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STRONGLY CLEAN MATRIX RINGS OVER NONCOMMUTATIVE LOCAL RINGS

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Abstract. An element of a ring R with identity is called strongly clean if it is the sum of an idempotent and a unit that commute, and R is called strongly clean if every element of R is strongly clean. Let R be a noncommutative local ring, a criterion in terms of solvability of a simple quadratic equation in R is obtained for $M_2(R)$ to be strongly clean.

1. Introduction

Let R be an associative ring with identity. R is called clean if every element of R is the sum of an idempotent and a unit, and this notion was introduced by Nicholson in [11] as a sufficient condition for a ring to be have the exchange property. Semiperfect ring and unit-regular rings are examples of clean ring as shown by Camillo and Yu [3], and Camillo and Khurana [4].

Nicholson [12] also defined the notion of strong cleanness. An element of a ring R is strongly clean if it is the sum of an idempotent and a unit that commute. A ring R is strongly clean if every element of R is strongly clean. Local rings are obviously strongly clean rings. It is a result of Burgess and Menal [2] that every strongly π-regular ring is strongly clean, where a ring R is strongly π-regular if for every $a \in R$, the chain $aR \supseteq a^2R \supseteq a^3R \supseteq \cdots$ terminates (or equivalently, the chain $Ra \supseteq Ra^2 \supseteq Ra^3 \supseteq \cdots$ terminates).

Han and Nicholson [8] showed that for a clean ring R the matrix ring $M_n(R)$ is clean for every $n \geq 1$. However, the analog for strongly clean rings fails to hold, even when R is a commutative local ring. The example is the localization of Z at the prime ideal generated by 2 (see Wang and Chen [13]). More generally, Chen, Yang, and Zhou showed that for each prime $p, M_2(Z(p))$ is not strongly clean, where $Z(p)$ is the localization of Z at the prime ideal generated by p.

This motivates us determine when is $M_n(R)$ strongly clean for a local ring R. Recently, Chen, Yang, and Zhou [5] and Li [10] investigated when a 2×2 matrix ring $M_2(R)$ over a commutative local ring R is strongly clean, and

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obtained some criterions for such a matrix ring to be strongly clean. Li [10] also got several equivalent conditions for an individual 2×2 matrix A over a commutative local ring to be strongly clean. More recently, the authors of [1, 7] have proved that for a commutative local ring R, $M_n(R)$ is strongly clean if and only if every monic polynomial of degree n in $R[x]$ has a so called SRC factorization, hence completely characterized the commutative local ring for which $M_n(R)$ is strongly clean.

In this paper, we will drop the crucial assumption that R is a commutative ring. Our main result is Theorem 2.7, which provided a necessary and sufficient condition for a 2×2 matrix ring over a noncommutative local ring to be strongly clean. Moreover, we give some examples of strongly clean rings.

As usual, we use $U(R)$ and $J(R)$ to denote the group of units and Jacobson radical of R, respectively. $\pi : R \to R/J(R)$ will denote the natural quotient ring homomorphism from R to $R/J(R)$, and we will write $\pi(r) = \overline{r}$ and $\overline{R} = R/J(R)$. The notation R_n always stands for the set $\left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mid x_1, \dots, x_n \in R \right\}$, which is a $(M_n(R), R)$-bimodule, and R^n stands for the set $\{(x_1, \dots, x_n) \mid x_1, \dots, x_n \in R\}$, which is a $(R, M_n(R))$-bimodule. The $n \times n$ upper triangular matrix ring over R is denoted by $T_n(R)$. Recall that two matrices A, B are similar if $A = P^{-1}BP$ for some invertible matrix P.

2. Results

To develop a criteria for 2×2 matrix ring over a noncommutative local ring to be strongly clean, we start with some useful lemmas:

Nicholson [12] provided the following characterization of a strongly clean endomorphism.

Lemma 2.1. *Let M_R be a right module and $f \in \mathrm{End}(M_R)$. The following are equivalent:*

(1) *f is strongly clean.*
(2) *There exist f-invariant submodules M_1, M_2 such that $M = M_1 \oplus M_2$, f is an isomorphism on M_1 and $1 - f$ is an isomorphism on M_2.*

We denote by $M_n(R)$ the ring of all $n \times n$ matrices with entries in R. It is well known that $\mathrm{End}_R(R^n) \simeq M_n(R)$, so we can treat any matrix $A \in M_n(R)$ as an endomorphism of ${}_R(R^n)$.

