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STRONGLY NONLINEAR WAVES

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1. INTRODUCTION

As steep waves have recently come to be described with increasing accuracy, a number of unexpected physical and mathematical phenomena have been revealed. Until ten years ago it had been assumed that accurate solutions for high waves would hold few surprises. Examples of such suppositions are that deep-water solutions would converge for all waves short of the highest, that important integral quantities such as speed, energy, and momentum would increase with wave height until the highest is reached, that the solutions for periodic waves would be unique, and that if one solitary wave overtakes another any change of wave height would be a decrease. It is now known that all these suppositions are false, having been disproved in the last decade. The nonlinearity of the describing equations produces a complexity of solution structure that is only now beginning to be appreciated.

This review will deal with effectively exact solutions for nonlinear waves and the phenomena revealed by such solutions. The governing equation within the fluid is taken to be Laplace's equation, corresponding to irrotational flow of an incompressible fluid. Excluded are the physical effects of viscosity, density gradients, compressibility, and rotation. This model of the flow is the simplest, but one which is an excellent approximation in many cases of wave motion, and is the traditional avenue of approach to most problems of fluid flow. Throughout this review, however, the problems and solutions described are those where the complete nonlinear boundary conditions have been included. It has been the nonlinearity of these conditions which has made the accurate solution of water-wave problems so difficult.

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The fluid is assumed to be inviscid and the flow irrotational, such that the velocity $\mathbf{q}$ may be expressed as the gradient of a potential ϕ, $\mathbf{q} = \nabla\phi$. If the fluid is assumed to be incompressible, such that $\nabla\cdot\mathbf{q} = 0$, the equation that holds throughout the fluid is Laplace's equation

$$\nabla^2\phi = \phi_{xx} + \phi_{yy} + \phi_{zz} = 0\,, \tag{1.1}$$

where the subscripts denote partial differentiation. The x and z coordinates are taken to be in a horizontal plane, the y axis vertically upwards. If the fluid is partly bounded by solid boundaries that are free to move such that $y = h(x, z, t)$ on the boundary, it can be shown that the condition that no fluid pass through the boundary is

$$h_t = \phi_y - \phi_x h_x - \phi_z h_z$$

on $y = h$. In many situations the boundary can be taken as stationary, $h_t = 0$, and horizontal, $h_x = h_z = 0$, so that the boundary condition becomes

$$\phi_y(x, h, z, t) = 0\,. \tag{1.2}$$

The free-surface boundary conditions are to be satisfied on $y = \eta\,(x, z, t)$, which is also unknown. The kinematic requirement that a particle on the surface remain on it is expressed by

$$\frac{D}{Dt}(y - \eta) = -\eta_t - \phi_x\eta_x - \phi_z\eta_z + \phi_y = 0\,, \tag{1.3}$$

on $y = \eta$. The dynamic boundary condition can be written

$$\phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + g\eta + p_s/\rho + T(1/R_1 + 1/R_2) = C(t) \tag{1.4}$$

on $y = \eta$, where g is gravitational acceleration, p_s is the pressure at the surface, ρ is fluid density, T is surface tension, R_1 and R_2 are principal radii of surface curvature, and $C(t)$ is a function only of time. In many situations, and throughout the rest of this review, air motion above the surface is neglected and the surface pressure taken to be a constant. The equations may be simplified in any or all of the cases of (*a*) steady flow, $\partial/\partial t \equiv 0$, (*b*) two-dimensional flow, $\partial/\partial z \equiv 0$, and (*c*) surface tension relatively unimportant, T set to zero. However, in all these physical simplifications, nonlinear terms remain in the free-surface boundary conditions.

2. THE CANONICAL PROBLEM : STEADY WAVES

The problem of a periodic train of waves propagating without change of form allows a considerable simplication by the addition of a suitable horizontal velocity to the reference frame, so that the fluid motion may be made steady and all time dependence and time derivatives vanish from

(1.1–1.4). By considering a coordinate frame in which one axis is parallel to the direction of propagation the problem is made two-dimensional. Surface tension will be neglected in this section, and its inclusion described in Section 3. The problem now formulated, the two-dimensional steady periodic gravity wave, is the simplest of all, and has often been the avenue by which solutions to more difficult and general problems have been approached. Despite its relative simplicity, it contains the full nonlinearity of the surface boundary conditions and has succumbed to accurate solution only in the last decade, during which several interesting phenomena have been discovered.

The existence of solutions to this problem has been studied by several authors. Krasovskii (1960) has devised the most significant proof. A recent discussion of his work has been given by Keady & Norbury (1978), who also established existence of a set of solutions of which his are a subset. Toland (1978) has shown that a solution exists for a wave of greatest height, and that this wave is the uniform limit of waves of almost extreme form.

A number of general theorems for the motions due to steady wave trains have been established, a summary of which is given in Wehausen & Laitone (1960, Section 32). These include theorems on the decrease of motion with depth, as well as relations between energy and momentum integrals. Longuet-Higgins (1974, 1975) has established a number of relations between the integral quantities of a wave train, and these have been generalized by Cokelet (1977a) to allow for the wave train moving at arbitrary speed relative to the frame of reference.

A steady wave train can be uniquely specified by three lengths—the peak-to-trough wave height H, the wavelength λ, and the mean depth D. From these, two independent dimensionless ratios can be formed, so that the steady-wave problem has a two-parameter family of solutions, although recently (see Section 2.2.1) it has been shown that for waves near the highest there may be more than one solution. Special limiting cases of the steady-wave problem are (*a*) $D/\lambda \to \infty$, λ and H finite, called the deep-water wave, and (*b*) $D/\lambda \to 0$, D and H finite, called the solitary wave.

