Strong uniqueness for SDEs in Hilbert spaces with non-regular drift

G. Da Prato, F. Flandoli, M. Röckner, A. Yu. Veretennikov

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Abstract

We prove pathwise uniqueness for a class of stochastic differential equations (SDE) on a Hilbert space with cylindrical Wiener noise, whose non-linear drift parts are sums of the subdifferential of a convex function and a bounded part. This generalizes a classical result by one of the authors to infinite dimensions. Our results also generalize and improve recent results by N. Champagnat and P. E. Jabin, proved in finite dimensions, in the case where their diffusion matrix is constant and non-degenerate and their weakly differentiable drift is the (weak) gradient of a convex function. We also prove weak existence, hence obtain unique strong solutions by the Yamada–Watanabe theorem. The proofs are based in part on a recent maximal regularity result in infinite dimensions, the theory of quasi-regular Dirichlet forms and an infinite dimensional version of a Zvonkin-type transformation. As a main application we show pathwise uniqueness for stochastic reaction diffusion equations perturbed by a Borel measurable bounded drift. Hence such SDE have a unique strong solution.

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1 Introduction

In a separable Hilbert space H, with inner product $\langle .,. \rangle$ and norm $|.|$, we consider the SDE

$$\begin{aligned} dX_t &= (AX_t - \nabla V(X_t) + B(X_t))\, dt + dW_t \\ X_0 &= z \end{aligned} \tag{1}$$

where we assume:

(H1) $A : D(A) \subset H \to H$ is a selfadjoint and strictly negative definite operator (i.e. $A \leq -\omega I$ for some $\omega > 0$), with A^{-1} of trace class.

(H2) $V : H \to (-\infty, +\infty]$ is a *convex, proper, lower-semicontinuous, lower bounded* function; denote by D_V the set of all $x \in \{V < \infty\}$ such that V is Gâteaux differentiable at x.

(H3) for the Gateaux derivative ∇V we have for some $\epsilon > 0$

$$\gamma(D_V) = 1,$$

$$\int_H \left(|V(x)|^{2+\epsilon} + |\nabla V(x)|^2\right) \gamma(dx) < \infty, \ \int_H \left\|D^2 V(x)\right\|_{\mathcal{L}(H)} \nu(dx) < \infty, \tag{2}$$

where γ is the centered Gaussian measure in H with covariance $Q = -\frac{1}{2}A^{-1}$ and ν is the probability measure on H defined as

$$\nu(dx) = \frac{1}{Z} e^{-V(x)} \gamma(dx), \qquad Z = \int_H e^{-V(x)} \gamma(dx).$$

Clearly, γ and ν have the same zero sets. Here the second assumption in (2) means that there exists $u_n \in \mathcal{F}C_b^2(H),\ n \in \mathbb{N}$, such that $V = \lim_{n\to\infty} u_n$ in $L^2(H, \nu)$ and $D^2V := \lim_{n\to\infty} D^2 u_n$ in $L^2(H, \nu; L(H))$, where $\mathcal{F}C_b^2(H)$ denotes the set of all C_b^2-cylindric functions on H (see below for the precise definition) and $L(H)$ the set of all bounded linear operators from H to H.

(H4) $B : H \to H$ is Borel measurable and bounded.

(H5) W is an $(\mathcal{F}_t)$- cylindrical Brownian motion in H, on some pobability space $(\Omega, \mathcal{F}, P)$ with normal filtration $(\mathcal{F}_t),\ t \geq 0$.
Formally W is a process of the form $W_t = \sum_{i=1}^\infty W_t^i e_i$ where W_t^i are independent real valued Brownian motions defined on a probability space $(\Omega, \mathcal{F}, P)$ and $\{e_i\}_{i\in\mathbb{N}}$ is a complete orthonormal system in H; for every $h \in H$, the series $\langle W_t, h\rangle = \sum_{i=1}^\infty W_t^i \langle e_i, h\rangle$ converges in $L^2(\Omega)$.

Remark 1 Since A is strictly negative definite, we may assume $V(x) \geq \epsilon |x|^2,\ x \in H$, for some $\epsilon > 0$ and all $x \in H$. Otherwise, replace A by $A + \frac{\omega}{2} I$ and V by $V + \frac{\omega}{2}|x|^2 + |\inf_{x\in H} V(x)|$. In particular, without loss of generality we have that $|x|^p e^{-V(x)}$ is bounded in $x \in H$ for all $p \in (0, \infty)$.

Remark 2 (i) We note that if $x \in D_V$ by definition

$$\lim_{s\to 0} \frac{1}{s}\left(V(x+sh) - V(x)\right) = \langle \nabla V(x), h\rangle$$

for all $h \in H$ where a priori the limit is taken in the Alexandrov topology on $(-\infty, +\infty]$, since $V(x+sh)$ could be $+\infty$ for some s. On the other hand, the limit $\langle \nabla V(x), h\rangle \in \mathbb{R}$, so $V(x+sh) \in \mathbb{R}$ for $s \leq s_0$ for some small enough $s_0 > 0$.

(ii) If $\{V < \infty\}$ is open, then $\gamma(D_V) = 1$. Indeed, if $\{V < \infty\}$ is open, then V is continuous on $\{V < \infty\}$ see e.g. [P93, Proposition 3.3]. Since furthermore, V is then locally Lipschitz on $\{V < \infty\}$ (see e.g. [P93, Proposition 1.6]), it follows by the fundamental result in [A76], [P78], see also [Bo10, Section 10.6], that $\gamma(\{V < \infty\} \setminus D_V) = 0$. But $\gamma(\{V < \infty\}) = 1$, since $V \in L^2(H, \gamma)$.

It turns out that the condition on the second (weak) derivative in (2) in Hypothesis (H3) is too strong for some applications (see Section 7 below). Therefore, we shall also consider the following modified version of (H3):

(H3)' V and ∇V satisfy (H3) with the condition on the second derivative of V replaced by the following: there exists a separable Banach space $E \subset H$, continuously and densely embedded, such that $E \subset D(V)$, $\gamma(E) = 1$ and on E the function V is twice Gâteaux-differentiable such that for all $x \in E$ its second Gâteaux-derivative $V''_E(x) \in L(E, E')$ (with E' being the dual of E) extends by continuity to an element in $L(H, E')$ such that

$$\|V''_E(x)\|_{L(H,E')} \leq \Psi(|x|_E)$$

for some convex function $\Psi : [0, \infty) \to [0, \infty)$. Furthermore, for γ-a.e. initial condition $z \in E$ there exists a (probabilistically) weak solution $X^V = X^V(t)$, $t \in [0, T]$, to SDE (1) with $B = 0$ so that

$$\mathbb{E} \int_0^T \Psi(|X^V(s)|_E)\, ds < \infty. \tag{3}$$

Though(H3)' is quite complicated to formulate, it is exactly what is fulfilled if ∇V is a polynomial. We refer to Section 7.1 below.

Remark 3 We would like to stress at this point that the conditions on the second derivative of V both in (H3) and in (H3)' are only used to be able to apply the mean value theorem in the proof of Lemma 39 below. For the rest of this paper we assume that (H1), (H2), (H4), (H5) and (H3) or (H3)' are in force.

Definition 4 *A solution of the SDE (1) in H is a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$ on H, an H-cylindrical $(\mathcal{F}_t)$-Brownian motion $(W_t)_{t\geq 0}$ on this space, a continuous $(\mathcal{F}_t)$-adapted process $(X_t)_{t\geq 0}$ on this space such that:*

i) $X_s(\omega) \in D_V$ for $dt \otimes P$ a.e. (s, ω) and $\int_0^T |\langle \nabla V(X_s), h\rangle|\, ds < \infty$ with probability one, for every $T > 0$ and $h \in D(A)$;

ii) for every $h \in D(A)$ and $t \geq 0$, one has

$$\langle X_t, h\rangle = \langle z, h\rangle + \int_0^t \left(\langle X_s, Ah\rangle + \langle B(X_s) - \nabla V(X_s), h\rangle\right) ds + \langle W_t, h\rangle$$

with probability one.

If X is $\mathcal{F}^W$-adapted, where $\mathcal{F}^W = \left(\mathcal{F}^W_t\right)_{t\geq 0}$ is the normal filtration generated by W, we say that X is a strong solution.

The Gaussian measure γ is invariant for the linear equation

$$dZ_t = AZ_t dt + dW_t$$

while ν is invariant for the non-linear equation

$$dX_t = \left(AX_t - \nabla V\left(X_t\right)\right) dt + dW_t.$$

They are equivalent, since $V < \infty$ (hence $e^{-V} > 0$) at least on D_V and $\gamma\left(D_V\right) = 1$. Hence the full measure sets in H are the same with respect to γ or ν. Our main uniqueness result is:

Theorem 5 *There is a Borel set $\Xi \subset H$ with $\gamma(\Xi) = 1$ having the following property. If $z \in \Xi$ and X, Y are two solutions of* (1) *with initial condition z (in the sense of Definition 4), defined on the same filtered probability space $\left(\Omega, \mathcal{F}, \left(\mathcal{F}_t\right)_{t\geq 0}, P\right)$ and w.r.t. the same cylindrical Brownian motion W, then X and Y are indistinguishable processes. Hence by the Yamada–Watanabe theorem they are (probabilistically) strong solutions and have the same law.*

The proof is given in Section 5. This result was first proved in [DFPR13] in the case $V = 0$ (see also the more recent [DFPR14], where also the case $V = 0$, but with B only bounded on balls was treated) with a rather complex proof based on the very non trivial maximal regularity results in $L^p(H, \gamma)$ for the Kolmogorov equation

$$\left(\lambda - \mathcal{L}_{A,B}\right) u = f$$

associated to the SDE, where $\mathcal{L}_{A,B}$ is the operator formally defined as

$$\mathcal{L}_{A,B}u\left(x\right) = \frac{1}{2}\operatorname{Tr}\left(D^2u\left(x\right)\right) + \left\langle Ax + B\left(x\right), Du\right\rangle$$

on suitable functions u, for $x \in D(A)$. Here we present a much simpler proof which covers also the case $V \neq 0$, based on several new ingredients.

First, in order to perform a suitable change of coordinates (analogous to [DFPR13] and [DFPR14]) we use the family of Kolmogorov equations

$$\left(\lambda + \lambda_i - \mathcal{L}_{A,B,V}\right) u = f$$

or in vector form

$$\left(\lambda - A - \mathcal{L}_{A,B,V}\right) U = F \tag{4}$$

where $\mathcal{L}_{A,B,V}$ is the operator formally defined as

$$\mathcal{L}_{A,B,V}u\left(x\right) = \frac{1}{2}\operatorname{Tr}\left(D^2u\left(x\right)\right) + \left\langle Ax - \nabla V\left(x\right) + B\left(x\right), Du\right\rangle$$

on suitable functions u. The presence of the term $\lambda_i u$ in the equation adds the advantages of the resolvent of A (given by $(\lambda - A)^{-1}$) to those of the elliptic regularity theory (given by $\mathcal{L}_{A,B}$). Moreover, we use the recent maximal regularity results in $L^2(H,\nu)$ for the Kolmogorov equation

$$(\lambda - \mathcal{L}_{A,B,V})\, u = f$$

proved in [DL14].

Second, thanks to the previous new Kolmogorov equation, we may apply a trick based on Itô's formula and the multiplication by the factor e^{-A_t} (see below the definition of A_t) which greatly simplifies the proof.

Third, we use Girsanov's theorem in a better form in the proof of the main Lemma 39. The new proof of the lemma along with the previous two innovations allow us to use only the L^2 theory of the Kolmogorov equation, which is much simpler.

Fourth, we heavily use the theory of classical (gradient type) Dirichlet forms on infinite dimensional state spaces.

For more background literature in the finite dimensional case following the initiating work [V80], we refer to [DFPR13], [DFPR14]. We only mention here the recent work [CJ13], where SDEs with weakly differentiable drifts are studied. In the case when in [CJ13] the diffusion matrix is constant and non-degenerate and if the weakly differentiable drift is the (weak) gradient of a convex function, our results generalize those in [CJ13] from $\mathbb{R}^d$ to a separable Hilbert space as state space, and to the case when a bounded merely measurable drift part is added. Finally, we mention the paper [CD13] which concerns pathwise uniqueness for some Hölder perturbation of reaction-diffusions equations studied in spaces of continuous functions instead of square integrable function.

The organization of the paper is as follows: Section 2 is devoted to existence of solutions and Section 3 to the regularity theory of the Kolmogorov operator (4) above. The mentioned change of coordinates is performed in Section 4. Sections 5 and 6 contain the proof of our main Theorem 5. In Section 7 we present applications.

We end this section by giving the definition of Sobolev spaces and some notation. We consider an orthonormal basis $\{e_k : k \in \mathbb{N}\}$ of H which diagonalizes A with corresponding eigenvalues $-\lambda_k \in (-\infty, 0)$, $k \in \mathbb{N}$. We set $x_k = \langle x, e_k \rangle$ for each $x \in H$, $k \in \mathbb{N}$. We denote by P_n the orthogonal projection on the linear span of $e_1, \dots, e_n$. For each $k \in \mathbb{N} \cup \{+\infty\}$ we denote by $\mathcal{FC}_b^k(H)$ the set of the cylindrical functions $\varphi(x) = \phi(x_1, \dots, x_n)$ for some $n \in \mathbb{N}$, with $\phi \in C_b^k(\mathbb{R}^n)$.

For $\mu = \gamma$ or $\mu = \nu$ the Sobolev spaces $W^{1,2}(H,\mu)$ is the completion of $\mathcal{FC}_b^1(H)$ in the norm

$$\|\varphi\|^2_{W^{1,2}(H,\mu)} := \int_H (|\varphi|^2 + \|D\varphi\|^2) d\mu = \int_H \left(|\varphi|^2 + \sum_{k=1}^{\infty} (D_k\varphi)^2 \right) d\mu.$$

The Sobolev spaces $W^{2,2}(H,\mu)$ is the completion of $\mathcal{FC}_b^2(H)$ in the norm

$$\begin{aligned}\|u\|^2_{W^{2,2}(H,\mu)} &= \|u\|^2_{W^{1,2}(H,\mu)} + \int_H \mathrm{Tr}\ ([D^2u]^2) d\mu \\ &= \|u\|^2_{W^{1,2}(H,\mu)} + \sum_{h,k\in\mathbb{N}} \int_H (D_{hk}u)^2 d\mu.\end{aligned}$$

We denote the Borel σ-algebra on H by $\mathcal{B}(H)$ and by $B_b(H)$ the set of all bounded $\mathcal{B}(H)$-measurable functions $\varphi : H \to \mathbb{R}$. We set for a function $\varphi : H \to \mathbb{R}$

$$\|\varphi\|_\infty := \sup_{x\in H} |\varphi(x)|.$$

$I : H \to H$ denotes the identity operator on H. For $k \in \mathbb{N}$, $C_b^k(H)$ denotes the set of all $\varphi : H \to \mathbb{R}$ of class C^k, which together with all their derivatives up to order k are bounded and uniformly continuous. Furthermore, we reserve the symbol D for the closure of the derivative for $u \in \mathcal{F}C_b^1$ in $L^2(H,\mu;H)$ for $\mu = \gamma$ or $\mu = \nu$. For the Gâteaux derivative we use the symbol ∇. Since they coincide on convex and Lipschitz functions u, in the sense that ∇u is a γ- or ν-version of Du, we shall write ∇u, whenever we want to stress that we consider that special version.

