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Strongly q-Additive Functions and Algebraic Independence

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1. Introduction.

Let $q\geq 2$ be a fixed integer. A complex-valued function $a(n)$ is said to be *q-additive* or *q-multiplicative* if

(1) $$a(kq^t+r)=a(kq^t)+a(r), \qquad a(0)=0\,,$$

or

(2) $$a(kq^t+r)=a(kq^t)a(r)\,, \qquad a(0)=1$$

for any integer $k\geq 0$, $t\geq 0$, and $0\leq r<q^t$, respectively. Furthermore, if

(3) $$a(kq)=a(k)\,,$$

$a(n)$ is said to be *strongly q-additive or strongly q-multiplicative*, respectively. We note that the strongly q-additive or q-multiplicative function $a(n)$ is determined completely by the initial values $a(1), \cdots, a(q-1)$. This paper concerns mainly with q-additive functions.

Let $a_1(n), \cdots, a_m(n)$ be m strongly q-additive functions. For each $a_k(n)$, we define a power series $f_k(z)$ by

(4) $$f_k(z):=\sum_{n\geq 0} a_k(n)z^n \in \mathbf{C}[[z]] \qquad (1\leq k\leq m)\,.$$

It follows from (1) and (3) that each $f_k(z)$ converges in $|z|<1$ and satisfies the functional equation

(5) $$f_k(z)=\frac{1-z^q}{1-z}f_k(z^q)+\frac{1}{1-z^q}\sum_{r=0}^{q-1} a_k(r)z^r \qquad (1\leq k\leq m)\,,$$

since

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$$f_k(z)=\sum_{n\geq 0} a_k(n)z^n=\sum_{t\geq 0}\sum_{r=0}^{q-1}\{a_k(t)+a_k(r)\}z^{tq+r}$$

$$=\sum_{t\geq 0} a_k(t)z^{tq}\sum_{r=0}^{q-1} z^r+\sum_{t\geq 0} z^{tq}\sum_{r=0}^{q-1} a_k(r)z^r$$

$$=\frac{1-z^q}{1-z}f_k(z^q)+\frac{1}{1-z^q}\sum_{r=0}^{q-1} a_k(r)z^r .$$

Putting

$$g_k(z)=(1-z)f_k(z) \qquad (1\leq k\leq m),$$

we have

$$g_k(z)=g_k(z^q)+\frac{1}{\varphi(z)}\sum_{r=0}^{q-1} a_k(r)z^r , \tag{6}$$

where $\varphi(z)=(1-z^q)/(1-z)\in\mathbf{C}[z]$. Because of the functional equations (6), we see by a theorem of Mahler [4] that if the functions $a_1(n), \cdots, a_m(n)$ are algebraic-valued and $g_1(z), \cdots, g_m(z)$ are algebraically independent over $\mathbf{C}(z)$, then the values $g_1(\alpha), \cdots, g_m(\alpha)$ are algebraically independent, where α is an algebraic number with $0<|\alpha|<1$.

An example of a strongly q-additive function is the *sum of digits functions* $s^{(q)}(n)$, that is,

$$s^{(q)}(n)=\sum_{j=0}^{t} b_j ,$$

where

$$n=b_0+b_1q+\cdots+b_tq^t , \tag{7}$$

$$b_t\neq 0 , \quad b_j\in\{0, 1, \cdots, q-1\} \quad (0\leq j\leq t)$$

is the q-adic expansion of an integer $n\geq 0$. Another example is the function $e_i^{(q)}(n)$ defined for each $i\in\{1, 2, \cdots, q-1\}$ by the number of the b_j's $(0\leq j\leq t)$ that are equal to i.

We state our results.

THEOREM 1. *Let* $a_k(n)$ $(1\leq k\leq m)$ *be* m *strongly* q*-additive functions and* $f_k(z)$ *be defined by* (4). *The following statements are equivalent*:

(i) *The functions* $f_1(z), \cdots, f_m(z)$ *are algebraically dependent over* $\mathbf{C}(z)$.

(ii) *The functions* $a_1(n), \cdots, a_m(n)$ *are linearly dependent over* $\mathbf{C}$.

