus U_\varepsilon$, the set $\{v < 0\} \cup U_\varepsilon$ is open. Hence, the set $(\{v < 0\} \cup U_\varepsilon) \cap (O \cup U_\varepsilon) = (\{v < 0\} \cap O) \cup U_\varepsilon$ is open.

Let a function $g \in H_0^1(\Omega)$ with $g \geq 1$ a.e. on U_ε be given. Then, $g \geq 1$ a.e. on $(\{v < 0\} \cap O) \cup U_\varepsilon$, since $\{v < 0\} \cap O$ has measure zero. By the definition of the capacity, this implies (note that both involved sets are open and hence neighborhoods of themselves)

$$\operatorname{cap}\big((\{v < 0\} \cap O) \cup U_\varepsilon\big) \leq \operatorname{cap}(U_\varepsilon).$$

Using the monotonicity and subadditivity of the capacity, see [Bonnans and Shapiro, 2000, Lem. 6.48], we obtain

$$\operatorname{cap}(\{v < 0\} \cap O) \leq \operatorname{cap}\big((\{v < 0\} \cap O) \cup U_\varepsilon\big) \leq \operatorname{cap}(U_\varepsilon) \leq 2\,\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have

$$\operatorname{cap}(\{v < 0\} \cap O) = 0.$$

By applying this lemma to $O = \Omega$, we find that $v \geq 0$ a.e. (on Ω) is equivalent to $v \geq 0$ q.e. (on Ω).

Finally, we recall some results on the relation between non-negative functionals in $H^{-1}(\Omega)$ and capacity theory, see [Bonnans and Shapiro, 2000, pp. 564–565].

Lemma 2.4. Let $\xi \in H^{-1}(\Omega)$ be a non-negative functional (i.e. ξ takes non-negative values on non-negative functions). Then, ξ can be identified with a regular Borel measure on Ω which is, in addition, finite on compact sets. Moreover, for every Borel set $D \subset \Omega$, $\operatorname{cap}(D) = 0$ implies $\xi(D) = 0$.

Finally, the quasi-continuous representative of every $v \in H_0^1(\Omega)$ is ξ-integrable and we have

$$\langle v, \xi \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} = \int_\Omega v \, \mathrm{d}\xi. \tag{2.2}$$

Note that, in particular, $v \geq 0$ q.e. implies $v \geq 0$ ξ-a.e. for all non-negative $\xi \in H^{-1}(\Omega)$.

Now, we are able to give an expression for the normal cone of K at a point $y \in K$, see also [Bonnans and Shapiro, 2000, Thm. 6.57] for the same result in the case $\psi = 0$.

Proposition 2.5. For $y \in K$ we have

$$\mathcal{T}_K(y) = \{v \in H_0^1(\Omega) : v \le 0 \text{ q.e. on } y = \psi\},$$
$$\mathcal{T}_K(y)^\circ = \{\xi \in H^{-1}(\Omega) : \xi \text{ is non-negative and } y - \psi = 0 \ \xi\text{-a.e. on } \Omega.\}$$

In particular, we have $\xi(\{x \in \Omega : y(x) < \psi(x)\}) = 0$ for $\xi \in \mathcal{T}_K(y)^\circ$.

Proof. The first identity is given in [Mignot, 1976, Lem. 3.2].

Let us prove the second identity.

"$\subset$": Let $\xi \in \mathcal{T}_K(y)^\circ$ be given. We start by proving that ξ is non-negative. For $w \in H_0^1(\Omega)$ with $w \ge 0$ a.e. in Ω, we have $v_w := y - w \le \psi$ a.e. in Ω. This implies $v_w \in K$ and, hence,

$$\langle \xi, w\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \langle \xi, y - v_w\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \ge 0 \quad \text{for all } w \in H_0^1(\Omega) : w \ge 0 \text{ a.e. in } \Omega. \tag{2.3}$$

By $y \in K$, we have $y - \psi \le 0$ a.e. in Ω and Lemma 2.3 implies $y - \psi \le 0$ q.e. in Ω. After modification of y on a set of capacity zero, the sets $\{y = \psi\}$, $\{y < \psi\}$ and $\{y > \psi\}$ are Borel sets, see Lemma 2.2. Now, Lemma 2.4 implies $y - \psi \le 0$ ξ-a.e. in Ω.

Now, let a smooth cut-off function $\chi \in C_0^\infty(\Omega)$ with $1 \ge \chi \ge 0$ and $\chi = 1$ on some compact $C \subset \Omega$ be given. By defining $w := \chi\,\psi + (1 - \chi)\,y \le \psi$ we obtain $w \in H_0^1(\Omega)$ and, in particular, $w \in K$. This yields

$$0 \ge \langle \xi, w - y\rangle = \int_\Omega \chi\,\psi + (1-\chi)\,y - y \,\mathrm{d}\xi = \int_\Omega \chi\,(\psi - y)\,\mathrm{d}\xi.$$

Since $\chi \ge 0$ everywhere and $\psi - y \ge 0$ ξ-a.e., we infer $\chi\,(y - \psi) = 0$ ξ-a.e., and in particular, $y - \psi = 0$ ξ-a.e. on C. Since Ω can be written as a countable union of compact sets and since ξ is countable additive, we have $y - \psi = 0$ ξ-a.e. on Ω. Finally,

$$\xi(\{y < \psi\}) \le \xi(\{y \ne \psi\}) = 0.$$

"$\supset$": Let ξ be non-negative with $y - \psi = 0$ ξ-a.e. on Ω, hence $\xi(\{y \ne \psi\}) = 0$.

For arbitrary $v \in \mathcal{T}_K(y)$ we obtain

$$\langle v, \xi\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} = \int_\Omega v\,\mathrm{d}\xi = \int_{\{y=\psi\}} v\,\mathrm{d}\xi + \int_{\{y\ne\psi\}} v\,\mathrm{d}\xi \le 0$$

since $\xi(\{y \ne \psi\}) = 0$ and $v \le 0$ q.e. on $\{y = \psi\}$ implies $v \le 0$ ξ-a.e. on $\{y = \psi\}$.

3. Linearization of the problem

We denote by $S : H^{-1}(\Omega) \to H_0^1(\Omega), u \mapsto y$ the solution operator of the variational inequality (VI)

$$\text{Find } y \in K, \quad \text{such that } \langle \mathcal{A}\, y - u - f,\, v - y\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \ge 0 \quad \text{for all } v \in K.$$

The unique solvability of this VI follows from [Kinderlehrer and Stampacchia, 1980, Thm. II.2.1]. It is known, that for given (y, u), there exists $\xi \in H^{-1}(\Omega)$ such that (y, u, ξ) is feasible for (**P**) if and only if $y = S(u)$ and $u \in U_{\text{ad}}$.

