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Strongly Closed Geodetic Numbers of Graphs

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Abstract

In this paper, we introduce the strongly closed geodetic number of a graph and determine its values for some graphs including graphs that resulted from the join, corona and composition of connected graphs. We also solve a realization problem involving geodetic number, strongly closed geodetic number, order and diameter of a graph.

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1 Introduction

Let G be a connected graph with $V(G)$ and $E(G)$ denoting the sets of all its vertices and edges, respectively. It is customary to denote by $|S|$ the cardinality of S for every $S \subseteq V(G)$, and call, in particular, $|V(G)|$ the **order** of G. For any vertices u and v in G, a u-v **geodesic** is any shortest path joining u and v. The symbol $d_G(u, v)$ denotes the length of any u-v geodesic. The length $diam(G)$ of the longest geodesic in G is called the **diameter** of G. The **closed interval** $I_G[u, v]$ is the union of all vertices in G lying in any u-v geodesic.

The **interior** of the interval $I_G[u, v]$ is the set $I_G(u, v) = I_G[u, v] \setminus \{u, v\}$. For any nonempty $S \subset V(G)$, the set $I_G[S] = \cup\{I_G[u, v] : u, v \in S\}$ is the **closure** of S. The **interior** of S is the set $I_G(S) = \cup\{I_G(u, v) : u, v \in S\}$. If $I_G[S] = V(G)$, S is called a **geodetic cover** of G. And the **geodetic number** $g(G)$ of G is the minimum cardinality among all geodetic covers of G. That is, $g(G) = min\{|S| : S \subset V(G) \text{ and } I_G[S] = V(G)\}$. Any geodetic closure S in G with $|S| = g(G)$ is called **geodetic basis**. A good number of interesting results related to this invariant are given in [3], [5] and [7].

The **2-path closure** $P_2[S]_G$ of $S \subseteq V(G)$ in G is defined by $P_2[S]_G = S \cup \{w \in V(G) : w \in I_G(u, v) \text{ for some } u, v \in S \text{ with } d_G(u, v) = 2\}$. S is called **2-path closure absorbing set** in G if ever $P_2[S]_G = V(G)$. Formulas in [1], [8] and [5] for the geodetic numbers of the join of graphs are given in terms of 2-path closure absorbing sets. A 2-path closure absorbing set S in G is **minimal** if S does not have a proper subset that is itself a 2-path closure absorbing set in G.

Inspired by two classes of graphical games, called Achievement and Avoidance games (see [10]), the concept of closed geodetic cover is introduced in [2]. A geodetic cover S of a connected graph G is a **closed geodetic cover** if it satisfies the property: S can be written in canonical form as $S = \{v_1, v_2, \dots, v_k\}$ such that $v_j \notin I_G[S_{j-1}]$, where $S_{j-1} = \{v_1, v_2, \dots, v_{j-1}\}$, for all j with $2 \leq j \leq k$. In [1], Aniversario et.al. characterized some graphs in terms of their closed geodetic covers. Some meaningful relationships were also established involving the associated invariant and the geodetic number of G.

The present paper considers requiring a geodetic cover S of G so that for each $v \in S$, $v \notin I_G[S \setminus \{v\}]$. Such a geodetic cover exists when $diam(G) = 2$. More interestingly, however, a connected graph G can be constructed so that $diam(G) > 2$ and that G has a geodetic cover satisfying this imposed requirement.

2 Strongly Closed Geodetic Numbers of Graphs

Let G be a connected graph. A geodetic cover S of G is a **strongly closed geodetic cover** of G if for each $v \in S$, $v \notin I_G[S \setminus \{v\}]$. Thus for a geodetic cover S of G, S is a strongly closed if and only if $S \cap I_G(S) = \emptyset$. The **strongly closed geodetic number** of G, denoted by $\hat{g}(G)$, is defined by

$$\hat{g}(G) = \begin{cases} min\{|S| : S \in \Sigma(G)\}, & \Sigma(G) \neq \emptyset, \\ \infty, & \Sigma(G) = \emptyset \end{cases},$$

where $\Sigma(G)$ denotes the collection of all strongly closed geodetic covers of G. If $\hat{g}(G)$ is finite, any strongly closed geodetic cover S of G with $|S| = \hat{g}(G)$ is called **strongy closed geodetic set** in G. Note that, in any case, $g(G) \leq \hat{g}(G)$.

By a **neighborhood** of a vertex v in a connected graph G is meant the set $N(v) = \{u \in V(G) : d_G(u, v) = 1\}$. Any vertex v in G is called an **extreme** vertex if for every pair of distinct vertices u and w in $N(v)$, $d_G(u, w) = 1$. The symbol $Ext(G)$ denotes the set of all extreme vertices in G. Accordingly, $Ext(G) \subseteq S$ for all geodetic covers S of G. If G is an extreme geodesic graph, i.e. $Ext(G)$ is a geodetic basis of G, then $\hat{g}(G) = |Ext(G)|$. More precisely, $\hat{g}(G) = |V(G)|$ if and only if G is complete. Moreover,