Lemma 2.2. *If R is a ring, M_R is a finitely generated R-module, and $x_1, x_2, \dots, x_n$ generate M if and only if their images $\overline{x}_1, \overline{x}_2, \dots, \overline{x}_n$ generate $M/MJ(R)$ as an $R/J(R)$-module.*

Proof. The only if statement is trivial. Suppose $\overline{x}_1, \overline{x}_2, \dots, \overline{x}_n$ generate $M/MJ(R)$. Let $N = x_1R + x_2R + \cdots + x_nR \subseteq M$. Then M/N is finitely generated and $(M/N)J(R) = M/N$. By Nakayama's lemma, $M/N = 0$ and so $M = N$. □

A basic result about local rings is the following:

Lemma 2.3. *If R is a local ring, not necessary commutative, then every projective finitely generated R-module is free with a unique rank. In particular, if $R^n = A \oplus B$ (as right modules), then A and B are free right modules, and* $\operatorname{rank} A + \operatorname{rank} B = n$.

Let R be a local ring and $M_2(R)$ be the 2×2 matrix ring R with an identity I_2. For any $A \in M_2(R)$, if A is invertible matrix, then A has a trivial strongly clean expression; if $I_2 - A$ is invertible matrix, then we can write $A = I_2 + (A - I_2)$, which is also a strongly clean expression.

Therefore, to determine A to be strongly clean, it suffices to consider the case when neither A nor $I_2 - A$ is a unit.

It is not difficult to see that if A is similar to B, then A is strongly clean if and only if B is strongly clean.

Lemma 2.4. *Let R be a noncommutative local ring. Assume that $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(R)$ such that neither A nor $I_2 - A$ is a unit. Then A is similar to the matrices $\left(\begin{smallmatrix} a_1 & 1 \\ c_1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & c_1 \\ 1 & a_1 \end{smallmatrix}\right)$ for some $a_1 \in 1 + J(R)$ and $c_1 \in J(R)$.*

To prove the above lemma, we first recall the following well known results:

Proposition 2.5. *Let R be a ring. Then an element $x \in R$ is invertible in R if and only if $\overline{x}$ is invertible in $\overline{R} = R/J(R)$.*

Proposition 2.6. *Let R be a ring and $M_n(R)$ denote the $n \times n$ full matrix of R. Then $J(M_n(R)) = M_n(J(R))$, hence we have the following ring isomorphism*

$$f: \quad M_n(R)/J(M_n(R)) \to M_n(R/J(R)),$$

where $f(\overline{A}) = (\overline{a_{ij}})$ for any $A = (a_{ij}) \in M_n(R)$.

Proof of Lemma 2.4. Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(R)$.

Case I. $b \in U(R)$. Let $P = \left(\begin{smallmatrix} 1 & 0 \\ db^{-1} & b^{-1} \end{smallmatrix}\right)$. Then

$$P^{-1}AP = \begin{pmatrix} 1 & 0 \\ -bdb^{-1} & b \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ db^{-1} & b^{-1} \end{pmatrix} = \begin{pmatrix} a + bdb^{-1} & 1 \\ bc - bdb^{-1}a & 0 \end{pmatrix}.$$

Set $a_1 = a + bdb^{-1}$, $c_1 = bc - bdb^{-1}a$. We get A is similar to $B = \left(\begin{smallmatrix} a_1 & 1 \\ c_1 & 0 \end{smallmatrix}\right)$.

Firstly, we claim that $c_1 \in J(R)$. Assume the contrary. Then $c_1 \in U(R)$ since R is local.

Let $B_1 = \left(\begin{smallmatrix} 0 & c_1^{-1} \\ 1 & -a_1 c^{-1} \end{smallmatrix}\right) \in M_2(R)$. It is easy to check that $BB_1 = B_1 B = I_2$, hence B is invertible. This implies that A is a unit, contradicting the assumption. So $c_1 \in J(R)$.

