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# STRONGLY DISSIPATIVE OPERATORS AND NONLINEAR EQUATIONS IN A FRÉCHET SPACE

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Abstract. Suppose that $X$ is a Fréchet space, $Y$ is a Banach subspace of $X$, and $A$ is a function from $Y$ into $X$. Sufficient conditions are determined to insure that the equation $Ax=y$ ($y\in Y$) has a unique solution $x_y$ which depends continuously on $y$. The techniques of this paper use the theory of dissipative operators in a Banach space, and the results are associated with the idea of admissibility of the space $Y$. Also, the equation $Ax=Cx+y$ is considered where $C$ is completely continuous.

Let $X$ be a Fréchet space over the real or complex field (i.e., $X$ is a locally convex, complete, metrizable topological vector space—see e.g., [10, p. 85]). In this paper we assume the following:

(X1) $(q_n)_1^\infty$ is an increasing family of continuous seminorms on $X$ which defines the topology of $X$ (i.e., $q_n \leqq q_{n+1}$ and $X-\lim_{k\to\infty} x_k=x$ if and only if $\lim_{k\to\infty} q_n(x_k-x)=0$ for each $n$).

(X2) $Y=\{x\in X: \sup_n\{q_n(x)\}<\infty\}$ and $|x|=\sup_n\{q_n(x)\}$ for each $x$ in $Y$.

Note that a sequence of seminorms satisfying (X1) always exists, and the space $Y$ in (X2) is a complete normed space with norm $|\cdot|$. However, the members of $Y$ depend on how the sequence $(q_n)_1^\infty$ is chosen.

In this paper, some results on strongly dissipative operators in a Banach space are used to establish analogous results for a class of operators which map $Y$ into $X$. Recently, some fixed point theorems for completely continuous perturbations of Lipschitz continuous functions in locally convex spaces have been obtained by Cain and Nashed [1]. In this paper, a class of functions $A$ from $Y$ into $X$ is considered which can be "approximated" by functions which are defined on $X$. Sufficient conditions are established to insure that the equation $Ax=y$ has a unique solution $x_y$ for each $y$ in $Y$, and the function $B$ defined by $By=x_y$ for each $y$ in $Y$ has certain continuity properties. This result is closely associated with the notion of admissibility introduced by Massera and Schäffer [6]. Recently, admissibility has been used in studying existence and stability of solutions to Volterra integral equations—see Corduneanu [2], [3] and Miller [8].

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We also consider the solvability of the equation $Ax=Cx+y$ where $C$ is a function from $Y$ into $Y$ which has certain compact properties and growth properties. Finally, two examples of integral equations on the half-line are given to illustrate when these techniques may be applied.

DEFINITION 1. If $A$ is a function from $X$ into $X$, then $A$ is said to be compatible with $q_n$ if whenever $x$ is in $X$ and $(x_k)_1^\infty$ is a sequence in $X$ such that $\lim_{k\to\infty} q_n(x_k-x)=0$, it follows that $\lim_{k\to\infty} q_n(Ax_k-Ax)=0$.

For each positive integer $n$ let $M_n=\{x\in X: q_n(x)=0\}$ and let $X/M_n=\{\phi(x): x\in X \text{ and } \phi(x)=x+M_n\}$. If $q_n^*(\phi(x))=q_n(x)$ for each $\phi(x)$ in $X/M_n$, then $q_n^*$ is well defined and is a norm on the quotient space $X/M_n$. If $A$ is a function from $X$ into $X$ which is compatible with $q_n$, then define $A^*\phi(x)=\phi(Ax)$ for each $\phi(x)$ in $X/M_n$. It is easy to see that $A^*$ is well defined and continuous on $X/M_n$.

DEFINITION 2. Suppose that $X/M_n$, $A$ and $A^*$ are as in the above paragraph and let $E_n$ denote the completion of the normed space $X/M_n$ with the norm on $E_n$ denoted by $q_n'$. Then $A$ is said to be strongly dissipative with respect to $q_n$ if there is a continuous function $\alpha_n$ from $[0,\infty)$ into $[0,\infty)$ and a continuous function $A'$ from $E_n$ into $E_n$ such that $\alpha_n(0)=0$, $\alpha_n(s)>0$ if $s>0$, $A^*\phi(x)=A'\phi(x)$ if $\phi(x)$ is in $X/M_n$, and

$$\lim_{h\to 0+} [q_n'(x-y+h(A'x-A'y))-q_n'(x-y)]/h \leqq -\alpha_n(q_n'(x-y))$$

for all $x$ and $y$ in $E_n$.