$$\hat{g}(G) = k \text{ if and only if } g(G) = k$$

for $k = 2, 3, |V(G)| - 1$. In the case where $k = |V(G)| - 1$, the unique graph G is given by $G = K_1 + \bigcup m_j K_j$, where m_j is the number of copies of K_j and $2 \leq \sum m_j$. For the Petersen graph P (see Figure 2.1), we have $g(P) = \hat{g}(P) = 4$. To see this, note that since the set $\{v_1, v_2, v_3, v_4\}$ is a strongly closed geodetic cover of P, $\hat{g}(P) \leq 4$. On the other hand, since $g(P) = 4$, the above equivalence implies that $\hat{g}(P) \geq 4$. Thus $\hat{g}(P) = 4$. However, for the graph G in Figure 2.1, $g(G) = 4$ but $\hat{g}(G) = 7$. Here we note that the set $\{u_1, u_2, x, y\}$ is the unique geodetic basis of G. The set $\{u_1, u_2, u_3, u_5, u_6, u_7\}$ is a strongly closed geodetic cover of G so that $\hat{g}(G) \leq 7$. On the other hand, since u_1 and u_2 are extreme vertices and $x \in I_G[u_1, u_2]$, x is not a vertex in a strongly closed geodetic cover of G. Moreover, the set $\{u_1, u_2, y, u_j\}$, $j = 3, 4, 5, 6, 7$, is not a geodetic cover, and $y \in I_G[u_i, u_j]$ for $3 \leq i, j \leq 7$. This shows that, in fact, $\{u_1, u_2, u_3, u_5, u_6, u_7\}$ is a unique strongly closed geodetic cover of G. Therefore the equivalence given above does not necessarily hold for $k \geq 4$.

Petersen Graph P G

Figure 2.1: Graphs with geodetic numbers equal to 4

In [2], it is said that a discussion of graphs of small diameter includes most graphs. In what follows, we investigate $\hat{g}(G)$ for graphs G with $diam(G) \leq 2$.

Let G be a connected graph with $diam(G) \leq 2$. If $diam(G) = 1$, then G is complete, and accordingly, $\hat{g}(G) = |V(G)|$. Suppose that $diam(G) = 2$. We consider two cases:

Case 1. Suppose that $Ext(G) = \emptyset$. Then each vertex of G lies in an induced K_2-gluing of (K_3, K_3) (i.e., the graph obtained after deleting an edge from K_4) or in an induced cycle C_n with $n \geq 4$. Choose $v_1, v_2 \in V(G)$ such that

$d_G(v_1, v_2) = 2$, and let $S_1 = \{v_1\}$ and $S_2 = \{v_1, v_2\}$. For $k \geq 3$, choose $v_k \in V(G)$ such that the following hold:

1. $v_k \notin I_G[S_{k-1}]$, where $S_{k-1} = \{v_1, v_2, \ldots, v_{k-1}\}$.
2. If v_k and some $v_i \in S_{k-1}$ belong to the same induced cycle C_n, $n \geq 4$, then v_k and v_i are vertices in a minimal 2-path closure absorbing set in C_n.
3. If v_k and some $v_i \in S_{k-1}$ belong to the same induced K_2-gluing G_o of (K_3, K_3) and $d_G(v_k, v_i) = 2$, then $G_o \cap S_k = \{v_k, v_i\}$.

Since $|V(G)|$ is finite, there exists a smallest positive integer N such that $I_G[S_N] = V(G)$. We claim that S_N is a strongly closed geodetic cover of G. Let x, y, and z be distinct vertices in S_N. There exist positive integers i, j, k with $1 \leq i, j, k \leq N$ such that $x = v_i, y = v_j, z = v_k$. We assume that $i < j < k$. By property 1, $v_k \notin I_G[v_i, v_j]$. By property 2 and 3, $v_i \notin I_G[v_j, v_k]$ and $v_j \notin I_G[v_i, v_k]$. Hence, S_N is a strongly closed geodetic cover of G. Consequently, $\hat{g}(G) \leq N$.

Case 2. Suppose that $Ext(G) \neq \emptyset$, and suppose further that $Ext(G) = \{v_1, v_2, \ldots, v_j\}$. For each $i = 1, 2, \ldots, j$, let $S_i = \{v_1, v_2, \ldots, v_i\}$. Choose $v_{j+1} \in V(G)$ such that the following hold:

1. $v_{j+1} \notin I_G[S_j]$.
2. If $v_{j+1} \in S \subseteq V(G) \backslash I_G[S_j]$ and S is a 2-path closure closure absorbing set in an induced cycle C_n with $n \geq 4$, then S is minimal.
3. If $v_{j+1} \in S \subseteq V(G) \backslash I_G[S_j]$ and S is a 2-path closure closure absorbing set in an induced K_2-gluing of (K_3, K_3), then S is minimal.

Put $S_{j+1} = \{v_1, v_2, \ldots, v_{j+1}\}$. For $k > j+1$, choose $v_k \in V(G)$ such that the following hold:

1. $v_k \notin I_G[S_{k-1}]$, where $S_{k-1} = \{v_1, v_2, \ldots, v_{k-1}\}$.
2. If v_k and some $v_i \in S_{k-1}$ belong to the same induced cycle C_n, $n \geq 4$, then v_k and v_i are vertices in a minimal 2-path closure absorbing set in C_n.
3. If v_k and some $v_i \in S_{k-1}$ belong to the same induced K_2-gluing G_o of (K_3, K_3) and $d_G(v_k, v_i) = 2$, then $G_o \cap S_k = \{v_k, v_i\}$.