2 Existence

In this section we shall prove that under conditions (H1)–(H4) from Section 1, which will be in force in all of this paper, that the SDE (1) has a solution in the sense of Definition 4. We start with the following proposition showing that the gradient DV in $L^2(H,\gamma;H)$ and the Gâteaux derivative ∇V coincide γ-a.e.

Proposition 6 *We have $V \in W^{1,2}(H,\gamma)$ and*

$$DV = \nabla V, \quad \gamma\text{-a.e.}$$

The proof of Proposition 6 requires a numbers of lemmas.

Lemma 7 *Let $k \in Q^{1/2}H$. Then*

$$\lim_{s\to 0} \frac{V(\cdot + sk) - V(\cdot)}{s} = \langle \nabla V, k\rangle \quad \text{in} \quad L^2(H,\gamma)$$

Proof. Let $x \in \{V < \infty\}$. Then by convexity for $s \in (0,1)$

$$V(x+sk) \le sV(x+k) + (1-s)V(x),$$

hence

$$\frac{V(x+sk) - V(x)}{s} \le V(x+k) - V(x). \tag{5}$$

Since $k \in Q^{1/2}H$, by the Cameron–Martin theorem (see e.g. [D04, Section 1.2.3]) the function on the right as a function of x is in $L^2(H,\gamma)$, since by assumption (H3) $V \in L^{2+\epsilon}(H,\gamma)$.

Furthermore for $x \in D_V$ taking the limit $s \to 0$ in (5) we find that

$$\langle \nabla V(x), k\rangle \le V(x+k) - V(x).$$

Replacing k by sk which is also in $Q^{1/2}H$, and dividing by s, we obtain

$$\langle \nabla V(x), k\rangle \le \frac{V(x+sk)-V(x)}{s}. \tag{6}$$

But the left hand side as a function of x is in $L^{2+\epsilon}(H,\gamma)$ by assumption (H3). Hence (5) and (6) imply the assertion of the lemma by Lebesgue's dominated convergence theorem, since $\gamma(D_V) = 1$. ■

Before we proceed to Lemma 8 we need to introduce the following space:

$$\begin{aligned} \mathcal{D}_0 := \Big\{ & u \in L^2(H,\gamma) :\ \exists F_u \in L^2(H,\gamma;H) \text{ such that} \\ & \lim_{s\to 0} \frac{1}{s}\,[u(\cdot + se_i) - u(\cdot)] = \langle \nabla F_u, e_i\rangle \text{ in } L^2(H,\gamma),\ \forall\, i \in \mathbb{N} \Big\} \end{aligned} \tag{7}$$

Set $\widetilde{D}u := F_u$ for $u \in \mathcal{D}_0$. Then obviously $\mathcal{F}C_b^2 \subset \mathcal{D}_0$ and $D\varphi = \widetilde{D}\varphi$ for all $\varphi \in \mathcal{F}C_b^2$.

Lemma 8 *(i) Let $u \in \mathcal{D}_0$, $\varphi \in \mathcal{F}C_b^2$ and $i \in \mathbb{N}$. Then*

$$\begin{aligned} \int_H \langle \widetilde{D}u(x), e_i\rangle \varphi(x)\,\gamma(dx) &= -\int_H u(x)\langle D\varphi(x), e_i\rangle\,\gamma(dx) \\ &\quad + 2\lambda_i \int_H u(x)\langle e_i, x\rangle \varphi(x)\,\gamma(dx) \end{aligned}$$

(ii) The operator $\widetilde{D} : \mathcal{D}_0 \subset L^2(H,\gamma) \to L^2(H,\gamma;H)$ is closable.

Proof. (i). We have

$$\begin{aligned} & \int_H \langle \widetilde{D}u(x), e_i\rangle \varphi(x)\,\gamma(dx) \\ & = \lim_{s\to 0} \frac{1}{s}\left[\int_H u(x+se_i)\,\varphi(x)\,\gamma(dx) - \int_H u(x)\varphi(x)\,\gamma(dx)\right]. \end{aligned} \tag{8}$$

But by the Cameron–Martin theorem the image measure $T_{se_i}(\gamma)$ of γ under the translation $x \mapsto x + se_i$ is absolutely continuous with respect to γ with density (cf. [D04, Section 1.2.3])

$$a_{se_i}(x) = e^{2s\lambda_i\langle e_i, x\rangle - s^2\lambda_i}.$$

Hence the difference of the two integrals on the right hand side of (8) can be written as

$$\int_H u(x)[\varphi(x-\lambda_i e_i)-\varphi(x)]a_{se_i}(x)\,\gamma(dx)+\int_H u(x)\varphi(x)(a_{se_i}(x)-1)\,\gamma(dx).$$

Hence letting $s\to 0$ in (8) assertion (i) follows.

(ii). Suppose $u_n\in\mathcal{D}_0,\ n\in\mathbb{N}$, such that $u_n\to 0$ in $L^2(H,\gamma)$ and $\widetilde{D}u_n\to F$ in $L^2(H,\gamma;H)$. Then for all $\varphi\in\mathcal{F}C_b^2,\ i\in\mathbb{N}$, by (i)

$$\int_H\langle F(x),e_i\rangle\varphi(x)\,\gamma(dx)=\lim_{n\to\infty}\int_H\langle\widetilde{D}u_n,e_i\rangle\varphi(x)\,\gamma(dx)=0.$$

Hence $F=0$. ■

Let us denote the closure of $(\widetilde{D},\mathcal{D}_0)$ again by $\widetilde{D}$ and its domain by $\widetilde{W}^{1,2}(H,\gamma)$. Clearly, since $\mathcal{F}C_b^2\subset\mathcal{D}_0$ with $D\varphi=\widetilde{D}\varphi$ for all $\varphi\in\mathcal{F}C_b^2$, it follows that $W^{1,2}(H,\gamma)$ is a closed subspace of $\widetilde{W}^{1,2}(H,\gamma)$. But in fact, they coincide.

Lemma 9

$\mathcal{F}C_b^2$ is dense in $\widetilde{W}^{1,2}(H,\gamma)$, hence

$$W^{1,2}(H,\gamma)=\widetilde{W}^{1,2}(H,\gamma)$$

and thus $Du=\widetilde{D}u$ for all $u\in W^{1,2}(H,\gamma)$.
Proof. Let $u\in\widetilde{W}^{1,2}(H,\gamma)$ such that

$$\int_H\langle\widetilde{D}\varphi,\widetilde{D}u\rangle d\gamma+\int_H\varphi\,u\,d\gamma=0,\quad\forall\,\varphi\in\mathcal{F}C_b^2. \tag{9}$$

Since $\varphi(x)=\Phi(x_1,...,x_N)$ for some $\Phi\in C_b^2(\mathbb{R}^N)$ and $x_i:=\langle x,e_i\rangle$, $1\le i\le N$, we have that

$$\langle\widetilde{D}\varphi,\widetilde{D}u\rangle=\sum_{i=1}^N\langle D\varphi,e_i\rangle\langle\widetilde{D}u,e_i\rangle$$

and that $\langle D\varphi,e_i\rangle\in\mathcal{F}C_b^2$. Hence by Lemma 8, (9) is equivalent to

$$-\int_H(2\mathcal{L}^{OU}-1)\varphi\,u\,d\gamma=0,\quad\forall\,\varphi\in\mathcal{F}C_b^2, \tag{10}$$

where

$$\mathcal{L}^{OU}\varphi(x)=\frac{1}{2}\,\mathrm{Tr}\,D^2\varphi(x)+\langle x,ADx\rangle.$$

But (10) implies that $u=0$, since it is well known that $\lambda-\mathcal{L}^{OU}$ has dense range in $L^2(H,\gamma)$ for $\lambda>0$. For the convenience of the reader we recall the argument: The C_0– semigroup generated by the Friedrichs extension of the symmetric operator $(\mathcal{L}^{OU},\mathcal{F}C_b^2)$ on $L^2(H,\gamma)$

is easily seen to be given by the following Mehler formula on bounded, Borel functions $f : H \to \mathbb{R}$

$$P_t f(x) = \int_H f(e^{tA}x + y) N_{Q_t}(dy), \quad t > 0, \tag{11}$$

where N_{Q_t} is the centred Gaussian measure on H with covariance operator

$$Q_t := \int_0^t e^{2sA} ds, \quad t > 0.$$

Obviously, $P_t(\mathcal{F}C_b^2) \subset \mathcal{F}C_b^2$, and also

$$\left(\int_0^\infty e^{-\lambda t} P_t \, dt \right) (\mathcal{F}C_b^2) \subset \mathcal{F}C_b^2.$$

But

$$\left(\lambda - \mathcal{L}^{OU} \right)^{-1} = \int_0^\infty e^{-\lambda t} P_t \, dt$$

as operators on $L^2(H, \gamma)$. Hence

$$\left(\lambda - \mathcal{L}^{OU} \right)^{-1} (\mathcal{F}C_b^2) \subset \mathcal{F}C_b^2$$

and so

$$\mathcal{F}C_b^2 \subset \left(1 - \mathcal{L}^{OU} \right) (\mathcal{F}C_b^2).$$

But $\mathcal{F}C_b^2$ is dense in $L^2(H, \gamma)$. ∎

Now we turn back to SDE (1).

Proof of Proposition 6. By (H2) and Lemma 7 we have that $V \in \widetilde{W}^{1,2}(H, \gamma)$ with $\nabla V = \widetilde{D} V$, γ–a.e. Hence Lemma 9 implies the assertion. ∎

Let us consider the case when in SDE (1) we have that $B = 0$, i.e.

$$\begin{aligned} dX_t &= (AX_t - \nabla V(X_t))dt + dW(t) \\ X_0 &= z, \end{aligned} \tag{12}$$

where for convenience we extend $\nabla V : D_V \to H$ by zero to the whole space D_V. The case for general B then follows easily from Girsanov's theorem.

To solve (12) in the (probabilistically) weak sense we shall use [AR91], i.e. the theory of Dirichlet forms, more precisely the so-called "classical (gradient type)" Dirichlet forms, which for the measure ν from Section 1 is just

$$\mathcal{E}_\nu(u, v) := \int_H \langle Du(x), Dv(x) \rangle \, \nu(dx), \quad u, v \in D(\mathcal{E}_\nu) := W^{1,2}(H, \nu).$$

But the whole theory has been developed for arbitrary finite measures m on $(H, \mathcal{B}(H))$ which satisfy an integration by parts formula (see [AR91], [MR92] and the references therein) or even more generally for finite measures m for which $D : \mathcal{F}C_b^\infty \subset L^2(H,m) \to L^2(H,m;H)$ is closable (see [AR89], [AR90],[MR92]). In particular, we can also take $m := \gamma$. Let us recall the following result which is crucial for the theory of classical Dirichlet forms which we shall formulate for ν, but holds for every m as above. For its formulation we need the notion of an "$\mathcal{E}_\nu$–nest": Let $F_n \subset H$, $n \in \mathbb{N}$, be an increasing sequence of closed sets and define for $n \in \mathbb{N}$

$$D(\mathcal{E}_\nu)|_{F_n} := \{u \in D(\mathcal{E}_\nu) : \ u = 0, \quad \nu\text{-a.e. on } H \setminus F_n\}.$$

Then $(F_n)_{n\in\mathbb{N}}$ is called an $\mathcal{E}_\nu$*–nest* if

$$\bigcup_{k=1}^{\infty} D(\mathcal{E}_\nu)|_{F_n} \textit{ is dense in } D(\mathcal{E}_\nu),$$

with respect to the norm

$$\mathcal{E}_{\nu,1}(u,u)^{1/2} := \Big(\mathcal{E}_\nu(u,u) + |u|^2_{L^2(H,\nu)}\Big)^{1/2}, \quad u \in D(\mathcal{E}_\nu),$$

i.e. with respect to the norm in $W^{1,2}(H,\nu)$.

Then the crucial result already mentioned is the following:

Theorem 10 *There exists an $\mathcal{E}_\nu$–nest consisting of compact sets.*

Proof. See [RS92] and [MR92, Chap. IV, Proposition 4.2]. ■

Let us denote $(K_n)_{n\in\mathbb{N}}$ this $\mathcal{E}_\nu$–nest consisting of compacts. This theorem says that $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ is completely determined in a K_σ set of H. Then it follows from the general theory that $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ is *quasi-regular*, hence has an associate Markov process which solves (SDE) (12) and this Markov process also lives on this K_σ set $\bigcup_{n=1}^\infty K_n$, i.e. the first hitting times $\sigma_{H\setminus K_n}$ of $H \setminus K_n$ converge to infinity as $n \to \infty$.

The precise formulations of these facts is the contents of Theorems 11 and 13 below. We need one more notion: A set $N \subset H$ is called $\mathcal{E}_\nu$-*exceptional*, if it is contained in the complement of an $\mathcal{E}_\nu$–nest. Clearly, this complement has ν-measure zero, hence $\nu(N) = 0$ if $N \in \mathcal{B}(H)$.

Theorem 11 *There exists $S \in \mathcal{B}(H)$ such that $H \setminus S$ is $\mathcal{E}_\nu$- exceptional (hence $\nu(H\setminus S) = 0$) and for every $z \in S$ there exists a probability space $(\Omega, \mathcal{F}, P_z)$ equipped with a normal filtration $(\mathcal{F}_t)_{t\geq 0}$, independent real valued $(\mathcal{F}_t)$–Brownian motions W_t^k, $t \geq 0$, $k \in \mathbb{N}$, on $(\Omega, \mathcal{F}, P_z)$ and a continuous H-valued $(\mathcal{F}_t)$- adapted process X_t, $t \geq 0$, such that P_z-a.s.:*

(i) $X_t \in S \quad \forall\, t \geq 0$,

(ii) $\displaystyle\int_H \mathbb{E}_{P_z}\left[\int_0^t |\nabla V(X_s)|^2 ds\right] \nu(dz) < \infty$ *and*

$$\mathbb{E}_{P_z}\left[\int_0^t 1_{H\setminus D_V}(X_s)\, ds\right] = 0 \quad \forall\, t \geq 0,$$

(iii) $\langle e_k, X_t\rangle = \langle e_k, z\rangle + \int_0^t (\langle Ae_k, X_s\rangle + \langle e_k, \nabla V(X_s)\rangle)ds + W_t^k,\ t \ge 0,\ k \in \mathbb{N}.$

Hence (by density) we have a solution of (12) *in the sense of Definition 4. Furthermore, up to completing $\mathcal{F}_t$ w.r. to P_z, $(\Omega, \mathcal{F})$, X_t, $t \ge 0$, and $(\mathcal{F}_t)$ can be taken canonical, independent of $z \in S$ and then*

$$\boldsymbol{M} := (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, (X_t)_{t\ge 0}, (P_z)_{z\in S}),$$

forms a conservative Markov process, with invariant measure ν.