Secondly, we will show that a_1 has the form $a_1 = 1 + j$ for some $j \in J(R)$. Since $I_2 - A$ is not a unit, $B_2 = P^{-1}(I_2 - A)P = \left(\begin{smallmatrix} 1-a_1 & -1 \\ -c_1 & 1 \end{smallmatrix}\right)$ is not a unit. If $1 - a_1$ is invertible, then $u = 1 - a_1 + c_1$ is a unit since $c_1 \in J(R)$. Let $B_3 = \left(\begin{smallmatrix} u^{-1} & -u^{-1} \\ -c_1 u^{-1} & -1+c_1 u^{-1} \end{smallmatrix}\right) \in M_2(R)$, one can check that $B_2 B_3 = B_3 B_2 = I_2$.

This implies that $I_2 - A$ is a unit since $I_2 - A$ is similar to B_2. So $1 - a_1$ is not a unit, hence $1 - a_1 = j_1$ for some $j_1 \in J(R)$, as desired.

Case II. $c \in U(R)$. It is easy to see that A is similar to $\begin{pmatrix} d & c \\ b & a \end{pmatrix}$. We are back to case I.

Case III. Assume that both b and c are in $J(R)$. We claim that either $a \in U(R)$ and $d \in J(R)$, or $d \in U(R)$ and $a \in J(R)$.

Subcase I. Both a and d are in $U(R)$. Let f be defined as Proposition 2.6. Then $f(\overline{A}) = \begin{pmatrix} \overline{a}_1 & 0 \\ 0 & \overline{d}_1 \end{pmatrix}$ is a unit in $M_n(R/J(R))$, hence $\overline{A}$ is a unit in $M_n(R)/J(M_n(R))$ since f is isomorphic. Therefore, A is invertible by Lemma 2.3, contradicting the assumption.

Subcase II. Both a and d are in $J(R)$.

Let $A_1 = I_2 - A = \begin{pmatrix} 1-a & -b \\ -c & 1-d \end{pmatrix}$. Then $1-a, 1-d \in U(R)$ and $-b, -c \in J(R)$. The same argument as Subcase I shows that A_1 is invertible, a contradiction.

So we conclude that either $a \in U(R)$ and $d \in J(R)$, or $d \in U(R)$ and $a \in J(R)$. Since $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is similar to $\begin{pmatrix} d & c \\ b & a \end{pmatrix}$, without loss of generality we may assume that $a \in U(R)$ and $d \in J(R)$.

Let $P_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then we have $P_1^{-1}AP_1 = \begin{pmatrix} a-c & a+b-c+d \\ c & c+d \end{pmatrix}$. Note that $a \in U(R)$ and $b, c, d \in J(R), a + b - c + d$ is a unit in R. Again we back to Case I.

Therefore, A is similar to the matrix $\begin{pmatrix} a_1 & 1 \\ c_1 & 0 \end{pmatrix}$ for some $a_1 \in 1 + J(R)$ and $c_1 \in J(R)$.

Note that $\begin{pmatrix} a_1 & 1 \\ c_1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & c_1 \\ 1 & a_1 \end{pmatrix}$ are similar, the result follows. □

The main result is the next theorem.

Theorem 2.7. *Let R be a local ring, not necessary commutative. Then the following are equivalent:*

(1) *$M_2(R)$ is strongly clean.*
(2) *For any $\mu \in 1 + J(R)$ and $\omega \in J(R)$, the equation $x^2 - x\mu + \omega = 0$ has two solutions: one in $J(R)$, the other in $1 + J(R)$.*

Proof. (1) $\Rightarrow$ (2) Let $M = \begin{pmatrix} 0 & -\omega \\ 1 & \mu \end{pmatrix}$ with $\omega \in J(R)$ and $\mu \in 1+J(R)$. Then $M \in M_2(R) = \mathrm{End}_R(R^2)$ is strongly clean by assumption. In view of Lemma 2.1, there exist M-invariant submodules ${}_RA$ and ${}_RB$ such that ${}_RR^2 = A \oplus B$ and M is an isomorphism on A and $I_2 - M$ is an isomorphism on B.

It is easy to prove that neither $\overline{M}$ nor $\overline{I_2 - M}$ is a unit in $M_2(\overline{R})$, so neither M nor $I_2 - M$ is invertible in $M_2(R)$ by Propositions 2.5 and 2.6. Therefore, both A and B are nontrivial submodules of ${}_R(R^2)$. Note that R is a local ring. We get that A and B are free R-modules of rank 1 by Lemma 2.2. Let $e_1 = (1\ 0), e_2 = (0\ 1)$.