There exists a smallest positive integer N such that $I_G[S_N] = V(G)$. By the definition, $v_{j+1} \notin I_G[v_i, v_k]$ for all $i, k > j+1$. Finally, following the arguments in Case 1, S_N is a strongly closed geodetic cover of G. Consequently, $\hat{g}(G) \leq N$.

The following theorem formalizes the above result.

Theorem 2.1 *Let G be a connected graph. If $diam(G) \leq 2$, then $\hat{g}(G) < +\infty$.*

3 Join of Graphs

The **join** $H+K$ of graphs H and K is the graph with $V(H+K) = V(H) \cup V(K)$ and $E(H+K) = E(H) \cup E(K) \cup \{uv : u \in V(H), u \in V(K)\}$.

In view of Theorem 2.1, since $diam(H+K) = 2$, we have $\hat{g}(H+K) < \infty$ for any connected graphs H and K.

Theorem 3.1 *Let $G = H + K_m$ where H is a nontrivial connected graph, and $m \geq 1$. Then*

$$\hat{g}(G) = min\{|S| : S \subseteq V(H), P_2[S]_H = V(H) \text{ and } S \in \sum(G)\}.$$

Proof: Let $S \subseteq V(G)$ be a strognly closed geodetic cover of G. Then $I_G[S] = V(G)$. Since G is noncomplete, there exist $x, y \in S$ such that $d_G(x, y) = 2$. Necessarily, $x, y \in V(H)$. Since $V(K_m) \subseteq I_G(x, y) \subseteq I_G(S)$ we have $S \cap V(K_m) = \emptyset$, by the definition of a strongly closed geodetic cover. Therefore, $S \subseteq V(H)$. Now, let $u \in V(H) \backslash S$. Then there exist $x, y \in S$ such that $d_G(x, y) = 2$ and $u \in I_G(x, y)$. Since $diam(G) = 2$, the $[x, u, y]$ path in G is a $[x, u, y]$ path in H. Thus, $d_H(x, y) = 2$. Consequently, $I_H[S] = V(H)$ and S is a 2-path closure absorbing set in H. The desired conclusion immediately follows. ■

We remark that we cannot replace $\sum(G)$ in Theorem 3.1 by $\sum(H)$. For example, any strongly closed geodetic cover of $C_{10} + K_1$ cannot be a strongly closed geodetic cover of C_{10}.

Corollary 3.2 *Let $G = H + K_m$, where H is a connected graph, and $m \geq 1$. If $diam(H) = 2$, then $\hat{g}(G) = \hat{g}(H)$.*

If K_m in Theorem 3.1 is replaced by any noncomplete connected K, then we have for the vertices x and y in the given proof either $x, y \in V(H)$ or $x, y \in V(K)$. Consequently, either

1. $S \subseteq V(H)$ and S is a 2-path closure absorbing in H; or
2. $S \subseteq V(K)$ and S is a 2-path closure absorbing in K,

and so follows the following theorem.

Theorem 3.3 *Let $G = H + K$, where H and K are noncomplete connected graphs. Then*

$$\hat{g}(G) = min\ \{\alpha_2(H), \alpha_2(K)\},$$

$$\text{where } \alpha_2(H) = \ min\ \{|S| : S \subseteq V(H), P_2[S]_H = V(H), S \in \sum(G)\}.$$

Corollary 3.4 *Let $G = H + K$, where H and K are noncomplete connected graphs. If $diam(H) = 2 = diam(K)$, then*

$$\hat{g}(G) = min\ \{\hat{g}(H), \hat{g}(K)\}.$$

Example 3.5 *1.* $\hat{g}(K_{n,k} + K_m) = min\,\{n, k\}$ *for* $n, k \geq 2$

2. $\hat{g}(P_n + P_m) = min\{\lceil \frac{n+1}{2} \rceil, \lceil \frac{m+1}{2} \rceil\}$, *for* $m, n > 2$.

3. $\hat{g}(C_n + C_m) = min\{\lceil \frac{n}{2} \rceil, \lceil \frac{m}{2} \rceil\}$, *for* $m, n > 3$.

4. $\hat{g}(K_{m,n} + K_{p.r}) = min\{m, n, p, r\}$ *for* $m, n, p, r \geq 2$.

4 Corona of Graphs

The **corona** $H \circ K$ of graphs H and K is the graph obtained by taking one copy of H and $|V(H)|$ copies of K, and then joining the ith vertex of H to every vertex in the ith copy of K.

Let $G = H \circ K$, where H and K are connected graphs. It is customary to denote by K_v that copy of K whose vertices are adjoined with the vertex v of H. In effect, G is composed of the subgraphs $K_v + v$ joined together by the edges of H. Moreover, $V(G) = \bigcup_{v \in V(H)} V(K_v + v)$.

In is worth noting that by Theorem 3.1, $\hat{g}(K_v + v) < \infty$.