Proof. The assertion follows from [AR91, Theorem 5.7]. ∎

For later use we define the Borel set

$$H_V := \left\{ z \in H :\ \mathbb{E}_{P_z}\left[\int_0^T |\nabla V(X_s)|^2 ds\right] < \infty \right\} \tag{13}$$

and note that by Theorem 11(ii) we have $\nu(H_V) = \gamma(H_V) = 1$.

In fact, by the convexity of V we also have uniqueness for the solutions to (12). We recall that the subdifferential ∂V of V is monotone (which is trivial to prove see e.g.[P93, Example 2.2 a]) and that for $x \in D_V$, $\partial V(x) = \nabla V(x)$ see e.g. [Ba10, page 8]. Hence we have

$$\langle \nabla V(x) - \nabla V(y), x - y\rangle \ge 0, \quad x, y \in D_V. \tag{14}$$

Theorem 12 *Let S be as in Theorem 11 and $z \in S$. Then pathwise uniqueness holds for all solutions in the sense of Definition 4 for SDE* (12)*. In particular, uniqueness in law holds for these solutions.*

Proof. The first assertion is an immediate consequence of the monotonicity (14), since a part of our Definition 4 requires that the solutions are in D_V $dt \otimes P$-a.e., see e.g. [DRW09, proof of the claim p.1008/1009] or [MR10, Section 4] for details. The second assertion then follows by the Yamada–Watanabe theorem (see e.g. [RSZ08] which easily can be adapted to apply to our case here). ∎

Theorem 13 *Let $\boldsymbol{M}$ be as in Theorem 11 and let $(F_n)_{n\in\mathbb{N}}$ be an $\mathcal{E}_\nu$-nest. Then*

$$P_z\left[\lim_{n\to\infty} \sigma_{H\setminus F_n} = \infty\right] = 1,$$

for all $z \in S \setminus N$, for some $\mathcal{E}_\nu$- exceptional set N, where for a closed set $F \subset H$

$$\sigma_{H\setminus F} := \inf\{t > 0 :\ X_t \in H \setminus F\}$$

is the first hitting time of $H \setminus F$.

Proof. Since $\boldsymbol{M}$ is conservative its lifetime ζ is infinity. So, the assertion follows from [MR92, Chapter V, Proposition 5.30]. ∎

Below we shall use the following simple lemma.

Lemma 14 *Let $(E, \|\cdot\|)$ be a Banach space and $V : E \to (-\infty, \infty]$ a convex function.*

(i) Let $K \subset E$ be convex and compact such that $V(K)$ is a bounded subset of $\mathbb{R}$. Then the restriction of V to K is Lipschitz.

(ii) Assume that V is lower semi-continuous and $K \subset E$ compact such that $V(K)$ is an upper bounded subset of $\mathbb{R}$. Then the restriction of V to K is Lipschitz.

Proof. (i). The proof is a simple modification of the classical proof that a continuous convex function on an open subset of E is locally Lipschitz (see [P93, Proposition 1.6]). For the convenience of the reader we give the argument:
Define

$$M = \sup_{x \in K} |V(x)|$$

and

$$d := \text{diam }(K) \; (:= \sup\{\|x - y\| : \; x, y \in K\}.$$

Let $x, y \in K$. Set $\alpha := \|x - y\|$ and

$$z := y + \frac{d}{\alpha}\,(y - x).$$

Then $\|x - y\| \le d$, hence $z \in K$ since K is convex. Furthermore,

$$y = \frac{\alpha}{\alpha + d}\, z + \frac{d}{\alpha + d}\, x,$$

hence

$$f(y) \le \frac{\alpha}{\alpha + d}\, f(z) + \frac{d}{\alpha + d}\, f(x),$$

so,

$$f(y) - f(x) \le \frac{\alpha}{\alpha + d}\,(f(z) - f(x)) \le \frac{2M}{d}\,\|x - y\|.$$

Interchanging x and y in this argument, implies the assertion.

(ii). This is an easy consequence of (i). Let K_1 be the closed convex hull of K. Then by Mazur's theorem K_1 is still compact and by convexity $V(K_1)$ is an upper bounded subset of $\mathbb{R}$. But $V(K_1)$ is also lower bounded, since V is lower semi-continuous. Hence by (i) V is Lipschitz on K_1, hence on K. ■

Now let us come back to our convex function $V : H \to (-\infty, \infty]$ satisfying (H2) and (H3). We know by Proposition 6 that $V \in W^{1,2}(H, \nu) = D(\mathcal{E}_\nu)$. Since $(\mathcal{E}_\nu, D(\mathcal{E}_\nu))$ is quasi-regular, it follows by [MR92, Chapter IV, Proposition 3.3] that there exists an $\mathcal{E}_\nu$-nest $(F_n)_{n \in \mathbb{N}}$ and a $\mathcal{B}(H)$-measurable function $\widetilde{V} : H \to \mathbb{R}$ such that

$$\widetilde{V} = V \;\; \nu\text{-a.e. and } \widetilde{V}|_{F_n} \text{ is continuous for every } n \in \mathbb{N}, \tag{15}$$

where $\widetilde{V}|_{F_n}$ denotes the restriction of $\widetilde{V}$ to F_n. By [MR92, Chapter III,Theorem 2.11], $(F_n \cap K_n)_{n \in \mathbb{N}}$ is again an $\mathcal{E}_\nu$-nest, where $(K_n)_{n \in \mathbb{N}}$ is the $\mathcal{E}_\nu$-nest of compacts from Theorem

10. Sinc $\nu(U) > 0$ for every non empty open set $U \subset H$, by [MR92, Chapter III, Proposition 3.8] we can find an $\mathcal{E}_\nu$-nest $(\widetilde{F}_n)_{n\in\mathbb{N}}$ such that $\widetilde{F}_n \subset F_n \cap K_n$ and the restriction of ν to $\widetilde{F}_n$ has full topological support on $\widetilde{F}_n$ for every $n \in \mathbb{N}$, i.e. $\nu(U \cap \widetilde{F}_n) > 0$ for all open $U \subset H$ with $U \cap \widetilde{F}_n \neq \varnothing$. (Such an $\mathcal{E}_\nu$ set is called *regular*.) Since we want to fix this special regular $\mathcal{E}_\nu$-nest of compacts depending on V below, we assign to it a special notation and set

$$K_n^V := \widetilde{F}_n, \quad n \in \mathbb{N}. \tag{16}$$

Now we can prove the following result which will be crucially used in Section 6.

Proposition 15 *(i) Let $n \in \mathbb{N}$ and K_n^V as in (16). Then $V|_{K_n^V}$ is real valued, Lipschitz continuous and bounded. Furthermore, $V(x) = \widetilde{V}(x)$ for every $x \in \bigcup_{n=1}^\infty K_n^V$.*

(ii) There exists $S_V \in \mathcal{B}(H)$ such that $H \setminus S_V$ is $\mathcal{E}_\nu$-exceptional, Theorem 11 holds with S_V replacing S and for every $z \in S_V$

$$P_z[\lim_{n\to\infty} \sigma_{H\setminus K_n^V} = \infty] = 1. \tag{17}$$

Proof. (i). Since $K_n^V \subset F_n$, we have for $\widetilde{V}$ from (15), that $V|_{K_n^V} - \widetilde{V}|_{K_n^V}$ is lower semi-continuous on K_n^V with respect to the metric on K_n^V induced by $|\cdot|$. Hence $\{V|_{K_n^V} - \widetilde{V}|_{K_n^V} > 0\} = K_n^V \cap U$ for some open subset $U \subset H$. Since $V|_{K_n^V} = \widetilde{V}|_{K_n^V}$ ν-a.s., it follows, since $(K_n^V)_{n\in\mathbb{N}}$ is a regular $\mathcal{E}_\nu$-nest, that

$$V(x) \leq \widetilde{V}(x) \text{ for every } x \in K_n^V.$$

But $\widetilde{V}|_{K_n^V}$ is continuous, hence bounded, because K_n^V is compact, so $V(K_n^V)$ is an upper bounded subset of $\mathbb{R}$, so by Lemma 14(ii) $V|_{K_n^V}$ is Lipschitz. But then $\{V|_{K_n^V} \neq \widetilde{V}|_{K_n^V}\} = K_n^V \cap U$ for some open subset $U \subset H$. Since $(K_n^V)_{n\in\mathbb{N}}$ is a regular $\mathcal{E}_\nu$-nest, we conclude that

$$V(x) = \widetilde{V}(x) \text{ for every } x \in K_n^V.$$

Hence assertion (i) is proved.

(ii). By Theorem 13 we know that there exists an $\mathcal{E}_\nu$-nest $(F_n)_{n\in\mathbb{N}}$ such that

$$P_z[\lim_{n\to\infty} \sigma_{H\setminus K_n^V} = \infty] = 1, \quad \forall\, z \in \bigcup_{n=1}^\infty F_n.$$

Then by a standard procedure (see e.g. [MR92, p. 114]) one can construct the desired set $S_V \in \mathcal{B}(H)$. ■

For the rest of this section we fix S_V as in Proposition 15.

Corollary 16 *Let $z \in S_V$ and $(X_t)_{t\geq 0}$ be a solution to (12) on some probability space $(\Omega, \mathcal{F}, P)$ with normal filtration $(\mathcal{F}_t)_{t\geq 0}$ and cylindrical $(\mathcal{F}_t)$-Brownian motion $W = W_t$, $t \geq 0$. Then (17) holds with P replacing P_z.*

Proof. This follows from the last part of Theorem 12. ∎

It is now easy to prove existence of (probabilistic) weak solutions to SDE (1) and uniqueness in law

Theorem 17 *For every $z \in S_V$ there exists a solution $Y = Y_t,\ t \in [0,T]$, to SDE (1) on some probability space $(\Omega', \mathcal{F}', P')$ in the sense of Definition 4 and this solution is unique in law. Furthermore, (17) holds with P' replacing P_z and Y replacing X, where $X = X_t,\ t \geq 0$, is the process from Theorem 11 and if $z \in S_V \cap H_V$ (with H_V as in (13)), then*

$$\int_0^T |\nabla V(Y_s)|^2 ds < \infty \quad P'\text{-a.s.}. \tag{18}$$

Proof. This is now an easy consequence of Theorem 11, Theorem 12 and Girsanov's theorem (see e.g. [DFPR13, Appendix A1]) which easily extends to the present case since uniqueness in law holds for SDE (12). To prove the last part we note that by Girsanov's theorem there exists a probability density $\rho : \Omega \to (0, \infty)$ such that

$$(\rho \cdot P') \circ Y^{-1} = P_z \circ X^{-1}.$$

Hence $P_z \circ X^{-1} = \rho_0 \, P' \circ Y^{-1}$, where ρ_0 is the $P' \circ Y^{-1}$-a.s. unique function such that $\rho_0(Y) = \mathbb{E}_{P'}[\rho|\sigma(Y)]$ P-a.s. and $\sigma(Y))$ denotes the σ-algebra generated by $Y_t,\ t \in [0,T]$. So, (17) and (18) follow, if $\rho_0 > 0$ $P' \circ Y^{-1}$-a.e. To show the latter we first note that

$$P' \circ Y^{-1}(\{\rho_0 = 0\}) = P'(\{\mathbb{E}_{P'}[\rho|\sigma(Y)] = 0\}).$$

But since

$$\rho = e^{-\int_0^T \langle B(Y_s), dW_s\rangle - \frac{1}{2}\int_0^T |B(Y_s)|^2 \, ds}$$

and W is $\sigma(Y)$-measurable by SDE (1), it follows that $\mathbb{E}_{P'}[\rho|\sigma(Y)] = \rho$. But $\rho > 0$. ∎

3 Regularity theory for the corresponding Kolmogorov operator

3.1 Uniform estimates on Lipschitz norms

First we are concerned with the scalar equation

$$\lambda u - \mathcal{L}u - \langle B, Du\rangle = f, \tag{19}$$

where $\lambda > 0$, $f \in B_b(H)$ and $\mathcal{L}$ is the Kolmogorov operator

$$\mathcal{L}u(x) = \frac{1}{2} \operatorname{Tr}[D^2 u(x)] + \langle Ax - DV(x), Du(x)\rangle, \quad x \in H. \tag{20}$$

Since the corresponding Dirichlet form

$$\mathcal{E}_B(v,w) := \frac{1}{2}\int_H \langle Dv, Dw\rangle\, d\nu - \int_H \langle B, Dv\rangle\, w\, d\nu + \lambda \int_H v\, w\, d\nu,$$

$v, w \in W^{1,2}(H,\nu)$, is weakly sectorial for λ big enough, it follows by [MR92, Chapter 1 and Subsection 3e) in Chapter II] that (19) has a unique solution $u \in L^2(H,\nu)$ such that $u \in D(\mathcal{L})$. We need, however, Lipschitz regularity for u and an estimate for its ν-a.e. defined Gâteaux derivative ∇V in terms of $\|f\|_\infty$. To prove this we also need the Kolmogorov operator associated to the linear equation that one obtains, when $B = V = 0$, in SDE (1), i.e. the Ornstein–Uhlenbeck operator

$$\mathcal{L}^{OU} u(x) = \frac{1}{2}\,\mathrm{Tr}\,[D^2 u(x)] + \langle Ax, Du(x)\rangle, \quad x \in H. \tag{21}$$

As initial domains of $\mathcal{L}$, $\mathcal{L}^{OU}$ and $\mathcal{L} + \langle B, D\rangle$ we take the set $\mathcal{E}_A(H)$ defined to consist of the linear span of all real parts of functions $\varphi : H \to \mathbb{R}$ of the form $\varphi(x) = e^{i\langle h,x\rangle}$, $x \in H$, with $h \in D(A)$. It is easy to check that $\mathcal{E}_A(H) \subset W^{1,2}(H,\gamma)$ densely and $\mathcal{E}_A(H) \subset W^{1,2}(H,\nu)$ densely. Then rewriting the last term in the above expression as $\langle Ax, Du(x)\rangle$, the above operators are well defined for $u \in \mathcal{E}_A(H)$. Below we are going to use results from [DR02] in a substantial way with $F := \partial V$, the subdifferential of V, which is maximal monotone (see e.g. [Ba10]) and which is in general multivalued, but single-valued on $D_V \subset D(F)$ because $\partial V(x) = \nabla V(x)$ for $x \in D_V$.

