STRONGLY BOUNDED GROUPS AND INFINITE POWERS OF FINITE GROUPS

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Abstract. We define a group as strongly bounded if every isometric action on a metric space has bounded orbits. This latter property is equivalent to the so-called uncountable strong cofinality, recently initiated by Bergman.

Our main result is that G^I is strongly bounded when G is a finite, perfect group and I is any set. This strengthens a result of Koppelberg and Tits. We also prove that ω_1-existentially closed groups are strongly bounded.

1. Introduction

Let us say that a group is strongly bounded if every isometric action on a metric space has bounded orbits.

We observe that the class of discrete, strongly bounded groups coincides with a class of groups which has recently emerged since a preprint of Bergman [Ber04], sometimes referred as groups with "uncountable strong cofinality", or "groups with uncountable cofinality and Bergman's Property". This class contains no countably infinite group, but contains symmetric groups over infinite sets [Ber04], and various automorphism groups of infinite structures such as 2-transitive chains [DG05]; see [Ber04] for more references.

In Section 3, we prove that ω_1-existentially closed groups are strongly bounded. This strengthens a result of Sabbagh [Sab75], who proved that they have cofinality $\neq \omega$.

In Section 4, we prove that if G is any finite perfect group, and I is any set, then G^I, endowed with the discrete topology, is strongly bounded. This strengthens a result of Koppelberg and Tits [KT74], who proved that this group has Serre's Property (FA). This group has finite exponent and is locally finite, hence amenable. In contrast, all previously known infinite strongly bounded groups contain a non-abelian free group.

2. Strongly bounded groups

Definition 2.1. We say that a group G is *strongly bounded* if every isometric action of G on a metric space has bounded orbits.

Remark 2.2. Let G be a strongly bounded group. Then every isometric action of G on a nonempty complete CAT(0) space has a fixed point; in particular, G has Property (FH) and Property (FA), which mean, respectively, that every isometric action of G on a Hilbert space (resp. simplicial tree) has a fixed point. This follows from the Bruhat-Tits fixed point lemma, which states that every action of a group on a complete CAT(0) space which has a bounded orbit has a fixed point (see [BH]).

It was asked in [W01] whether the equivalence between Kazhdan's Property (T) and Property (FH), due to Delorme and Guichardet (see [BHV]) holds for more general classes of groups than locally compact σ-compact groups; in particular, whether it holds for general locally compact groups.

The answer is negative, even if we restrict to discrete groups: this follows from the existence of uncountable strongly bounded groups, combined with the fact that Kazhdan's Property (T) implies finite generation [BHV].

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Definition 2.3. We say that a group G is *Cayley bounded*[1] if, for every generating subset $U \subset G$, there exists some n (depending on U) such that every element of G is a product of n elements of $U \cup U^{-1} \cup \{1\}$. This means every Cayley graph of G is bounded.

A group G is said to have cofinality ω if it can be expressed as the union of an increasing sequence of proper subgroups; otherwise it is said to have cofinality $\neq \omega$.

The combination of these two properties, sometimes referred as "uncountable strong cofinality", has been introduced and is extensively studied in Bergman's preprint [Ber04]; see also [DG05]. Note that an uncountable group with cofinality $\neq \omega$ is not necessarily Cayley bounded: the free product of two uncountable groups of cofinality $\neq \omega$, or the direct product of an uncountable group of cofinality $\neq \omega$ with $\mathbf{Z}$, are obvious counterexamples.

The following result can be compared to Lemma 10 in [Ber04]:

Proposition 2.4. *A group G is strongly bounded if and only if it is Cayley bounded and has cofinality $\neq \omega$.*

Proof: Suppose that G is not Cayley bounded. Let U be a generating subset such that G the corresponding Cayley graph is not bounded. Since G acts transitively on it, it has an unbounded orbit.

Suppose that G has cofinality ω. Then G acts on a tree with unbounded orbits [Ser, Chap I, §6.1].