Since the obstacle $\psi \in H^1(\Omega)$ has a quasi-continuous representative, we can apply [Mignot, 1976, Thm. 3.2] to infer the polyhedricity of K. Hence, [Mignot, 1976, Thm. 2.1] yields the directional differentiability of S. Using the Lipschitz continuity of S, we find that S is even Hadamard-differentiable by [Shapiro, 1990, Prop. 3.5], see also [Bonnans and Shapiro, 2000, Thm. 6.58] for a similar argument in the case $\psi = 0$. The derivative $S'(\bar u; h)$ in the direction $h \in H^{-1}(\Omega)$ is the solution of the VI

$$\text{Find } y_h \in \mathcal{K}(\bar u), \quad \text{such that } \langle \mathcal{A}\, y_h - h,\, v - y_h\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \ge 0 \text{ for all } v \in \mathcal{K}(\bar u), \tag{3.1}$$

where

$$\mathcal{K}(\bar u) := \mathcal{T}_K(\bar y) \cap \bar\xi^\perp = \big\{y_h \in H_0^1(\Omega) : y_h \le 0 \text{ q.e. in } A \text{ and } \langle y_h,\, \bar\xi\rangle_{H_0^1,H^{-1}} = 0\big\}, \tag{3.2}$$

see [Mignot, 1976, Lem. 3.2]. The VI (3.1) is equivalent to the complementarity system

$$\mathcal{A}\, y_h - h + \xi_h = 0, \tag{3.3a}$$
$$y_h \in \mathcal{K}(\bar u), \tag{3.3b}$$
$$\xi_h \in \mathcal{K}(\bar u)^\circ, \tag{3.3c}$$
$$\langle \xi_h,\, y_h\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = 0. \tag{3.3d}$$

The following lemma provides a useful characterization of the closed convex cone $\mathcal{K}(\bar u)$ in terms of q.e.-(in)equalities.

Lemma 3.1. Let $\bar u \in H^{-1}(\Omega)$ be given and denote $\bar y = S(\bar u)$, $\bar\xi = \bar u - \mathcal{A}\,\bar y + f$. Then, there exists a set $\tilde A_s$, such that $\tilde A_s \subset A$ and

$$\begin{aligned} \mathcal{K}(\bar u) &= \big\{v \in H_0^1(\Omega) : v \le 0 \text{ q.e. in } A \text{ and } v = 0 \text{ q.e. in } \tilde A_s\big\} \\ &= \big\{v \in H_0^1(\Omega) : v \le 0 \text{ q.e. in } \tilde B \text{ and } v = 0 \text{ q.e. in } \tilde A_s\big\}, \end{aligned} \tag{3.4}$$

where $\tilde B := A \setminus \tilde A_s$ is the biactive set. In particular, we could choose $\tilde A_s$ to be quasi-closed.

Proof. Since $\bar\xi$ is a non-negative functional on $H_0^1(\Omega)$, we identify it with a regular Borel measure, see Lemma 2.4. Now, let $v \in H_0^1(\Omega)$ be given, satisfying $v \le 0$ q.e. on A. Lemma 2.4 implies $v \le 0$ $\bar\xi$-a.e. on A. Using (2.2), we have

$$\langle v, \bar\xi\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} = \int_\Omega v \,\mathrm{d}\bar\xi = \int_A v \,\mathrm{d}\bar\xi,$$

since $\bar\xi(\Omega \setminus A) = 0$, see Proposition 2.5. By using $v \le 0$ $\bar\xi$-a.e. on A,

$$\langle v, \bar\xi\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} = \int_A v \,\mathrm{d}\bar\xi = 0$$

is equivalent to $v = 0$ $\bar\xi$-a.e. on A. Since $\bar\xi(\Omega \setminus A) = 0$, this is in turn equivalent to $v = 0$ $\bar\xi$-a.e. on Ω.

The above reasoning shows

$$\mathcal{K}(\bar u) = \big\{v \in H_0^1(\Omega) : v \le 0 \text{ q.e. in } A \text{ and } v = 0\ \bar\xi\text{-a.e.}\big\}, \tag{3.5}$$

compare (3.2). Finally, [Stollmann, 1993, Thm. 1] implies the existence of a quasi-closed set $\tilde A_s$, such that

$$\big\{v \in H_0^1(\Omega) : v = 0\ \bar\xi\text{-a.e.}\big\} = \big\{v \in H_0^1(\Omega) : v = 0 \text{ q.e. in } \tilde A_s\big\}.$$

It remains to show $\tilde A_s \subset A$. Since $\bar y - \psi = 0$ $\bar\xi$-a.e., we have $\bar y - \psi = 0$ q.e. on $\tilde A_s$, hence $\operatorname{cap}(\tilde A_s \setminus A) = 0$. Replacing $\tilde A_s$ by $\tilde A_s \cap A$ yields the claim.

Note that we give a more explicit characterization of the strictly active set $\tilde A_s$ in Appendix A, see in particular Lemma A.5. Moreover, we do not denote the strictly active set by A_s in order to remind the reader that our definition of it differs from the usual definition in the literature.

We consider the reduced formulation of (**P**)

$$\begin{aligned} \text{Minimize} \quad & j(S(u)) + \frac{\alpha}{2}\,\|u\|^2_{L^2(\Omega)} \\ \text{and} \quad & u \in U_{\mathrm{ad}}. \end{aligned} \tag{$\mathbf{P}_{\mathrm{red}}$}$$

Due to the continuity of S, $(\bar y, \bar u, \bar\xi)$ is a local solution of (**P**) if and only if $\bar u$ is a local solution of ($\mathbf{P}_{\mathrm{red}}$). The local optimality of $\bar u$ for ($\mathbf{P}_{\mathrm{red}}$) and the Hadamard-differentiability of S imply that $h = 0$ is a global solution of the "linearized" problem

$$\begin{aligned} \text{Minimize} \quad & j'(S(\bar u))\, S'(\bar u; h) + \alpha\,(\bar u,\, h)_{L^2(\Omega)} \\ \text{such that} \quad & h \in \mathcal{T}_{U_{\mathrm{ad}}}(\bar u), \end{aligned} \tag{$\mathbf{P}_{\mathrm{red}}^{\mathrm{lin}}$}$$

where $\mathcal{T}_{U_{\mathrm{ad}}}(\bar u) \subset L^2(\Omega)$ is the tangent cone of U_{ad}.

In the sequel, we will consider different restrictions of this linearized problem in order to prove properties of the minimizer $\bar u$, see Section 4, and to prove the strong stationarity of $\bar u$ in Section 5.

4. Properties of local solutions

We are going to prove properties of a local minimizer $\bar u$ by evaluating optimality conditions of a certain restriction of $(\mathbf{P}^{\text{lin}}_{\text{red}})$. From now on, we assume the additional regularity $\bar u \in H^1_0(\Omega)$. This property can be shown by penalization arguments, see, e.g., Mignot and Puel [1984], Schiela and Wachsmuth [2013], or by limiting variational calculus, see [Hintermüller et al., 2013, Rem. 1]. In order to keep the presentation simple, we just *assume* this regularity, keeping in mind that it can be achieved under rather mild assumptions on the data, in particular one uses $u_a, u_b \in H^1_0(\Omega)$.

We interpret $L^2(\Omega)$ as an subspace of $H^{-1}(\Omega)$ via the canonical embedding $E : L^2(\Omega) \to H^{-1}(\Omega)$, $h \mapsto (v \mapsto \int_\Omega h\, v \,\mathrm{d}x)$. Up to now we did not mention this embedding in favor of a clearer presentation. In order to get sharper optimality conditions for $(\mathbf{P}^{\text{lin}}_{\text{red}})$ we are going to enlarge the feasible set. The closeness of the constraint set in $H^{-1}(\Omega)$ in (4.1) will be crucial for rewriting (4.5) into (4.6).