Let us first check that the assumptions (H1) and (H2) in there are satisfied.

First, Hypothesis 1.1 in [DR02] is satisfied since we are in the special case $A = A^*$ and $C = I$. Hypothesis 1.2(ii) is satisfied for $\mathcal{L}$ defined above, replacing N_0 in [DR02] with $F_0 := \nabla V$, since by integrating by parts we have

$$\int_H \mathcal{L}\,\varphi\,\psi\, d\nu = -\frac{1}{2}\int_H \langle D\varphi, D\psi\rangle\, d\nu, \quad \forall\, \varphi,\ \psi \in \mathcal{E}_A(H)$$

and thus, taking $\psi = 1$,

$$\int_H \mathcal{L}\,\varphi\, d\nu = 0, \quad \forall\, \varphi,\ \psi \in \mathcal{E}_A(H). \tag{22}$$

Here F_0 is the minimal section of F in [DR02] and hence $\nabla V = F_0$ on $D(V) \subset D(F)$, so Hypothesis 1.2(iii) holds. Hypothesis 1.2(i) follows from Remark 1.

The first result we now deduce from [DR02] is the following:

Proposition 18 *$(\mathcal{L}, \mathcal{E}_A(H))$ is closable on $L^2(H,\nu)$ and its closure $(\mathcal{L}, D(\mathcal{L}))$ is m–dissipative on $L^2(H,\nu)$.*

Proof. This is a special case of [DR02, Theorem 2.3]. ■

For later use we need to replace $\mathcal{E}_A(H)$ in Proposition 18 above by $\mathcal{F}C_b^2$ (defined in Section 1 of this paper). We need the following easy lemma.

Lemma 19 *Let $\varphi \in C_b^2(\mathbb{R}^d)$. Then there exists a sequence φ_n, $n \in \mathbb{N}$, each φ_n consisting of linear combinations of functions of type $x \to \cos\langle a, x\rangle_{\mathbb{R}^d}$, $a \in \mathbb{R}^d$, such that $\sup\limits_{n\in\mathbb{N}} \left\{ \|\varphi_n\|_\infty + \|D\varphi_n\|_\infty + \|D^2\varphi_n\|_\infty \right\} < \infty$ and*

$$\lim_{n\to\infty} \varphi_n(x) = \varphi(x), \quad \lim_{n\to\infty} D\varphi_n(x) = D\varphi(x), \quad \lim_{n\to\infty} D^2\varphi_n(x) = D^2\varphi(x),$$

for all $x \in \mathbb{R}^d$.

Proof. First assume that $\varphi \in C_b^\infty(\mathbb{R}^d)$ with compact support. Then we have

$$\varphi(x) = \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle_{\mathbb{R}^d}} \, \widehat{\varphi}(\xi) \, d\xi, \quad x \in \mathbb{R}^d,$$

where $\widehat{\varphi}$ is in the Schwartz test function space, with the corresponding integral representations for $D\varphi$ and $D^2\varphi$.

Discretizing the integrals immediately implies the assertion since $x \mapsto (1 + |x|^2)\widehat{\varphi}$ is Lebesgue integrable. Replacing φ by $\chi_n\varphi$ where χ_n, $n \in \mathbb{N}$, is a suitable sequence of localizing functions (*bump functions*), the result follows for all $\varphi \in C_b^\infty(\mathbb{R}^d)$ by regularization through convolution with a Dirac sequence. ∎

As an immediate consequence of Proposition 18 and Lemma 19 we get:

Proposition 20 *$(\mathcal{L}, \mathcal{F}C_b^2)$ is closable on $L^2(H, \nu)$ and the closure $(\mathcal{L}, D(\mathcal{L}))$ is the same as that in Proposition 18, hence it is m–dissipative on $L^2(H, \nu)$. Furthermore,*

$$\mathcal{L}u = \mathcal{L}^{OU}u - \langle \nabla V, Du\rangle, \quad \forall\, u \in \mathcal{F}C_b^\infty.$$

Since $(\mathcal{L}, D(\mathcal{L}))$ is an m–dissipative operator on $L^2(H, \nu)$ by Proposition 18, every $\lambda > 0$ is in its resolvent set, hence $(\lambda - \mathcal{L})^{-1}$ exists as a bounded operator on $L^2(H, \nu)$. The following is one of the main results in [DR02].

Theorem 21 *Let $\lambda > 0$ and $f \in B_b(H)$. Then there exists a ν-version of $(\lambda - \mathcal{L})^{-1}f$ denoted by $R_\lambda f$, which is Lipschitz on H, more precisely*

$$|R_\lambda f(x) - R_\lambda f(y)| \le \sqrt{\frac{\pi}{\lambda}} \, \|f\|_\infty \, |x - y|, \quad \forall\, x, y \in H. \tag{23}$$

Proof. We first notice that H_0, defined in [DR02] to be the topological support of ν, in our case is equal to H, since ν has the same zero sets as the (non degenerate) Gaussian measure γ on H. Hence the assertion follows from the last sentence of [DR02, Proposition 5.2]. ∎

Remark 22 In fact, each R_λ is a kernel of total mass λ^{-1}, absolutely continuous with respect to ν and $(R_\lambda)_{\lambda>0}$ forms a resolvent of kernels on $(H, \mathcal{B}(H))$. We refer to [DR02, Section 5] for details.

Now we are going to solve (19) for each $f \in B_b(H)$ if λ is large enough, and show that the solution $u \in L^2(H,\nu)$ has a ν-version which is Lipschitz continuous, with Lipschitz constant dominated up to a constant by $\|f\|_\infty$.

First we need the following.

Lemma 23 *Let $g : H \to \mathbb{R}$ be Lipschitz. Then $g \in W^{1,2}(H,\gamma)$, hence also in $W^{1,2}(H,\nu)$ and $\|Dg\|_\infty \le \|g\|_{Lip}$ (=Lipschitz norm of g). Furthermore, $Dg = \nabla g$, γ-a.e. where ∇g is the Gâteaux derivative of g which exists γ-a.e..*

Proof. By the fundamental result in [A76], [P78] the set D_g of all $x \in H$ where g is Gâteaux-(even Fréchet-) differentiable has γ measure one. Let ∇g denote its Gâteaux derivative. Since $|\nabla g| \in L^\infty(H,\mu)$, it follows trivially that $g \in \mathcal{D}_0$ defined in (7). Hence by Lemma 9 the assertion follows. ■

Lemma 24 *Consider the operator $T_\lambda : L^\infty(H,\nu) \to L^\infty(H,\nu)$ defined by*

$$T_\lambda\varphi = \langle B, \nabla R_\lambda \varphi\rangle, \quad \varphi \in L^\infty(H,\nu).$$

Then for $\lambda \ge 4\pi\|B\|_\infty^2$

$$\|T_\lambda\varphi\|_{L^\infty(H,\nu)} \le \frac{1}{2}\,\|\varphi\|_{L^\infty(H,\nu)}, \quad \forall\, \varphi \in L^\infty(H,\nu).$$

Proof. We have by (23) and Lemma 23 that for $\varphi \in L^\infty(H,\mu)$

$$\|T_\lambda\varphi\|_{L^\infty(H,\nu)} \le \|B\|_\infty \sqrt{\frac{\pi}{\lambda}}\,\|\varphi\|_{L^\infty(H,\nu)},$$

and the assertion follows. ■

Proposition 25 *Let $f \in B_b(H)$ and $\lambda \ge 4\pi\|B\|_\infty^2$. Then (19) has a unique solution given by the Lipschitz function*

$$u := R_\lambda((I - T_\lambda)^{-1} f).$$

This solution is Lipschitz on H with Lipschitz norm

$$\|u\|_{Lip} \le 2\sqrt{\frac{\pi}{\lambda}}\,\|f\|_\infty.$$

Furthermore,

$$\|u\|_\infty \le \frac{2}{\lambda}\,\|f\|_\infty.$$

Proof. Since the operator norm of T_λ is less than $\frac{1}{2}$, the operator $(I - T_\lambda)^{-1}$ exists as a continuous operator on $L^\infty(H,\nu)$ with operator norm less than 2. Furthermore by Theorem 21 and Lemma 23

$$(\lambda - \mathcal{L})R_\lambda((I - T_\lambda)^{-1}f) - \langle B, DR_\lambda((I - T_\lambda)^{-1}f)\rangle$$

$$= (I - T_\lambda)^{-1}f - T_\lambda((I - T_\lambda)^{-1}f) = f.$$

The final part follows from (23) ■

Having established the result for the scalar equation (19) for $\lambda \geq 4\pi \|B\|_\infty^2$, we may prove it for the vector equation (4), whose solution U has components u^i satisfying the equation

$$(\lambda + \lambda_i) u^i - \mathcal{L} u^i - \langle B(x), Du^i \rangle = f^i \tag{24}$$

where f^i are the components of the vector function $F : H \to H$. ($U(x) = \sum_{i=1}^\infty u^i(x) e_i$, $F(x) = \sum_{i=1}^\infty f^i(x) e_i$).

We have by Proposition 25

$$\left|u^i(x) - u^i(y)\right|^2 \leq \frac{4\pi}{\lambda + \lambda_i} \|f^i\|_\infty^2 |x-y|^2 \leq \frac{4\pi}{\lambda + \lambda_i} \|F\|_\infty^2 |x-y|^2,$$

hence

$$\sum_{i=1}^\infty \left|u^i(x) - u^i(y)\right|^2 \leq c(\lambda)^2 \|F\|_\infty^2 |x-y|^2$$

where $c(\lambda) := \sum_{i=1}^\infty \frac{4\pi}{\lambda+\lambda_i}$. This series converges and $\lim_{\lambda\to\infty} c(\lambda) = 0$. Moreover, $|U(x) - U(y)|^2 = \sum_{i=1}^\infty \left|u^i(x) - u^i(y)\right|^2$, hence we have proved:

Lemma 26 *$U(= U_\lambda)$ defined above satisfies*

$$|U(x) - U(y)| \leq c(\lambda) \|F\|_\infty |x-y| \qquad x, y \in H$$

with $\lim_{\lambda\to\infty} c(\lambda) = 0$.

3.2 An Itô–formula for the Lipschitz continuous solutions

Below we want to apply Itô's formula to $u(X_t)$, $t \geq 0$, where u is as in Proposition 25 and $(X_t)_{t\geq 0}$ are the paths of the Markov process $\boldsymbol{M}$ from Theorem 11. Since u is only Lipschitz and we are on the infinite dimensional state space H, this is a delicate issue. To give a technically clean proof we need a specific approximation of the solution u in Proposition 25 by functions $u_n \in \mathcal{F}C_b^2$, $n \in \mathbb{N}$. More precisely, we shall prove the following result:

Proposition 27 *Let $\lambda > 0$ and $g \in B_b(H) \cap D(\mathcal{L}^{OU}, C_{b,2}^1(H))$ (for the definition of the latter see below). Set*

$$w := R_\lambda g.$$

Then there exists a sequence $u_n \in \mathcal{F}C_b^2$, $n \in \mathbb{N}$, such that

$$\begin{gathered}\sup_{n\in\mathbb{N}}(\|u_n\|_\infty + \|\nabla u_n\|_\infty) \leq 2\sqrt{\pi}\ \max\{\lambda^{-1}, \lambda^{-1/2}\}\, \|g\|_\infty, \\ \lim_{n\to\infty} \int_H \left[|\mathcal{L}^{OU}(w - u_n)|^2 + |\nabla(w-u_n)|^2 + (w-u_n)^2\right] d\nu = 0.\end{gathered} \tag{25}$$

In particular, $u_n \to w$ as $n \to \infty$ in $\mathcal{L}$-graph norm (on $L^2(H, \nu)$) and

$$\mathcal{L}w = \mathcal{L}^{OU} w - \langle \nabla V, \nabla w \rangle.$$

For the proof we need some more details from [DR02].

Define for $\lambda > 0$ and $\varphi \in B_b(H)$

$$R(\lambda, \mathcal{L}^{OU})\varphi(x) = \int_0^\infty e^{-\lambda t} P_t \varphi(x) dt, \tag{26}$$

where P_t is defined as in (11). Then

$$R(\lambda, \mathcal{L}^{OU})(C^1_{b,2}(H)) \subset C^1_{b,2}(H),$$

where $C^1_{b,2}(H)$ denotes the set of all $\varphi \in C^1_b(H)$ such that

$$\sup_{x \in H} \frac{|\varphi(x)|}{1 + |x|^2} < \infty \text{ and } \sup_{x \in H} \frac{|D\varphi(x)|}{1 + |x|^2} < \infty.$$

As in [DR02] we set

$$D(\mathcal{L}^{OU}, C^1_{b,2}(H)) := R(\lambda, \mathcal{L}^{OU})(C^1_{b,2}(H)),$$

which by this resolvent equation is independent of $\lambda > 0$ and is a natural domain for the operator $\mathcal{L}^{OU}$.

