Proceedings of the Edinburgh Mathematical Society (1980), **23**, 27–35 ©

STRONGLY REGULAR NEAR-RINGS

by GORDON MASON

(Received 15th September 1978)

1. Introduction

A strongly regular ring R is one in which for all $x \in R$, there is an $a \in R$ with $x = x^2a$. Equivalently, for all x there is an a with $x = ax^2$. Such a ring is regular, duo, biregular, and a left and right V-ring. Moreover since R is reduced, all nilpotent elements are central (vacuously) and so all idempotent elements are central.

In this paper we examine these definitions for right near-rings. In the first place it is necessary to distinguish between left and right strong regularity. Then, by specialising to zero-symmetric unital near-rings, analogues of the ring-theoretic results are obtained. We also briefly discuss weak regularity. The basic reference for near-ring concepts is (**7**) and following him, all near-rings N will be right near-rings.

2. The general case

Definitions. N is *left regular* if for all $x \in N$, there exists $a \in N$ with $x = ax^2 = xax$, and N is *left strongly regular* if for all x there is $a \in N$ with $x = ax^2$. *Right regularity* and *right strong regularity* are defined in a symmetric way and the definition of regularity shall be the same as for rings. Thus a left regular near-ring is both regular and left strongly regular. These definitions are not consistent with terminology in (**8**), but have been chosen to coincide with ring-theoretic usage. N is *reduced* if it has no nilpotent elements and N has *I.F.P.* (insertion of factors property) if $ab = 0$ implies $axb = 0$ for all $x \in N$.

Lemma 1. (a) *If N is left or right strongly regular, it is reduced.*
(b) *In a zero-symmetric reduced near-ring, $ab = 0 \Rightarrow ba = 0$, and I.F.P. holds.*

Proof. (a) The right strongly regular case is trivial. If $x^2 = 0$ and $x = ax^2 = a \cdot 0$, then $0 = x^2 = (a \cdot 0)x = a(0x) = a \cdot 0 = x$.
(b) This is well known. See, e.g. (**2**, Lemma 1).

Lemma 2. *A left regular near-ring with I.F.P. is right regular.*

Proof. If $x = ax^2 = xax$, then $(ax - xa)x = 0$ so by I.F.P. $(ax - xa)ax = 0$ or $axax = xa^2x$. Thus $ax = axax = xa^2x$ so $x = xax = x^2a^2x$ i.e. $x = x^2b$ as required (where $b = a^2x$). Moreover $xbx = xa^2x^2 = xax = x$.

https://doi.org/10.1017/S0013091500003564 Published online by Cambridge University Press

Remarks. 1. If N is left and right strongly regular, when $\forall x \neq 0 \exists\ a, b$ with $x = ax^2 = x^2b$ and, unlike the ring-theoretic case, it is apparently not necessarily true that b can be chosen equal to a. As the proof of the lemma shows, adding regularity does not appear to help. However there are two cases when N is left and right regular that b can be chosen equal to a; namely when N is zero-symmetric and unital (see next Proposition) or when N has no zero divisors. In fact, if N has no zero-divisors and is left strongly regular, then $x = ax^2$ implies $(x - x^2a)x^2 = x^3 - x^2ax^2 = x^3 - x^3 = 0$. Thus since $x^2 \neq 0$, $x = x^2a$ as required.

2. The converse to the lemma is open. However we have

Proposition 1. *If N is zero symmetric, left regularity is equivalent to left strong regularity, and these imply right regularity. Moreover if N is unital, all three conditions are equivalent.*

Proof. If N is left strongly regular, then for all x there is an a with $x = ax^2$. It follows that $(x - xax)x = x^2 - xax^2 = x^2 - x^2 = 0$ so by Lemma 1, $x(x - xax) = 0$ also. Then $(x - xax)^2 = x(x - xax) - xax(x - xax) = 0$. Since N is reduced, $x = xax$. Thus N is left regular. Then N is right regular by the two lemmas. Conversely if N is unital and $x = x^2a = xax$, then xa and ax are idempotents. Therefore from (2) (or Corollary to Proposition 3) these are central and so $x = ax^2$.

Corollary 1. *If N is unital with I.F.P., the following are equivalent* (1) *regular* (2) *right regular* (3) *left regular* (4) *left strongly regular.*

Proof. (1)$\rightarrow$(2) Since $x = xax$ implies $(1 - xa)x = 0$, by I.F.P. $ax = xa^2x$. Then $x = xax = x^2(a^2x)$. The rest follows from the proposition since if N is unital with I.F.P., $1 \cdot 0 = 0$ implies $1 \cdot x \cdot 0 = 0$ for all x i.e. $x \cdot 0 = 0$ so N is zero symmetric.