Lemma 4.1. The functional $h = 0 \in H^{-1}(\Omega)$ is a global minimizer of

$$\begin{aligned} \text{Minimize} \quad & j'(S(\bar u))\, S'(\bar u; h) + \alpha\, \langle \bar u,\, h\rangle_{H^1_0(\Omega, H^{-1}(\Omega))} \\ \text{such that} \quad & h \in \overline{E\mathcal{T}_{U_{\text{ad}}}(\bar u)}^{H^{-1}(\Omega)}. \end{aligned} \tag{4.1}$$

Proof. By using the canonical embedding $E : L^2(\Omega) \to H^{-1}(\Omega)$ in $(\mathbf{P}^{\text{lin}}_{\text{red}})$, we obtain that $h = 0 \in H^{-1}(\Omega)$ is a global solution of

$$\begin{aligned} \text{Minimize} \quad & j'(S(\bar u))\, S'(\bar u; h) + \alpha\, \langle \bar u,\, h\rangle_{H^1_0(\Omega, H^{-1}(\Omega))} \\ \text{such that} \quad & h \in E\mathcal{T}_{U_{\text{ad}}}(\bar u). \end{aligned} \tag{4.2}$$

We proceed by contradiction and assume that 0 is not a global solution of (4.1). This yields the existence of $h \in \overline{E\mathcal{T}_{U_{\text{ad}}}(\bar u)}^{H^{-1}(\Omega)}$ with

$$j'(S(\bar u))\, S'(\bar u; h) + \alpha\, \langle \bar u,\, h\rangle_{H^1_0(\Omega, H^{-1}(\Omega))} < 0.$$

Since $E\mathcal{T}_{U_{\text{ad}}}(\bar u)$ is dense in $\overline{E\mathcal{T}_{U_{\text{ad}}}(\bar u)}^{H^{-1}(\Omega)}$ and since the objective in (4.1) is continuous w.r.t. $h \in H^{-1}(\Omega)$, this gives the existence of $\tilde h \in E\mathcal{T}_{U_{\text{ad}}}(\bar u)$ with

$$j'(S(\bar u))\, S'(\bar u; \tilde h) + \alpha\, \langle \bar u,\, \tilde h\rangle_{H^1_0(\Omega, H^{-1}(\Omega))} < 0.$$

This, however, is a contradiction to the fact that $h = 0$ is a global minimizer of (4.2). ∎

Using the equivalent reformulation (3.3) of the linearized VI (3.1), the problem (4.1) can be written as

$$\begin{aligned}
\text{Minimize}\quad & j'(S(\bar u))\, y_h + \alpha\, \langle \bar u,\, h\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} \\
\text{with respect to}\quad & (y_h, h, \xi_h) \in H_0^1(\Omega) \times H_0^1(\Omega) \times H^{-1}(\Omega), \\
\text{such that}\quad & \mathcal{A}\, y_h - h + \xi_h = 0, \\
& y_h \in \mathcal{K}(\bar u), \\
& \xi_h \in \mathcal{K}(\bar u)^\circ, \\
& \langle y_h,\, \xi_h\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} = 0, \\
& h \in \overline{E\mathcal{T}_{U_{\mathrm{ad}}}(\bar u)}^{H^{-1}(\Omega)}.
\end{aligned} \tag{4.3}$$

Since $h = 0$ is a global minimizer of (4.1), $(y_h, h, \xi_h) = (0, 0, 0)$ is a global minimizer of (4.3). Note that (4.3) still contains a complementarity constraint (as long as $\mathcal{K}(\bar u)$ is not a subspace). By restricting y_h to zero, we obtain that $(h, \xi_h) = (0, 0)$ is a global solution of the auxiliary problem

$$\begin{aligned}
\text{Minimize}\quad & \alpha\, \langle \bar u,\, h\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} \\
\text{with respect to}\quad & (h, \xi_h) \in H^{-1}(\Omega) \times H^{-1}(\Omega), \\
\text{such that}\quad & -h + \xi_h = 0, \\
& \xi_h \in \mathcal{K}(\bar u)^\circ, \\
& h \in \overline{E\mathcal{T}_{U_{\mathrm{ad}}}(\bar u)}^{H^{-1}(\Omega)}.
\end{aligned} \tag{4.4}$$

Making use of the constraint $h = \xi_h$, $h = 0$ is a global solution of

$$\begin{aligned}
\text{Minimize}\quad & \alpha\, \langle \bar u,\, h\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} \\
\text{such that}\quad & h \in \mathcal{K}(\bar u)^\circ \cap \overline{E\mathcal{T}_{U_{\mathrm{ad}}}(\bar u)}^{H^{-1}(\Omega)}.
\end{aligned} \tag{4.5}$$

The optimality condition reads (note that this requires no constraint qualification)

$$\alpha\, \bar u \in -\Big[\mathcal{K}(\bar u)^\circ \cap \overline{E\mathcal{T}_{U_{\mathrm{ad}}}(\bar u)}^{H^{-1}(\Omega)}\Big]^\circ,$$

where the polar cones are to be evaluated w.r.t. the $H^{-1}(\Omega)$-$H_0^1(\Omega)$ duality. By using $(K_1 \cap K_2)^\circ = \overline{K_1^\circ + K_2^\circ}$ for closed, convex cones K_1, K_2 in a reflexive Banach space, see, e.g., [Bonnans and Shapiro, 2000, (2.32)], we obtain

$$\Big[\mathcal{K}(\bar u)^\circ \cap \overline{E\mathcal{T}_{U_{\mathrm{ad}}}(\bar u)}^{H^{-1}(\Omega)}\Big]^\circ = \overline{\mathcal{K}(\bar u) + \Big(\overline{E\mathcal{T}_{U_{\mathrm{ad}}}(\bar u)}^{H^{-1}(\Omega)}\Big)^\circ}^{H_0^1(\Omega)}.$$

Since $A^\circ = (\,\overline{A}\,)^\circ$ holds for all sets A, we get

$$\alpha\, \bar u \in -\Big[\mathcal{K}(\bar u)^\circ \cap \overline{E\mathcal{T}_{U_{\mathrm{ad}}}(\bar u)}^{H^{-1}(\Omega)}\Big]^\circ = -\overline{\mathcal{K}(\bar u) + (E\mathcal{T}_{U_{\mathrm{ad}}}(\bar u))^\circ}^{H_0^1(\Omega)}. \tag{4.6}$$

It remains to evaluate the right-hand side.