Proposition 28 *Let $u \in D(\mathcal{L}^{OU}, C^1_{b,2}(H))$. Then there exists $\varphi_n \in \mathcal{E}_A(H)$, $n \in \mathbb{N}$, such that $\varphi_n \to u$ in ν-measure and for some $C \in (0, \infty)$*

$$|\varphi_n(x)| + |D\varphi_n(x)| + |\mathcal{L}^{OU}\varphi_n(x)| \le C(1 + |x|^2), \quad \forall\, x \in H,\ n \in \mathbb{N}.$$

In particular, $u \in D(\mathcal{L})$ and $\varphi_n \to u$ in the graph norm of $\mathcal{L}$ on $L^2(H, \nu)$ and

$$\mathcal{L}u = \mathcal{L}^{OU}u - \langle \nabla V, Du \rangle.$$

Proof. Since convergence in measure comes from a metrizable topology this follows from [DR02, Lemma 2.2], Lebesgue's dominatd convergence theorem, Remark 0 and the fact that $(\mathcal{L}, \mathcal{E}_A(H))$ is closable on $L^2(H, \nu)$. ∎

Now let us recall the approximation procedure for ∂V, more precisely for its sub-differential $F := -\partial V$ with domain $D(F)$, performed in [DR02] (We recall that ∇V is maximal monotone (see e.g [Ba10]), hence we can consider its Yosida approximations.). For $\alpha \in (0, \infty)$ we set

$$F_\alpha(x) := \frac{1}{\alpha}\,(J_\alpha(x) - x), \quad x \in H,$$

where

$$J_\alpha(x) := (I - \alpha F)^{-1}(x), \quad x \in H.$$

It is well known (see e.g [Ba10]) that

$$\begin{aligned} &\lim_{\alpha \to 0} F_\alpha(x) = F_0(x), \quad \forall\, x \in D(F) \\ &|F_\alpha(x)| \le F_0(x), \quad \forall\, x \in D(F), \end{aligned} \tag{27}$$

where

$$F_0(x) := \inf_{y \in F(x)} |y|.$$

(Recall that $F(x) = \partial V(x)$ is in general multivalued unless $x \in D_V$, when $\partial V(x) = \nabla V(x)$.) We need a further standard regularization by setting

$$F_{\alpha,\beta}(x) := \int_H e^{\beta B} F_\alpha(e^{\beta B} + y) N_{\frac{1}{2}B^{-1}(e^{2\beta B}-1)}(dy), \quad \alpha, \beta \in (0,\infty), \tag{28}$$

where $B : D(B) \subset H \to H$ is a self-adjoint negative definite operator such that B^{-1} is of trace class. Then $F_{\alpha,\beta}$ is dissipative, of class C^∞ has bounded derivatives of all orders and

$$F_{\alpha,\beta} \to F_\alpha \text{ pointwise as } \beta \to 0 \tag{29}$$

(see [DZ92, Theorem 9.19]).

Now let us fix $\lambda > 0$ and consider the equation for $v \in C_b^2(H)$

$$\lambda u - \mathcal{L}^{OU} u - \langle F_{\alpha,\beta}, Du \rangle = v. \tag{30}$$

Then by [DR02, p. 268] there exists a linear map

$$R_\lambda^{\alpha,\beta} : C_b^2(H) \to D(\mathcal{L}^{OU}, C_{b,2}^1(H)) \cap C_b^2(H))$$

(in fact given by the resolvent of the SDE corresponding to the Kolmogorov operator on the left hand side of (30)) such that $R_\lambda^{\alpha,\beta} v$ is a solution to (30) for each $v \in C_b^2(H)$. In particular,

$$\|R_\lambda^{\alpha,\beta} v\|_\infty \le \frac{1}{\lambda} \|v\|_\infty, \quad \lambda > 0, \ v \in C_b^2(H). \tag{31}$$

We also have by [DR02, (4.7)] that

$$\sup_{x \in H} |\nabla R_\lambda^{\alpha,\beta} v(x)| \le \sqrt{\frac{\pi}{\lambda}} \|v\|_\infty, \quad \lambda > 0, \ v \in C_b^2(H). \tag{32}$$

Now the proof of Proposition 27 will be the consequence of the following two lemmas.

Lemma 29 *Let $\alpha_n \in (0,\infty)$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} \alpha_n = 0$. Then there exists $\beta_n \in (0,\infty)$, $n \in \mathbb{N}$, such that for all $v \in C_b^2(H)$ we have that*

$$\lim_{n\to\infty} R_\lambda^{\alpha_n,\beta_n} v = R_\lambda v,$$

in $\mathcal{L}$-graph norm (on $L^2(H,\nu)$).

Proof. (cf. the proof of [DR02, Theorem 2.3]). Since $D(F) \supset D_V$, so $\nu(D(F)) \ge \nu(D_V) = 1$, it follows by (27) and Lebesgue's dominated convergence theorem that

$$\lim_{n\to\infty} \int_H |F_{\alpha_n} - \nabla V|^2 d\nu = 0. \tag{33}$$

Since by the definition of $F_{\alpha,\beta}$ we have that for each $\alpha > 0$ there exists $c_\alpha \in (0,\infty)$ such that

$$|F_{\alpha,\beta}(x)| \leq c_\alpha(1+|x|), \quad \forall\, x \in H,$$

it follows by (29) that for $n \in \mathbb{N}$ there exists $\beta_n \in (0, \frac{1}{n})$, such that

$$\int_H |F_{\alpha_n,\beta_n}(x) - F_{\alpha_n}(x)|\, \nu(dx) \leq \frac{1}{n}.$$

Hence by (33)

$$\lim_{n\to\infty} \int_H |F_{\alpha_n,\beta_n} - \nabla V|^2 d\nu = 0. \tag{34}$$

Now let $v \in C_b^2(H)$. Then $R_\lambda^{\alpha_n,\beta_n} v \in C_b^2(H) \cap D(\mathcal{L}^{OU}, C_{b,2}^1(H))$, hence by Proposition 28 and, because $R_\lambda^{\alpha_n,\beta_n} v$ solves (30), we have

$$(\lambda - \mathcal{L}) R_\lambda^{\alpha_n,\beta_n} v = v + \langle F_{\alpha_n,\beta_n} - \nabla V, \nabla R_\lambda^{\alpha_n,\beta_n} v\rangle, \tag{35}$$

consequently,

$$R_\lambda^{\alpha_n,\beta_n} v = (\lambda - \mathcal{L})^{-1} v + (\lambda - \mathcal{L})^{-1} \left(\langle F_{\alpha_n,\beta_n} - \nabla V, \nabla R_\lambda^{\alpha_n,\beta_n} v\rangle \right). \tag{36}$$

But by (32) and (34)

$$\lim_{n\to\infty} \int_H |\langle F_{\alpha_n,\beta_n} - \nabla V, \nabla R_\lambda^{\alpha_n,\beta_n} v\rangle|^2\, d\nu = 0.$$

Hence (35) and (36) imply the assertion, because $(\lambda - \mathcal{L})^{-1}$ is continuous on $L^2(H,\nu)$ ad $R_\lambda v$ is a ν-version of $(\lambda - \mathcal{L})^{-1} v$. ■

Lemma 30 *Let λ, g and w be as in Proposition 27. Then there exist $u_n \in C_b^2(H) \cap D(\mathcal{L}^{OU}, C_{b,2}^1(H))$, $n \in \mathbb{N}$, such that*

$$\sup_{n\in\mathbb{N}} (\|u_n\|_\infty + \|\nabla u_n\|_\infty) \leq 2\sqrt{\pi}\, \max\{\lambda^{-1}, \lambda^{-1/2}\}\, \|g\|_\infty,$$

and (25) *holds for these u_n, $n \in \mathbb{N}$.*

Proof. Since $C_b^2(H) \subset L^2(H,\nu)$ densely, we can find $v_k \in C_b^2(H)$, $k \in \mathbb{N}$, such that

$$\sup_{k\in\mathbb{N}} \|v_k\|_\infty \leq 2\|g\|_\infty$$

and

$$\lim_{k\to\infty} \int_H |g - v_k|^2\, d\nu = 0,$$

hence by the continuity of $(\lambda - \mathcal{L})^{-1}$,

$$\lim_{k\to\infty} \int_H |R_\lambda(g - v_k)|^2 \, d\nu = 0.$$

Therefore, $R_\lambda v_k \to R_\lambda g$ in $\mathcal{L}$-graph norm (on $L^2(H,\nu)$) as $k \to \infty$. Hence by Lemma 29 we can choose a subsequence $(k_n)_{n\in\mathbb{N}}$ such that

$$R_\lambda^{\alpha_{n_k},\beta_{n_k}} v_k \to R_\lambda g \text{ in } \mathcal{L}\text{-graph norm (on } L^2(H,\nu)) \text{ as } k \to \infty.$$

Taking $u_k := R_\lambda^{\alpha_{n_k},\beta_{n_k}} v_k$, $k \in \mathbb{N}$, the assertion follows from (31) and (32), recalling that convergence in $\mathcal{L}$-graph norm implies convergence in $W^{1,2}(H,\nu)$. ■

Proof of Proposition 27. Let $u \in C_b^2(H) \cap D(\mathcal{L}^{OU}, C_{b,2}^1(H))$ and define $u_n := u \circ P_n \in \mathcal{F}C_b^2$, $n \in \mathbb{N}$. Then $\|u_n\|_\infty \leq \|u\|_\infty$ and $\|\nabla u_n\|_\infty \leq \|\nabla u\|_\infty$. Furthermore, $u_n \to u$, $\nabla u_n \to \nabla u$ and $\mathcal{L}^{OU} u_n \to \mathcal{L}^{OU} u$ pointwise on H as $n \to \infty$. Furthermore, $\mathcal{L}^{OU} u_n \to \mathcal{L}^{OU} u$ in $L^2(H,\gamma)$, hence in $L^2(H,\nu)$ as $n \to \infty$. Now the assertion follows by Lemma 30. ■

Corollary 31 *Let $f \in B_b(H), \lambda \geq 4\pi\|B\|_\infty^2$ and u as in Proposition 25, i.e.*

$$u := R_\lambda((I - T_\lambda)^{-1} f).$$

Let $u_n \in \mathcal{F}C_b^2 \cap D(\mathcal{L}^{OU}, C_{b,2}^1(H))$, $n \in \mathbb{N}$, be as in Proposition 27 with $g := (I - T_\lambda)^{-1} f$ ($\in B_b(H)$, with $\|g\|_\infty \leq 2\|f\|_\infty$ by the proof of Proposition 25). Consider the Markov process

$$\boldsymbol{M} := (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_z)_{z\in S_V})$$

from Theorem 11, with S_V defined in Proposition 15. Then there exists an $\mathcal{E}_\nu$–nest $(F_k^{\lambda,f})_{k\in\mathbb{N}}$ of compacts such that for every $k \in \mathbb{N}$, $F_k^{\lambda,f} \subset S_V$ and some subsequence $n_l \to \infty$

(i) $u_{n_l}(z) \to u(z)$,

(ii) $\mathbb{E}_{P_z} \displaystyle\int_0^\infty e^{-\lambda s} |\nabla u - \nabla u_{n_l}|^2(X_s) ds = R_\lambda(|\nabla u - \nabla u_{n_l}|^2)(z) \to 0,$

(iii) $\mathbb{E}_{P_z} \displaystyle\int_0^\infty e^{-s} |\mathcal{L}(u - u_{n_l})(X_s)| ds \to 0,$

uniformly in $z \in F_k^{\lambda,f}$. In particular, for all $z \in \bigcup_{k=1}^\infty F_k^{\lambda,f} \setminus N$ with an $\mathcal{E}_\nu$-exceptional set N , we have that P_z–a.e. the following Itô–formula holds

$$u(X_t) - z - \int_0^t \mathcal{L}u(X_s) ds = \int_0^t \langle \nabla u(X_s), dW(s) \rangle, \quad \forall\, t \geq 0. \tag{37}$$

Proof. Since the convergence of all three sequences in (i)–(iii) takes place in $W^{1,2}(H,\nu)$, the existence of such an $\mathcal{E}_\nu$-nest and subsequence $(n_l)_{l\in\mathbb{N}}$ follows from [MR92, Chapter III, Proposition 3.5] and Theorem 10 above. By Theorem 13 for $z \in \bigcup_{k=1}^\infty F_k^{\lambda,f} \setminus N$, for some

$\mathcal{E}_\nu$-exceptional set N we know that $P_z[\bigcup_{k=1}^\infty \{\tau_{H\setminus F_k^{\lambda,f}} > t\}] = 1$ for all $t \geq 0$. So, fix $z \in \bigcup_{k=1}^\infty F_k^{\lambda,f} \setminus N$. Then by the classical Itô–formula on finite dimensional Euclidean space and by Theorem 11(iii) we have P_z-a.s.

$$\begin{aligned} &u_{n_l}(X_t) - z - \int_0^t (\mathcal{L}^{OU} u_{n_l} - \langle \nabla V, \nabla u_{n_l} \rangle)(X_s) ds \\ &= \int_0^t \langle \nabla u_{n_l}(X_s), dW(s) \rangle, \quad \forall\, t \geq 0. \end{aligned} \tag{38}$$

Fix $t > 0$. Then on $\{\tau_{H\setminus F_k^{\lambda,f}} > t\}$ we have by (ii) above that $u_{n_l}(X_t) \to u(X_t)$ as $n \to \infty$ and by the last part of Proposition 20 and (iii) above

$$\begin{aligned} &\mathbb{E}_{P_z} \int_0^t |(\mathcal{L}u - (\mathcal{L}^{OU} u_{n_l} - \langle \nabla V, \nabla u_{n_l} \rangle)(X_s))| ds \\ &\leq e^t\, \mathbb{E}_{P_z} \int_0^\infty e^{-s} |\mathcal{L}(u - u_{n_l})(X_s)| ds \to 0, \quad \text{as } l \to \infty, \end{aligned}$$

and also that by Itô's isometry and by (ii) above

$$\begin{aligned} &\mathbb{E}_{P_z} \left| \int_0^t \langle \nabla u(X_s) - \nabla u_{n_l}(X_s), dW_s \rangle \right|^2 \\ &\leq \mathbb{E}_{P_z} \int_0^t |\nabla u(X_s) - \nabla u_{n_l}(X_s)|^2 ds \\ &\leq e^{\lambda t} \int_0^\infty e^{-\lambda s} \mathbb{E}_{P_z} \left(|\nabla u(X_s) - \nabla u_{n_l}(X_s)|^2 \right) ds \\ &= e^{\lambda t} R_\lambda (|\nabla u - \nabla u_{n_l}|^2)(z) \to 0 \quad \text{as } l \to \infty. \end{aligned}$$

Hence on $\bigcup_{k=1}^\infty \{\tau_{H\setminus F_k^{\lambda,f}} > t\}$ we can pass to the limit in (38) to get (37). $\blacksquare$

Remark 32 By the same standard procedure already mentioned at the end of the proof of Proposition 15 we can find $S_V^{\lambda,f}$ such that $H \setminus S_V^{\lambda,f}$ is $\mathcal{E}_\nu$–exceptional and Theorem 11, Proposition 15, Theorem 17 hold with$S_V^{\lambda,f}$ replacing S_V and for all $z \in S_V^{\lambda,f}$, (i)–(iii) in Corollary 31 hold and (37) holds P_z-a.s. .

3.3 Maximal regularity estimates

Let us first consider again the solution u of the scalar equation (19). The following result is the main technical ingredient of this paper, on the Kolmogorov equation, see [DL14, Proposition 4.2].

Lemma 33 *We have that $u \in W^{2,2}(H,\nu)$ and there is a constant $C > 0$ such that, for all $\lambda \geq 1$,*

$$\int_H |Du(x)|^2 \nu(dx) \leq \frac{C}{\lambda} \int_H |f(x)|^2 \nu(dx) \tag{39}$$

$$\int_H \left\| D^2 u(x) \right\|_{HS}^2 \nu(dx) \leq C \int_H |f(x)|^2 \nu(dx). \tag{40}$$

We then apply this result componentwise to equation (4).