We also have $\overline{B} = \overline{BM} \subseteq \overline{R}^2\overline{M} = \mathrm{im}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \overline{R}(\overline{e}_1 + \overline{e}_2)$. Similarly, $\overline{A} \subseteq \mathrm{im}(\overline{I_2 - M}) = \mathrm{im}\begin{pmatrix} \overline{1} & 0 \\ -\overline{1} & 0 \end{pmatrix} = \overline{R}\overline{e_1}$.

Since $A \oplus B = R^2$, we have $\overline{A} + \overline{B} = \overline{R}^2$. Hence we conclude that $\overline{A} = \overline{R}\overline{e}_1$ and $\overline{B} = \overline{R}(\overline{e}_2 + \overline{e}_2)$ since $\overline{R}\overline{e}_1 \oplus \overline{R}(\overline{e}_1 + \overline{e}_2) = \overline{R}^2$.

Assume that A is free on $\eta = (\eta_1\ \eta_2)$ and B is free on $\tau = (\tau_1\ \tau_2)$. Then $\overline{r}_1 (\overline{\eta}_1\ \overline{\eta}_2) = (\overline{1}\ 0)$ and $\overline{r}_2 (\overline{\tau}_1\ \overline{\tau}_2) = (\overline{1}\ \overline{1})$ for some $r_1, r_2 \in R$.

Comparing the coordinates, we get that $r_1\eta_1, r_2\tau_1 \in 1 + J(R) \subseteq U(R)$, and $r_1\eta_2 \in J(R)$. Therefore, $\eta_1, \tau_1 \in U(R)$ and $\eta_2 \in J(R)$. Let $\eta' = \eta_1^{-1}\eta, \tau' = \tau_1^{-1}\tau$. Then η', τ' are also the basis of A and B, respectively.

Let $\eta' = (1\ j_1), \tau' = (1\ u)$ for some $j_1 \in J(R)$ and $u \in U(R)$. Then $\eta' M = (1\ j_1)\left(\begin{smallmatrix} 0 & -\omega \\ 1 & \mu \end{smallmatrix}\right) = (j_1\ \ -\omega + j_1\mu)$ and $\tau' M = (1\ u)\left(\begin{smallmatrix} 0 & -\omega \\ 1 & \mu \end{smallmatrix}\right) = (u\ \ -\omega + u\mu)$.

Note that $\overline{\tau'}$ must have the form $(\overline{1}\ \overline{1})$. Hence we get $u \in 1 + J(R)$.

Since $A = R\eta', B = R\tau'$ are M-invariant submodules, we have $(j_1\ \ -\omega + j_1\mu) = r_3 (1\ j_1)$ and $(u\ \ -\omega + u\mu) = r_4 (1\ u)$ for some $r_3, r_4 \in R$.

Comparing all the corresponding coordinates, we find $j_1 = r_3,\ -\omega + j_1\mu = r_3 j_1$ and $u = r_4,\ -\omega + +u\mu = r_4 u$.

Hence $r_3^2 - r_3\mu + \omega = 0$ and $r_4^2 - r_4\mu + \omega = 0$ with $r_3 \in J(R), r_4 \in 1 + J(R)$.

Therefore, the equation $x^2 - \mu x + \omega = 0$ has two solutions: one in $J(R)$, the other in $1 + J(R)$.

(2) $\Rightarrow$ (1) For any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(R)$, it suffices to prove that $\left(\begin{smallmatrix} 0 & -\omega \\ 1 & \mu \end{smallmatrix}\right)$ is strongly clean for any $\mu \in 1 + J(R), \omega \in J(R)$ by Lemma 2.4. Let $M = \left(\begin{smallmatrix} 0 & -\omega \\ 1 & \mu \end{smallmatrix}\right)$ where $\mu \in 1 + J(R)$ and $\omega \in J(R)$. The the equation $x^2 - x\mu + \omega = 0$ has two solutions by assumption. Say $j^2 - j\mu + \omega = 0$ and $u^2 - u\mu + \omega = 0$ with $j \in J(R), u \in 1 + J(R)$. Let $\alpha = (1\ j), \beta = (1\ u)$, and $A = R\alpha, B = R\beta$.

Firstly, we claim that $R^2 = A \oplus B$.