Lemma 4.2. The polar cone of $E\mathcal{T}_{U_{\text{ad}}}(\bar u)$ w.r.t. the $H^{-1}(\Omega)$-$H_0^1(\Omega)$ duality is given by

$$(E\mathcal{T}_{U_{\text{ad}}}(\bar u))^\circ = \big\{v \in H_0^1(\Omega) : v \le 0 \text{ q.e. on } A_a,\ v \ge 0 \text{ q.e. on } A_b, \text{ and } v = 0 \text{ q.e. on } \Omega \setminus (A_a \cup A_b)\big\}.$$

Proof. A simple calculation shows

$$\begin{aligned}(E\mathcal{T}_{U_{\text{ad}}}(\bar u))^\circ &= \big\{v \in H_0^1(\Omega) : \langle v,\, h\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} \le 0 \text{ for all } h \in E\mathcal{T}_{U_{\text{ad}}}(\bar u)\big\}\\ &= \big\{v \in H_0^1(\Omega) : \int_\Omega v\, u \,\mathrm{d}x \le 0 \text{ for all } u \in \mathcal{T}_{U_{\text{ad}}}(\bar u)\big\}\\ &= \{v \in H_0^1(\Omega) : v \le 0 \text{ a.e. on } A_a,\ v \ge 0 \text{ a.e. on } A_b, \text{ and}\\ &\qquad v = 0 \text{ a.e. on } \Omega \setminus (A_a \cup A_b)\}.\end{aligned}$$

Now, the inclusion "$\supset$" of the assertion follows easily, since $v \le 0$ q.e. on A_a implies $v \le 0$ a.e. on A_a, and analogous arguments for the other conditions.

Let $v \in (E\mathcal{T}_{U_{\text{ad}}}(\bar u))^\circ$ be given. By the above calculation, we have $v \ge 0$ a.e. on $\Omega \setminus A_a$. The set $\Omega \setminus A_a = \{x \in \Omega : \bar u(x) < u_a(x)\}$ is quasi-open. Using Lemma 2.3, we find that $v \ge 0$ a.e. on $\Omega \setminus A_a$ implies $v \ge 0$ q.e. on $\Omega \setminus A_a$ and, in particular, $v \ge 0$ q.e. on A_b. Similarly, we obtain $v \le 0$ q.e. on $\Omega \setminus A_b \supset A_a$ and $v = 0$ q.e. on $\Omega \setminus (A_a \cup A_b)$. This shows the claim.

Now, we obtain the announced properties of the local minimizer $\bar u$. We recall the definition of the biactive set $\tilde B = A \setminus \tilde A_s$ from Lemma 3.1.

Lemma 4.3. If $\bar u$ belongs to $H_0^1(\Omega)$, we have the sign conditions

$$\begin{aligned}\bar u &= 0 \quad \text{q.e. on } \tilde A_s \cap [\Omega \setminus (A_a \cup A_b)],\\ \bar u &\le 0 \quad \text{q.e. on } \tilde A_s \cap A_b,\\ \bar u &\ge 0 \quad \text{q.e. on } (\tilde A_s \cap A_a) \cup [\tilde B \cap (\Omega \setminus A_b)].\end{aligned}$$

In particular, $u_b \ge 0$ q.e. on A and $u_a \le 0$ q.e. on $\tilde A_s$ imply $\bar u \ge 0$ q.e. on $\tilde B$ and $\bar u = 0$ q.e. on $\tilde A_s$.

Proof. By using (4.6), there are sequences $\{v_1^{(i)}\} \subset -\mathcal{K}(\bar u)$ and $\{v_2^{(i)}\} \subset -(E\mathcal{T}_{U_{\text{ad}}}(\bar u))^\circ$, such that

$$\bar u = \lim_{i\to\infty} \big(v_1^{(i)} + v_2^{(i)}\big) \quad \text{in } H_0^1(\Omega).$$

After passing to a subsequence, we have the pointwise convergence

$$\bar u = \lim_{i\to\infty} \big(v_1^{(i)} + v_2^{(i)}\big) \quad \text{q.e. in } \Omega, \tag{4.7}$$

see [Bonnans and Shapiro, 2000, Lem. 6.52]. By using Lemma 3.1 we know

$$v_1^{(i)} \in -\mathcal{K}(\bar{u}) = \{v \in H_0^1(\Omega) : v \geq 0 \text{ q.e. in } \tilde{B} \text{ and } v = 0 \text{ q.e. in } \tilde{A}_s\}$$

and by Lemma 4.2 we have

$$-v_2^{(i)} = -(E\mathcal{T}_{U_{\text{ad}}}(\bar{u}))^\circ = \{v \in H_0^1(\Omega) : v \geq 0 \text{ q.e. on } A_a,\, v \leq 0 \text{ q.e. on } A_b, \text{ and } v = 0 \text{ q.e. on } \Omega \setminus (A_a \cup A_b)\}.$$

That is, we have the following q.e. sign conditions:

$$\begin{aligned} &\text{on } \tilde{A}_s \cap [\Omega \setminus (A_a \cup A_b)] : && v_1^{(i)} = 0 \quad \text{and} \quad v_2^{(i)} = 0, \\ &\text{on } \tilde{A}_s \cap A_b : && v_1^{(i)} = 0 \quad \text{and} \quad v_2^{(i)} \leq 0, \\ &\text{on } \tilde{A}_s \cap A_a : && v_1^{(i)} = 0 \quad \text{and} \quad v_2^{(i)} \geq 0, \\ &\text{on } \tilde{B} \cap (\Omega \setminus A_b) : && v_1^{(i)} \geq 0 \quad \text{and} \quad v_2^{(i)} \geq 0. \end{aligned}$$

Together with (4.7), this gives the desired sign conditions of $\bar{u}$.

5. Strong stationarity

We use the results of the previous section together with the KKT-conditions of a restriction of ($\mathbf{P}_{\text{red}}^{\text{lin}}$) in order to prove necessity of the strong stationarity system (1.3). In addition to $\bar{u} \in H_0^1(\Omega)$, we assume

$$u_b \geq 0 \quad \text{q.e. in } \tilde{B}, \tag{5.1a}$$

$$\operatorname{cap}(A_a \cap \tilde{B}) = 0, \tag{5.1b}$$

$$\bar{u} = 0 \quad \text{q.e. on } \tilde{A}_s. \tag{5.1c}$$

We refer to Lemma 5.3 for a simple condition which implies that this assumption is satisfied.

We start by restating ($\mathbf{P}_{\text{red}}^{\text{lin}}$) in a subspace of $H_0^1(\Omega)$. Therefore, we recall the characterization

$$\mathcal{K}(\bar{u}) = \{v \in H_0^1(\Omega) : v \leq 0 \text{ q.e. in } \tilde{B} \text{ and } v = 0 \text{ q.e. in } \tilde{A}_s\}$$

from Lemma 3.1. We define

$$V = \{v \in H_0^1(\Omega) : v = 0 \text{ q.e. in } \tilde{A}_s\}. \tag{5.2}$$

Note that the subspace V is closed, since sequences converging in $H_0^1(\Omega)$ contain a pointwise quasi-everywhere convergent subsequence, see [Bonnans and Shapiro, 2000, Lem. 6.52]. Since $\mathcal{K}(\bar{u})$ is a subset of the closed subspace V, we can restate the VI

(3.1) characterizing the derivative $S'(\bar u; h)$ in the space V. To this end, we introduce the canonical injection

$$I : V \to H_0^1(\Omega),\ v \mapsto v. \tag{5.3}$$

The action of its adjoint $I^\star : H^{-1}(\Omega) \to V'$ is the restriction of the domain of a functional from $H_0^1(\Omega)$ to V. Further, we introduce the bounded, linear operator

$$\mathcal{A}_V = I^\star \mathcal{A}\, I : V \to V',$$

which inherits the ellipticity from $\mathcal{A}$, and the closed convex cone

$$\mathcal{K}_V(\bar u) = \big\{ v \in V : v \le 0 \text{ q.e. in } \tilde B \big\}.$$