Conversely, suppose that G has has cofinality $\neq \omega$ and is Cayley bounded. Let G act isometrically on a metric space. Let $x \in X$, let $K_n = \{g \in G,\ d(x, gx) < n\}$, and let H_n be the subgroup generated by K_n. Then $G = \bigcup K_n = \bigcup H_n$. Since G has cofinality $\neq \omega$, $H_n = G$ for some n, so that K_n generates G. Since G is Cayley bounded, and since K_n is symmetric, $G \subset (K_n)^m$ for some m. This easily implies that $G \subset K_{nm}$, so that the orbit of x is bounded. ■

Remark 2.5. It follows that a countably infinite group Γ is not strongly bounded: indeed, either Γ has a finite generating subset, so that the corresponding Cayley graph is unbounded, or else Γ is not finitely generated, so is an increasing union of a sequence of finitely generated subgroups, so has cofinality ω.

Definition 2.6. If G is a group, and $X \subset G$, define

$$\mathcal{G}(X) = X \cup \{1\} \cup \{x^{-1},\ x \in X\} \cup \{xy,\ x, y \in X\}.$$

The following proposition is immediate and is essentially contained in Lemma 10 of [Ber04].

Proposition 2.7. *The group G is strongly bounded if and only if, for every increasing sequence (X_n) of subsets such that $\bigcup_n X_n = G$ and $\mathcal{G}(X_n) \subset X_{n+1}$ for all n, one has $X_n = G$ for some n.* ■

Remark 2.8. The first Cayley bounded groups with uncountable cofinality have been constructed by Shelah [She80, Theorem 2.1]. They seem to be the only known to have a uniform bound on the diameter of Cayley graphs. They are torsion-free. These groups are highly non-explicit and their construction, which involves small cancellation theory, rests on the Axiom of Choice.

The first explicit examples, namely, symmetric groups over infinite sets, are due to Bergman [Ber04]. The first explicit torsion-free examples, namely, automorphism groups of double transitive chains, are due to Droste and Göbel [DG05].

3. ω_1-existentially closed groups

Recall that a group G is ω_1-existentially closed is every countable set of equations and inequations with coefficients in G which has a solution is a group containing G, has a solution in G. Sabbagh [Sab75] proved that every ω_1-existentially closed group has cofinality $\neq \omega$. We give a stronger result:

Theorem 3.1. *Every ω_1-existentially closed group G is strongly bounded.*

[1]In the literature, Cayley bounded is sometimes referred as "Bergman's Property".

Proof: Let G act isometrically on a nonempty metric space X. Fix $x \in X$, and define $l(g) = d(gx, x)$ for all $x \in X$. Then l is a length function, i.e. satisfies $l(1) = 0$ and $l(gh) \leq l(g) + l(h)$ for all $g, h \in G$. Suppose by contradiction that l is not bounded. For every n, fix $c_n \in G$ such that $l(c_n) \geq n^2$. Let C be the group generated by all c_n. By the proof of the HNN embedding Theorem [LS, Theorem 3.1], C embeds naturally in the group

$$\Gamma = \langle C, a, b, t\ ;\ c_n = t^{-1}b^{-n}ab^n ta^{-n}b^{-1}a^n\ (n \in \mathbf{N})\rangle,$$

which is generated by a, b, t. Since G is ω_1-existentially closed, there exist $\bar{a}, \bar{b}, \bar{t}$ in G such that the group generated by C, $\bar{a}$, $\bar{b}$, and $\bar{t}$ is naturally isomorphic to Γ. Set $M = \max(l(\bar{a}), l(\bar{b}), l(\bar{t}))$. Then, since l is a length function and c_n can be expressed by a word of length $4n+4$ in a, b, c, we get $l(c_n) \leq M(4n+4)$ for all n, contradicting $l(c_n) \geq n^2$. ■

It is known [Sco51] that every group embeds in a ω_1-existentially closed group. Thus, we obtain:

Corollary 3.2. *Every group embeds in a strongly bounded group.* ■

Note that this was already a consequence of the strong boundedness of symmetric groups [Ber04], but provides a better cardinality: if $|G| = \kappa$, we obtain a group of cardinality $\kappa^{\aleph_0}$ rather that 2^κ.