(iii) *The* $(q-1)$*-dimensional vectors* $(a_1(1), \cdots, a_1(q-1)), \cdots, (a_m(1), \cdots, a_m(q-1))$ *are linearly dependent over* $\mathbf{C}$.

Theorem 1 implies that $\{e_1^{(q)}(n), \cdots, e_{q-1}^{(q)}(n)\}$ is a base of the vector space of strongly q-additive functions over $\mathbf{C}$, since $e_i^{(q)}(j)=\delta_{ij}$ $(1\leq i, j\leq q-1)$.

THEOREM 2. *Let α be an algebraic number with $0<|\alpha|<1$. Then the values $\{\sum_{n\geq 0} s^{(q)}(n)\alpha^n\}_{q\geq 2}$ are algebraically independent. For any fixed positive integer i, the values $\{\sum_{n\geq 0} e_i^{(q)}(n)\alpha^n\}_{q>i}$ are algebraically independent.*

COROLLARY. *The functions $\{\sum_{n\geq 0} s^{(q)}(n)z^n\}_{q\geq 2}$ are algebraically independent over $\mathbf{C}(z)$. For any fixed positive integer i, the functions $\{\sum_{n\geq 0} e_i^{(q)}(n)z^n\}_{q>i}$ are algebraically independent over $\mathbf{C}(z)$.*

In Theorem 2 and Corollary, if i is not fixed, then the functions $\{\sum_{n\geq 0} e_i^{(q)}(n)z^n\}_{q>i}$ are algebraically dependent over $\mathbf{C}(z)$. In fact, the functions

$$\left\{\sum_{n\geq 0} e_i^{(q)}(n)z^n\right\}_{(q,i)=(2,1),(4,1),(4,2),(4,3)}$$

are algebraically dependent over $\mathbf{C}(z)$, since these functions are strongly 4-additive and the dimension of the vector space of the strongly 4-additive functions is 3.

THEOREM 3. *If a function $a(n)$ is strongly q_1-additive and also strongly q_2-additive, where $\log q_1/\log q_2$ is not a rational number, then $a(n)$ must be identically zero.*

THEOREM 4. *If a function $a(n)$ is strongly q_1-multiplicative and also strongly q_2-multiplicative, where $\log q_1/\log q_2$ is not a rational number, then $a(n)=0$ $(n\geq 1)$ or $a(n)=\gamma^n$ $(n\geq 1)$, where $\gamma^{q_1-1}=\gamma^{q_2-1}=1$.*

2. Lemmas.

LEMMA 1 (Kubota [1], Loxton and van der Poorten [2], see Nishioka [6]). *Suppose $f_{i,j}(z)\in\mathbf{C}[[z]]$ $(1\leq i\leq n,\ 1\leq j\leq M_i)$ satisfy the functional equations*

$$f_{i,j}(z^d)=a_i f_{i,j}(z)+b_{i,j}(z) \qquad (1\leq i\leq n,\ 1\leq j\leq m_i)\,,$$

where $a_i\in\mathbf{C}$, $a_i\neq a_j$ $(i\neq j)$, and $b_{i,j}(z)\in\mathbf{C}(z)$. If $f_{i,j}(z)$ $(1\leq i\leq n,\ 1\leq j\leq M_i)$ are algebraically dependent over $\mathbf{C}(z)$, then there exists an integer i_0 $(1\leq i_0\leq n)$ and $c_1,\cdots,c_{M_{i_0}}\in\mathbf{C}$, not all zero, such that

$$c_1 f_{i_0,1}(z)+\cdots+c_{M_{i_0}} f_{i_0,M_{i_0}}(z)\in\mathbf{C}(z)\,.$$

LEMMA 2 (Toshimitsu [7]). *Let N be a positive integer and put $\varphi(z)=(1-z^q)/(1-z)$. If a polynomial $P(z)\in\mathbf{C}[z]$ with $\deg P(z)\leq N(q-1)$ and a rational function $c(z)\in\mathbf{C}(z)$ satisfy*

$$c(z)+\frac{P(z)}{\varphi(z)^N}=c(z^q)\,,$$

then $c(z)\in\mathbf{C}$ and $P(z)$ is identically zero.