Note that $\mathcal{K}(\bar u) = I\,\mathcal{K}_V(\bar u)$. Let us recall the VI (3.1) characterizing the derivative $S'(\bar u; h)$ of S in direction $h \in H^{-1}$

$$\text{Find } y_h \in \mathcal{K}(\bar u), \quad \text{such that } \langle \mathcal{A}\, y_h - h,\, v - y_h \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \ge 0 \text{ for all } v \in \mathcal{K}(\bar u).$$

We define $S_V'(\bar u; h)$ for $h \in V'$ as the unique solution of

$$\text{Find } y_h \in \mathcal{K}_V(\bar u), \quad \text{such that } \langle \mathcal{A}_V\, y_h - h,\, v - y_h \rangle_{V', V} \ge 0 \text{ for all } v \in \mathcal{K}_V(\bar u), \tag{5.4}$$

see [Kinderlehrer and Stampacchia, 1980, Thm. II.2.1] for the unique solvability. An immediate consequence is

$$S'(\bar u; h) = I\, S_V'(\bar u; I^\star h)$$

for all $h \in H^{-1}(\Omega)$. Using this equivalence and $\bar u \in V$ by (5.1c), we obtain from ($\mathbf{P}^{\text{lin}}_{\text{red}}$) that $h = 0 \in V'$ is a global solution of

$$\begin{aligned} \text{Minimize} \quad & j'(S(\bar u))\, I\, S_V'(\bar u; h) + \alpha\, \langle \bar u,\, h \rangle_{V,V'} \\ \text{such that} \quad & h \in I^\star E \mathcal{T}_{U_{\text{ad}}}(\bar u). \end{aligned} \tag{5.5}$$

Arguing similarly as in Lemma 4.1, we obtain that $h = 0 \in V'$ is a global solution of

$$\begin{aligned} \text{Minimize} \quad & j'(S(\bar u))\, I\, S_V'(\bar u; h) + \alpha\, \langle \bar u,\, h \rangle_{V,V'} \\ \text{such that} \quad & h \in \overline{I^\star E \mathcal{T}_{U_{\text{ad}}}(\bar u)}^{V'}. \end{aligned} \tag{5.6}$$

Similar to (3.3), we can rewrite the VI (5.4) as a complementarity system and obtain that $(y_h, h, \xi_h) = (0, 0, 0)$ is a global solution of

$$\begin{aligned} \text{Minimize} \quad & j'(S(\bar u))\, I\, y_h + \alpha\, \langle \bar u,\, h \rangle_{V,V'} \\ \text{such that} \quad & \mathcal{A}_V\, y_h - h + \xi_h = 0, \\ & y_h \in \mathcal{K}_V(\bar u), \\ & \xi_h \in \mathcal{K}_V(\bar u)^\circ, \\ & \langle y_h,\, \xi_h \rangle_{V,V'} = 0, \\ & h \in \overline{I^\star E \mathcal{T}_{U_{\text{ad}}}(\bar u)}^{V'}. \end{aligned} \tag{5.7}$$

We restrict the slack variable ξ_h to 0. This enables us to drop the complementarity condition. We obtain that $(h, y_h) = (0, 0)$ is a global solution of the auxiliary problem

$$\begin{aligned} \text{Minimize} \quad & j'(S(\bar u))\, I\, y_h + \alpha\, \langle \bar u,\, h\rangle_{V,V'} \\ \text{such that} \quad & \mathcal{A}_V\, y_h - h = 0, \\ & y_h \in \mathcal{K}_V(\bar u), \\ & h \in \overline{I^\star E \mathcal{T}_{U_{\text{ad}}}(\bar u)}^{V'}. \end{aligned} \tag{5.8}$$

Due to this restriction of ξ_h, the optimality system (5.10) of the problem (5.8) will not contain any information of p on $\tilde B$. However, this information can be recovered by the gradient equation (5.10b) and the signs of $\bar u$ from Lemma 4.3. Note that this relies heavily on the fact that the control lives on the same domain as the constraint $y \le \psi$.

Lemma 5.1. The polar cone of $I^\star E \mathcal{T}_{U_{\text{ad}}}(\bar u) \subset V'$ is given by

$$\begin{aligned} (I^\star E \mathcal{T}_{U_{\text{ad}}}(\bar u))^\circ = \big\{ v \in V : v \le 0 \text{ q.e. on } A_a,\ v \ge 0 \text{ q.e. on } A_b, \text{ and} \\ v = 0 \text{ q.e. on } \Omega \setminus (A_a \cup A_b) \big\}. \end{aligned} \tag{5.9}$$

Proof. A simple calculation, see also [Aubin and Frankowska, 2009, Lem. 2.4.3], shows

$$(I^\star E \mathcal{T}_{U_{\text{ad}}}(\bar u))^\circ = I^{-1} (E \mathcal{T}_{U_{\text{ad}}}(\bar u))^\circ,$$

where the right-hand side denotes the preimage of $(E \mathcal{T}_{U_{\text{ad}}}(\bar u))^\circ$ w.r.t. the injection $I : V \to H_0^1(\Omega)$. Now, Lemma 4.2 yields the claim.

We show that the CQ of Robinson-Zowe-Kurcyusz is satisfied at the solution $(y_h, h) = (0, 0)$ of (5.8). Let an arbitrary $\mu \in V'$ be given. We have to show the existence of $y_h \in \mathcal{K}_V(\bar u)$, $h \in \overline{I^\star E \mathcal{T}_{U_{\text{ad}}}(\bar u)}^{V'}$, such that $\mathcal{A}_V\, y_h - h = \mu$. We set $y_h = S_V'(\bar u, \mu) \in \mathcal{K}_V(\bar u)$. Then, there exists $h \in -\mathcal{K}_V(\bar u)^\circ$ such that

$$\mathcal{A}\, y_h - h = \mu$$

and $\langle y_h,\, h\rangle_{V,V'} = 0$ (we do not use this condition) are satisfied. Note that we have

$$\begin{aligned} (I^\star E \mathcal{T}_{U_{\text{ad}}}(\bar u))^\circ &\subset \{ v \in V : v \le 0 \text{ q.e. on } A_a,\ v \ge 0 \text{ q.e. on } \Omega \setminus A_a \} \\ &\subset \{ v \in V : v \ge 0 \text{ q.e. on } \tilde B \} = -\mathcal{K}_V(\bar u) \end{aligned}$$

by (5.1b) and the definition of $\mathcal{K}_V(\bar u)$. Hence, $h \in -\mathcal{K}_V(\bar u)^\circ \subset (I^\star E \mathcal{T}_{U_{\text{ad}}}(\bar u))^{\circ\circ} = \overline{I^\star E \mathcal{T}_{U_{\text{ad}}}(\bar u)}^{V'}$. This shows that the CQ of Robinson-Zowe-Kurcyusz is satisfied by the problem (5.8).