Theorem 4.1 *Let $G = H \circ K_m$, where H is a nontrivial connected graph. Then $S = \bigcup_{v \in V(H)} V((K_m)_v)$ is a unique strongly closed geodetic cover of G.*

Proof: For each $v \in V(H)$, we let $S_v = V((K_m)_v)$. Put $S = \bigcup_{v \in V(H)} S_v$. We claim that S is a strongly closed geodetic cover of G. Let $v \in V(G) \backslash S$. Then $v \in V(H)$. Let $u \in V(H) \backslash \{v\}$ and take $x \in S_v$ and $y \in S_u$. Then $v \in I_G(x, y) \subseteq I_G(S)$. Thus, $I_G[S] = V(G)$. Incidentally, $V(H) = I_G(S)$, so that $I_G(S) \cap S = \emptyset$. Thus, S is a strongly closed geodetic cover of G.

Now, suppose that S is a strongly closed geodetic cover of G. Since $V((K_m)_v) \subseteq Ext(G)$, $S^* = \bigcup_{v \in V(H)} V((K_m)_v) \subseteq S$. Moreover, since $V(H) \subseteq I_G(S^*) \subseteq I_G(S)$, we have $S \cap V(H) = \emptyset$. This means that $S \subseteq V(G) \backslash V(H) = S^*$. Thus, $S^* = S$. ■

In view of Theorem 4.1, $\hat{g}(H \circ K_m) < +\infty$ for any nontrivial connected graph H.

Corollary 4.2 *Let $G = H \circ K_m$, where H is any connected graph. Then*

$$\hat{g}(G) = \begin{cases} m+1, & \text{if } H \text{ is trivial} \\ m \cdot |V(H)|, & \text{if } H \text{ is nontrivial} \end{cases}$$

Lemma 4.3 *Let $G = H \circ K$ where H is a nontrivial connected graph and K a noncomplete graph. If for each $v \in V(H)$, S_v is a strongly closed geodetic cover of $K_v + v$, then $S_v \subseteq V(K_v)$ and $\bigcup_{v \in V(H)} S_v$ is a closed geodetic cover of G.*

Proof: For each $v \in V(H)$, let S_v be a strongly closed geodetic cover of $K_v + v$. Since $K_v + v$ is not complete, there exist $u, w \in S_v$ such that $d_{K_v+v}(u, w) = 2$. Since $v \in I_{K_v+v}(u, w)$, $v \notin S_v$. That is, $S_v \subseteq V(K_v)$. Now, let $S = \bigcup_{v \in V(H)} S_v$. We will show that S is a strongly closed geodetic cover of G. Let $v \in V(G) \backslash S$. Then $v \in V(K_u + u)$ for some $u \in V(H)$. There exist $x, y \in S_u$ such that $v \in I_{K_u+u}(x, y) = I_G(x, y) \subseteq I_G(S)$. Since v is arbitrary, $V(G) \backslash S \subseteq I_G(S)$. This means that $I_G[S] = V(G)$ and $S \cap I_G(S) = \emptyset$. That is, S is a strongly closed geodetic cover of G. ■

In view of Theorem 3.1 , such S_v in Lemma 4.3 exists. Consequently, Lemma 4.3 implies that $\hat{g}(H \circ K) < +\infty$ for any nontrivial connected graph H and a noncomplete graph K.

Lemma 4.4 *Let $G = H \circ K$, where H is a nontrivial connected graph and K a noncomplete graph, and let $S \subseteq V(G)$. If S is a strongly closed geodetic cover of G, then $S \cap V(K_v)$ is a strongly closed geodetic cover of $K_v + v$ for every $v \in V(H)$.*

Proof: Let S be a strongly closed geodetic cover of G, and let $v \in V(H)$. Put $S_v = S \cap V(K_v)$. Since $V(K_v) \cap I_G[x, y] = \emptyset$, for every pair $x, y \in V(H)$, $S_v \cap V(H) = \emptyset$. Moreover, since K_v is noncomplete, $|S_v| \geq 2$ and there exist $u, w \in S_v$ such that $d_G(u, w) = d_{K_v+v}(u, w) = 2$. Let $x \in V(K_v + v) \backslash S_v$. If $x = v$, then $x \in I_{K_v+v}(u, w) \subseteq I_{K_v+v}(S_v)$. Suppose that $x \neq v$. There exist $y, z \in S$ such that $x \in I_G[y, z]$. Obviously, $y, z \in S_v$. Thus, $x \in I_G[y, z] = I_{K_v+v}[y, z] \subseteq I_{K_v+v}[S_v]$. Therefore, $V(K_v + v) = I_{K_v+v}[S_v]$. Moreover, $S_v \cap I_{K_v+v}(S_v) = S_v \cap I_G(S_v) \subseteq S \cap I_G(S) = \emptyset$. This means that S_v is a strongly closed geodetic cover of $K_v + v$. ■

Theorem 4.5 *Let $G = H \circ K$ where H is a nontrivial connected graph and K a noncomplete graph, and let $S \subseteq V(G)$. Then S is a strongly closed* *geodetic cover of* G *if and only if* $S = \bigcup_{v \in V(H)} S_v$, *where* $S_v \subseteq V(K_v)$ *is a strongly closed geodetic cover of* $K_v + v$.