2.1 *Solution by Perturbation Expansion Methods*

2.1.1 STOKES EXPANSIONS Stokes (1847) first used a systematic perturbation technique to solve the steady-wave problem. He assumed that the free-surface elevation may be represented by an infinite Fourier series

$$\eta(x) = a_1 \cos x + a_2 \cos 2x + \dots \tag{2.1a}$$

and that the velocity potential may be similarly represented by

$$\phi(x, y) = cx + b_1 \cosh (y + y_0) \sin x + b_2 \cosh 2(y + y_0) \sin 2x + \dots \tag{2.1b}$$

where the problem has been made dimensionless by referring lengths to $\lambda/2\pi$ and velocities to $c_0 = (g\lambda/2\pi)^{1/2}$, the speed of an infinitesimal wave in deep water. The coefficient c and the a_n and b_n are functions of the water depth. In some situations y_0 has been assumed to be D, while in inverse formulations of the problem (see Section 2.2.1) it is more convenient to let y_0, the height of the origin above the bottom, be $d = Q/c$ where Q is the area flow rate under the stationary wave. The leading coefficient c is the phase speed of the wave train in another frame through which the waves propagate such that the time-mean fluid velocity at all points is zero. The coefficients c, a_n, and b_n are assumed to be power-series expansions in a_1, such that the leading orders of a_n and $b_n \sim O(a_1^n)$. When these series are substituted into (2.1a and b), which are then substituted into the steady free-surface conditions, an ordered set of equations is obtained from which the coefficients in the power series can be found successively. The complexity of the manipulations makes a manual high-order calculation impractical. Fifth-order solutions have been obtained by De (1955), Chappelear (1961), who claimed mistakes in De's solution, and by Skjelbreia & Hendrickson (1961).

If the inverse problem is studied, where ϕ and ψ rather than x and y are the independent variables, the calculations can be greatly reduced. The free surface becomes a known boundary, $\psi = 0$.

Stokes (1880) computed the deep-water case to $O(a_1^5)$ and found results for finite depth to $O(a_1^3)$. In the most ambitious manual computations using the inverse method, Wilton (1914) carried the infinite depth calculation to $O(a_1^{10})$ (but has errors at the eighth order) and De (1955) has published a fifth-order solution for general depth.

In order to reveal details of highly nonlinear waves by the series method, solutions of much higher order must be obtained. Schwartz (1972, 1974) simplified the formalism of the inverse method by using complex functions. Each wave cycle in the physical $z = x + iy$ plane is mapped onto the interior of an annulus in the ζ plane, where $\zeta = \exp(-if/c)$ where f is the complex potential $f = \phi + i\psi$. The mapping function is an infinite polynomial in ζ with coefficients a_n, each being taken to be a power series in ϵ, a parameter that is zero for the undisturbed stream and assumed to increase monotonically with wave height. By substituting into the dynamic boundary condition, a set of recurrence relations is found, and the coefficients in the power series determined successively by computer. The system of equations is closed by defining ϵ. Choosing $\epsilon = a_1$ reproduces the procedure of Stokes (1880).

Stokes (1847) showed that the highest wave, assumed to be sharp-crested, must have an included angle of 120° at the crest. He conjectured that this limiting wave would correspond to the critical value of a_1 in the series expansion. In fact this is not the case, for a_1 is not a monotonic function of wave height but achieves its maximum value before the highest wave is attained, about 10% before for the deep-water wave. To surmount this difficulty Schwartz introduced the wave height as a new parameter, in fact, $\epsilon = H/2$. He found that all the coefficients in the expansions reached maxima before the highest wave is achieved. Thus when a_1 is used as the independent parameter, the maximum of a_1 as a function of H becomes a square-root singularity, which limits the convergence of Stokes' expansion in a_1. Schwartz subsequently used series-analysis techniques, including Padé approximants and Domb-Sykes plots, to estimate the limiting wave height in deep water. He found this to be $(H/\lambda)_{max} = 0.1412$. From the accurate results of the enhanced series, Schwartz showed that the mass of the wave has a maximum in H. This has important implications for other integral quantities of the wave train such as energy and momentum.

Longuet-Higgins (1975) used Schwartz's program for the deep-water wave, recomputed to 32 decimal places by Cokelet, and re-expressed the series in terms of the parameter $\omega = 1 - (q_c q_t/cc_0)^2$, where q_c is the fluid speed at the crest and q_t that at the trough. The parameter ω has the useful property that its range is known ab initio, undisturbed flow corresponding to $\omega = 0$ and the limiting wave to $\omega = 1$. Longuet-Higgins found maxima in each of the integral quantities : wave speed, momentum, and potential and kinetic energy, *before* the highest wave was attained, as had been found by Longuet-Higgins & Fenton (1974) for the solitary wave. The variation of the integral quantities with wave steepness H/λ is shown in Figure 1 for the deep-water case. It is clear that the highest wave is not

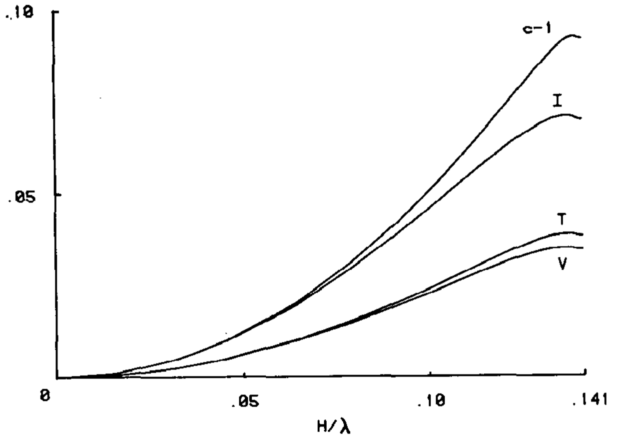

Figure 1 Dimensionless wave speed, impulse and kinetic and potential energies versus wave height for deep-water progressive gravity waves.