4. Powers of finite groups

Theorem 4.1. *Let G be a finite perfect group, and I a set. Then the (unrestricted) product G^I is strongly bounded.*

Remark 4.2. Conversely, if I is infinite and G is not perfect, then G^I maps onto the direct sum $\mathbf{Z}/p\mathbf{Z}^{(\mathbf{N})}$ for some prime p, so has cofinality ω and is not Cayley bounded, as we see by taking as generating subset the canonical basis of $\mathbf{Z}/p\mathbf{Z}^{(\mathbf{N})}$.

Remark 4.3. By Theorem 4.1, every Cayley graph of G^I is bounded. If I is infinite and $G \neq 1$, one can ask whether we can choose a bound which does not depend on the choice of the Cayley graph. The answer is negative: indeed, for all $n \in \mathbf{N}$, observe that the Cayley graph of G^n has diameter exactly n if we choose the union of all factors as generating set. By taking a morphism of G^I onto G^n and taking the preimage of this generating set, we obtain a Cayley graph for G^I whose diameter is exactly n.

Our remaining task is to prove Theorem 4.1. The proof is a adequate modification of the original proof of the (weaker) result of Koppelberg and Tits [KT74], which states that G^I has cofinality $\neq \omega$.

If A is a ring with unity, and $X \subset A$, define

$$\mathcal{R}(X) = X \cup \{-1, 0, 1\} \cup \{x + y,\ x, y \in X\} \cup \{xy,\ x, y \in X\}.$$

It is clear that $\bigcup_{n \in \mathbf{N}} \mathcal{R}^n(X)$ is the subring generated by X.

Recall that a Boolean algebra is an associative ring with unity which satisfies $x^2 = x$ for all x. Such a ring has characteristic 2 (since $2 = 2^2 - 2$) and is commutative (since $xy - yx = (x+y)^2 - (x+y)$). The ring $\mathbf{Z}/2\mathbf{Z}$ is a Boolean algebra, and so are all its powers $\mathbf{Z}/2\mathbf{Z}^E = \mathscr{P}(E)$, for any set E.

Proposition 4.4. *Let E be a set, and $(\mathscr{X}_i)_{i \in \mathbf{N}}$ an increasing sequence of subsets of $\mathscr{P}(E)$. Suppose that $\mathcal{R}(\mathscr{X}_i) \subset \mathscr{X}_{i+1}$ for all i. Suppose that $\mathscr{P}(E) = \bigcup_{i \in \mathbf{N}} \mathscr{X}_i$. Then $\mathscr{P}(E) = \mathscr{X}_i$ for some i.*

Remark 4.5. 1) We could have defined, in analogy of Definition 2.3, the notion of strongly bounded ring (although the terminology “uncountable strong cofinality” seems more appropriate in this context). Then Proposition 4.4 can be stated as: if E is infinite, the ring $\mathscr{P}(E) = \mathbf{Z}/2\mathbf{Z}^E$ is strongly bounded. If E is infinite, note that, as a *additive group*, it maps onto $\mathbf{Z}/2\mathbf{Z}^{(\mathbf{N})}$, so has cofinality ω and is not Cayley bounded.

Proof of Proposition 4.4. Suppose the contrary. If $X \subseteq E$, denote by $\mathscr{P}(X)$ the power set of X, and view it as a subset of $\mathscr{P}(E)$. Define $\mathscr{L} = \{X \in \mathscr{P}(E),\ \forall i,\ \mathscr{P}(X) \nsubseteq \mathscr{X}_i\}$. The assumption is then: $E \in \mathscr{L}$.

Observation: if $X \in \mathscr{L}$ and $X' \subset X$, then either X' or $X - X'$ belongs to $\mathscr{L}$. Indeed, otherwise, some $\mathscr{X}_i$ would contain $\mathscr{P}(X')$ and $\mathscr{P}(X - X')$, and then $\mathscr{X}_{i+1}$ would contain $\mathscr{P}(X)$.

We define inductively a decreasing sequence of subsets $B_i \in \mathscr{L}$, and a non-decreasing sequence of integers (n_i) by:

$$B_0 = E;$$
$$n_i = \inf\{t,\ B_i \in \mathscr{X}_t\};$$
$$B'_{i+1} \subset B_i \quad \text{and} \quad B'_{i+1} \notin \mathscr{X}_{n_i+1};$$
$$B_{i+1} = \begin{cases} B'_{i+1}, & \text{if } B'_{i+1} \in \mathscr{L}, \\ B_i - B'_{i+1}, & \text{otherwise.} \end{cases}$$

Define also $C_i = B_i - B_{i+1}$. The sets C_i are pairwise disjoint.