Proof: If S is a strongly closed geodetic cover of G, then by Lemma 4.4, $S \cap V(K_v)$ is a strongly closed geodetic cover of G. Since $V(H) \subseteq I_G(\bigcup_{v \in V(H)} [S \cap V(K_v)])$, $S = \bigcup_{v \in V(H)} [S \cap V(K_v)]$. Put $S_v = S \cap V(K_v)$, and we are done.

Conversely, suppose that $S = \bigcup_{v \in V(H)} S_v$, where $S_v \subseteq V(K_v)$ is a strongly closed geodetic cover of $K_v + v$. By Lemma 4.3, S is a strongly closed geodetic cover of G. ■

Corollary 4.6 *Let* $G = H \circ K$ *where* K *is a noncomplete graph. Then*

$$\hat{g}(G) = |V(H)| \cdot \hat{g}(K + K_1).$$

Corollary 4.7 *Let* $G = H \circ K$ *where* H *and* K *are connected graphs. If* K *is noncomplete, then*

$$\hat{g}(G) = |(V(H)| \cdot min\left\{|S| : S \subseteq V(K), P_2[S]_K = V(K), I_G(S) \cap S = \emptyset\right\}.$$

Example 4.8 *1.* $\hat{g}(P_m \circ C_n) = m \cdot \left\lceil \frac{n}{2} \right\rceil, n > 3$

2. $\hat{g}(C_n \circ P_m) = n \cdot \left\lceil \frac{m+1}{2} \right\rceil, m > 2$

5 Composition of Graphs

The **composition** $G[H]$ of graphs G and H is the graph with $V(G[H]) = V(G) \times V(H)$ and $(u, v)(u', v') \in E(G[H])$ if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$.

Theorem 5.1 **[4]** *If* A *is a geodetic cover of* G, *then* $A \times V(K_m)$ *is a geodetic cover of* $G[K_m]$.

Theorem 5.2 **[4]** *If* S *is a geodetic cover of* $G[K_m]$, *then* S_f *is a geodetic cover of* G, *where* $S_f = \{u : (u, v) \in S \;\; for \;\; some \;\; v \in V(K_m)\}$.

Lemma 5.3 **[9]** *If* $[u_1, u_2, \cdots, u_k], k \geq 3$ *is a* u_1-u_k *geodesic in* G, *then for any* $v_1, v_2, \cdots, v_k \in V(K_m)$, *the path* $[(u_1, v_1), (u_2, v_2), \cdots, (u_k, v_k)]$ *is a* (u_1, v_1)-(u_k, v_k) *geodesic in* $G[K_m]$.

Lemma 5.4 **[9]** *If* $[(u_1, v_1), (u_2, v_2), \cdots, (u_k, v_k)], k \geq 3$, *is a* (u_1, v_1)-(u_k, v_k) *is a geodesic in* $G[K_m]$, *then* $[u_1, u_2, \cdots, u_k]$ *is a* u_1-u_k *geodesic in* G

Theorem 5.5 *Let* G *be a noncomplete connected graph, and let* $S \subseteq V(G[K_m])$. *Then* S *is a strongly closed geodetic cover of* $G[K_m]$ *if and only if*

$$S = A \times V(K_m),$$

for some strongly closed geodetic cover A *of* G.

Proof: Let A be a strongly closed geodetic cover of G, and put $S = A \times V(K_m)$. By Theorem 5.1, $I_{G[S]} = V(G[K_m])$. Suppose that $S \cap I_{G[K_m]}(S) \neq \emptyset$, and let $(x, y) \in S \cap I_{G[K_m]}(S)$. Then there exist $u_1, u_2 \in A$ and $v_1, v_2 \in V(K_m)$ such that $(x, y) \in I_{G[K_m]}((u_1, v_1), (u_2, v_2))$. By Lemma 5.3, $x \in I_G(u_1, u_2) \subseteq I_G(A)$. Since $x \in A$ and A is a strongly closed geodetic cover of G, this is a contradiction. Thus, $S \cap I_{G[K_m]}(S) = \emptyset$. This makes S a strongly closed geodetic cover of $G[K_m]$.

Conversely, let $S \subseteq V(G[K_m])$ be a strongly closed geodetic cover of $G[K_m]$.
Claim 1: S_f is a strongly closed geodetic cover of G.

By Theorem 5.2, $I_G[S_f] = V(G)$. Now, suppose that $S_f \cap I_G(S_f) \neq \emptyset$, and let $x \in S_f \cap I_G(S_f)$. Then there exist $u_1, u_2 \in S_f$ such that $x \in I_G(u_1, u_2)$. By Lemma 5.3, for any $y, v_1, v_2 \in V(K_m)$, $(x, y) \in I_{G[K_m]}((u_1, v_1), (u_2, v_2))$. This means that $S \cap I_{G[K_m]}(S) \neq \emptyset$, which is a contradiction to the definition of S. Therefore, $S_f \cap I_G(S_f) = \emptyset$, and S_f is a strongly closed geodetic cover of G.
Claim 2: $S = S_f \times V(K_m)$.