LEMMA 3 (Nishioka [5]). *Let $d_1,\cdots,d_t$ be integers greater than 1 and suppose that $\log d_i/\log d_j\notin\mathbf{Q}$ if $i\neq j$. Let K be an algebraic number field. Assume that $f_{i,j}(z)\in K[[z]]$ $(1\leq i\leq t,\ 1\leq j\leq M_i)$ satisfy the functional equations*

$$f_{i,j}(z^{d_i})=a_{i,j}(z)f_{i,j}(z)+b_{i,j}(z) \qquad (1\le i\le t,\ 1\le j\le M_i)\,,$$

where $a_{i,j}(z)$, $b_{i,j}(z)\in K(z)$, $a_{i,j}(0)=1$, *and* $f_{i,1}(z),\cdots,f_{i,M_i}(z)$ *are algebraically independent over* $K(z)$ *for each* i $(1\le i\le t)$. *If* α *is an algebraic number with* $0<|\alpha|<1$, $a_{i,j}(\alpha^{d^k})\ne 0$ $(k\ge 0)$ *and all* $f_{i,j}(z)$ *converge at* α, *then the values*

$$f_{i,j}(\alpha) \qquad (1\le i\le t,\ 1\le j\le M_i)$$

are algebraically independent.

3. Proof of the theorems.

PROOF OF THEOREM 1. (ii)→(i) is trivial. We prove (i)→(iii). Assume that $f_1(z),\cdots,f_m(z)$ are algebraically dependent over $\mathbf{C}(z)$. Then $g_1(z),\cdots,g_m(z)$ are algebraically dependent over $\mathbf{C}(z)$. By the functional equation (6) and Lemma 1 there exist complex numbers c_j $(1\le j\le m)$, not all zero, such that

$$c_1g_1(z)+\cdots+c_mg_m(z)=c(z)\in\mathbf{C}(z)\,. \tag{8}$$

We have by (6) and (8)

$$c(z)=c(z^q)+\frac{1}{\varphi(z)}P(z)\,,$$

where $P(z)=\sum_{j=1}^{m}c_j\sum_{r=0}^{q-1}a_j(r)z^r$ is a polynomial of $\deg P(z)\le q-1$. Using Lemma 2, we see that $P(z)$ is identically zero. Hence for any r with $1\le r\le q-1$, $\sum_{j=1}^{m}c_ja_j(r)=0$, which implies the third statement.

Finally we prove (iii)→(ii). By (1), (3) and (7) we get

$$a_j(n)=a_j(b_0)+a_j(b_1)+\cdots+a_j(b_t)\,, \qquad b_j\in\{0,1,\cdots,q-1\} \tag{9}$$

for each function $a_j(n)$ $(1\le j\le m)$. By the assumption,

$$\sum_{j=1}^{m}c_ja_j(r)=0 \qquad (1\le r\le q-1)\,, \tag{10}$$

where $c_j\in\mathbf{C}$ $(1\le j\le q-1)$ are not all zero. The second statement follows from (9) and (10).

PROOF OF THEOREM 2. Define a subset of the positive integers by

$$D=\{d\in\mathbf{N}\mid d\ne a^n \text{ for any } a,n\in\mathbf{N},n\ge 2\}\,.$$

Then we have

$$\mathbf{N}\setminus\{1\}=\bigcup_{d\in D}\{d,d^2,\cdots\}=\{d^j\in\mathbf{N}\mid d\in D,j\ge 1\}\,.$$

Here we note that if $d,d'\in D$ are distinct, then $\log d/\log d'\notin\mathbf{Q}$. We prove the first assertion of the theorem. By Lemma 3, it is enough to prove the algebraic independence of the functions $\{f_{d^j}(z)=\sum_{n\geq 0} s^{(d^j)}(n)z^n\}_{1\leq j\leq h}$ over $\mathbf{C}(z)$ for a fixed $d\in D$. Putting $H=h!$, we see that all the functions $s^{(d^j)}(n)$ $(1\leq j\leq h)$ are strongly d^H-additive. Assume that $f_d(z), \cdots, f_{d^h}(z)$ are algebraically dependent over $\mathbf{C}(z)$. Then by Theorem 1, there exist complex numbers c_i $(1\leq i\leq h)$, not all zero, such that

$$c_1 s^{(d)}(n)+c_2 s^{(d^2)}(n)+\cdots+c_h s^{(d^h)}(n)=0 \qquad (n=1, 2, \cdots). \tag{11}$$