the fastest nor the most impulsive nor the most energetic. The physical implications of this are not well understood, but it may be responsible for the instability of the crest of high waves (see Section 4.1), for the multiplicity of solutions (see Section 2.2.1) and be relevant to the observed intermittency of spilling breakers (Longuet-Higgins & Turner 1974). As the maxima occur for high waves, when surface tension and viscosity would also be important, especially on a laboratory scale, experimental confirmation would be difficult.

Cokelet (1977a) used a method very similar to that of Schwartz, and for a wide range of finite water depths produced a number of accurate results for integral quantities of the wave train. Each showed a maximum before the highest wave was reached. Finite-depth results for wave speed are displayed on Figure 2, using data from Schwartz (1972), Cokelet (1977a), and Longuet-Higgins & Fenton (1974).

2.1.2. SHALLOW-WATER EXPANSIONS It has been shown by Ursell (1953) that the linear theory of periodic waves (the first term in Stokes' expansion) is valid only if the shallowness parameter $H\lambda^2/D^3$ as well as the wave steepness H/λ, is small. That is, the waves must not be too long relative to the water depth. This is shown by the results given by Schwartz (1974), where the radii of convergence of the Stokes expansions become smaller for longer waves. Analytical theories show the dependence on wavelength more explicitly; it can be shown that whereas the nominal Stokes expansion parameter is a_1k, where k is the wavenumber $2\pi/\lambda$, the ratios of successive terms in the expansion actually behave like $a_1k/\sinh^3 kD$. For shallow water

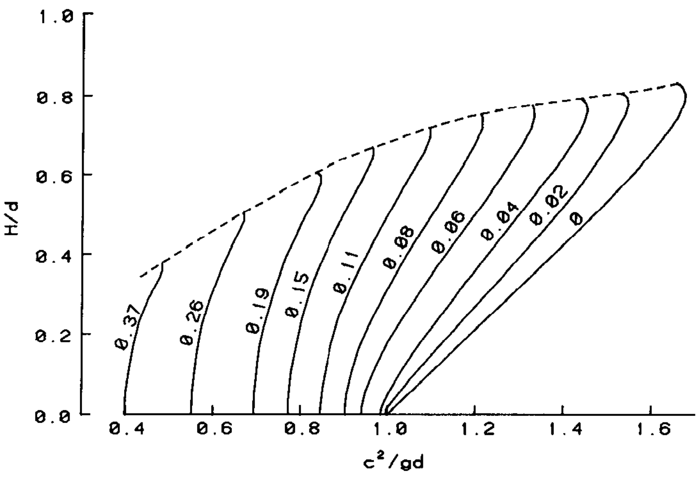

Figure 2 Wave speed versus amplitude and depth for progressive gravity waves. Numbers on curves are values of d/λ.--sharp-crested waves.

kD becomes small so that the effective expansion parameter varies like $a_1k/(kD)^3$, proportional to Ursell's parameter.

A theory for waves in shallow water was put forward by Korteweg & de Vries (1895), who showed that to first order, the free surface of steady waves is of the form of a Jacobian elliptic *cn* function squared, giving rise to the term "cnoidal" waves which show the long flat troughs and short crests characteristic of waves in shallow water. Higher-order cnoidal wave theories have been obtained by Laitone (1960) and Chappelear (1962) to second and third order respectively. A systematic method using power-series expansions in terms of shallowness, $(D/\lambda)^2$, was used by Fenton (1979), who used computer manipulation of the long series to produce a ninth-order solution. It was shown that the most appropriate depth scale is h, the water depth under the troughs, and that the natural expansion parameter when the series are recast in terms of wave height is H/mh, where m is the parameter of the elliptic functions. For the long-wavelength limit of the solitary wave $m \rightarrow 1$; however, for shorter waves m becomes smaller and the expansion parameter larger, complementary to the manner in which the $\sinh^3 kD$ denominator in Stokes' expansion invalidates their application to shallow water. The cnoidal-wave solutions were found to be not accurate for very high waves, for reasons associated with the maxima of integral quantities as a function of wave height. While the cnoidal-wave results should be used instead of Stokes's wave solutions in shallow water, for physical applications where accurate solutions are necessary, both have been somewhat superseded by the numerical method of Rienecker & Fenton (1981), described in Section 2.2.2.

2.1.3 SOLITARY WAVE The solitary wave is a steady wave of infinite wavelength. A recent specialized article on solitary waves has been written by Miles (1980), so that the treatment here will be brief. The only high-order series results are those presented by Fenton (1972) to ninth order, and by Longuet-Higgins & Fenton (1974) to fourteenth order. It was found that the series in terms of wave height H/h did not give accurate solutions for very high waves. The series was recast in terms of the parameter $\omega = 1 - q_c^2/gh$, where ω has the known range (0,1). Using these series, Padé methods were implemented, and convergent results were obtained for all wave heights. It was found that the integral quantities of the wave all showed maxima as functions of wave height.