For any $\overline{r} = (r_1\ r_2) \in \overline{R}^2$, $\overline{r} = (r_1 - r_2)(\overline{1}\ 0) + r_2(\overline{1}\ \overline{1})$, so $\overline{\alpha}, \overline{\beta}$ generate $\overline{R}^2$. By Lemma 2.2, α, β generate R^2. If $h \in R\alpha \cap R\beta$, then $h = (h_1\ h_2) = x_1 (1\ j) = x_2 (1\ u)$ for some $x_1, x_2 \in R$. We find that $x_1 = x_2, x_1 j = x_2 u$, hence $x_2(u - j) = 0$. This implies $x_2 = 0$ since $u - j$ is a unit in R, so $R\alpha \cap R\beta = \{0\}$. Therefore, $R^2 = A \oplus B$.

Secondly, we will show that A, B are M-invariant and M is an isomorphism on B and $I_2 - M$ is an isomorphism on A. Let $s_1 = j, s_2 = u$. One can check that $\alpha M = (1\ j)\left(\begin{smallmatrix} 0 & -\omega \\ 1 & \mu \end{smallmatrix}\right) = s_1 (1\ j) \in A$ and $\beta M = (1\ u)\left(\begin{smallmatrix} 0 & -\omega \\ 1 & \mu \end{smallmatrix}\right) = s_2 (1\ u) \in B$.

Hence A, B are M-invariant. It is also straightforward to check that

$$\alpha(I_2 - M) = \begin{pmatrix} 1 & j \end{pmatrix} \begin{pmatrix} 1 & \omega \\ -1 & 1 - \mu \end{pmatrix} = (1 - j) \begin{pmatrix} 1 & j \end{pmatrix}.$$

Noting that $1 - j \in 1 + J(R) \subseteq U(R)$ and $u \in U(R)$, we get that M maps B isomorphically to B; and $I_2 - M$ maps A isomorphically to A.

In view of Lemma 2.1, M is strongly clean, and $M_2(R)$ is a strongly clean ring. □

Remark. From Theorem 2.7, $M_2(R)$ is a strongly clean ring for any division ring R.

For a commutative local ring, we have the following:

Corollary 2.8. *Let R be a commutative local ring. Then the following are equivalent:*

(1) *$M_2(R)$ is strongly clean.*
(2) *The polynomial $f(x) = x^2 - (1+j)x + \omega$ has a root in $J(R)$ for any $j, \omega \in J(R)$.*
(3) *The polynomial $f(x) = x^2 - x + \omega$ has a root in $J(R)$ for any $\omega \in J(R)$.*

Proof. (1) $\Rightarrow$ (2) Note that R is commutative, this implication is a weakening by Theorem 2.7.

(2) $\Rightarrow$ (1) If the polynomial $f(x) = x^2 - (1+j)x + \omega$ has a root $\lambda_1 \in J(R)$, then there is a factorization $f(x) = (x - \lambda_1)(x - \lambda_2)$ for some $\lambda_2 \in J(R)$.

λ_2 is also a root of $f(x)$ and $\lambda_2 \in U(R)$ since $\lambda_1 + \lambda_2 = 1 + j$ and $j \in J(R)$. so $M_2(R)$ is strongly clean by Theorem 2.7.

(2) $\Rightarrow$ (3) Let $j = 0 \in J(R)$.

(3) $\Rightarrow$ (2) Consider the polynomial $g(y) = y^2 - y + \omega/(1+j)^2, g(y) = 0$ has a solution $\lambda_0 \in J(R)$ by assumption. It is easy to check that $r = (1+j)\lambda_0 \in J(R)$ is a root of $f(x) = x^2 - (1+j)x + \omega = 0$. □

3. Some examples

For commutative local rings, Chen, Yang, and Zhou [5] proved that the strongly cleanness of $M_2(R), M_2(R[[x]])$ and $M_2(R(x))/(x)^n$ for any $n \geq 1$ are equivalent. In this section, we will show that this equivalence is also hold for noncommutative local rings without imposing any further assumptions. The following proposition is due to Dorsey [7, Proposition 5.1.5].

Lemma 3.1. *Let R be a local ring. If $M_2(R)$ is strongly clean, then $T_2(R)$ is also strongly clean.*

Theorem 3.2. *Let R be a local ring, not necessary commutative. Then the following are equivalent:*

(1) *$M_2(R)$ is strongly clean.*
(2) *$M_2(R[[x]])$ is strongly clean.*
(3) *$M_2(R(x)/(x)^n)$ is strongly clean for any $n \geq 1$.*

Proof. Note that $M_2(R)$ is a homomorphic image of

$$M_2(R[[x]]) \quad \text{and} \quad M_2(R(x)/(x)^n),$$

so (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) are obvious.