REMARK. In many cases the quotient spaces $X/M_n$ are complete—for example, the spaces considered in Examples 1 and 2 below have this property. If $X/M_n$ is complete and $A$ is compatible with $q_n$, note that $A$ is strongly dissipative with respect to $q_n$ only in case

$$\lim_{h\to 0+} [q_n(x-y+h(Ax-Ay))-q_n(x-y)]/h \leqq -\alpha_n(q_n(x-y))$$

for all $x$ and $y$ in $X$. Also, if the function $A^*$ is uniformly continuous on bounded subsets of $X/M_n$, then $A^*$ can be extended to a continuous function $A'$ on $E_n$, and it can be shown that $A$ is strongly dissipative with respect to $q_n$.

LEMMA 1. *Suppose that $A$ is a function from $X$ into $X$ which is strongly dissipative with respect to $q_n$. Then, for each $z$ in $X$ and each $\varepsilon>0$, there is a $x_z^\varepsilon$ in $X$ such that $q_n(Ax_z^\varepsilon-z)\leqq\varepsilon$.*

INDICATION OF PROOF. If the function $A'$ is as in Definition 2, it is easy to see that $A'$ satisfies each of the suppositions of Theorem 2 in [7], and so there is a unique $y'$ in $E_n$ such that $q_n'(A'y'-\phi(z))=0$. Since $X/M_n$ is dense in $E_n$ and $A'$ is continuous, there is an $x_z^\varepsilon$ in $X$ such that

$q_n'(A'\phi(x_z^\varepsilon)-\phi(z))\leqq\varepsilon$. Thus, $q_n(Ax_x^\varepsilon-z)=q_n^*(A^*\phi(x_z^\varepsilon)-\phi(z))\leqq\varepsilon$ and the lemma is true.

We now prove our main result.

THEOREM 1. *Suppose that conditions* (X1)—(X2) *are fulfilled*, $\alpha$ *is a continuous, increasing function from* $[0,\infty)$ *into* $[0,\infty)$ *such that* $\alpha(0)=0$ *and* $\lim_{s\to\infty}\alpha(s)=\infty$, *A is a function from Y into X, and* $(A_n)_1^\infty$ *is a sequence of functions from X into X such that*

(i) $A_n$ *is strongly dissipative with respect to* $q_n$ *for each n*;

(ii) $\lim_{h\to 0+}[q_n(x-y+h(A_nx-A_ny))-q_n(x-y)]/h\leqq-\alpha(q_n(x-y))$ *for each x and y in X and each n*;

(iii) *there is a number* $L\geqq 0$ *such that* $q_n(A_n0)\leqq L$ *for each n*;

(iv) *for each pair of positive numbers K and* $\delta$ *there is a positive integer* $N(K,\delta)$ *such that if* $j\geqq i\geqq N(K,\delta)$ *and x is a member of X such that* $q_j(x)\leqq K$, *then* $q_i(A_ix-A_jx)\leqq\delta$; *and*

(v) *if K is a positive number, then* $\lim_{i\to\infty}q_i(A_ix-Ax)=0$, *uniformly for x in Y with* $|x|\leqq K$.

*Then, for each z in Y, there is a unique* $x_z$ *in Y such that* $Ax_z=z$. *Furthermore, if the function B from Y into Y is defined by* $Bz=x_z$ *for each z in Y, then* $|Bz-Bw|\leqq\alpha^{-1}(|z-w|)$ *for all z and w in Y.*

REMARK. Note that the function $A$ in Theorem 1 need not be defined on all of $X$ and does not necessarily map $Y$ into $Y$. Furthermore, by supposition (ii) if the quotient spaces $X/M_n$ are complete for each $n$, then we need only assume that $A_n$ is compatible with $q_n$ in supposition (i)—see remark following Definition 2.