Fact 4.6. For all i, $B_{i+1} \notin \mathscr{X}_{n_i}$ and $C_i \notin \mathscr{X}_{n_i}$.

Proof: Observe that $\{B_{i+1}, C_i\} = \{B'_{i+1}, B_i - B'_{i+1}\}$. We already know $B'_{i+1} \notin \mathscr{X}_{n_i+1}$, so it suffices to check $B_i - B'_{i+1} \notin \mathscr{X}_{n_i}$. Otherwise, $B'_{i+1} = B_i - (B_i - B'_{i+1}) \in \mathcal{R}(\{B_i, B_i - B'_{i+1}\}) \subset \mathcal{R}(\mathscr{X}_{n_i}) \subset \mathscr{X}_{n_i+1}$; this is a contradiction. □

This fact implies that the sequence (n_i) is strictly increasing. We now use a diagonal argument. Let $(N_j)_{j\in\mathbf{N}}$ be a partition of $\mathbf{N}$ into infinite subsets. Set $D_j = \bigsqcup_{i\in N_j} C_i$ and $m_j = \inf\{t,\ D_j \in \mathscr{X}_t\}$, and let l_j be an element of N_j such that $l_j > \max(m_j, j)$.

Set $X = \bigsqcup_j C_{l_j}$. For all j, $D_j \cap X = C_{l_j} \notin \mathscr{X}_{l_j}$. On the other hand, $D_j \in \mathscr{X}_{m_j} \subset \mathscr{X}_{l_j-1}$ since $l_j \geq m_j + 1$. This implies $X \notin \mathscr{X}_{l_j-1} \supset \mathscr{X}_j$ for all j, contradicting $\mathscr{P}(E) = \bigcup_{i\in\mathbf{N}} \mathscr{X}_i$. ■

The following corollary, of independent interest, was suggested to me by Romain Tessera.

Corollary 4.7. *Let A be a finite ring with unity (but not necessarily associative or commutative). Let E be a set, and $(\mathscr{X}_i)_{i\in\mathbf{N}}$ an increasing sequence of subsets of A^E. Suppose that $\mathcal{R}(\mathscr{X}_i) \subset \mathscr{X}_{i+1}$ for all i. Suppose that $A^E = \bigcup_{i\in\mathbf{N}} \mathscr{X}_i$. Then $A^E = \mathscr{X}_i$ for some i.*

Proof: Upon extracting, we can suppose that $\mathscr{X}_0$ contains the constants. Write $\mathscr{Y}_i = \{J \subset E,\ 1_J \in \mathscr{X}_{3i}\}$. If $J, K \in \mathscr{Y}_i$, $1_{J\cap K} = 1_J 1_K \in \mathscr{X}_{3i+1} \subset \mathscr{X}_{3i+3}$, so that $J \cap K \in \mathscr{Y}_{i+1}$, and $1_{J\vartriangle K} = 1_J + 1_K - 2.1_J 1_K \in \mathscr{X}_{3i+3}$, so that $J \vartriangle K \in \mathscr{Y}_{i+1}$. By Proposition 4.4, $\mathscr{Y}_m = \mathscr{P}(E)$ for some m. It is then clear that $A^E = \mathscr{X}_n$ for some n (say, $n = 3m + 1 + \lceil \log_2 |A| \rceil$). ■

If A is a Boolean algebra, and $X \subset A$, we define

$$\mathcal{D}(X) = X \cup \{0,1\} \cup \{x+y,\ x,y \in X \text{ such that } xy = 0\} \cup \{xy,\ x,y \in X\}.$$
$$\mathcal{I}_k(X) = \{x_1 x_2 \ldots x_k,\ x_1, \ldots, x_k \in X\}.$$
$$\mathcal{V}_k(X) = \{x_1 + x_2 + \ldots x_k,\ x_1, \ldots, x_k \in X \text{ such that } x_i x_j = 0\ \ \forall i \neq j\}.$$

The following lemma contains some immediate facts which will be useful in the proof of the main result.