Hence, there exists multipliers $(p, \tilde\mu, \nu) \in V \times V' \times V$ satisfying the optimality system

$$\mathcal{A}_V^\star p + I^\star j'(S(\bar u)) + \tilde\mu = 0 \text{ in } V', \tag{5.10a}$$
$$\alpha \bar u - p + \nu = 0 \text{ in } V, \tag{5.10b}$$
$$\tilde\mu \in \{y \in V : y \le 0 \text{ q.e. in } \tilde B\}^\circ, \tag{5.10c}$$
$$\nu \in \big(\overline{I^\star E \mathcal{T}_{U_{\text{ad}}}(\bar u)}^{V'}\big)^\circ = (I^\star E \mathcal{T}_{U_{\text{ad}}}(\bar u))^\circ. \tag{5.10d}$$

Now we show that the system (1.3) is satisfied, where $\mu \in H^{-1}(\Omega)$ is defined by

$$\mu = -\mathcal{A}^\star p - j'(S(\bar u)).$$

Due to this definition of μ, (1.3a) holds. The gradient equation (1.3b) follows from (5.10b), since $\bar u$, p, and ν are zero on $\tilde A_s$.

By definition of p, we have $p = 0$ q.e. on $\tilde A_s$. By the gradient equation (5.10b), we obtain

$$p = \alpha \bar u + \nu \ge \alpha \bar u \ge 0 \quad \text{q.e. on } \tilde B.$$

The first inequality follows from (5.1b) and (5.9), whereas the second one follows from (5.1a) and Lemma 4.3. Hence, (1.3c') is satisfied.

In order to show the sign condition (1.3d') on μ, let $v \in H_0^1(\Omega)$, $v \le 0$ q.e. on $\tilde B$ and $v = 0$ q.e. on $\tilde A_s$ be given. Using $v \in V$, we obtain from the definition of μ and (5.10c)

$$\begin{aligned}\langle \mu, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} &= \langle -\mathcal{A}^\star p - j'(S(\bar u)), v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ &= \langle -\mathcal{A}_V^\star p - I^\star j'(S(\bar u)), v\rangle_{V',V} = \langle \tilde\mu, v\rangle_{V',V} \le 0.\end{aligned}$$

This is the desired sign condition on μ.

Since $V \subset L^2(\Omega)$, we have (the following inequalities are to be understood in the a.e.-sense)

$$\begin{aligned}&\nu \in (I^\star E \mathcal{T}_{U_{\text{ad}}}(\bar u))^\circ \\ &= \big\{v \in V \quad\quad : v \le 0 \text{ on } A_a,\ v \ge 0 \text{ on } A_b, \text{ and } v = 0 \text{ on } \Omega \setminus (A_a \cup A_b)\big\} \\ &\subset \big\{v \in L^2(\Omega) : v \le 0 \text{ on } A_a,\ v \ge 0 \text{ on } A_b, \text{ and } v = 0 \text{ on } \Omega \setminus (A_a \cup A_b)\big\} \\ &= \mathcal{N}_{U_{\text{ad}}}(\bar u),\end{aligned}$$

which is the sign condition (1.3e) on ν.

Altogether, we have proven the following theorem.

Theorem 5.2. Let $(\bar y, \bar u, \bar\xi) \in H_0^1(\Omega) \times H_0^1(\Omega) \times H^{-1}(\Omega)$ be a local solution of (**P**), such that (5.1) holds. Then, there exist multipliers $(p, \mu, \nu) \in H_0^1(\Omega) \times H^{-1}(\Omega) \times H_0^1(\Omega)$, such that the strong stationarity conditions (1.3) are satisfied.

Note that the uniqueness of multipliers does not simply follow from (1.3) (as in the case without control constraints).

The arguments leading to Theorem 5.2 remain valid in the cases $u_a = -\infty$ or $u_b = +\infty$, with the obvious modifications.

As announced, we remark that assumption (5.1) is implied by a simple assumption on the control bounds, which can be checked a-priori.

Lemma 5.3. If the bounds $u_a, u_b \in H^1(\Omega)$ fulfill

$$u_a < 0 \le u_b \text{ q.e. in } \Omega, \tag{5.11}$$

then (5.1) is satisfied.

Note that we do not need to assume that u_a is uniformly negative in (5.11).

Proof. It is clear that (5.1a) holds. Lemma 4.3 implies $\bar{u} = 0$ q.e. on $\tilde{A}_s$, i.e. (5.1c), and $\bar{u} \ge 0$ q.e. on A. Hence, we have $\bar{u} = 0 > u_a$ q.e. on A. This shows (5.1b).

Finally, we give a remark on the condition (5.1c). By inspecting the calculation leading to Theorem 5.2, we find that this assumption could be replaced by the following weaker one: assume that

$$\tilde{u} = \begin{cases} 0 & \text{on } \tilde{A}_s \\ \bar{u} & \text{on } \Omega \setminus \tilde{A}_s \end{cases} \quad \text{belongs to } H_0^1(\Omega). \tag{5.12}$$

Moreover, we could drop the assumption $\bar{u} = 0$ on $\tilde{A}_s$ if we could discuss an auxiliary problem similar to (5.8) directly in $H^{-1}(\Omega) \times H_0^1(\Omega)$. However, we were not able to provide a CQ for such an auxiliary problem.

6. Counterexamples

In this section we present two counterexamples, which show that strong stationarity may not hold if $u_a < 0$ or $u_b \ge 0$ are violated. Note that we do not have a counterexample if $\bar{u} = 0$ on $\tilde{A}_s$ is violated. In both examples, the domain is $\Omega = (0, 1)$ and $\mathcal{A} = -\Delta$, i.e., $\mathcal{A} y = -y''$.

6.1. The lower bound is zero and active

This counterexample, which was constructed by the author, can already be found in Schiela and Wachsmuth [2013]. We consider

$$\begin{aligned} \text{Minimize} \quad & \frac{1}{2} \|y+1\|^2_{L^2(\Omega)} + \frac{\alpha}{2} \|u\|^2_{L^2(\Omega)}, \\ \text{such that} \quad & \mathcal{A} y = u - \xi, \\ & 0 \ge y \;\perp\; \xi \ge 0, \\ \text{and} \quad & u \ge 0. \end{aligned}$$

For all feasible u, the solution of the complementarity system is $(y, \xi) = (0, u)$. Hence, the unique global solution of this problem is $(\bar y, \bar u, \bar\xi) = (0, 0, 0)$. Using $A = A_a = \Omega$ and $\tilde A_s = \emptyset$, the system of strong stationarity (1.3) reads

$$\begin{aligned} \mathcal{A} p + 1 + \mu &= 0 && \text{in } H^{-1}(\Omega), \\ -p + \nu &= 0 && \text{a.e. in } \Omega, \\ p &\ge 0 && \text{q.e. in } \Omega, \\ \langle \mu,\, v\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} &\ge 0 && \text{for all } v \in H_0^1(\Omega), v \ge 0 \text{ q.e. in } \Omega, \\ \nu &\le 0 && \text{a.e. in } \Omega. \end{aligned}$$

This directly implies $p = \nu = 0$ and $\mu = -1$, which is a contradiction.