PROOF OF THEOREM 1. Note that supposition (ii) implies that

$$q_n(x-y)\leqq\alpha^{-1}(q_n(A_nx-A_ny)) \tag{1}$$

for all $x$ and $y$ in $X$. By Lemma 1 and suppositions (i) and (ii), there is an $x_z^n$ in $X$ such that $q_n(A_nx_z^n-z)\leqq n^{-1}$ for each $z$ in $Y$ and positive integer $n$. Since $\alpha^{-1}$ is increasing, we have from (1) and supposition (iii) that

$$q_n(x_z^n)\leqq\alpha^{-1}(q_n(A_nx_z^n-A_n0))\leqq\alpha^{-1}(|z|+1+L) \tag{2}$$

for each $n$. Let $\varepsilon$ be a positive number and let $\delta>0$ be such that $\alpha^{-1}(s)\leqq\varepsilon$ whenever $0\leqq s\leqq 2\delta$. If $N$ is as in (iv) and $j\geqq i\geqq N(\alpha^{-1}(|z|+1+L),\delta)$, then $q_j(x_z^j)\leqq\alpha^{-1}(|z|+1+L)$, so

$$\begin{aligned} q_i(A_ix_z^i-A_ix_z^j)&\leqq q_i(z-A_ix_z^j)+i^{-1}\\ &\leqq q_i(z-A_jx_z^j)+q_i(A_jx_z^j-A_ix_z^j)+i^{-1}\\ &\leqq 2i^{-1}+q_i(A_jx_z^j-A_ix_z^j)\leqq 2i^{-1}+\delta. \end{aligned} \tag{3}$$

Now let $m$ be a positive integer such that

$$m \geqq \max\{N(\alpha^{-1}(|z|+1+L), \delta), 2/\delta\}.$$

If $j \geqq i \geqq m$, then by (1), (3), and the choice of $\delta$,

$$q_i(x_z^i - x_z^j) \leqq \alpha^{-1}(q_i(A_i x_z^i - A_i x_z^j)) \leqq \alpha^{-1}(2\delta) \leqq \varepsilon. \tag{4}$$

It now follows easily from (4) and (X1) that $(x_z^k)_1^\infty$ is a Cauchy sequence in $X$. Since $X$ is complete, there is an $x_z$ in $X$ such that $X-\lim_{k\to\infty} x_z^k = x_z$. By (2),

$$q_n(x_z) = \lim_{k\to\infty} q_n(x_z^k) \leqq \limsup_{k\to\infty} q_k(x_z^k) \leqq \alpha^{-1}(|z| + 1 + L)$$

for each $n$, so $x_z$ is in $Y$ with $|x_z| \leqq \alpha^{-1}(|z|+1+L)$. Letting $j\to\infty$ in the term on the left side of (3) shows that $\lim_{i\to\infty} q_i(A_i x_z^i - A_i x_z) = 0$. Thus, from supposition (v), if $n$ is a positive integer,

$$\begin{aligned} q_n(Ax_z - z) &\leqq \lim_{i\to\infty} q_i(Ax_z - z) = \lim_{i\to\infty} q_i(A_i x_z - z) \\ &\leqq \lim_{i\to\infty} q_i(A_i x_z - A_i x_z^i) + i^{-1} = 0. \end{aligned}$$

Hence $Ax_z = z$. If $y$ is in $Y$ and $Ay = z$, then by (1) and supposition (v),

$$\begin{aligned} q_n(y - x_z) &\leqq \lim_{i\to\infty} q_i(y - x_z) \leqq \lim_{i\to\infty} \alpha^{-1}(q_i(A_i y - A_i x_z)) \\ &\leqq \lim_{i\to\infty} \alpha^{-1}(q_i(A_i y - z) + q_i(z - A_i x_z)) = \alpha^{-1}(0) = 0 \end{aligned}$$

for each $n$. Consequently $y = x_z$ and the function $B$ defined in the statement of the theorem is well defined. Furthermore, if $z$ and $w$ are in $Y$ and $(x_z^i)_1^\infty$ and $(x_w^i)_1^\infty$ are as constructed above, then by (1) and the fact that $q_i(A_i x_z^i - A_i x_w^i) \leqq q_i(z-w) + 2i^{-1}$,

$$\begin{aligned} q_n(x_z - x_w) &= \lim_{i\to\infty} q_n(x_z^i - x_w^i) \leqq \limsup_{i\to\infty} q_i(x_z^i - x_w^i) \\ &\leqq \limsup_{i\to\infty} \alpha^{-1}(q_i(A_i x_z^i - A_i x_w^i)) \\ &\leqq \limsup_{i\to\infty} \alpha^{-1}(q_i(z - w) + 2i^{-1}) \leqq \alpha^{-1}(|z - w|). \end{aligned}$$

Thus,

$$|Bz - Bw| = \sup\{q_n(x_z - x_w) : n = 1, 2, \cdots\} \leqq \alpha^{-1}(|z - w|)$$

and the proof of Theorem 1 is complete.