Clearly, $S \subseteq S_f \times V(K_m)$. Suppose that $S \neq S_f \times V(K_m)$. Then there exists $x \in S_f$ and there exists $y \in V(K_m)$ such that $(x, y) \notin S$. There exist $(u_1, v_1), (u_2, v_2) \in S$ such that $(x, y) \in I_{G[K_m]}((u_1, v_1), (u_2, v_2))$. By Lemma 5.4, $x \in I_G(u_1, u_2) \subseteq I_G(S_f)$, a contradiction to Claim 1 above. Therefore, $S = S_f \times V(K_m)$. Put $A = S_f$, and the proof of the theorem is complete. ■

Corollary 5.6 *Let* G *be a connected graph. Then* $\hat{g}(G[K_m]) = m \cdot \hat{g}(G)$.

Proof: If G is complete, then $G[K_m]$ is the complete graph $K_{m|V(G)|}$, and thus $\hat{g}(G[K_m]) = m \cdot |V(G)| = m \cdot \hat{g}(G)$. Suppose that G is noncomplete. In view of Theorem 5.5, $\sum(G) = \emptyset$ if and only if $\sum(G[K_m]) = \emptyset$. In this case, Corollary 5.6 obviously holds. Finally, if $\sum(G) \neq \emptyset$, then by Theorem 5.5,

$$\hat{g}(G[K_m]) = |V(K_m)| \cdot min\left\{|A| : A \in \sum(G)\right\} = m \cdot \hat{g}(G).$$

■

Now we consider $K_m[G]$. Let G be a noncomplete connected graph, $m \geq 2$, and let $u \in V(K_m)$ and $v_1, v_2 \in V(G)$. If $d_G(v_1, v_2) > 1$, then $d_{K_m[G]}((u, v_1), (u, v_2)) = 2$. To see this, let $v \in V(K_m) \setminus \{u\}$. Since

$$d_{K_m[G]}((u, v_1), (v, v_1)) = 1 = d_{K_m[G]}((u, v_2), (v, v_1))$$

we have

$$d_{K_m[G]}((u,v_1),(u,v_2)) = d_{K_m[G]}((u,v_1),(v,v_1)) + d_{K_m[G]}((v,v_1),(u,v_2)) = 2.$$

This implies that if G is a noncomplete connected graph, and $m \geq 2$, then $diam(K_m[G]) = 2$. Consequently, $\hat{g}(K_m[G]) < +\infty$.

Theorem 5.7 *Let G be a noncomplete connected graph, and $m \geq 2$. Let $S \subseteq V(K_m[G]$, If S is a strongly closed geodetic cover of $K_m[G]$, then*

$$S = \{w\} \times T$$

for some 2-path closure absorbing set T in G and some $w \in V(K_m)$.

Proof: Let S be a strongly closed geodetic cover of $K_m[G]$. Since $K_m[G]$ is not complete, there exist $(u_1,v_1),(u_2,v_2) \in S$ such that $d_{K_m[G]}((u_1,v_1),(u_2,v_2)) = 2$. Necessarily, $u_1 = u_2 = w \in S_f$ and $v_1 \neq v_2$. If $x \in V(K_m)\setminus\{w\}$ and $y \in V(G)$, then $(x,y) \in I_{K_m[G]}((w,v_1),(w,v_2)) \subseteq I_{K_m[G]}(S)$. Hence,

$$[V(K_m)\setminus\{w\}] \times V(G) \subseteq I_{K_m[G]}(S).$$

By the definition of S, $S = \{w\} \times T$, for some $T \subseteq V(G)$. Let $v \in V(G)\setminus T$. Then $(w,v) \notin S$. There exist $x,y \in T$ such that

$$d_{K_m[G]}((w,x),(w,y)) = 2 \text{ and } (w,v) \in I_{K_m[G]}((w,x),(w,y)).$$

This is possible only if $d_G(x,y) = 2$ and $v \in I_G(x,y)$. Thus, $P_2[T]_G = V(G)$, and T is a 2-path closure absorbing set in G. ■

T in Theorem 5.7 need not be a strongly closed geodetic cover of G. Consider, for example, the composition $K_2[C_9]$. A 2-path closure absorbing set in C_9 must have at least 5 vertices. However, a strongly closed geodetic cover of C_9 has at most 3 vertices only.

Corollary 5.8 *Let G be a noncomplete graph, and $m \geq 2$. Then*

$$L \leq \hat{g}(K_m[G]) \leq |V(G)|,$$

where $L = min\{|T| : T \subseteq V(G), P_2[T]_G = V(G)\}$.

Proof: Let $S \in \sum(K_m[G])$. By Theorem 5.7, $S = \{w\} \times T$ for some $w \in V(K_m)$ and for some 2-path closure absorbing set $T \subseteq V(G)$ in G. Then $L \leq |T| = |S| \leq |V(G)|$. Since S is arbitrary, we get the desired inequality. ■

Theorem 5.9 *Let G be a noncomplete connected graph, and $m \geq 2$. Suppose that $diam(G) = 2$, and $S \subseteq V(K_m[G])$. Then S is a strongly closed geodetic cover of $K_m[G]$ if and only if*

$$S = \{w\} \times T$$

for some $w \in V(K_m)$ and some strongly closed geodetic cover T of G.