Let h_0 be the smallest index with $c_{h_0}\neq 0$. Substituting $n=1$ in (11), we have $c_{h_0}+\cdots+c_h=0$, since $s^{(d^{h_0})}(1)=\cdots=s^{(d^h)}(1)=1$. Putting $n=d^{h_0}$ in (11), we get

$$c_{h_0}+d^{h_0}c_{h_0+1}+\cdots+d^{h_0}c_h=0 .$$

Hence $c_{h_0}=0$, a contradiction.

We prove the second assertion of the theorem. We may prove the algebraic independence of the functions $\{g_{d^j}(z)=\sum_{n\geq 0} e_i^{(d^j)}(n)z^n\}_{\log_d i<j\leq h}$. By the same argument as above, we have

$$c_{h_0}e_i^{(d^{h_0})}(n)+\cdots+c_h e_i^{(d^h)}(n)=0 \qquad (n=1, 2, \cdots), \tag{12}$$

where i $(1\leq i<d^{h_0})$ is fixed and $c_{h_0}\neq 0$. Putting $n=id^{h_0}$ in (12), we get $c_{h_0}=0$, since $e_i^{(d^{h_0})}(id^{h_0})=1$, $e_i^{(d^{h_0+1})}(id^{h_0})=\cdots=e_i^{(d^h)}(id^{h_0})=0$; which is a contradiction and the proof is completed.

Proof of Theorem 3. First we assume that $a(n)$ takes algebraic-values and $a(n)$ is not identically zero. Since $a(n)$ is determined completely by $a(1), \cdots, a(q_1-1)$, we have $\{a(n)\}_{n\geq 0}\subset K$ for some algebraic number field K. Putting

$$f(z)=(1-z)\sum_{n\geq 0} a(n)z^n ,$$

we see that $f(z)$ is transcendental over $\mathbf{C}(z)$ by Theorem 1. Since $a(n)$ is strongly q_1-additive and strongly q_2-additive, we have

$$f(z)=f_i(z)=f_i(z^{q_i})+\frac{1-z}{1-z^{q_i}}\sum_{r=0}^{q_i-1} a(r)z^r \qquad (i=1, 2)$$

by (5). Applying Lemma 3, we see that the values $f_1(\alpha)$ and $f_2(\alpha)$ are algebraically independent for an algebraic number α with $0<|\alpha|<1$, which contradicts $f_1(z)=f(z)=f_2(z)$.

Next we treat general case. Suppose $a(n)$ is not identically zero. Then we have $a(r_0)\neq 0$ for some r_0 $(1\leq r_0\leq q_1-1)$. Let $s_1, \cdots, s_{q_1-1}, t_1, \cdots, t_{q_2-1}, u$ be variables. Let $P(s_1, \cdots, s_{q_1-1}; n)$ be a strongly q_1-additive function satisfying $P(s_1, \cdots, s_{q_1-1}; r)=s_r$ for $1\leq r\leq q_1-1$ and $Q(t_1, \cdots, t_{q_2-1}; n)$ be a strongly q_2-additive function satisfying $Q(t_1, \cdots, t_{q_2-1}; r)=t_r$ for $1\leq r\leq q_2-1$. For each non-negative integer n, $P(s_1, \cdots, s_{q_1-1}; n)$ and $Q(t_1, \cdots, t_{q_2-1}; n)$ are polynomials in $\mathbf{Q}[s_1, \cdots, s_{q_1-1}, t_1, \cdots, t_{q_2-1}]$. Let I be an ideal of $\mathbf{Q}[s_1, \cdots, s_{q_1-1}, t_1, \cdots, t_{q_2-1}, u]$ which is generated by