6.2. The upper bound is negative

We consider

$$\begin{aligned} \text{Minimize} \quad & \frac{1}{2} \|y+1\|^2_{L^2(\Omega)} + \frac{\alpha}{2} \|u\|^2_{L^2(\Omega)}, \\ \text{such that} \quad & \mathcal{A} y = u - \xi + 1, \\ & 0 \ge y \;\perp\; \xi \ge 0, \\ \text{and} \quad & u \le -1. \end{aligned}$$

Since $u - \xi + 1 \le 0$ for all admissible controls u and all multipliers $\xi \ge 0$, $0 \ge y$ is satisfied trivially by the maximum principle. Since ξ is unique, $\xi = 0$ follows for all admissible u. Hence, the problem is equivalent to the control constrained problem

$$\begin{aligned} \text{Minimize} \quad & \frac{1}{2} \|y+1\|^2_{L^2(\Omega)} + \frac{\alpha}{2} \|u\|^2_{L^2(\Omega)}, \\ \text{such that} \quad & \mathcal{A} y = u + 1, \\ \text{and} \quad & u \le -1. \end{aligned}$$

In the case $\alpha \ge 1/8$, $(\bar y, \bar u) = (0, -1)$ is the unique global solution (this can be proven by checking the first order necessary and sufficient conditions) and hence the solution of the original problem. Then, we have $A = A_b = \Omega$ and $\tilde{A}_s = \emptyset$. However, there are no multipliers p, μ, ν, such that the strong stationarity system (1.3)

$$\begin{aligned} \mathcal{A}\,p + 1 + \mu &= 0 && \text{in } H^{-1}(\Omega), \\ \alpha\,\bar{u} - p + \nu &= 0 && \text{a.e. in } \Omega, \\ p &\geq 0 && \text{q.e. in } \Omega, \\ \langle \mu,\, v\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} &\geq 0 && \text{for all } v \in H_0^1(\Omega), v \geq 0 \text{ q.e. in } \Omega, \\ \nu &\geq 0 && \text{a.e. in } \Omega \end{aligned}$$

is satisfied, since if p satisfies the first equation with some $\mu \geq 0$, we have $p < 0$ by the maximum principle.

A. Discussion of the strictly active set

The aim of this section is to show that the strictly active set $\tilde{A}_s$ defined in Lemma 3.1 can chosen to be the fine support (to be defined, see Lemma A.4) of $\bar{\xi}$, in contrast to the implicit definition in the proof of Lemma 3.1.

In order to use some results from the literature, we have to define a capacity for arbitrary sets $A \subset \mathbb{R}^n$ by

$$\operatorname{cap}_{\mathbb{R}^n}(A) = \inf\big\{\|(v, \nabla v)\|^2_{L^2(\mathbb{R}^n)^{n+1}} : v \in H^1(\mathbb{R}^n), v \geq 1 \text{ a.e. in a neighbourhood of } A\big\},$$

compare [Heinonen et al., 1993, Sec. 2.35]. Note that there are two differences to Definition 2.1: $H_0^1(\Omega)$ is replaced by $H^1(\mathbb{R}^n)$ and we use a different norm. Following the proof of [Attouch et al., 2006, Prop.5.8.3 (a)], we find that this definition is equivalent to [Adams and Hedberg, 1996, Def. 2.2.1–2.2.4].

For sets $A \subset \Omega$, $\operatorname{cap}_{\mathbb{R}^n}(A)$ can be estimated from above by $\operatorname{cap}(A)$:

Lemma A.1. There exists a constant $C > 0$, such that

$$\operatorname{cap}_{\mathbb{R}^n}(A) \leq C \operatorname{cap}(A) \tag{A.1}$$

holds for all $A \subset \Omega$.

Proof. Let a function $v \in H_0^1(\Omega)$ satisfying $v \geq 1$ in a neighbourhood of A be given. Then, $v \in H^1(\mathbb{R}^n)$ and

$$\|(v, \nabla v)\|^2_{L^2(\mathbb{R}^n)^{n+1}} \leq C\,\|\nabla v\|^2_{L^2(\Omega)^n}$$

for some $C > 0$ by Poincaré's inequality. Taking the infimum over all such v, we obtain

$$\inf\big\{\|(v, \nabla v)\|^2_{L^2(\mathbb{R}^n)^{n+1}} : v \in H_0^1(\Omega), v \geq 1 \text{ a.e. in a neighbourhood of } A\big\} \leq C \operatorname{cap}(A).$$

This implies the claim.

Note that the reverse estimate to (A.1) does not hold in the general case, in particular we have $\operatorname{cap}(\Omega) = \infty$, but $\operatorname{cap}_{\mathbb{R}^n}(\Omega) < \infty$. However, we have the following important lemma.

Lemma A.2 ([Heinonen et al., 1993, Lem. 2.9, Cor. 2.39])**.** For a set $A \subset \Omega$, we have

$$\operatorname{cap}(A) = 0 \quad \Longleftrightarrow \quad \operatorname{cap}_{\mathbb{R}^n}(A) = 0. \tag{A.2}$$

Finally, we need the concept of the so-called fine topology in $\mathbb{R}^n$, which is closely related to the notion of capacities. The fine topology is defined as the coarsest topology such that all sub-harmonic functions are continuous. We refer to [Adams and Hedberg, 1996, Def. 6.4.1] or [Heinonen et al., 1993, Chap. 12] for more details. For our purposes it is enough to know that the fine topology possesses the following properties.

- The fine topology is finer than the usual topology on $\mathbb{R}^n$.
- Every $\operatorname{cap}_{\mathbb{R}^n}$-quasi-open set O (defined similarly to Definition 2.1) is equivalent to a finely open set $\tilde{O}$, in the sense that $\operatorname{cap}_{\mathbb{R}^n}\big((O \setminus \tilde{O}) \cup (\tilde{O} \setminus O)\big) = 0$, and every finely open set is $\operatorname{cap}_{\mathbb{R}^n}$-quasi-open, see [Adams and Hedberg, 1996, Prop. 6.4.12, 6.4.13].
- The fine topology has the quasi-Lindelöf property, i.e., for every family $\{A_\alpha\}$ of finely open sets, there exists a countable subfamily $\{A_{\alpha_i}\}_{i\in\mathbb{N}}$, such that

  $$\operatorname{cap}_{\mathbb{R}^n}\Big(\bigcup_\alpha A_\alpha \setminus \bigcup_{i\in\mathbb{N}} A_{\alpha_i}\Big) = 0,$$

  see [Adams and Hedberg, 1996, Rem. 6.5.11].

The induced topology on Ω is also called the fine topology. Since Ω is open, it is finely open. Therefore, a set $A \subset \Omega$ is finely open in $\mathbb{R}^n$ if and only if it is finely open in Ω.

Now, we are going to define the support w.r.t. the fine topology of a non-negative $\xi \in H^{-1}(\Omega)$, which is identified with a Borel measure by Lemma 2.4. To this end, we have to extend the Borel measure ξ to finely open sets. This requires the definition of a σ-algebra which contains the finely open sets.