Proof: Let S be a strongly closed geodetic cover of $K_m[G]$. By Theorem 5.7, there exists $w \in V(K_m)$ and a 2-path closure absorbing set $T \subseteq V(G)$ in G such that $S = \{w\} \times T$. We claim that T is, in fact, a strongly closed geodetic cover of G. Let x, y and z be distinct vertices in T with $z \in I_G(x, y)$. This means that $[(w, x), (w, z), (w, y)]$ is a (w, x)-(w, y) geodesic in $K_m[G]$. Consequently, $(w, z) \in I_{K_m[G]}((w, x), (w, y))$. Since $(w, x), (w, z), (w, y) \in S$, this is a contradiction. Therefore, T is a strongly closed geodetic cover of G.

Conversely, let $(w, x), (w, z), (w, y) \in S = \{w\} \times T$ such that

$$(w, z) \in I_{K_m[G]}((w, x), (w, y)).$$

Since $diam(K_m[G] = 2$, $[(w, x), (w, z), (w, y)]$ is a (w, x)-(w, y) geodesic in $K_m[G]$. This means that $[x, y, z]$ is a x-y geodesic in G, and $z \in I_G(x, y)$. Since $x, y, z \in T$ and T is a strongly closed geodetic cover of G, this is a contradiction. Therefore, $S \cap I_{K_m[G]} = \emptyset$. Finally, to show that $I_{K_m[G]}[S] = V(K_m[G])$, let $u, x \in V(K_m[G] \backslash S$. Then either $u \neq w$ or $u = w$ and $x \notin T$. Suppose that $u \neq w$. Note that since $diam(G) = 2$ and T is a geodetic cover of G, there exist $x, y \in T$ such that $d_G(x, y) = 2$. By the previous remark, $d_{K_m[G]}((w, x), (w, y)) = 2$. In fact, $(u, x) \in I_{K_m[G]}((w, x), (w, y))$. Thus, in this case, $(u, x) \in I_{K_m[G]}[S]$.

Now, suppose that $u = w$ and $x \notin T$. Since T is a 2-path closure absorbing set in G, there exist $u, v \in T$ such that $d_G(u, v) = 2$ and $x \in I_G(u, v)$. Consequently, $d_{K_m[G]}((w, u), (w, v)) = 2$ and $[(w, u), (u, x), (w, v)]$ is a $w, u)$-(w, v) geodesic in $K_m[G]$. Threfore, $(u, x) \in I_{K_m[G]}((w, u), (w, v)) \subseteq I_{K_m[G]}[S]$. The above results yield $V(K_m[G]) = I_{K_m[G]}[S]$, and S is a strongly closed geodetic cover of $K_m[G]$. ■

Corollary 5.10 *Let G be a noncomplete connected graph, and $m \geq 2$. Suppose that $diam(G) = 2$. Then*

$$\hat{g}(K_m[G]) = \hat{g}(G).$$

Example 5.11 *1.* $\hat{g}(K_n[K_m]) = mn$

2. $\hat{g}(P_n[K_m]) = 2m$ *for* $n \geq 2$

3. $\hat{g}(C_n[K_m]) = m \cdot \hat{g}(C_n) = \begin{cases} 2m, & \text{if } n \text{ is even} \\ 3m, & \text{if } n \text{ is odd} \end{cases}$

4. $\hat{g}(K_n[W_m]) = \hat{g}(C_m + K_1) = \begin{cases} 4, & \text{if } m = 3 \\ \lceil \frac{m}{2} \rceil, & \text{if } m > 3 \end{cases}$

6 Realization Problem

Theorem 6.1 *For any positive integers* m,n,k *and* d *such that* $4 \leq n \leq m$, $2 \leq d \leq k$ *and* $k \geq 2m - n + d + 1$, *there exists a connected graph* G *such that* $|V(G)| = k$, $diam(G) = d$, $g(G) = n$ *and* $\hat{g}(G) = m$.

Proof: Suppose that $d = 2$ and $m = n$. Then $k \geq m + 3$. Put $l = k - m - 2$. Let P_5 be the path $[u_1, u_2, u_3, u_4, u_5]$, and let K_l and K_{m-3} be complete graphs with vertex sets $V(K_l) = \{v_1, v_2, \ldots, v_l\}$ and $V(K_{m-3}) = \{x_1, x_2, \ldots, x_{m-3}\}$. Let $G_1 = P_5 + K_l$ and $G_1 = K_l + \overline{K_{m-3}}$. Form a connected graph G, as shown in Figure 6.1, consisting of G_1 and G_2. Then $|V(G)| = k$ with $Ext(G) = \{x_1, x_2, \ldots, x_{m-3}\}$. Since the set $\{u_1, u_3, u_5, x_1, x_2, \ldots, x_{m-3}\}$ is a strongly closed geodetic cover of G, $\hat{g}(G) \leq m$. On the other hand, for any vertices $u, v \in V(P_5)$, $Ext(G) \cup \{u, v\}$ is not a geodetic cover of G. This means that $g(G) \geq m$. Therefore $g(G) = \hat{g}(G) = m$.