(13) $$\{P(s_1, \cdots, s_{q_1-1}; n) - Q(t_1, \cdots, t_{q_2-1}; n)\}_{n\geq 1} \cup \{s_{r_0}u-1\}.$$

Since $(a(1), \cdots, a(q_1-1), a(1), \cdots, a(q_2-1), a(r_0)^{-1}) \in \mathbf{C}^{q_1+q_2-1}$ is a zero of the ideal I, there exists an algebraic zero $(b_1, \cdots, b_{q_1-1}, c_1, \cdots, c_{q_2-1}, d) \in \bar{\mathbf{Q}}^{q_1+q_2-1}$ of I. We note that $b_{r_0} \neq 0$ by $b_{r_0}d=1$. Let $b(n)$ be a strongly q_1-additive function satisfying $b(r)=b_r$ $(1 \leq r \leq q_1-1)$ and $c(n)$ be a strongly q_2-additive function satisfying $c(r)=c_r$ $(1 \leq r \leq q_2-1)$. Noting

$$P(b_1, \cdots, b_{q_1-1}; n) = b(n), \qquad Q(c_1, \cdots, c_{q_2-1}; n) = c(n),$$

and (13), we have $b(n)=c(n)$ $(n \geq 1)$. Hence $b(n)$ is an algebraic-valued q_1-additive and q_2-additive function and so $b(n)$ must be identically zero. This contradicts $b_{r_0} \neq 0$ and the proof is completed.

PROOF OF THEOREM 4. Putting $f(z)=\sum_{n\geq 0} a(n)z^n$, we have

$$f(z) = \prod_{k\geq 0} (1 + a(1)z^{q_i^k} + \cdots + a(q_i-1)z^{(q_i-1)q_i^k}) \qquad (i=1, 2),$$

$$f(z^{q_i}) = \frac{f(z)}{1 + a(1)z + \cdots + a(q_i-1)z^{q_i-1}} \qquad (i=1, 2)$$

by (2) and (3). First we assume that $a(n)$ takes algebraic-values. If $f(z)$ is transcendental over $\mathbf{C}(z)$, we get a contradiction by the same argument as in the proof of Theorem 3. Hence $f(z)$ is algebraic over $\mathbf{C}(z)$. By Example 1.3.1 in Nishioka [6], we have $a(n)=0$ $(n \geq 1)$ or $a(n)=\gamma^n$ $(n \geq 1)$, where $\gamma^{q_1-1}=\gamma^{q_2-1}=1$.

Next we treat general case. Define polynomials $P(s_1, \cdots, s_{q_1-1}; n)$, $Q(t_1, \cdots, t_{q_2-1}; n)$ in the same way as in the proof of Theorem 3. Assume that

$$1 + a(1)z + \cdots + a(q_1-1)z^{q_1-1} \neq 1 + \gamma z + \cdots + z^{q_1-1},$$

$$1 + a(1)z + \cdots + a(q_1-1)z^{q_1-1} \neq 1.$$

Then $(a(1), \cdots, a(q_1-1))$ is not a zero of at least one of the polynomials $s_1^{q_1}-s_1$, $s_r-s_1^r$ $(r=2, \cdots, q_1-1)$, say $R(s_1, \cdots, s_{q_1-1})$. Considering $R(s_1, \cdots, s_{q_1-1})u-1$ in place of $s_{r_0}u-1$ in the proof of Theorem 3, we can prove the theorem in the similar way.

As for the algebraic case, a similar argument is found in Loxton [3]. The deduction of the general case from the algebraic case is suggested by Kumiko Nishioka.

REMARK (by Kumiko Nishioka). By considering the rational function field $\mathbf{C}(t)$ in place of an algebriac number field K, the similar argument as in [5] leads to the following: if $f(z) \in \mathbf{C}[[z]]$ satisfies the functional equations

$$f(z^{q_i}) = a_i(z)f(z) + b_i(z) \qquad (i=1, 2),$$

where $\log q_1/\log q_2$ is not a rational number and $a_i(z)$, $b_i(z) \in \mathbf{C}(z)$, $a_i(0)=1$, then $f(z) \in \mathbf{C}(t)(z)$, and so $f(z) \in \mathbf{C}(z)$.

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