(1) $\Rightarrow$ (2) Let $S = R[[x]]$. Then S is also a local ring with $J(S) = J(R) + Sx$. For any $\omega_1, \omega_2 \in J(S)$, say $\omega_1 = b_0 + b_1 x + b_2 x^2 + \cdots$ and $\omega_2 = c_0 + c_1 x + c_2 x^2 + \cdots$ with $b_0, c_0 \in J(R)$. We will show that there exists $x = a_0 + a_1 x + a_2 x^2 + \cdots \in S$ such that $x^2 - x(1 + \omega_1) + \omega_0 = 0$. This is equivalent to:

$$(*) \qquad a_0^2 - a_0(1 + b_0) + c_0 = 0$$

and

$$a_0a_{k+1} + a_1a_k + \cdots + a_ka_1 + a_{k+1}a_0 - a_{k+1}(1 + b_0) \\ - a_kb_1 - \cdots - a_1b_k - a_0b_{k+1} + c_{k+1} = 0 \quad \text{for } k = 0, 1, 2, \ldots.$$

That is, $a_0^2 - a_0(1 + b_0) + c_0 = 0$ and $a_0a_{k+1} - a_{k+1}(1 + b_0 - a_0) = (b_1a_k + \cdots + b_ka_0) - c_{k+1} - a_1a_k + \cdots + a_ka_1 + a_{k+1}a_0$ for $k = 0, 1, 2, \ldots$. In view of Theorem 2.7, $(*)$ has two solution: one in $J(R)$, the other in $1+J(R)$. Assume that $a_0 \in J(R)$.

Let $r_{k+1} = (b_1a_k + \cdots + b_ka_0) - c_{k+1} - a_1a_k + \cdots + a_ka_1 + a_{k+1}a_0$. By Lemma 3.1, $T_2(R)$ is also a strongly clean rings, hence $\left(\begin{smallmatrix} a_0 & r_{k+1} \\ 0 & 1+b_0-a_0 \end{smallmatrix}\right)$ has a strongly clean expression. Since R is local, the idempotents in $T_2(R)$ are $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & t \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & t \\ 0 & 0 \end{smallmatrix}\right)$ for some $t \in R$.

The strong cleanness of $\left(\begin{smallmatrix} a_0 & r_{k+1} \\ 0 & 1+b_0-a_0 \end{smallmatrix}\right)$ implies that there exists some $a_{k+1} \in R$ such that $\left(\begin{smallmatrix} a_0 & r_{k+1} \\ 0 & 1+b_0-a_0 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 1 & a_{k+1} \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a_0-1 & r_{k+1}-a_{k+1} \\ 0 & 1+b_0-a_0 \end{smallmatrix}\right)$ is a unit in $T_2(R)$ and commutes with $\left(\begin{smallmatrix} 1 & a_{k+1} \\ 0 & 0 \end{smallmatrix}\right)$. One can easily check that $a_0a_{k+1} - a_{k+1}(1+b_0-a_0) = r_{k+1}$. Therefore, we can get all a_k by induction, hence we can find a root $x = a_0 + a_1x + a_2x^2 + \cdots \in J(R)$ for the equation $x^2 - x(1 + \omega_1) + \omega_0 = 0$. Similarly, if we assume that $a_0 \in 1 + J(R)$, we can find the other root in $1 + J(R)$. Thus (2) holds by Theorem 2.7.

(1) $\Rightarrow$ (3) is similar to the proof of [5, Theorem 9], so we omitted it here. □

Question 3.3. Zhou and Chen [6] proved that if either r_0 or $1-r_0$ is a strongly π-regular element of R, then $r = \sum r_ix^i$ is a strongly clean element in $R[[x]]$.

Following Lam [9, Example 1.6], for any ring R, we can define the ring $R((x))$ of Laurent series in one variable x over R to be the set of formal Laurent series $F = \sum_{-\infty}^{+\infty} f_ix^i$, where, among the coefficients $f_i \in R$ with $i < 0$, only finite many can be nonzero. These Laurent series are multiplied formally, with the elements of R commuting with x. Hence we ask:

When is $f = \sum_{-\infty}^{+\infty} f_ix^i \in R((x))$ strongly clean and when is Laurent series ring a strongly clean ring?

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