The other approach has been through an integral-equation formulation, the history of which is given by Miles (1980). Accuracy of the alternative equations and numerical methods has usually been measured by their ability to describe the wave of greatest height. Some of the more notable results for the maximum height $(H/h)_{max}$ are 0.827 ± 0.008 (Yamada 1957b), 0.827 (Lenau 1966), 0.8262 (Yamada et al. 1968), and the result of 0.827 from Longuet-Higgins & Fenton (1974). All these results seem to support one another. However, Witting & Bergin, in an unpublished work mentioned by Witting (1975), obtained a value of 0.8332, precisely the result obtained by Fox (1977) in an unpublished dissertation. This agreement is rather striking, despite the use of some extrapolation in both cases. Finally, it should be noted that Witting (1975) has suggested that the method developed by Fenton (1972), on which the high-order series results are based, is defective in that the assumed expansions are incomplete.

2.2. *Solution by Numerical Methods*

Recent numerical solutions of the steady-wave problem are as accurate as the series results, are often easier to implement computationally, and generally do not need the convergence improvement techniques of the series. They do suffer from the usual disadvantage of numerical solutions in that they reveal less about the nature of the problem and its solution. In the case of high waves, however, so much numerical smoothing and extending of the series solutions is necessary that they too suffer from this disadvantage. Numerical methods can be divided into two categories depending on whether the authors choose to solve the problem in the physical (x,y) plane or the inverse (ϕ,ψ) plane.

2.2.1 INVERSE PLANE METHODS In this case the problem is to be solved on a region which is known a priori. This huge advantage is offset by the fact that as the sharp-crested wave is approached, the singularities near the crest are stronger in the inverse plane. Also, the calculation of local quantities such as pressure and velocity as functions of position becomes a separate problem to be solved subsequently if the inverse plane is used.

Schwartz & Vanden-Broeck (1979) used an algorithm that is typical of inverse plane methods to solve the capillary-gravity wave problem on infinitely-deep fluid and that was generalized to the finite-depth case in Vanden-Broeck & Schwartz (1979). One wavelength of the flow was mapped onto an annulus, the dynamic boundary condition becoming a nonlinear differential equation for x and y on the unit circle. To satisfy Laplace's equation within the annulus, a Cauchy integral was written, valid on the unit circle and satisfying the bottom boundary condition identically. The equations were approximated by finite-difference expressions to give a system of nonlinear algebraic equations which were solved by Newton's method. The spacing of computational points at equal intervals of velocity potential was found to work well for capillary-gravity waves where the points tended to be clustered around the narrow troughs where velocities were greatest. For pure gravity waves, on the other hand, the sparse spacing occurred in the vicinity of the sharp crests and produced poor results for waves near the highest. This was overcome by a simple transformation which could be used to cluster points near the crest. The results given for some wave speeds and energies are probably the most accurate to date.

A remarkable result has recently been found by Chen & Saffman (1980a), providing convincing evidence that solutions for permanent gravity waves of finite amplitude are not unique when they are sufficiently high! They formulated the deep-water steady-wave problem as a nonlinear integro-differential equation, and approximated it by finite-difference methods to give a system of nonlinear algebraic equations which were solved by Newton's method. Having noted the existence of the "premature" maxima of the various integral properties of periodic waves, and its possible analogy with the analytic structure of the capillary-gravity wave problem, where multiple solutions were known to exist, they carefully monitored the determinant of the Jacobian matrix used in the Newton's method solution. Zeroes were found, identifying bifurcation points in the solutions, and the separate branches were followed. The new families of solutions, found to occur for very steep waves, $H/\lambda \approx 0.13$, corresponded to a doubling and tripling of the fundamental wavelength. In the doubling case, for example, the solution obtained was a train of steep waves in which alternate waves differed slightly in wave height.

2.2.2 PHYSICAL PLANE METHODS These have an important role to play in practical applications where, despite an approximate doubling of the number of unknowns because the free surface is also unknown, it is considerably easier to solve the problem from the beginning in the physical plane. Such solutions include the recent work of Rottman & Olfe (1979) and Rienecker & Fenton (1981). In the first of these, a boundary-integral technique was used to formulate an integro-differential equation, Newton iteration being used to find the vector of surface points. The method worked well for steep gravity waves in that the now well-known speed maximum was found. It is, however, not well suited to its nominal objective, the computation of capillary-gravity waves, since it fails when $\eta(x)$ becomes double-valued.

Rienecker & Fenton (1981) used a method in which the stream function is represented by a truncated Fourier series similar to that of Stokes in (2.1b). However, for a given wave the coefficients of the expansion are found by numerical means, obviating the introduction of general power series with their finite radius of convergence and breakdown in the inappropriate depth limit. The numerical method depends for its accuracy on the ability of a Fourier series to describe the wave train. This approach was originated by Chappelear (1961) and Dean (1965), but the method of Rienecker & Fenton is substantially different in that, for example, the solution method is simpler because Newton's method is used directly, the only approximation is in the truncation of the Fourier series, and the method recognizes that the waves may propagate at speeds determined by quantities such as mass flux. In comparing results for fluid velocity with experimental results, good agreement was found. Very close agreement for all waves including the highest was found between the results for phase speed and those reported by Cokelet (1977a) and Vanden-Broeck & Schwartz (1979).