Theorem 34 *Let $U(x) = \sum_{i=1}^{\infty} u^i(x) e_i$ be the solution of equation (4) with $F(x) = \sum_{i=1}^{\infty} f^i(x) e_i$, namely $u = u^i$ satisfies equation (24) with $f = f^i$, for every $i \in \mathbb{N}$. Then*

$$\int_H \sum_{i=1}^{\infty} \left(\lambda_i \left| Du^i(x) \right|^2 + \left\| D^2 u^i(x) \right\|_{HS}^2 \right) \nu(dx) \leq C \int_H (|F(x)|^2 + |B(x)|^2) \nu(dx).$$

Proof. We apply the lemma and get

$$\begin{aligned} \int_H \left| Du^i(x) \right|^2 \nu(dx) &\leq \frac{C}{\lambda + \lambda_i} \int_H (\left| f^i(x) \right|^2 + |\langle B(x), e_i \rangle|^2) \nu(dx) \\ &\leq \frac{C}{\lambda_i} \int_H (\left| f^i(x) \right|^2 + |\langle B(x), e_i \rangle|^2) \nu(dx) \end{aligned}$$

$$\int_H \left\| D^2 u^i(x) \right\|_{HS}^2 \nu(dx) \leq C \int_H (\left| f^i(x) \right|^2 + |\langle B(x), e_i \rangle|^2) \nu(dx).$$

Therefore

$$\begin{aligned} &\int_H \sum_{i=1}^{\infty} \left(\lambda_i \left| Du^i(x) \right|^2 + \left\| D^2 u^i(x) \right\|_{HS}^2 \right) \nu(dx) \\ &\leq 2C \int_H (|F(x)|^2 + |B(x)|^2) \nu(dx) < \infty. \end{aligned}$$

The proof is complete. ■

Remark 35 Consider the situation of Lemma 33 and let $(u_l)_{l \in \mathbb{N}}$ be the sequence $(u_{n_l})_{l \in \mathbb{N}}$ from Corollary 31. Then it follows by Proposition 27 and Corollary 31 that as $n \to \infty$

$$f_n := (\lambda - \mathcal{L}) u_n + \langle B, Du_n \rangle \to f \quad \text{in } L^2(H,\nu).$$

Hence by (40)

$$\lim_{n \to \infty} \int_H \| D^2(u - u_n) \|_{HS}^2 \, d\nu = 0.$$

This will be crucially used to justify the application of mean value theorem in the proof of Lemma 39 below.

4 New formulation of the SDE

In this section we fix U, u^i as in Theorem 34 with $f^i := \langle B, e_i\rangle$ and $F := B$. Let $\lambda \geq 4\pi \|B\|_\infty^2$ so large that $c(\lambda) \leq \frac{1}{2}\|B\|_\infty^{-1}$ where $c(\lambda)$ is as in Lemma 25. Again, we write x^i for $\langle x, e_i\rangle$, $u^i(x)$ for $\langle U(x), e_i\rangle$, and so on. Below we shall apply Corollary 31 with f replaced by B^i and u^i replacing u, for $i \in \mathbb{N}$.

Remark 36 As the corresponding sets of allowed starting points $S_V^{B^i,\lambda}$, $i \in \mathbb{N}$, are concerned, as in Remark 32, by a standard diagonal procedure we can find $S_V \subset \cap_{i\in\mathbb{N}} S_V^{B^i,\lambda}$ such that $H \setminus S_V$ is $\mathcal{E}_\nu$-exceptional and Theorem 11, Proposition 15, Theorem 17 hold with this (smaller) S_V and for all $z \in S_V$ (i)–(iii) in Corollary 31 hold and (37) holds P_z-a.s.

Below we fix this set $S_V (\subset H)$.

Lemma 37 *Let $z \in S_V$ and set*

$$\varphi(x) = x + U(x), \quad x \in H,$$

namely $\varphi^i(x) = x^i + u^i(x)$ and let X be a solution of the SDE (1). Then for each $i \in \mathbb{N}$

$$d\varphi^i(X_t) = \left(-\lambda_i X_t^i - D_i V(X_t)\right) dt + (\lambda + \lambda_i)\, u^i(X_t)\, dt + \left\langle Du^i(X_t), dW_t \right\rangle + dW_t^i. \quad (41)$$

Proof. Fix $i \in \mathbb{N}$. Let us first prove the following

Claim: We have P_z-a.e.

$$\begin{aligned} u^i(X_t) &= u^i(z) + \int_0^t (\mathcal{L}^{OU} u^i(X_s) - \langle \nabla V(X_s) - B(X_s), Du^i(X_s)\rangle)\, ds \\ &\quad + \int_0^t \langle Du^i(X_s), dW_s\rangle, \quad t \geq 0. \end{aligned} \quad (42)$$

Indeed, considering the set Ω_0 of all $\omega \in \Omega$ such that (42) holds, we have to prove that $P(\Omega_0) = 1$. But by Girsanov's theorem this is equivalent to (37) with u^i replacing u. Hence the claim is proved.

As a consequence we obtain that

$$\begin{aligned} du^i(X_t) &= \mathcal{L}u^i(X_t)\, dt + \langle B(X_t), Du^i(X_t)\rangle dt + \left\langle Du^i(X_t), dW_t\right\rangle \\ &= -B^i(X_t)\, dt + (\lambda + \lambda_i)\, u^i(X_t)\, dt + \left\langle Du^i(X_t), dW_t\right\rangle \end{aligned}$$

and thus

$$\begin{aligned} dX_t^i &= (-\lambda_i X_t^i - D_i V(X_t))\, dt - du^i(X_t) \\ &\quad + (\lambda + \lambda_i)\, u^i(X_t)\, dt + \langle Du^i(X_t), dW_t\rangle + dW_t^i. \end{aligned}$$

Then

$$d\varphi^i(X_t) = \left(-\lambda_i X_t^i - D_i V(X_t)\right) dt + (\lambda + \lambda_i)\, u^i(X_t)\, dt + \left\langle Du^i(X_t), dW_t\right\rangle + dW_t^i.$$

■

In vector form we could formally rewrite (41) as

$$dX_t = (AX_t - \nabla V(X_t))\,dt - dU(X_t) + (\lambda - A)\,U(X_t)\,dt + DU(X_t)\,dW_t + dW_t$$

and

$$d\varphi(X_t) = (AX_t - \nabla V(X_t))\,dt + (\lambda - A)\,U(X_t)\,dt + DU(X_t)\,dW_t + dW_t.$$

But this is not rigorous, since X_t is typically not in the domain of A.

5 Proof of Theorem 5

Consider the situation described at the beginning of Section 4 with S_V being the set of all allowed starting points from Remark 36. In particular, by Lemma 26 we can choose λ big enough so that

$$\|U\|_{Lip} \leq \frac{1}{2}.$$

Lemma 38 *For every $x, y \in H$, we have*

$$\frac{1}{2}\,|x-y| \leq |\varphi(x) - \varphi(y)| \leq \frac{3}{2}\,|x-y|.$$

In particular, φ is injective and its inverse is Lipschitz continuous.

Proof. One has

$$\begin{aligned}|x-y| &\leq |x + U(x) - y - U(y)| + |U(x) - U(y)| \\ &\leq |\varphi(x) - \varphi(y)| + \frac{1}{2}\,|x-y|\end{aligned}$$

where we have used

$$|U(x) - U(y)| \leq \sup_{x\in H} \|DU(x)\|\,|x-y| \leq \frac{1}{2}\,|x-y|\,.$$

The claim follows. ■

Let X and Y be two solutions with initial condition x, defined on the same filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$ and w.r.t. the same cylindrical $(\mathcal{F}_t)$-Brownian motion W.

Lemma 39 *There is a Borel set $\Xi \subset S_V$ with $\gamma(\Xi) = 1$ having the following property: If $z \in \Xi$ and X, Y are two solutions with initial condition z (in the sense of Definition 4), defined on the same filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$ and w.r.t. the same $(\mathcal{F}_t)$-cylindrical Brownian motion W, then*

$$A_{t,z} < \infty$$

with probability one, for every $t \geq 0$, where the process $A_{t,z}$ is defined as

$$\begin{aligned} A_{t,z} &= 2\int_0^t \frac{|\nabla V(X_s) - \nabla V(Y_s)|}{|\varphi(X_s) - \varphi(Y_s)|}\, 1_{\varphi(X_s)\neq\varphi(Y_s)}\, ds \\ &\quad +2\sum_{i=1}^{\infty} \lambda_i \int_0^t \frac{(u^i(X_s) - u^i(Y_s))^2}{|\varphi(X_s) - \varphi(Y_s)|^2}\, 1_{\varphi(X_s)\neq\varphi(Y_s)}\, ds \\ &\quad +\sum_{i=1}^{\infty} \int_0^t \frac{|Du^i(X_s) - Du^i(Y_s)|^2}{|\varphi(X_s) - \varphi(Y_s)|^2}\, 1_{\varphi(X_s)\neq\varphi(Y_s)}\, ds. \end{aligned} \tag{43}$$

Proof. Let us first treat the case when (H3) holds. By the mean value theorem and Lemma 38 we have for ν-a.e. $z \in S_V$

$$A_t \leq 4N_{t,z}$$

where

$$\begin{aligned} N_{t,z} : &= 2\int_0^1 \int_0^t \|D^2 V(Z_s^\alpha)\|_{\mathcal{L}(H)}\, d\alpha\, ds \\ &\quad +\sum_{i=1}^{\infty} \int_0^1 \int_0^t \left(2\lambda_i |Du^i(Z_s^\alpha)|^2 + \|D^2 u^i(Z_s^\alpha)\|_{HS}^2\right)\, d\alpha\, ds, \end{aligned}$$

where

$$Z_t^\alpha = \alpha X_t + (1-\alpha)\, Y_t.$$

Let us briefly show why we can indeed use the mean value theorem here. We do it separately for all three differences under the integrals in (43). However, we only explain it for the last difference. The other two can be treated analogously. So, fix $i \in \mathbb{N}$. We want to prove that for γ-a.e. starting point $z \in H$ we have $P \otimes dt$-a.e.

$$Du^i(X_s) - Du^i(Y_s) = \int_0^1 D^2 u^i(\alpha X_s + (1-\alpha) Y_s)(X_s - Y_s) d\alpha. \tag{44}$$

We know by Corollary 31 and Remark 35 that there exists $u_n \in \mathcal{F}C_b^2$, $n \in \mathbb{N}$, such that for $\lambda \geq 4\pi \|B\|_\infty^2$ and all $z \in S_V$

$$\lim_{n\to\infty} \mathbb{E}_{P_z}\left[\int_0^\infty e^{-\lambda s} |Du^i - Du_n|^2 (X_s^V) ds\right] = 0 \tag{45}$$

and

$$\lim_{n\to\infty} \int_H \|D^2 u^i - D^2 u_n\|_{HS}^2\, d\nu = 0. \tag{46}$$

Here P_z is from the Markov process

$$\boldsymbol{M} := (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t^V)_{t\geq 0}, (P_z)_{z\in S_V})$$

in Corollary 31 (and we changed notation and used $(X_t^V)_{t\geq 0}$ instead of $(X_t)_{t\geq 0}$ in Corollary 31 to avoid confusion with our fixed solution $(X_t)_{t\in[0,T]}$ above).

Recalling that by Girsanov's theorem both X and Y have laws which are equivalent to the law of $X^V := X_t^V$, $t \in [0,T]$, it follows by (45) that as $n \to \infty$

$$\int_0^T |Du^i(X_s) - Du_n(X_s)|^2 ds \to 0, \quad \int_0^T |Du^i(Y_s) - Du_n(Y_s)|^2 ds \to 0,$$

in probability. If we can show that also

$$\int_0^T \int_0^1 \|D^2u^i - D^2u_n\|_{HS}\,(\alpha X_s + (1-\alpha)Y_s)|X_s - Y_s| d\alpha\, ds \to 0 \tag{47}$$

in probability as $n \to \infty$, (44) follows, since it trivially holds for u_n replacing u^i.

But the expression in (47) is bounded by

$$\sup_{s\in[0,T]} |X_s - Y_s| \int_0^T \int_0^1 \|D^2u^i - D^2u_n\|_{HS}\,(\alpha X_s + (1-\alpha)Y_s) d\alpha\, ds$$

and by the continuity of sample paths

$$\sup_{s\in[0,T]} |X_s - Y_s| < \infty, \quad P\text{-a.s.}$$

Furthermore, it follows from (46) and the proof of Lemma 40 and Corollary 41 below that for ν–a.e. $z \in S_V$

$$\int_0^T \int_0^1 \|D^2u^i - D^2u_n\|_{HS}\,(\alpha X_s + (1-\alpha)Y_s) d\alpha\, ds \to 0$$

as $n \to \infty$ P-a.s. Hence (47) follows.

By assumption (2) in (H3) we know that

$$\int_H \left\|D^2V(x)\right\|_{\mathcal{L}(H)} \nu(dx) < \infty$$

and by Theorem 34 we know that

$$\int_H \sum_{i=1}^{\infty} \left(\lambda_i \left|Du^i(x)\right|^2 + \left\|D^2u^i(x)\right\|_{HS}^2\right) \nu(dx) < \infty.$$

Thus we may apply Corollary 41 below with

$$f(x) = \left\|D^2V(x)\right\|_{\mathcal{L}(H)} + \sum_{i=1}^{\infty} \left(2\lambda_i \left|Du^i(x)\right|^2 + \left\|D^2u^i(x)\right\|_{HS}^2\right)$$

and get that $\int_0^1 \int_0^t f\left(Z_s^\alpha\right) d\alpha ds < \infty$ with probability one, for every $t \geq 0$ and ν-a.e.$z \in S^V$, i. e.

$$N_{t,z} < \infty$$

with probability one, for every $t \geq 0$, which concludes the proof since $A_t \leq 4N_{t,z}$.