Let us remark that every finely open set $O \subset \Omega$ is a Borel set up to a set of zero capacity, compare also Lemma 2.2: since O is $\operatorname{cap}_{\mathbb{R}^n}$-quasi-open, there exists, for any $\varepsilon > 0$, an open set G_ε such that $\operatorname{cap}_{\mathbb{R}^n}(G_\varepsilon) \le \varepsilon$ and $O \cup G_\varepsilon$ is open. Since Ω is open, we can assume that $G_\varepsilon \subset \Omega$. Then, we have $O \subset \bigcap_{i\in\mathbb{N}}(O \cup G_{1/i})$ and

$$\operatorname{cap}_{\mathbb{R}^n}\Big(\Big[\bigcap_{i\in\mathbb{N}} O \cup G_{1/i}\Big] \setminus O\Big) \le \operatorname{cap}_{\mathbb{R}^n}\Big(\Big[\bigcap_{i\in\mathbb{N}} G_{1/i}\Big]\Big) = 0.$$

Hence, O differs from the Borel set $\bigcap_{i\in\mathbb{N}}(O \cup G_{1/i})$ by a set $M \subset \Omega$ with $\operatorname{cap}_{\mathbb{R}^n}(M) = 0$. By (A.2) we have $\operatorname{cap}(M) = 0$.

This motivates the following definition.

Definition A.3. We define the set

$$\mathcal{C} = \{G \cup H \subset \Omega : G \text{ is a Borel set and } \operatorname{cap}(H) = 0\}.$$

Then, $\mathcal{C}$ is a σ-algebra and it contains the finely open sets and all Borel sets.

Proof. We have to prove that $\mathcal{C}$ is a σ-algebra. It is easy to see that $\mathcal{C}$ is closed under countable unions, since the countable union of sets of zero capacity still has zero capacity. In order to show that $\mathcal{C}$ is closed under countable intersections, we remark that

$$\bigcap_{i\in\mathbb{N}} G_i \subset \bigcap_{i\in\mathbb{N}} (G_i \cup H_i) \subset \Big(\bigcap_{i\in\mathbb{N}} G_i\Big) \cup \Big(\bigcup_{i\in\mathbb{N}} H_i\Big).$$

Hence, for $\{G_i \cup H_i\} \subset \mathcal{C}$, the intersection differs from the Borel set $\bigcap_{i\in\mathbb{N}} G_i$ only by a set of zero capacity.

In the following, we simply say "$G \cup H \in \mathcal{C}$", instead of "$G \subset \Omega$ is a Borel set and $H \subset \Omega$ has zero capacity".

Now, let $\xi \in H^{-1}(\Omega)$ be a non-negative functional, which is identified with a Borel measure, see Lemma 2.4. Since $\xi(A) = 0$ for Borel sets A with $\operatorname{cap}(A) = 0$, we can extend ξ to $\mathcal{C}$ in a well-defined way by letting

$$\xi(G \cup H) = \xi(G) \quad \text{for all } G \cup H \in \mathcal{C}. \tag{A.3}$$

It is easy to show that ξ is additive on $\mathcal{C}$. Moreover, for all $\{G_i \cup H_i\} \subset \mathcal{C}$ we have

$$\xi\Big(\bigcup_{i\in\mathbb{N}} (G_i \cup H_i)\Big) = \xi\Big(\bigcup_{i\in\mathbb{N}} G_i \cup \bigcup_{i\in\mathbb{N}} H_i\Big) = \xi\Big(\bigcup_{i\in\mathbb{N}} G_i\Big) \le \sum_{i\in\mathbb{N}} \xi(G_i) = \sum_{i\in\mathbb{N}} \xi(G_i \cup H_i).$$

Hence, ξ is countably subadditive on $\mathcal{C}$.

Now, we are in the position to define the fine support of ξ.

Lemma A.4. Let $\xi \in H^{-1}(\Omega)$ be a non-negative functional. There exists a largest finely open set $M \subset \Omega$ with $\xi(M) = 0$. Its complement $\Omega \setminus M$ is called the fine support of ξ and is denoted by $\operatorname{f-supp}(\xi)$.

Proof. Let $\{A_\alpha\}$ be the family of finely open sets in Ω, whose ξ-measure is zero. Let $\{A_{\alpha_i}\}_{i\in\mathbb{N}}$ be a subfamily given by the quasi-Lindelöf property. We define

$$M = \bigcup_{\alpha} A_\alpha, \quad \tilde{M} = \bigcup_{i\in\mathbb{N}} A_{\alpha_i}, \quad O = M \setminus \tilde{M}.$$

By the definition of $\{A_{\alpha_i}\}$, we have $\operatorname{cap}_{\mathbb{R}^n}(O) = 0$ and $O \subset \Omega$. Hence, $\operatorname{cap}(O) = 0$ by Lemma A.2. By definition (A.3) of ξ, this gives $\xi(O) = 0$. Using that ξ is countably additive, we have

$$\xi(M) = \xi(\tilde{M} \cup O) = \xi(\tilde{M}) \le \sum_{i\in\mathbb{N}} \xi(A_{\alpha_i}) = 0.$$

This shows that M is the desired finely open set.

With these tools at hand, we can prove a refinement of Lemma 3.1.

Lemma A.5. Let $\xi \in H^{-1}(\Omega)$ be a non-negative functional. Then, we have

$$\{v \in H_0^1(\Omega) : v = 0 \ \xi\text{-a.e.}\} = \{v \in H_0^1(\Omega) : v = 0 \text{ q.e. on f-supp}(\xi)\}.$$

In particular, we have

$$\mathcal{K}(\bar{u}) = \mathcal{T}_K(\bar{y}) \cap \bar{\xi}^\perp = \big\{v \in H_0^1(\Omega) : v \le 0 \text{ q.e. in } A \text{ and } v = 0 \text{ q.e. in } \tilde{A}_s\big\},$$

where $\tilde{A}_s = \text{f-supp}(\bar{\xi})$.

Proof. We only have to prove the first identity. The second one follows together with (3.5).

"$\subset$": Let $v \in H_0^1(\Omega)$, $v = 0$ ξ-a.e. be given. The set $O = \{x \in \Omega : v \ne 0\}$ is quasi-open, hence, $\text{cap}_{\mathbb{R}^n}$-quasi-open by (A.1). Therefore, there exists a finely open set F which differs by O only with capacity zero. Thus, $\xi(O) = \xi(F)$ and $\xi(O) = 0$ by assumption. Hence, $F \subset \Omega \setminus \text{f-supp}(\xi)$ and consequently, $\text{cap}(O \cap \text{f-supp}(\xi)) = 0$.

"$\supset$": Let $v \in H_0^1(\Omega)$, $v = 0$ q.e. on f-supp(ξ) be given. We define the set $O = \{x \in \Omega : v \ne 0\}$. By assumption we have $\text{cap}(O \cap \text{f-supp}(\xi)) = 0$. This implies $\xi(O \cap \text{f-supp}(\xi)) = 0$ and hence $\xi(O) = 0$ by $\xi(\Omega \setminus \text{f-supp}(\xi)) = 0$.

Note that the support of ξ (defined similarly by using the usual topology of $\mathbb{R}^n$ on Ω), is larger than the fine support of ξ (since every open set is finely open). Hence, we may not replace the fine support by the support of ξ in Lemma A.5.

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