2.3 *The Highest and Almost-Highest Waves*

2.3.1 THE HIGHEST WAVE A number of attempts have been made to solve the problem of the sharp-crested wave of greatest height, usually incorporating Stokes' discovery that it has an included crest angle of 120°. For a sharp-crested wave with its apex at $z = x + iy = 0$, it follows from the Bernoulli condition that the complex potential $f = \phi + i\psi$ varies like $z^{3/2}$ locally. Thus the complex velocity df/dz varies like $f^{1/3}$ near the crest. If the computation is restricted to deep water, an expansion can be assumed (Michell 1893):

$$\frac{df}{dz} = c(1 - \zeta)^{1/3}(1 + b_1\zeta + b_2\zeta^2 + \ldots), \tag{2.2}$$

where $\zeta = \exp(-if/c)$. Michell substituted this into the dynamic boundary condition and determined the first few coefficients of the expansion to give a result for the limiting steepness of $(H/\lambda)_{max} = 0.142$. The same method was used by Havelock (1919) who obtained a limiting steepness of 0.1418. He also calculated solutions for waves short of the highest by displacing the cube-root singularity above the crest. This technique is apparently defective, however, since Grant (1973) has shown that only square-root singularities are admissible in all cases but the limiting one. He showed that the singularity structure of the highest wave solution is much more complicated than had previously been assumed and that the sharp crest is not a regular singular point. The Stokes singularity is merely the first term in a local expansion about the crest in which irrational exponents occur.

Meanwhile Yamada (1957a) had assumed a solution equivalent to that of Michell, truncated after twelve terms, and obtained a limiting steepness of 0.1412. In a later paper, Yamada & Shiotani (1968), the method was extended to finite depth. McCowan (1894) and Lenau (1966) used comparable techniques for the solitary wave. Schwartz (1972, 1974) analyzed his high-order Stokes series and incorporated the inferred singularity structure in a recast form of the series. The highest wave was graphically indistinguishable from Yamada's and had the same steepness.

Michell's expansion (2.2) has been used again, but with a delightful difference, by Olfe & Rottman (1980). They observed that the nonlinear equation resulting from substitution of a one-term expansion into the dynamic boundary condition has multiple roots, one corresponding to Michell's solution but another real root almost eliminating the fundamental Fourier term so that the series is dominated by a higher harmonic—precisely the behavior found by Chen & Saffman (1980a) for irregular gravity waves. Subsequently, using (2.2) with up to 120 terms, Olfe & Rottman experimented with Newton's method and found other solutions in addition to Michell's, corresponding to every second, third, or fourth crest being sharp, the intermediate ones being lower and more rounded, as found by Chen & Saffman for waves lower than the highest.

Some simple and accurate irrational approximations, not part of systematic schemes, have been found by Longuet-Higgins (1973, 1974) for the highest steady, standing, and solitary waves.

2.3.2 ALMOST-HIGHEST WAVES A local expansion in the vicinity of the crest for waves just short of the highest was devised by Longuet-Higgins & Fox (1977). They found a class of self-similar flows with a length scale of $\ell = q_c^2/2g$, which have a smooth crest and whose free surface oscillates about the Stokes corner flow with a decaying amplitude like $(\ell/r)^{1/2}$, where r is the distance from the Stokes corner. The oscillations cause the maximum inclination of the surface to be greater than 30°, namely 30.37°, a result confirmed from extrapolated numerical results of Sasaki & Murakami (1973) and Byatt-Smith & Longuet-Higgins (1976). In a second paper, Longuet-Higgins & Fox (1978) matched the local-crest solution to a form of Michell's expansion for deep-water waves, valid far from the crest. A small parameter proportional to q_c was introduced, and a number of asymptotic expressions found for the height of the waves, the phase speed and other integral quantities. Unlike almost all other results for nonlinear waves, these are valid in the limit of the highest wave. These expressions have the unusual feature that they show an infinitude of local maxima and minima as the highest wave is approached. The global maximum and the first local minimum of speed and energy was found in the numerical solution of Schwartz & Vanden-Broeck (1979) for deep water and for a particular case of finite depth $D/\lambda = 0.110$. No doubt all finite-depth wave trains, and perhaps the solitary wave, show such behaviour.

Longuet-Higgins & Fox (1978) analytically continued their solution across the free surface in the crest neighborhood. They found a stagnation point above the crest corresponding to the square-root singularity found by Grant (1973) and Schwartz (1972, 1974). The apparent transition to the 2/3-power form for the sharp-crested wave was explained by Grant as the coalescence of several square-root singularities. Longuet-Higgins

(1979a) has subsequently developed a simpler approximation to the crest flow found by himself and Fox. This result, and other approximations, have been used to calculate fluid particle paths (Longuet-Higgins 1979b).

3. OTHER PERIODIC WAVES

There are two other important classes of spatially periodic surface waves that will be discussed here. The emphasis, as before, will be on strongly nonlinear effects.

3.1 *The Inclusion of Surface Tension*

In this case the waves considered are also two-dimensional, periodic, and the flow is steady, but the surface-tension term in (1.4) is retained. For two-dimensional waves, $1/R_2 = 0$, and $1/R_1 = \eta_{xx}/(1 + \eta_x^2)^{3/2}$ a highly nonlinear function of η. The parameter $\kappa = 4\pi^2 T/\rho g \lambda^2$ is used to measure the relative importance of surface tension and gravity. When κ is large, corresponding to short wavelengths, surface tension is the dominant restoring force. For longer waves gravity is most significant; however, capillarity becomes increasingly important as the wave steepens and the crest becomes more sharply rounded.

The problem of pure capillary waves in deep water has been resolved completely by Crapper (1957). In this most remarkable work, he obtained an exact closed-form solution for waves of arbitrary amplitude. More recently, closed-form solutions for finite depth were found by Kinnersley (1976), a result anticipated by Crapper. Unlike pure gravity waves, steep capillary waves are characterized by deep troughs and broad flat crests. The limiting wave occurs when the two sides of the trough meet, enclosing a pendant-shaped bubble. Steeper "waves" can be computed but they are physically impossible since the free surface crosses itself. Recently Vanden-Broeck & Keller (1980) discovered a new family of capillary waves by allowing a nonzero pressure within the enclosed pendant.