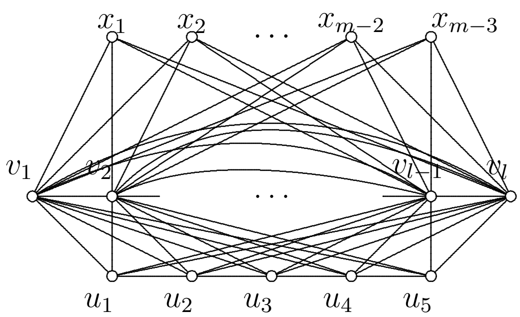

Figure 6.1: Graph G satisfying Theorem 6.1 where $d = 2$ and $m = n$

Suppose that $d = 2$ but $m > n$. Suppose further that $n > 4$. Put $l = k - m - 2$ and $r = m - n + 2$. Let $V(K_l) = \{v_1, v_2, \ldots, v_l\}$ and let $G_1 = ([u, v] \cup \overline{K_{r-1}} \cup [z, w]) + K_l$, where $\overline{K_{r-1}}$ is the complement of the complete graph K_{r-1} with vertices $u_1, u_2, \ldots, u_{r-1}$, and $[u, v]$ and $[z, w]$ being K_2 graphs with vertices u, v and z, w, respectively. Construct a connected graph G from G_1, as shown in Figure 6.2, by adjoining $(n - 4)$ triangles $[x_i, u, v_1]$, $i = 1, 2, \ldots, n - 4$, and the edges $[y, u]$, $[y, z]$, $[y, u_i]$, $i = 1, 2, \ldots, r - 1$.

Then $|V(G)| = l + r + n - 1 = k$ with $Ext(G) = \{x_1, x_2, \ldots, x_{n-4}, v, w\}$. Clearly, G is not an extreme geodesic graph. Moreover, For any $x \notin Ext(G)$, $Ext(G) \cup \{x\}$ is not a geodetic cover of G. Thus $g(G) \geq n$. Since the set $\{x_1, x_2, \ldots, x_{n-4}, v, w, y, v_l\}$ is a geodetic cover of G, $g(G) \leq n$, and so $g(G) = n$. Now the set $\{x_1, x_2, \ldots, x_{n-4}, v, z, w, u_1, u_2, \ldots, u_{r-1}\}$ is a unique strongly closed geodetic cover of G. Thus, $\hat{g}(G) = m$. If $n = 4$, we reconstruct G in Figure 6.2 by using K_{l+1} in the place of K_l, and using K_{r-2} instead of K_{r-1}. In this case, $g(G)$ and $\hat{g}(G)$ are determined by the geodetic covers $\{v, w, y, v_j\}$ and $\{u, v, z, w, u_1, u_2, \ldots, u_{r-2}\}$ in G, respectively.

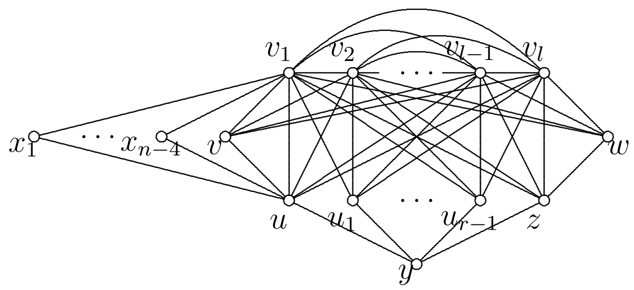

Figure 6.2: Graph G satisfying Theorem 6.1 where $d = 2$ and $4 < n < m$

Finally suppose that $d \geq 3$. Let $r = m - n + 3$ and $l = k - m - r - d + 3$. Let G be the graph in Figure 6.3, where $V(G) = \{v_1, v_2, \ldots, v_r\} \cup \{u_1, u_2, \ldots, u_{r+l}\} \cup \{w_1, w_2, \ldots, w_{d-3}\} \cup \{x_1, x_2, \ldots, x_{n-3}\}$. Note that $l \geq 1$. The sets $\{v_1, v_r, u_1, x_1, x_2, \ldots, x_{n-3}\}$ and $\{v_1, v_2, \ldots, v_r, x_1, x_2, \ldots, x_{n-3}\}$ are geodetic basis and strongly closed geodetic set of G, respectively. Thus, $g(G) = n$ and $\hat{g}(G) = m$. Also the diameter of G is the length of the path $[u_1, v_1, u_{r+l}, w_1, w_2, \ldots w_{d-3}, x_1]$, and thus $diam(G) = d$. And lastly, $|V(G)| = r + (r + l) + (d - 3) + (n - 3) = k$. ■

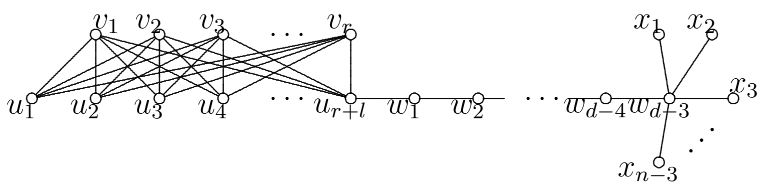

Figure 6.3: Graph G satisfying Theorem 6.1 where $d \geq 3$

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