Lemma 4.8. *Let A be a Boolean algebra, and $X \subset A$ a symmetric subset (i.e. closed under $x \mapsto 1 - x$) such that $0 \in X$. Then, for all $n \geq 0$,*
1) $\mathcal{R}^n(X) \subset \mathcal{D}^{2n}(X)$, and
2) $\mathcal{D}^n(X) \subset \mathcal{V}_{2^{2^n}}(\mathcal{I}_{2^n}(X))$.

Proof: 1) It suffices to prove $\mathcal{R}(X) \subset \mathcal{D}^2(X)$. Then the statement of the lemma follows by induction. Let $u \in \mathcal{R}(X)$. If $u \notin \mathcal{D}(X)$, then $u = x + y$ for some $x, y \in X$. Then $u = (1-x)y + (1-y)x \in \mathcal{D}^2(X)$.

2) Is an immediate induction. ■

Definition 4.9 ([KT74])**.** Take $n \in \mathbf{N}$, and let G be a group. Consider the set of functions $G^n \to G$; this is a group for pointwise multiplication. The elements $m(g_1, \ldots, g_n)$ in the subgroup generated by the constants and the canonical projections are called *monomials*. Such a monomial is *homogeneous* if $m(g_1, \ldots, g_n) = 1$ as soon as at least one g_i is equal to 1.

Lemma 4.10 ([KT74])**.** *Let G be a finite group which is not nilpotent. Then there exist $a \in G$, $b \in G - \{1\}$, and a homogeneous monomial $f : G^2 \to G$, such that $f(a,b) = b$.*

The proof can be found in [KT74], but, for the convenience of the reader, we have included the proof from [KT74] in the (provisional) Appendix below.

Remark 4.11. If G is a group, and $f(x_1,\dots,x_n)$ is a homogeneous monomial with $n \geq 2$, then $m(g_1,\dots,g_n)=1$ as soon as at least one g_i is central: indeed, we can then write, for all $x_1,\dots,x_n$ with x_i central, $m(x_1,\dots,x_i,\dots,x_n)=m'(x_1,\dots,\widehat{x_i},\dots,x_n)x_i^k$. By homogeneity in x_i, $m'(x_1,\dots,\widehat{x_i},\dots,x_n)=1$, and we conclude by homogeneity in x_j for any $j\neq i$.

Accordingly, if (C_α) denotes the (transfinite) ascending central series of G, an immediate induction on α shows that if $f(a,b)=b$ for some homogeneous monomial f, $a\in G$ and $b\in C_\alpha$, then $b=1$. In particular, if G is nilpotent (or even residually nilpotent), then the conclusion of Lemma 4.10 is always false.

Lemma 4.12. *Let G be a finite group, I a set, and $H=G^I$. Suppose that $f(a,b)=b$ for some $a,b\in G$, and some homogeneous monomial f, and let N be the normal subgroup of G generated by b. Let (X_m) be an increasing sequence of subsets of H such that $\mathcal{G}(X_m)\subset X_{m+1}$ (see Definition 2.6), and $\bigcup X_m=H$. Then $N^I\subset X_m$ for m big enough.*

Proof: Suppose the contrary. If $x\in G$ and $J\subset I$, denote by x_J the element of G^I defined by $x_J(i)=x$ if $i\in J$ and $x_J(i)=1$ if $i\notin J$.

Denote by $\bar f=f^I$ the corresponding homogeneous monomial: $H^2\to H$. Upon extracting, we can suppose that all c_I, $c\in G$, are contained in X_0. In particular, the "constants" which appear in $\bar f$ are all contained in X_0.