Wilton (1915) treated the combined capillary-gravity wave problem by using a Stokes expansion which he carried to fifth-order in amplitude, invoking Stokes' hypothesis that the nth Fourier coefficient is nth order. When the parameter $\kappa = 1/n$ for $n = 2,3, \ldots$, this expansion fails because certain series coefficients become infinite. For the particular case of $\kappa = 1/2$, Wilton was able to find two solutions, each of which he carried to third order, by revoking Stokes' hypothesis and re-ordering the terms in the series.

Recently the numerical methods described in Section 2.2.1 have been applied to the problem (Schwartz & Vanden-Broeck 1979, Chen & Saffman 1980b). Waves of maximum height can be computed without difficulty for most values of κ. In all cases for $\kappa > 0$, the highest waves are topologically limited, in that the surface encloses one or more bubbles, just as for pure capillary waves, and unlike pure gravity waves where the limiting wave is sharp-crested. The critical values of κ, where the Stokes-series solution fails, are indicative of multiple solutions. When $\frac{1}{n+1} < \kappa < \frac{1}{n}$, for moderate steepness the series yields a family of profiles with approximately n inflection points or "dimples." The numerical methods showed that each solution family can be analytically continued outside its "natural" domain. Thus there can be many different wave forms for a given wavelength. Certain of the multiple solutions arise via a finite-amplitude bifurcation from the regular wave train. As $\kappa \to 0$ the number of solution families increases. The structure of this highly singular limit, where surface tension should be important only in the neighborhood of the sharp crest, remains to be explored. A stability analysis for each of the several possible wave forms would be a useful contribution and might resolve the non-uniqueness in the ripple regime. Experimental results are not completely consistent, but Schooley (1960), for example, has published photographs of several multi-dimpled profiles. Analytical work, dealing with the genesis of multiple solutions, has been presented by Pierson & Fife (1961), Nayfeh (1970), and Chen & Saffman (1979).

Hogan (1979) obtained a number of relations between integral quantities of the steady wave train, which for Crapper's pure capillary wave reduced to very simple expressions. He then (Hogan 1980) extended the series technique of Schwartz (1972, 1974) to include surface tension for the deep-water case, and obtained members of the family of solutions corresponding to the gravity wave. For κ sufficiently small, each integral property was found to have a maximum as a function of wave height, but the effect of increasing κ was to move the maximum closer to the highest wave possible, showing how surface tension acts so as to make experimental verification of the maxima more difficult. Beyond a certain value of κ, the maximum wave height became limited by the wave geometry, and no maxima in the integral quantities were found before the limiting bubble was attained.

Figure 3 shows two steep wave profiles for the case $\kappa = 1/2$, taken from Schwartz & Vanden-Broeck (1979). The profile labelled 1 belongs to the family of Crapper's limiting wave and exhibits one trapped bubble per wave cycle; the other wave is from a different family which, while also symmetric, is limited by two bubbles per cycle.

3.2 *Finite-Amplitude Standing Waves*

A time-dependent relative of the steady gravity wave is the two-dimensional time and space periodic standing wave. Physically the problem is that of the periodic "sloshing" or "seiching" of water between two vertical side walls. To first order it corresponds to the reflection of a periodic wave train by a vertical barrier or the interaction of two oppositely propagating wave trains. Unlike the steady wave, no rigorous existence proof for finite amplitudes has been obtained.

In the spirit of Stokes, Rayleigh (1915) solved the problem by a third-order expansion in wave height. Penney & Price (1952) computed a fifth-order solution, and obtained a maximum steepness, for deep water, of 0.22. They also suggested that this highest wave has a 90° included angle at the crest, but the premises in their argument have been questioned by subsequent workers. Experiments of Taylor (1953) and Edge & Walters (1964) confirmed that the crest angle is close to 90°.

Recently Schwartz & Whitney (1977, 1981) have produced a 25th-order solution by a time-dependent conformal mapping method. They found that Penney & Price's procedure is defective in that it produces non-periodic secular time dependence if carried to higher order. This "resonance" may be suppressed by exploiting a degree of arbitrariness in certain of the series coefficients. Very steep waves were found to possess several inflection points near the crest, reminiscent of those for steady waves. The highest wave steepness was found to be about 0.208, with an included angle of about 90°. The oscillating water surface is never flat and has no nodes. The wave frequency, which in general decreases with increasing amplitude, appears to reach a minimum value just short of the limiting steepness.

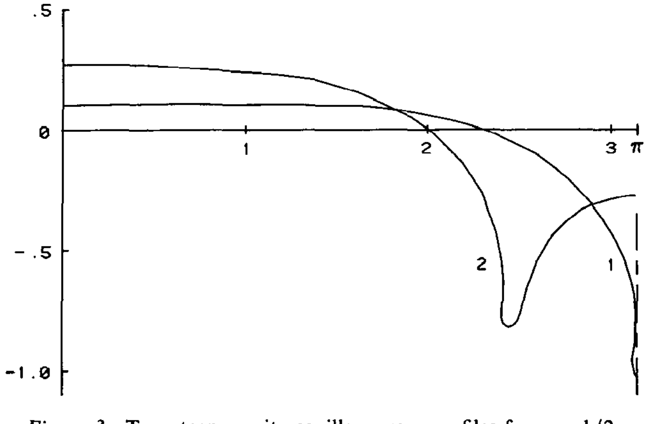

Figure 3 Two steep gravity-capillary wave profiles for $\kappa = 1/2$.