LEMMA 2. *Let the suppositions of Theorem 1 be fulfilled and for each $R>0$ let $Q_R = \{x \in Y : |x| \leqq R\}$. Then, considering $Q_R$ with the topology induced by $X$, the function $B$ defined in Theorem 1 is continuous from $Q_R$ into $X$.*

INDICATION OF PROOF. Let $x$ be in $Q_R$ and let $(x_k)_1^\infty$ be a sequence in $Q_R$ such that $\lim_{k\to\infty} q_n(x_k-x)=0$ for each $n$. Let $n$ be a positive integer, let $\varepsilon>0$, and let $\delta>0$ be such that $\alpha^{-1}(s)\leqq\varepsilon$ whenever $0\leqq s\leqq\delta$. Note that if $y$ is in $Q_R$, then $|By|\leqq|By-BO|+|BO|\leqq\alpha^{-1}(R)+|BO|$. By supposition (v) of Theorem 1, let the integer $m\geqq n$ be such that $q_m(A_my-Ay)\leqq\delta/3$ for all $y$ in $Y$ with $|y|\leqq\alpha^{-1}(R)+|BO|$. Let $p$ be a positive integer such that $q_m(x_k-x)\leqq\delta/3$ whenever $k\geqq p$. If $k\geqq p$, we have from (1), the choice of $\delta$, and the fact that $\alpha^{-1}$ is increasing, that

$$\begin{aligned} q_n(Bx_k - Bx) &\leqq q_m(Bx_k - Bx) \leqq \alpha^{-1}(q_m(A_mBx_k - A_mBx)) \\ &\leqq \alpha^{-1}(q_m(A_mBx_k - ABx_k) + q_m(ABx_k - ABx) \\ &\qquad + q_m(ABx - A_mBx)) \\ &\leqq \alpha^{-1}(\delta/3 + q_m(x_k - x) + \delta/3) \leqq \alpha^{-1}(\delta) \leqq \varepsilon. \end{aligned}$$

Thus $\lim_{k\to\infty} q_n(Bx_k-Bx)=0$ for each $n$, and the assertion of the lemma follows.

THEOREM 2. *Let the suppositions of Theorem* 1 *be fulfilled and suppose that C is a function from Y into Y such that*

$$\limsup_{|x|\to\infty} \alpha^{-1}(|Cx|)/|x| = \beta < 1.$$

*Suppose further that at least one of the following is satisfied*:

(i) *C is continuous from the Banach space Y into itself and maps bounded subsets of Y into relatively compact subsets of Y*; *or*

(ii) *if $Q_R$ is as in Lemma* 2 (*with the topology induced by X*), *then, for each $R>0$, C is continuous from $Q_R$ into X and the image of $Q_R$ under C is relatively compact in X*.

*Then, for each z in Y, there is a $y_z$ in Y such that $Ay_z-Cy_z=z$.*

INDICATION OF PROOF. Note that we need only show that there is an $x_0$ in $Y$ such that $Ax_0-Cx_0=0$, or equivalently, $B\cdot Cx_0=x_0$ where $B$ is as defined in Theorem 1. Since $\limsup_{|x|\to\infty} |B\cdot Cx|/|x|<1$, let $r_1>0$ be such that $|B\cdot Cx|\leqq|x|$ whenever $|x|\geqq r_1$ and let $r_2=\sup\{|B\cdot Cx|:|x|\leqq r_1\}$. If $R=\max\{r_1, r_2\}$, then $B\cdot C$ maps $Q_R$ into $Q_R$. If (i) holds the theorem follows from the Schauder fixed point theorem, and if (ii) holds the theorem follows from Lemma 2 and the Tychonov fixed point theorem (see e.g. [**5**, Theorem 5, p. 456]).