Hence we have, for all m, $\bar f(X_m,X_m)\subset X_{m+d}$, where d depends only on the length of f. For $J,K\subset I$, we have the following relations:

$$a_I.a_J^{-1}=a_{I-J}, \tag{4.1}$$

$$\bar f(a_J,b_K)=b_{J\cap K}, \tag{4.2}$$

$$\bar f(a_J,b_I)=b_J, \tag{4.3}$$

$$\text{If } J\cap K=\emptyset,\quad b_J.b_K=b_{J\sqcup K}. \tag{4.4}$$

For all m, write $\mathscr{W}_m=\{J\in\mathscr{P}(I),\ a_J\in X_m\}$, and let $\mathscr{A}_m$ be the Boolean algebra generated by $\mathscr{W}_m$. Then $\bigcup_m\mathscr{A}_m=\mathscr{P}(I)$. By Proposition 4.4, there exists some M such that $\mathscr{A}_M=\mathscr{P}(I)$. Set $\mathscr{X}_n=\mathcal{R}^n(\mathscr{W}_M)$. Then, since $\mathscr{A}_M=\mathscr{P}(I)$, $\bigcup_n\mathscr{X}_n=\mathscr{P}(I)$. Again by Proposition 4.4, there exists some N such that $\mathscr{X}_N=\mathscr{P}(I)$. So, by 1) of Lemma 4.8, we get

$$\mathcal{D}^{2N}(\mathscr{W}_M)=\mathscr{P}(I). \tag{4.5}$$

Define, for all m, $\mathscr{Y}_m=\{J\in\mathscr{P}(I),\ b_J\in X_m\}$. Then from (4.3) we get: $\mathscr{W}_m\subset\mathscr{Y}_{m+d}$; from (4.2) we get: if $J\in\mathscr{W}_m$ and $K\in\mathscr{Y}_m$, then $J\cap K\in\mathscr{Y}_{m+d}$; and from (4.4) we get: if $J,K\in\mathscr{Y}_{m+1}$ and $J\cap K=\emptyset$, then $J\sqcup K\in\mathscr{Y}_{m+1}$.

By induction, we deduce $\mathcal{I}_k(\mathscr{W}_m)\subset\mathscr{Y}_{m+kd}$ for all k, and $\mathcal{V}_k(\mathscr{Y}_m)\subset\mathscr{Y}_{m+k}$ for all k. Composing, we obtain $\mathcal{V}_k(\mathcal{I}_l(\mathscr{W}_m))\subset\mathcal{V}_k(\mathscr{Y}_{m+ld})\subset\mathscr{Y}_{m+ld+k}$. By 2) of Lemma 4.8, we get $\mathcal{D}^n(\mathscr{W}_m)\subset\mathscr{Y}_{m+2^nd+2^{2^n}}$. Hence, using (4.5), we obtain $\mathscr{P}(I)=\mathscr{Y}_D$, where $D=M+4^Nd+2^{4^N}$.

Let B denote the subgroup generated by b, so that N is the normal subgroup generated by B. Let r be the order of b. Then B^I is contained in X_{D+r}. Moreover, there exists R such that every element of N is the product of R conjugates of elements of B. Then, using that $c_I\in X_0$ for all $c\in G$, N^I is contained in X_{D+r+3R}. ■

Theorem 4.13. *Let G be a finite group, and let N the last term of its descending central series (so that $[G,N]=N$). Let I be any set, and set $H=G^I$. Let (X_m) be an increasing sequence of subsets of H such that $\mathcal{G}(X_m)\subset X_{m+1}$ and $\bigcup X_m=H$. Then $N^I\subset X_m$ for m big enough.*

Proof: Let G be a counterexample with $|G|$ minimal. Let W be a normal subgroup of G such that W^I is contained in X_m for large m, and which is maximal for this property. Since G is a counterexample, $N\not\subseteq W$. Hence G/W is not nilpotent, and is another counterexample, so that, by minimality, $W=\{1\}$. Since G is not nilpotent, there exists, by Lemma 4.10, $a\in G$, $b\in G-\{1\}$, and a homogeneous monomial $f:G^2\to G$, such that $f(a,b)=b$. So, if M is the normal subgroup generated by b, M^I is contained, by Lemma 4.12, in X_i for large i. This contradicts the maximality of W $(=\{1\})$. ■

In view of Proposition 2.7, Theorem 4.1 immediately follows from Theorem 4.13.