Unlike the deep-water case, shallow-water standing waves show an increase in frequency with amplitude, the critical value of the D/λ ratio at which the first nonlinear correction to the frequency changes sign being 0.17. This was obtained as part of the third-order series solution for finite depth obtained independently by Chabert-d'Hières (1960) and Tadjbakhsh & Keller (1960), and has been experimentally confirmed by Fultz (1962). An interesting feature of the series method is that certain values of D/λ must be excluded so as to suppress secular terms, which may indicate multiple solutions; however, the set of excluded depths becomes infinite as the order of the series solution becomes infinite (Concus 1964). Until this can be explained, the existence of standing-wave solutions for finite amplitude and depth remains in doubt.

A numerical method for finite depth, employing truncated Fourier series to represent space and time variation has been used by Vanden-Broeck & Schwartz (1981). While inappropriate for very steep waves because of computer storage limitations, the results confirmed the series results for moderate steepness and showed that, for some depths, the frequency is not a monotonic function of wave height.

Miche (1944) established that deep-water standing waves have a second-order contribution to the pressure field that is unattenuated with depth and varies with twice the frequency of the surface displacement. Longuet-Higgins (1950) proposed this effect as a likely cause of observed microseismic activity and explained the phenomenon very simply in terms of variation of the potential energy of the water mass. The phenomenon has been demonstrated experimentally by Cooper & Longuet-Higgins (1951) for low waves, but recent calculations by Schwartz (1980) indicate that the leading-order theory used by Longuet-Higgins may overpredict the pressures for very steep waves by as much as 40%.

4. UNSTEADY WAVES

The standing-wave problem described in the previous section is unsteady, but it is periodic in time and space and all flow quantities can be expanded in Fourier series in both variables. In this section, problems, methods and results are described in which the motion is more generally unsteady and where the effects of nonlinearities may bring about irreversible changes. Unlike the problems described in Sections 2 and 3, for which some of the most powerful methods were devised last century, almost all of the methods and results described here have appeared in the last decade.

4.1 *Stability of Steady Waves*

There has been much effort expended on the stability of weakly nonlinear waves, but very little on the stability of steep steady waves. The reason is simple enough—it was not until 1972 that accurate solutions for waves of arbitrary height became available. Longuet-Higgins (1978a, 1978b) examined the stability of steady waves in deep water for all heights, testing the linear stability of the nonlinear wave solutions to small harmonic perturbations. For low waves it was found that they were neutrally stable, but above a certain steepness the waves became unstable to subharmonic perturbations (disturbance wavelength greater than the fundamental). The growth rates obtained agreed quite well with experiment. For higher waves these modes became stable again, but for waves close to the highest, a very fast-growing superharmonic (small wavelength) instability was found. This was associated with the crest, and it seemed to owe its existence to the fact that the wave was steep enough that the first Fourier coefficient of the unperturbed wave had a maximum.

The results of this linear stability analysis were checked by Longuet-Higgins & Cokelet (1978), who followed the evolution of the perturbation by numerically solving the full nonlinear equations, and confirmed the accuracy of the linear analysis. It was found that the local superharmonic instability could quickly lead to overturning at the crest, providing the beginning of insight into the causes of that phenomenon. More remarkably, however, they found that the slow subharmonic instability, if followed for long enough, could also cause an overturning of the crest. Results from the linear theory have had some success in supporting a conjecture of Lake & Yuen (1978) that in very steep wind waves the modulation frequency of the wave amplitude may correspond to the fastest-growing subharmonic instability (Longuet-Higgins 1980d).

4.2 *Breaking Waves*

The plunging or overturning wave is the most important and dramatic of all breaker types and probably is the origin of the spilling breaker. It is one of the most difficult of all wave phenomena to analyze because of the rapidity with which it can occur, the large amplitudes and accelerations involved, and the contortions of the free surface. The origin of plunging in deep water remains unexplained; however, as described in Section 4.1, it has been shown that the wave crest is vulnerable to perturbations and that overturning of the crest is a common outcome even for waves that are considerably lower than the highest.

Some progress has been made in describing the overturning. Several methods are described in Cokelet (1977b), but the only method so far able to describe the overturning wave for most of its duration is that of Longuet-Higgins & Cokelet (1976, 1978), which will be outlined in Section 4.3. In the original 1976 paper they studied an idealized problem of a steady wave in deep water to which an asymmetric pressure distribution was applied for a finite time, after which the wave turned over. Cokelet (1977b) studied an initially sinusoidal wave train with an excess of energy and also found that the crest steepened and overturned. Subsequently Longuet-Higgins & Cokelet (1978) followed the motion of steady waves with a small perturbation and found the overturning as described in Section 4.1. A more detailed picture of the dynamics and kinematics inside a wave as it approaches breaking has been given by Peregrine, Cokelet & McIver (1980). The wave crest before breaking has been found to travel faster than the maximum phase speed for a wave of that length, while the whole front face of the wave had an acceleration several times greater than gravity!

Longuet-Higgins (1980e) has studied simple analytical models of the development of the overturning, and for the evolution of the tip of a plunging wave (Longuet-Higgins 1980b). In a very different study (Longuet-Higgins 1980c) he calculated the angular momentum of steady waves in deep water, and found that the relative persistence of wind-wave crests may be explained by the fact that for a force applied to the wave to have a minimum effect, it should act at about the wave-crest level; this is what actually occurs for both wind forces and drag forces due to whitecapping.