EXAMPLE 1. Let $X$ be the space $LL^2[0, \infty)$ of all measurable functions $x$ from $[0, \infty)$ into the real numbers such that $q_n(x)=(\int_0^n |x(s)|^2\, ds)^{1/2}<\infty$ for each positive integer $n$. Note that $Y$ is the space $L^2[0, \infty)$ and $|x|=(\int_0^\infty |x(s)|^2\, ds)^{1/2}$ for each $x$ in $Y$. Let $a$ and $b$ be symmetric, measurable, locally integrable functions defined on $[0, \infty)^2$ such that the operator

$Tx(t)=\int_0^t a(t, s)x(s)\,ds$ maps $LL^2[0, \infty)$ continuously into $LL^2[0, \infty)$ and the operator $Sx(t)=\int_0^\infty b(t, s)x(s)$ maps $L^2[0, \infty)$ into $L^2(0, \infty)$. Assume further that (a) $\int_0^n\int_0^n a(t, s)x(s)x(t)\,ds\,dt \geqq 0$ for each $x$ in $LL^2[0, \infty)$ and each $n$; (b) $\int_0^n\int_0^n b(t, s)x(s)x(t)\,ds\,dt \geqq 0$ for each $x$ in $LL^2[0, \infty)$ and each $n$; and (c) $\int_0^\infty\int_0^\infty |b(t, s)|^2\,ds\,dt < \infty$. Define $Ax=-x-Tx-Sx$ for each $x$ in $L^2[0, \infty)$. Then the suppositions of Theorem 1 are fulfilled with $\alpha(s)=-s$ for all $s \geqq 0$ and $A_n x=-x-Tx-S_n x$ where $S_n x(t)=\int_0^n b(t, s)x(s)\,ds$ for each $x$ in $LL^2[0, \infty)$. Suppositions (i) and (iii) of Theorem 1 are easily seen to be true. Supposition (ii) is immediate from (a) and (b) above and the fact that

$$2\int_0^n\left[\int_0^t a(t, s)x(s)\,ds\right]x(t)\,dt = \int_0^n\int_0^n a(t, s)x(s)x(t)\,ds\,dt,$$

which follows from the symmetry of $a$. Since condition (c) above implies $\lim_{p\to\infty}\int_0^p\int_p^\infty |b(t, s)|^2\,ds\,dt=0$, it is easy to see that suppositions (iv) and (v) are fulfilled. Thus, for each $z$ in $L^2[0, \infty)$, there is a unique $x_z$ in $L^2[0, \infty)$ such that

$$x_z(t) + \int_0^t a(t, s)x_z(s)\,ds + \int_0^\infty b(t, s)x_z(s)\,ds = z(t) \tag{5}$$

for almost all $t$ in $[0, \infty)$. Also, $\int_0^\infty |x_z(s)|^2\,ds \leqq \int_0^\infty |z(s)|^2\,ds$. Note the operator defined by the left side of (5) is not necessarily defined on all of $LL^2(0, \infty)$ and need not map $L^2[0, \infty)$ into itself.