Question 4.14. Let G be a finite group, and N a subgroup of G which satisfies the conclusion of Theorem 4.13 (I being infinite). Is it true that, conversely, N must be contained in the last term of the descending central series of G? We conjecture that the answer is positive, but the only thing we know is that N must be contained in the derived subgroup of G.

Remark 4.15. We could have defined relative definitions: if G is a group and N a normal subgroup, we say that (G,N) has cofinality $\neq \omega$ if for every increasing sequence of subgroups (H_n) such that $\bigcup H_n = G$, then $H_n \supset N$ for large n. We say that (G,N) is Cayley bounded if N is bounded in every Cayley graph of G. We say that the pair (G,N) is strongly bounded if, for every isometric action of G on any metric space, then N has bounded orbits. We can show, as in the non-relative case (Proposition 2.4), that (G,N) is strongly bounded if and only if (G,N) has cofinality $\neq \omega$ and is Cayley bounded. The proof uses the fact that if (G,N) has cofinality ω, then there exists an action of G on a tree such that N has no fixed point; the construction is the same as in the non-relative case ([Ser], p. 82): if $\mathcal{T}$ is the tree associated to the family (H_n), then N has a fixed point on $\mathcal{T}$ if and only if N is contained in a conjugate of H_n for large n (this is why we suppose N normal).

Theorem 4.13 has a consequence which is stronger than Theorem 4.1: if G is a finite group and N is the last term of its descending central series, and if I is any set, then the pair (G^I, N^I) is strongly bounded. In particular, it has relative Property (FH): for every isometric action of G^I on a affine Hilbert space, N^I has a fixed point. This shows that a solvable group can have have an infinite subgroup with relative Property (FH). We do not know if this can happen in a nilpotent group (see also Question 4.14). Note that an infinite solvable group cannot have Property (FH): indeed, it has a finite index subgroup with infinite abelianization, and Property (FH) is inherited by finite index subgroups ([BHV], Section 2.6).

Question 4.16. Does there exist a strongly bounded group with cardinality $\aleph_1$?

It seems likely that a variation of the argument in [She80] might provide examples.

Appendix A. Proof of Lemma 4.10

This Appendix is added for the convenience of the reader. It should be dropped in case of publication.

Lemma A.1 ([KT74]). *Let G be a group, $g \in G$, and g' an element of the subgroup generated by the conjugates of g. Then there exists a homogeneous monomial $f : G \to G$ such that $f(g) = g'$.*

Proof: Write $g' = \prod c_i g^{\alpha_i} c_i^{-1}$. Then $x \mapsto \prod c_i x^{\alpha_i} c_i^{-1}$ is a homogeneous monomial and $f(g) = g'$. ■

The following assertion is considered as "clear" in [KT74]. I did not see a more straightforward argument than the following proof, which uses Lemma A.3.

Lemma A.2. *Let G be a finite group. Suppose that G is not nilpotent. Then there exists $a \in G$ such that the normal subgroup of G generated by a is not nilpotent.*

Proof: Let $\mathcal{C}$ be the class of finite groups which do not contain such an element. The class $\mathcal{C}$ is clearly closed under taking subgroups, quotients, and contains nilpotent groups. It clearly contains no non-abelian simple group; hence every G in $\mathcal{C}$ is solvable. Let G be a minimal non-nilpotent group in $\mathcal{C}$. By Lemma A.3, G is the normal closure of a single element and cannot be in $\mathcal{C}$, contradiction. ■

Lemma A.3. *Let G be a finite solvable group. Suppose that G is not nilpotent, but that G is minimal for this property, namely: every proper subgroup or quotient of G is nilpotent. Then $G \simeq C_p \ltimes V$, where V is some vector space of dimension at least two over $\mathbf{F}_q$, q is prime, and the cyclic group C_p acts irreducibly and non-trivially on V. In particular, G is the normal closure of a single element.*