Now let us consider the case when (H3)' holds. Clearly, we then handle the second and the third term in the right hand side of (42) as above. For the first term the treatment is different, but simpler. Indeed, we have by (H3)', Lemma 37 and by the mean value theorem that

$$\int_0^T \frac{|\nabla V(X_s) - \nabla V(Y_s)|}{|\varphi(X_s) - \varphi(Y_s)|} 1_{\varphi(X_s)\neq\varphi(Y_s)}\, ds$$

$$\leq 2\int_0^T \int_0^1 \|V_E''(Z_s^\alpha)\|_{L(H,E')}\, d\alpha\, ds$$

$$\leq 2\int_0^T [\Psi(|X_s|_E) + \Psi(|Y_s|_E)]\, ds.$$

But again using Girsanov's theorem we know that the laws of X and Y are equivalent to that of X^V, hence the last expression is finite P-a.e. ■

We may now prove Theorem 5. Let $z \in \Xi$. By Lemma 37,

$$d(\varphi^i(X_t) - \varphi^i(Y_t)) = -(\lambda_i(X_t^i - Y_t^i) + D_iV(X_t) - D_iV(Y_t))dt$$

$$+(\lambda + \lambda_i)(u^i(X_t) - u^i(Y_t))dt + \langle Du^i(X_t) - Du^i(Y_t), dW_t\rangle.$$

Hence, by Itô's formula, we get

$$d(\varphi^i(X_t) - \varphi^i(Y_t))^2$$

$$= -2(\varphi^i(X_t) - \varphi^i(Y_t))(\lambda_i(X_t^i - Y_t^i) + D_iV(X_t) - D_iV(Y_t))dt$$

$$+2(\varphi^i(X_t) - \varphi^i(Y_t))(\lambda + \lambda_i)(u^i(X_t) - u^i(Y_t))dt$$

$$+2(\varphi^i(X_t) - \varphi^i(Y_t))\langle Du^i(X_t) - Du^i(Y_t), dW_t\rangle$$

$$+|Du^i(X_t) - Du^i(Y_t)|^2 dt.$$

By definition of φ in Lemma 37, in the lines above there are the terms $-2(u^i(X_t) - u^i(Y_t))\lambda_i(X_t^i - Y_t^i)$ and $2(X_t^i - Y_t^i)\lambda_i(u^i(X_t) - u^i(Y_t))$ which cancel each other. Moreover the term $-2(X_t^i - Y_t^i)\lambda_i(X_t^i - Y_t^i)$ is negative.

Thus we deduce

$$\begin{aligned} d\left(\varphi^{i}\left(X_{t}\right)-\varphi^{i}\left(Y_{t}\right)\right)^{2} \leq & -2\left(\varphi^{i}\left(X_{t}\right)-\varphi^{i}\left(Y_{t}\right)\right)\left(D_{i} V\left(X_{t}\right)-D_{i} V\left(Y_{t}\right)\right) dt \\ & +2 \lambda\left(\varphi^{i}\left(X_{t}\right)-\varphi^{i}\left(Y_{t}\right)\right)\left(u^{i}\left(X_{t}\right)-u^{i}\left(Y_{t}\right)\right) dt \\ & +2 \lambda_{i}\left(u^{i}\left(X_{t}\right)-u^{i}\left(Y_{t}\right)\right)^{2} dt \\ & +2\left(\varphi^{i}\left(X_{t}\right)-\varphi^{i}\left(Y_{t}\right)\right)\left\langle D u^{i}\left(X_{t}\right)-D u^{i}\left(Y_{t}\right), d W_{t}\right\rangle \\ & +\left|D u^{i}\left(X_{t}\right)-D u^{i}\left(Y_{t}\right)\right|^{2} dt . \end{aligned}$$

Let $A_t = A_{t,z}$ be the process introduced in Lemma 39. We have

$$\begin{aligned} & d(e^{-A_t}(\varphi^i(X_t)-\varphi^i(Y_t))^2) \\ & \leq -2e^{-A_t}(\varphi^i(X_t)-\varphi^i(Y_t))(D_iV(X_t)-D_iV(Y_t))\, dt \\ & \quad +2\lambda e^{-A_t}(\varphi^i(X_t)-\varphi^i(Y_t))(u^i(X_t)-u^i(Y_t))\, dt \\ & \quad +2e^{-A_t}(\varphi^i(X_t)-\varphi^i(Y_t))\langle Du^i(X_t)-Du^i(Y_t), dW_t\rangle \\ & \quad +2\lambda_i e^{-A_t}(u^i(X_t)-u^i(Y_t))^2\, dt \\ & \quad +e^{-A_t}|Du^i(X_t)-Du^i(Y_t)|^2\, dt - e^{-A_t}(\varphi^i(X_t)-\varphi^i(Y_t))^2\, dA_t \end{aligned}$$

and thus, for every $N > 0$, summing the previous inequality for $i = 1, ..., N$, we get

$$\begin{aligned} & d(e^{-A_t}|P_N(\varphi(X_t)-\varphi(Y_t))|^2) \\ & \leq -2e^{-A_t}\langle P_N(\varphi(X_t)-\varphi(Y_t)), P_N(\nabla V(X_t)-\nabla V(Y_t))\,\rangle dt \\ & \quad +2\lambda e^{-A_t}\langle P_N(\varphi(X_t)-\varphi(Y_t)), U(X_t)-U(Y_t)\rangle\, dt \\ & \quad +2e^{-A_t}\sum_{i=1}^{N}(\varphi^i(X_t)-\varphi^i(Y_t))\langle Du^i(X_t)-Du^i(Y_t), dW_t\rangle \\ & \quad +2e^{-A_t}\sum_{i=1}^{N}\lambda_i(u^i(X_t)-u^i(Y_t))^2\, dt \\ & \quad +e^{-A_t}\sum_{i=1}^{N}|Du^i(X_t)-Du^i(Y_t)|^2\, dt - e^{-A_t}|P_N(\varphi(X_t)-\varphi(Y_t))|^2\, dA_t. \end{aligned}$$

Substituting dA_t, taking expectation and using simple inequalities we get

$$\mathbb{E}\left[e^{-A_t}\left|P_N\left(\varphi\left(X_t\right)-\varphi\left(Y_t\right)\right)\right|^2\right]$$

$$\leq 2\lambda\int_0^t \mathbb{E}\left[e^{-A_s}\left|\varphi\left(X_s\right)-\varphi\left(Y_s\right)\right|\left|U\left(X_s\right)-U\left(Y_s\right)\right|\right]ds$$

$$+2\int_0^t \mathbb{E}\left[e^{-A_s}\left|\varphi\left(X_s\right)-\varphi\left(Y_s\right)\right|\left|P_N\left(\nabla V\left(X_s\right)-\nabla V\left(Y_s\right)\right)\right|\right]ds$$
$$-2\int_0^t \mathbb{E}\left[e^{-A_s}\left|P_N\left(\varphi\left(X_s\right)-\varphi\left(Y_s\right)\right)\right|^2\frac{2\left|\nabla V\left(X_s\right)-\nabla V\left(Y_s\right)\right|}{\left|\varphi\left(X_s\right)-\varphi\left(Y_s\right)\right|}1_{\varphi(X_s)\neq\varphi(Y_s)}\right]ds$$
$$+\int_0^t \mathbb{E}\left[e^{-A_s}g_s\right]ds-\int_0^t E\left[e^{-A_s}g_s\frac{\left|P_N\left(\varphi\left(X_s\right)-\varphi\left(Y_s\right)\right)\right|^2}{\left|\varphi\left(X_s\right)-\varphi\left(Y_s\right)\right|^2}1_{\varphi(X_s)\neq\varphi(Y_s)}\right]ds$$

where for shortness of notation we have written

$$g_s:=2\sum_{i=1}^{\infty}\lambda_i\left(u^i\left(X_s\right)-u^i\left(Y_s\right)\right)^2+\sum_{i=1}^{\infty}\left|Du^i\left(X_s\right)-Du^i\left(Y_s\right)\right|^2.$$

By monotone convergence we may take the limit as $N\to\infty$ and deduce

$$\mathbb{E}\left[e^{-A_t}\left|\varphi\left(X_t\right)-\varphi\left(Y_t\right)\right|^2\right]\leq 2\lambda\int_0^t \mathbb{E}\left[e^{-A_s}\left|\varphi\left(X_s\right)-\varphi\left(Y_s\right)\right|\left|U\left(X_s\right)-U\left(Y_s\right)\right|\right]ds$$

$$+2\int_0^t \mathbb{E}\left[e^{-A_s}\left|\varphi\left(X_s\right)-\varphi\left(Y_s\right)\right|\left|\nabla V\left(X_s\right)-\nabla V\left(Y_s\right)\right|\right]ds$$
$$-2\int_0^t \mathbb{E}\left[e^{-A_s}\left|\varphi\left(X_s\right)-\varphi\left(Y_s\right)\right|^2\frac{2\left|\nabla V\left(X_s\right)-\nabla V\left(Y_s\right)\right|}{\left|\varphi\left(X_s\right)-\varphi\left(Y_s\right)\right|}1_{\varphi(X_s)\neq\varphi(Y_s)}\right]ds$$
$$+\int_0^t \mathbb{E}\left[e^{-A_s}g_s\right]ds-\int_0^t \mathbb{E}\left[e^{-A_s}g_s\frac{\left|\varphi\left(X_s\right)-\varphi\left(Y_s\right)\right|^2}{\left|\varphi\left(X_s\right)-\varphi\left(Y_s\right)\right|^2}1_{\varphi(X_s)\neq\varphi(Y_s)}\right]ds.$$

Notice that by Lemma 38, $X_s=Y_s$ if and only if $\varphi(X_s)=\varphi(Y_s)$. Hence we may drop the indicator function $1_{\varphi(X_s)\neq\varphi(Y_s)}$ in all integrals in the above inequality.

Therefore, certain terms cancel in the previous inequality and we get

$$\mathbb{E}\left[e^{-A_t}\left|\varphi\left(X_t\right)-\varphi\left(Y_t\right)\right|^2\right]\leq 2\lambda\int_0^t \mathbb{E}\left[e^{-A_s}\left|\varphi\left(X_s\right)-\varphi\left(Y_s\right)\right|\left|U\left(X_s\right)-U\left(Y_s\right)\right|\right]ds.$$

Using Lemma 26 and Lemma 38 we get

$$\mathbb{E}\left[e^{-A_t}\left|\varphi\left(X_t\right)-\varphi\left(Y_t\right)\right|^2\right]\leq 2\lambda C\int_0^t \mathbb{E}\left[e^{-A_s}\left|\varphi\left(X_s\right)-\varphi\left(Y_s\right)\right|^2\right]ds,$$

whence $\mathbb{E}\left[e^{-A_t}\left|\varphi\left(X_t\right)-\varphi\left(Y_t\right)\right|^2\right]=0$ by Gronwall's lemma, and thus $\varphi\left(X_t\right)=\varphi\left(Y_t\right)$ with probability one (since $A_t<\infty$ a.s.), for all $t\geq 0$; the same is true for the identity $X_t=Y_t$ since φ is invertible and finally X and Y are also indistinguishable since they are continuous processes.

To complete the proof we have to prove Corollary 41 below, which was used in the proof of Lemma 39.

6 Main lemmata

Let S_V as in Remark 36 and H_V as in (13) and set

$$\Xi_V := S_V \cap H_V \tag{48}$$

.

Lemma 40 *Let $f: H\rightarrow[0,\infty)$ be a Borel measurable function such that*

$$\int_H f(x)\,\gamma(dx)<\infty. \tag{49}$$

Then there is a Borel set $\Xi\subset S_V\cap H_V$ with $\gamma(\Xi)=1$ having the following property. Given any $z\in\Xi$ and any two solutions X,Y with initial condition z (as in the statement of Theorem 5) for all $T>0$ we have

$$P\left(\int_0^1\int_0^T f\left(Z_s^\alpha\right)ds\,d\alpha<\infty\right)=1$$

where $Z_t^\alpha=\alpha X_t+(1-\alpha)\,Y_t$.

Proof. Step 1 (estimates on OU process). A number $T>0$ is fixed throughout the proof. From the assumption on f it follows that there is a Borel set $\Xi_f\subset H$, with Ξ_f^c of γ-measure zero, such that

$$\mathbb{E}\left[\int_0^T f\left(Z_s^{OU,z}\right)ds\right]=\int_0^T\left(\int_H f(x)\,p_{s,z}(dx)\right)ds<\infty$$

for all $z\in\Xi_f$, where $p_{s,z}(dx)$ is the law at time s of the Ornstein-Uhlenbeck process $Z_s^{OU,z}$, i.e. the solution of the equation

$$dZ_t=AZ_tdt+dW_t,\qquad Z_0=z. \tag{50}$$

Indeed, we have

$$\begin{aligned}&\int_H\left(\int_0^T\left(\int_H f(x)\,p_{s,z}(dx)\right)ds\right)\gamma(dz)\\&=\int_0^T\left(\int_H\int_H f(x)\,p_{s,z}(dx)\,\gamma(dz)\right)ds\\&=\int_0^T\left(\int_H f(z)\,\gamma(dz)\right)ds=T\int_H f(z)\,\gamma(dz).\end{aligned}$$

This implies $\int_0^T \left(\int_H f(x)\, p_{s,z}(dx)\right) ds < \infty$ for γ-a.e. z.

Step 2 (Girsanov transform). Let Ξ_V as in (48) and Ξ_f be given as in Step 1. Let $\Xi = \Xi_V \cap \Xi_f$, of full γ-measure. In the sequel, $z \in \Xi$ will be given, thus we avoid to index all quantities by z.

From Theorem 17 we have

$$\int_0^T |\nabla V(X_s)|^2 \, ds + \int_0^T |\nabla V(Y_s)|^2 \, ds < \infty$$

for all $T > 0$, with probability one.

Let us introduce the sequence $\{\tau^n\}$ of stopping times defined as

$$\tau^n = \tau_B^n \wedge \tau_{V,1}^n \wedge \tau_{V,2}^n$$

$$\tau_B^n := \inf\left\{ t \geq 0 : \ \left| \int_0^t B(X_s)\, dW_s \right| + \left| \int_0^t B(Y_s)\, dW_s \right| \geq n \right\} \wedge T$$

$$\tau_{V,1}^n := \inf\left\{ t \geq 0 : \left| \int_0^t \langle \nabla V(X_s), dW_s \rangle \right| + \left| \int_0^t \langle \nabla V(Y_s), dW_s \rangle \right| \geq n \right\} \wedge T$$
$$\tau_{V,2}^n := \inf\left\{ t \geq 0 : \int_0^t |\nabla V(X_s)|^2 \, ds + \int_0^t |\nabla V(Y_s)|^2 \, ds \geq n \right\} \wedge T$$

for $n \geq 1$ (an infimum is equal to $+\infty$ if the corresponding set is empty). All stochastic and Lebesgue integrals are well defined and continuous in t, hence we have $\tau^n = T$ eventually, with probability one. In order to prove the lemma it is sufficient to prove that $E\left[\int_0^1 \int_0^{T\wedge\tau^n} f(Z_s^\alpha)\, ds d\alpha\right] < \infty$ for each n.