4.3 *Solution of the Unsteady Equations*

An early approach to the numerical solution of the full nonlinear equations (1.1–1.4) was through a marker-and-cell technique. The furthest development of this method is that of Chan & Street (1970) who used it to study the reflection of a solitary wave by a wall. This problem was also considered by Byatt-Smith (1971), who obtained a second-order analytical solution from a pair of nonlinear integro-differential equations. Other methods for the full equations include those of Brennen & Whitney (1970) (summarized in Brennen 1970), Whitney (1970), Multer (1973), and Chan (1975). None, however, seems to have been adopted and exploited further.

In 1976 a method was introduced which seems to have considerable potential. Longuet-Higgins & Cokelet (1976) studied the evolution of waves on water of infinite depth. If the initial free surface and ϕ on it are known, then a Cauchy-type integral equation may be formulated and approximated using discrete computational points on the surface, to give subsequently the velocity of each point normal to the surface as the solution of a matrix equation. The tangential velocity can be found by numerical differentiation. The surface particles are allowed to move a finite distance in a small time step, giving a new surface location. The new values of ϕ may be calculated from the dynamic free-surface condition and the process repeated, with a predictor-corrector method, for a large number of time steps. Longuet-Higgins & Cokelet found that a slow instability developed, but this could be countered by regular smoothing. This method has proved capable of describing the evolution of high waves, including the overturning of the crest (see Sections 4.1 and 4.2). Fenton & Mills (1976) showed how the method may be applied to water of finite depth with arbitrary boundary geometry, but were unable to produce solutions.

In two recent works, the free-surface boundary conditions have been given in different forms. Longuet-Higgins (1978a) expressed them in inverse form with dependent variables x and y as functions of ϕ and ψ. Another technique (Longuet-Higgins 1980a) has been developed in which the usual dynamic boundary condition with $p = 0$ is used, but the kinematic condition is written in terms of the material derivative of pressure, $Dp/Dt = 0$. This has been used (Longuet-Higgins 1980b, 1980e) to find exact solutions which mimic the overturning breaker.

A Fourier method has been developed by Fenton & Rienecker (1980, but described in greater detail in a manuscript submitted for publication). The method is applicable to irrotational flows over arbitrary bed topography and makes use of Fourier approximation throughout to represent horizontal variation. The truncation of Fourier series for ϕ and η, similar to (2.1), is the only approximation. The solution may be advanced in time by a leapfrog scheme, although it is necessary to solve a matrix equation at each time step. The method was found to be stable and accurate in describing solitary wave interactions, the use of Fourier series automating a number of numerical operations as well as facilitating deductions about stability or spectral growth. This makes it well suited to studies of finite-wave interactions, although it loses accuracy if at any stage a wave becomes sharp-crested, and it cannot describe overturning waves because it depends upon the surface elevation's being a single-valued function of x.

4.4 *Solitary Wave Interactions*

The problem of two interacting solitary waves has come to be considered a classical problem of nonlinear waves because it is completely specified by only two parameters (the incident wave heights), and because of the fundamental nature of the first-order equations which describe it, the existence of an exact solution to one of these equations, and the fact that the solution shows that the waves emerge unchanged from the interaction.

4.4.1 OVERTAKING SOLITARY WAVES To first order in wave height and shallowness, the interaction of one solitary wave overtaking another is described by the Korteweg–de Vries (K–de V) equation, for which an exact solution has been obtained (see Miles 1980). The only change after interaction contained in the solution is that the high and fast wave has received a finite phase shift forward and the low wave has been shifted backward. Weidman & Maxworthy (1978) conducted a number of experiments and found generally good agreement between experiment and K–de V theory, but with some consistent differences. Nothing was reported on the waves after interaction; this has been studied numerically using the full nonlinear equations by Fenton & Rienecker in the report mentioned in Section 4.3. For the one interaction studied they found that the waves after interaction, contrary to expectation, had a larger height difference than the incident waves—the high wave had grown slightly higher, at the expense of the lower wave. After interaction the waves propagated almost without further change, and there was no trace of a trailing wave train.

4.4.2. COLLIDING SOLITARY WAVES The other form of interaction, when the waves travel in opposite directions, has received much attention, as described in Miles (1980). In this case the interaction, although brief, is more nonlinear. Nevertheless, to second order in wave height, theory predicts that the waves emerge from the interaction unchanged, with a finite backward phase shift (Oikawa & Yajima 1973, Byatt-Smith 1971). In recent work, a third-order calculation has been made by Su & Mirie (1980). The waves after interaction were shown to have the same height as before, but the profiles are tilted backwards, and each sheds dispersive waves. Fenton & Rienecker (1980, and the above-mentioned report) studied a number of collisions with the numerical method outlined in Section 4.3. They found that both waves were actually degraded by the interaction, but that this change was slightly greater than that for overtaking. Accordingly, they recommend that the adjectives "weak" and "strong" not be used for the colliding and overtaking interactions, as the former has more effect on the waves than the latter. In contradiction of the third-order theory, they found strong evidence that the change of wave height is actually of third order, and that the waves after interaction were travelling faster than before, also noted experimentally by Maxworthy (1976). This change of speed has important implications for the measurements of phase changes due to the interaction, for it means that the change depends on the measuring location, which may explain some unusual features of Maxworthy's results for the phase change. In view of the ambiguity of the spatial phase shift, Fenton & Rienecker recommend use of the temporal shift at the wall, and show that this is considerably underestimated by second- and third-order theories.

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