Proof: Since G is solvable, it has a proper normal subgroup H of prime index p, which must be nilpotent, so has nontrivial centre. Let n be the exponent of $Z(H)$ and write $n = qr$, with r prime. Let V be a minimal nonzero G-invariant subspace of the $\mathbf{F}_q$-vector space $rZ(H)$. By minimality, G/V is also nilpotent, so that the action of G on H/V is nilpotent. This implies that the action of G on the $\mathbf{F}_q$-vector space V is not nilpotent; by minimality, it is irreducible. Let $g \in G$ be the lift of a generator of G/H. The cyclic subgroup generated by g cannot intersect V: otherwise, it would provide an invariant vector for the action of G on V. So, by minimality, G is the semidirect product of $\langle g \rangle$ by V. Since the subgroup generated by g^p is normal, and even central (noting that $g^p \in H$), it is trivial, by minimality. Finally, note that the normal subgroup of G generated by g is all of G. ■

Proof of Lemma 4.10. Let G be a finite group which is not nilpotent. We must show that there exist $a \in G$, $b \in G - \{1\}$, and a homogeneous monomial $f : G^2 \to G$, such that $f(a, b) = b$.

Take a as in Lemma A.2, and A the normal subgroup generated by a. Let A_1 be the upper term of the ascending central series of A. We define inductively the sequences $(a_i)_{i\in\mathbf{N}}$ and $(b_i)_{i\in\mathbf{N}}$ such that

$$b_i \in A - A_1, \qquad a_i \in A \qquad \text{and} \qquad b_{i+1} = [a_i, b_i] \in A - A_1.$$

Since G is finite, there exist integers m, m' such that $m < m'$ and $b_m = b_{m'}$. Set $b = b_m$, and for all i, choose, using Lemma A.1, a homogeneous monomial f_i such that $f_i(a) = a_i$. Then the monomial

$$f : \quad (x, y) \mapsto [f_{m'-1}(x), [f_{m'-2}(x), \dots, [f_m(x), y], \dots]]$$

satisfies $f(a, b) = b$. ■

Appendix B. Groups with cardinality $\aleph_1$ and Property (FH)

This appendix should be dropped in case of publication.

Proposition B.1. *Let G be a countable group. Then G embeds in a group of cardinality $\aleph_1$ with Property (FH).*

The proof rests on two ingredients.

Theorem B.2 (Delzant)**.** *If G is any countable group, then G can be embedded in a group with Property (T).*

Sketch of proof: this is a corollary of the following result, independently proved by Delzant[2] and Olshanskii[3]: if H is any non-elementary word hyperbolic group, then H is SQ-universal, that is, every countable group embeds in a quotient of H. Thus, the result follows from the stability of Property (T) by quotients, and the existence of non-elementary word hyperbolic groups with Property (T); for instance, uniform lattices in $\mathrm{Sp}(n, 1)$, $n \geq 2$ (see [HV]). ■

Let $\mathcal{C}$ be any class of metric spaces, let G be a group. Say that G has Property (F$\mathcal{C}$) if for every isometric action of G on a space $X \in \mathcal{C}$, all orbits are bounded. For instance, if $\mathcal{C}$ is the class of all Hilbert spaces, then we get Property (FH).

Proposition B.3. *Let G be a group in which every countable subset is contained in a subgroup with Property (F$\mathcal{C}$). Then G has Property (F$\mathcal{C}$).*

Proof: Let us take an affine isometric action of G on a metric space X in $\mathcal{C}$, and let us show that it has bounded orbits. Otherwise, there exists $x \in X$, and a sequence (g_n) in G such that $d(g_n x, x) \to \infty$. Let H be a subgroup of G with Property (F$\mathcal{C}$) containing all g_n. Since Hx is not bounded, we have a contradiction. ■

Proof of Proposition B.1. We make a standard transfinite induction on ω_1 (as in [Sab75]), using Theorem B.2. For every countable group Γ, choose a proper embedding of Γ into a group $F(\Gamma)$ with Property (T) (necessarily finitely generated). Fix $G_0 = G$, $G_{\alpha+1} = F(G_\alpha)$ for every $\alpha < \omega_1$, and $G_\lambda = \varinjlim_{\beta<\lambda} G_\lambda$ for every limit ordinal $\lambda \leq \omega_1$. It follows from Proposition B.3 that G_{ω_1} has Property (FH). Since all embeddings $G_\alpha \to G_{\alpha+1}$ are proper, G_{ω_1} is not countable, hence has cardinality $\aleph_1$. ■

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