Let us also introduce the stochastic processes

$$b_s^\alpha := \alpha B(X_s) + (1-\alpha) B(Y_s)$$
$$v_s^\alpha := \alpha \nabla V(X_s) + (1-\alpha) \nabla V(Y_s)$$

and the stochastic exponentials

$$\rho_t^\alpha := \exp\left(-\int_0^t \langle b_s^\alpha - v_s^\alpha, dW_s \rangle - \frac{1}{2} \int_0^t |b_s^\alpha - v_s^\alpha|^2 \, ds \right).$$

Denote

$$\rho_t^{\alpha,n} := \rho_{t\wedge\tau^n}^\alpha = \exp\left(-\int_0^t \langle 1_{s\leq\tau^n} (b_s^\alpha - v_s^\alpha), dW_s \rangle - \frac{1}{2} \int_0^t 1_{s\leq\tau^n} |b_s^\alpha - v_s^\alpha|^2 \, ds \right).$$

By Novikov's criterium, this is a martingale (indeed $\int_0^T 1_{s\leq\tau^n} |b_s^\alpha - v_s^\alpha|^2 \, ds$ is a bounded r.v..
We may thus introduce the following new measures (and the corresponding expectations)

$$Q^{\alpha,n}(A) := \mathbb{E}\left[\rho_T^{\alpha,n} 1_A\right].$$

Girsanov's theorem implies that

$$
\begin{aligned}
\widetilde{W}_t^{n,\alpha} &:= W_t + \int_0^t 1_{s\leq\tau^n} \left(b_s^\alpha - v_s^\alpha\right) ds \\
&= W_t + \int_0^{t\wedge\tau^n} \left(b_s^\alpha - v_s^\alpha\right) ds
\end{aligned}
$$

is a new cylindrical Brownian motion.

Step 3 (Auxiliary process and conclusion). Recall also that Z_t^α (with the new notations) satisfies

$$dZ_t^\alpha = AZ_t^\alpha dt + \left(b_s^\alpha - v_s^\alpha\right) dt + dW_t.$$

Let us introduce the auxiliary process $Z_t^{\alpha,n}$ which solves, in the sense of Definition 4, the equation

$$Z_t^{\alpha,n} = z + \int_0^t AZ_s^{\alpha,n} ds + \int_0^t 1_{s\leq\tau^n} \left(b_s^\alpha - v_s^\alpha\right) ds + W_t.$$

It exists, by the explicit formula

$$Z_t^{\alpha,n} = e^{tA}z + \int_0^t e^{(t-s)A} 1_{s\leq\tau^n} \left(b_s^\alpha - v_s^\alpha\right) ds + \int_0^t e^{(t-s)A} dW_s$$

where e^{tA} is the analytic semigroup in H generated by A (taking inner product with the elements e_k of the basis, it is not difficult to check that this mild formula gives a solution in the weak sense of Definition 4). This process satisfies also

$$Z_t^{\alpha,n} = z + \int_0^t AZ_s^{\alpha,n} ds + \widetilde{W}_t^{n,\alpha}$$

by the definition of $\widetilde{W}_t^{n,\alpha}$, hence its law under $Q^{\alpha,n}$ is the same as the Gaussian law of Z_t^{OU} under P. Moreover,

$$Z_t^{\alpha,n} = Z_t^\alpha \text{ for } t \in [0,\tau^n]$$

(indeed, by the weak formulation, the process $Y_t = Z_t^{\alpha,n} - Z_t^\alpha$ verifies, pathwise, on $[0,\tau^n]$, the equation $Y_t' = AY_t$, $Y_0 = 0$, in the weak sense of Definition 4 and thus, taking inner product with the elements e_k of the basis, one proves $Y = 0$).

Therefore,

$$
\begin{aligned}
\mathbb{E}^{Q^{\alpha,n}}\left[\int_0^{T\wedge\tau^n} f\left(Z_s^\alpha\right) ds\right] &= \mathbb{E}^{Q^{\alpha,n}}\left[\int_0^{T\wedge\tau^n} f\left(Z_s^{\alpha,n}\right) ds\right] \\
&\leq \mathbb{E}^{Q^{\alpha,n}}\left[\int_0^{T} f\left(Z_s^{\alpha,n}\right) ds\right] \\
&= \mathbb{E}\left[\int_0^{T} f\left(Z_s^{OU}\right) ds\right] =: C' < \infty.
\end{aligned}
$$

But

$$\mathbb{E}^{Q^{\alpha,n}}\left[\int_0^{T\wedge\tau^n} f\left(Z_s^\alpha\right) ds\right] = \mathbb{E}\left[\rho_T^{\alpha,n}\int_0^{T\wedge\tau^n} f\left(Z_s^\alpha\right) ds\right]$$
$$\geq C_n\mathbb{E}\left[\int_0^{T\wedge\tau^n} f\left(Z_s^\alpha\right) ds\right]$$

where $C_n > 0$ is a constant such that $\rho_T^{\alpha,n} \geq C_n$: it exists because

$$\left(\rho_T^{\alpha,n}\right)^{-1} := \exp\left(\int_0^{T\wedge\tau^n} \langle b_s^\alpha - v_s^\alpha, dW_s\rangle + \frac{1}{2}\int_0^{T\wedge\tau^n} \left|b_s^\alpha - v_s^\alpha\right|^2 ds\right)$$

and τ^n includes the stopping of all these integrals. Therefore

$$\mathbb{E}\left[\int_0^{T\wedge\tau^n} f\left(Z_s^\alpha\right) ds\right] \leq \frac{C'}{C_n}$$

and thus also

$$\mathbb{E}\left[\int_0^1\int_0^{T\wedge\tau^n} f\left(Z_s^\alpha\right) ds d\alpha\right] \leq \frac{C'}{C_n}.$$

The proof is complete. ■

The next Corollary extends the previous result to the case when $\int_H f(x)\,\nu(dx) < \infty$. Clearly

$$\int_H f(x)\,\nu(dx) \leq \frac{1}{Z}\int_H f(x)\,\gamma(dx)$$

but not conversely, without additional assumptions on V. Hence Corollary 41 implies Lemma 40, but not conversely, in an obvious way. However, we may easily deduce Corollary 41 from Lemma 40 by assumptions (H1)–(H3).

Corollary 41 *Let $f : H \to [0,\infty)$ be a Borel measurable function such that*

$$\int_H f(x)\,\nu(dx) < \infty. \tag{51}$$

Then there is a Borel set $\Xi \subset S_V \cap H_V$ with $\nu(\Xi) = 1$ (equivalently $\gamma(\Xi) = 1$) having the property stated in Lemma 40.

Proof. Since $\int_H f(x)\,e^{-V(x)}\gamma(dx) < \infty$, we may apply Lemma 40 to the function $f(x)\,e^{-V(x)}$ instead of $f(x)$ and get, as a result, that there is a Borel set $\Xi \subset S_V \cap H_V$ with $\gamma(\Xi) = 1$ having the following property: given any $z \in \Xi$ and any two solutions X, Y as in the statement of Theorem 5), for all $T > 0$ we have

$$P\left(\int_0^1\int_0^T f\left(Z_s^\alpha\right) e^{-V\left(Z_s^\alpha\right)} ds d\alpha < \infty\right) = 1 \tag{52}$$

where $Z_t^\alpha = \alpha X_t + (1-\alpha) Y_t$. Take $z \in \Xi$. Since $V(Z_s^\alpha) \le V(X_t) + V(Y_t)$ (recall that $V \ge 0$ by Remark 1) and by Theorem 17

$$P\left(\bigcup_{n=1}^{\infty} \left\{\sigma^{X,Y}_{H\setminus K_n^V} > T\right\}\right) = 1,$$

where

$$\sigma^{X,Y}_{H\setminus K_n^V} := \min\left(\sigma^{X}_{H\setminus K_n^V}, \sigma^{Y}_{H\setminus K_n^V}\right)$$

and $\sigma^{X}_{H\setminus K_n^V}, \sigma^{Y}_{H\setminus K_n^V}$ are the first hitting times of $H \setminus K_n^V$ of X, Y respectively, we have by (52)

$$\int_0^1 \int_0^T f(Z_s^\alpha)\, e^{-V(X_s)}\, e^{-V(Y_s)}\, ds\, d\alpha < \infty \quad \text{on } \bigcup_{n=1}^{\infty} \left\{\sigma^{X,Y}_{H\setminus K_n^V} > T\right\}, \; P\text{-a.s.}$$

But for $\omega \in \{\sigma^{X,Y}_{H\setminus K_n^V} > T\}$ and $M_n := \sup\{V(z) :\; z \in K_n^V\}$

$$\int_0^1 \int_0^T f(Z_s^\alpha(\omega))\, ds\, d\alpha \le e^{2M_n} \int_0^1 \int_0^T f(Z_s^\alpha(\omega))\, e^{-V(X_s)}\, e^{-V(Y_s)}\, ds\, d\alpha < \infty.$$

Hence

$$P\left(\int_0^1 \int_0^T f(Z_s^\alpha(\omega))\, ds\, d\alpha < \infty\right) = 1.$$

■

7 Applications

7.1 Reaction–diffusion equations

Let $H := L^2((0,1), d\xi)$, with $d\xi =$ Lebesgue measure and $A = -\Delta$ with domain $H^2(0,1) \cap H_0^1(0,1)$, i.e. A is the Dirichlet Laplacian on $(0,1)$. Then clearly (H1) holds.

Let $m \in [1, \infty)$ and

$$V(x) := \begin{cases} \displaystyle\int_0^1 |x(\xi)|^{m+1}\, d\xi, & \text{if } x \in L^{m+1}((0,1), d\xi) \\ +\infty, & \text{else.} \end{cases} \tag{53}$$

V obviously satisfies (H2). Now we are going to verify (H3)' for this convex functional. (Of course, then according to Remark 1 we subsequently replace this V by $V + \frac{\omega}{2}|\cdot|_H^2$.)

For the separable Banach space E in (H3)' we take

$$E := L^{2m}((0,1), d\xi) =: L^{2m}. \tag{54}$$

Then by elementary calculations for $x \in E$

$$V_E'(x) = (m+1)|x|^{m-1}\, x \in H \subset L^{\frac{2m}{2m-1}} = E', \tag{55}$$

$$V_E''(x)(h_1, h_2) = m(m+1) \int_0^1 |x(\xi)|^{m-1}\, h_1(\xi)\, h_2(\xi)\, d\xi, \tag{56}$$

for $h_1, h_2 \in E$. Obviously, the right hand side of (56) is also defined for $h_1, h_2 \in H$ and by Hölder's inequality, continuous in $(h_1, h_2) \in E \times H$ with respect to the product topology. Hence for all $x \in E$

$$V_E''(x) \subset L(H, E')$$

and furthermore (again by Hölder's inequality)

$$\|V_E''(x)\|_{L(H,E')} \leq |x|_E^{m-1}.$$

(55) implies that $E \subset D_V$. But our Gaussian measure $\gamma = N_{-\frac{1}{2}A^{-1}}$ is known to have full mass even on $C([0,1];\mathbb{R})$ because it is the law of the Brownian Bridge, hence $\gamma(E) = 1$ and so, $\gamma(D_V) = 1$. Furthermore, then obviously by Fernique's theorem the first inequality in (2) is satisfied.

It remains to verify (3), i. e. for γ-a.e. initial condition $z \in H$

$$\mathbb{E} \int_0^T |X^V(s)|_E^{m-1}\, ds < \infty, \tag{57}$$

where $X^V(t),\ t \in [0,T]$, solves SDE (1) with $B = 0$. But the existence of such a process for γ-a.e. $z \in H$ follows from Theorem 10 in Section 2 above. That this process satisfies (57) follows from results in [BDR10]. Indeed, it follows by [BDR10, Theorem 3.6 and Proposition 4.1] and Fatou's lemma that even

$$\mathbb{E} \int_0^T |X^V(s)|_E^{2m}\, ds < \infty$$

for (γ-a.e.) $z \in E$.

Hence (H3)' is verified and our main result, Theorem 5, applies to this case.

7.2 Weakly differentiable drifts

The main motivation to also consider condition (H3), i. e. to assume that the (γ-weak) second derivative D^2V of V exists and is in $L^1(H, \gamma; L(H))$, was to make a connection between our results and those in finite dimensions by [CJ13]. As mentioned in the introduction our results generalize some of the results of [CJ13] in the special case when $H = \mathbb{R}^d$. In addition, since we work with respect to a Gaussian measure (and not Lebesgue measure on $\mathbb{R}^d$) our integrability conditions are generically weaker than those in [CJ13]. As far as the infinite dimensional case is concerned, one might ask what are examples of such functions V

satisfying condition (H3). There are plenty of them and let us briefly describe a whole class of such functions.

Let $\varphi : H \to [0,\infty]$ be convex, lower semicontinuous, $\varphi \in L^{2+\delta}(H,\gamma)$ for some $\delta > 0$, and Gâteaux differentiable, γ-a.e., i.e. $\gamma(D_\varphi) = 1$. Define

$$V(x) := R(\lambda, \mathcal{L}^{OU})\varphi(x), \quad x \in H, \tag{58}$$

with $R(\lambda, \mathcal{L}^{OU})$ defined as in (26), i.e. it is the resolvent of the Ornstein–Uhlenbeck operator $\mathcal{L}^{OU}$. Then it is elementary to check from the definition, that $V : H \to [0,\infty]$ is also convex and lower semicontinuous.

Furthermore, V is in the $L^2(H,\gamma)$-domain of $\mathcal{L}^{OU}$. Hence by the maximal regularity result of [DL14] (already recalled in Section 3.3 above) applied to the case when $U \equiv 0$, we conclude that $V \in W^{2,2}(H,\gamma)$, in particular we have

$$\int_H \|D^2V\|_{HS}^2 \, d\gamma < \infty,$$

which is stronger than the second part of condition (2) in (H3).

Of course, one needs additional, but obviously quite mild bounds on $\nabla\varphi$, to ensure that $\gamma(D_V) = 1$ and $\nabla V \in L^2(H,\gamma)$. But then the class of V defined in (58) satisfy (H3). To be concrete in choosing φ above, consider the situation of Section 7.1. Then if we take $\varphi := V$ as defined in (53), the new V given by (58) satisfy (H3).

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Giuseppe Da Prato, Scuola Normale Superiore, Pisa, Italy, e-mail: daprato@sns.it

Franco Flandoli, Università di Pisa, e-mail: flandoli@dma.unipi.pi

Michael Röckner, University of Bielefeld, e-mail: roeckner@math.uni-bielefeld.de

Alexander Veretennikov, University of Leeds, LS2 9JT, Leeds, UK, & Institute for Information Transmission Problems, Moscow, Russia, & National Research University Higher School of Economics, Moscow, Russia, email: a.veretennikov @ leeds.ac.uk