EXAMPLE 2. Let $C_c[0, \infty)$ denote the Fréchet space of all continuous functions $x$ from $[0, \infty)$ into the $m$ dimensional space $R^m$ (with $\|\cdot\|$ denoting an appropriate norm on $R^m$) with the topology generated by uniform convergence on compact subsets of $[0, \infty)$. Define $q_n(x)=\max\{\|x(t)\|: 0 \leqq t \leqq n\}$ for each $x$ in $C_c[0, \infty)$ and each positive integer $n$. Then $Y$ is the space $BC[0, \infty)$ of all $x$ in $C_c[0, \infty)$ such that $|x|=\sup\{\|x(t)\|: t \geqq 0\} < \infty$. Now let $f$ and $g$ be continuous functions from $[0, \infty)^2 \times R^m$ into $R^m$ such that (a) $f(t, s, 0)=g(t, s, 0)=0$ for all $(t, s)$ in $[0, \infty)^2$; (b) $f(t, s, \xi)=0$ for all $(t, s, \xi)$ in $[0, \infty)^2 \times R^m$ with $s>t$; and there are continuous functions $\theta$ and $\phi$ from $[0, \infty)^2$ into $[0, \infty)$ such that (c) $\sup\{\int_0^t \theta(t, s)\,ds: t \geqq 0\}=\lambda<1$; (d) $\sup\{\int_0^\infty \phi(t, s)\,ds: t \geqq 0\}=\gamma<1-\lambda$; (e) $\|f(t, s, \xi_1)-f(t, s, \xi_2)\| \leqq \theta(t, s)\|\xi_1-\xi_2\|$ for all $(t, s, \xi_1)$ and $(t, s, \xi_2)$ in $[0, \infty)^2 \times R^m$; and (f) $\|g(t, s, \xi)\| \leqq \phi(t, s)\,\|\xi\|$ for all $(t, s, \xi)$ in $[0, \infty)^2 \times R^m$. If $S$ is the integral operator defined on $C_c[0, \infty)$ by $Sx(t)=\int_0^t f(t, s, x(s))\,ds$, then the suppositions of Theorem 1 are easily seen to be fulfilled with $A=-x+Sx$ for each $x$ in $BC[0, \infty)$, $A_n x=-x+Sx$ for each $x$ in $C_c[0, \infty)$, and $\alpha(s)=(1-\lambda)s$ for each $s \geqq 0$. Let $T$ be the integral operator defined in $BC[0, \infty)$ by $Tx(t)=\int_0^\infty g(t, s, x(s))\,ds$ and suppose that $T$ maps $BC[0, \infty)$ into $BC[0, \infty)$. Note that conditions (d) and (f) above imply that $|Tx| \leqq \gamma |x|$ for all $x$ in $BC[0, \infty)$. Now let $z$ be in $BC[0, \infty)$ and let $Cx = Tx + z$ for each $x$ in $BC[0, \infty)$. If $B$ is as in Theorem 1 (i.e., $A \cdot Bx = x$ for all $x$ in $BC[0, \infty)$), then $BO = 0$ and we have that $|Bx| \leqq (1-\lambda)^{-1}|x|$ for all $x$ in $BC[0, \infty)$. It now follows easily that the operator $B \cdot C$ maps the ball $Q = \{x \in BC[0, \infty) : |x| \leqq |z|(1-\gamma-\lambda)^{-1}\}$ into itself. Thus, if $C$ is completely continuous for the $BC[0, \infty)$ topology on $Q$ or if $C$ is completely continuous for the $C_c[0, \infty)$ topology on $Q$ (note that this is the case if

$$\lim_{p \to \infty} \left[ \sup \left\{ \int_p^\infty \phi(t, s)\, ds : 0 \leqq t \leqq p \right\} \right] = 0,$$

where $\phi$ is as in (d)), then there is an $x_z$ in $BC[0, \infty)$ with

$$|x_z| \leqq |z|\,(1 - \gamma - \lambda)^{-1}$$

such that

$$x_z(t) - \int_0^t f(t, s, x_z(s))\, ds = \int_0^\infty g(t, s, x_z(s))\, ds + z(t) \tag{6}$$

for all $t$ in $[0, \infty)$. Furthermore, since $|B \cdot Cx| < |x|$ if $|x| > |z|(1-\gamma-\lambda)^{-1}$, all solutions $x_z$ to (6) satisfy $|x_z| \leqq |z|(1-\gamma-\lambda)^{-1}$. In particular $|x_z| \to 0$ as $|z| \to 0$, so we have a type of stability criteria for the zero solution of equation (6) when $z(t) = 0$ for all $t \geqq 0$.

REMARK. In Example 2, we need only assume that the inequalities (e) and (f) hold in some neighborhood of the origin in $R^m$. This follows from the fact that if $r > 0$ and $h$ is a function from $Q_r = \{\xi \in R^m : \|\xi\| \leqq r\}$ into $R^m$ such that $\|h(\xi_1) - h(\xi_2)\| \leqq k\|\xi_1 - \xi_2\|$ for all $\xi_1$ and $\xi_2$ in $Q_r$, then there is a function $h^*$ from $R^m$ into $R^m$ such that $h^*(\xi) = h(\xi)$ for all $\xi$ in $Q_r$ and $\|h^*(\xi_1) - h^*(\xi_2)\| \leqq k^*\|\xi_1 - \xi_2\|$ for all $\xi_1$ and $\xi_2$ in $R^m$ where $k \leqq k^* \leqq 2k$ (take $h^*(\xi) = h(r\xi/\|\xi\|)$ if $\|\xi\| > r$). Note that $k^* = k$ if $\|\cdot\|$ is the Euclidean norm on $R^m$. Thus the stability and existence criteria established in Example 2 give some improvements to those of Miller, Nohel and Wong [9].

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