Pilz (7, p. 277) defines N to have property P_0 if for all x, there is an integer $n = n(x)$ with $x^n = x$. These near-rings are clearly both left and right (strongly) regular. If N is a finite zero-symmetric left (or right) strongly regular near-ring, then it has P_0 by (7, 9.42) so is right (or left) (strongly) regular.

In fact the zero symmetry is not required and we can extend the result to periodic near-rings i.e. those in which $\forall x \exists n \neq m$ with $x^n = x^m$.

Proposition 2. *A periodic left (right) strongly regular near-ring has P_0.*

Proof. Since for all x there is an a with $x = ax^2$, then for all $n > k$, $x^k = a^{n-k}x^n$. Now N is periodic so there is some integer s minimal with respect to the property $x^s = x^t$ for some $t \neq s$. (In fact $t > s + 1$ or else $x^s = x^{s+1}$ and $x^s = ax^{s+1}$ imply $x^s = ax^s = x^{s-1}$ contradicting the minimality of s).

Now if $t - s > s - 1$ then $t - 2s + 1 > 0$ so $x^s = a^{t-s}x^t = a^{t-s}x^s$ implies $x = a^{t-s}x^{t-s+1} = a^{t-s}x^sx^{t-2s+1} = x^sx^{t-2s+1} = x^{t-s+1}$. Similarly $t - s < s - 1$ gives $0 < 2s - 1 - t$ and so $x = a^{s-1}x^s = a^{2s-1-t}a^{t-s}x^s = a^{2s-1-t}x^s = x^{t+1-s}$ as required.

Example 1. If N is a planar near-ring, it is zero-symmetric, and $nx = 0$ for some n if $nx = 0$ for all n (7, 8.87 and 8.88a). Thus if N has any right zero divisors, N is

https://doi.org/10.1017/S0013091500003564 Published online by Cambridge University Press

not regular or left strongly regular. On the other hand, if N is integral, then (7, 8.88b) $\forall n \neq 0$ and $\forall x \exists a$ with $an = x$. In particular, for $n = x^2$, this shows N is left strongly regular, and Proposition 1 applies.

Example 2. In (1, Thm 2.2) Adams constructs examples of zero-symmetric non-unital periodic near-rings N with no zero divisors, by defining a multiplication on a certain group of upper-triangular matrices over a field of characteristic zero. These examples are therefore both left and right regular. Note that $(N, +)$ is non-abelian (cf Theorem 1 below).

3. The zero-symmetric unital case

N has the property C.N. if all nilpotent elements are central and property C.I. if all idempotent elements are central. ("central" refers to the multiplicative centre.) It is easy to see that if N is regular with C.I., then it is left and right regular.

From now on N will be assumed to be zero-symmetric unless otherwise specified.

Lemma 3. *Let N have* C.N.

(i) *If $ab = 0$ then ba, axb and bxa are central for all x.*

(ii) *If e is an idempotent and $ae = 0$ then $exa = 0$ for all x.*

Proof. (i) If $ab = 0$, then $baba = 0$ so ba is central and similarly bxa is central for all x. Then $axbaxb = a(ba)x^2b = 0$ so axb is central.

(ii) If $ae = 0$ then (i) implies that exa is central, so commuting with e gives $exa = exae = 0$.

Generalising a result of (2) we have

Proposition 3. *If N has* C.N., *then for all idempotents e and for all x in N,*

(i) *ex-exe is central*;

(ii) *Every distributive idempotent is central*;

(iii) $x^2e = (xe)^2$;

(iv) *If N is unital, $xe = exe$.*

Proof. (i) $(ex - exe)e = 0$ so by Lemma 3 (ii), for all $r \in N$ $er(ex - exe) = 0$. Then $(ex - exe)^2 = ex(ex - exe) - exe(ex - exe) = 0$ so $ex - exe$ is central.

(ii) If e is distributive, $e(ex - exe) = ex - exe$ (since when d is distributive $d(-n) = -dn$ in general) and so using (i) $e(ex - exe)$ is central. Commuting with e shows it is zero so $ex = exe$. On the other hand, since $e(xe - exe) = 0$, Lemma 3 (i) shows that $(xe - exe)e$ is central. Thus it commutes with e showing it is zero and giving $xe = exe$.

(iii) $(x - xe)e = 0$ implies $e(x - xe) = 0$ by Lemma 3(ii). Then $[(x - xe)xe]^2 = 0$ so $(x - xe)xe$ is central. Commuting it with e shows it to be zero. Thus $x^2e = (xe)^2$.

(iv) Since $e^2 = e$, $(1 - e)e = 0$ when N is unital. By Lemma 3, $e(1 - e) = 0$ and $(1 - e)xe$ is central for all x. Commuting $(1 - e)xe$ with e implies it is zero.

Corollary 2. **(2)** *If N is reduced and unital, it has* C.I.

https://doi.org/10.1017/S0013091500003564 Published online by Cambridge University Press

Proof. As seen in the proof of (i) of the Proposition, $ex - exe$ is nilpotent so N reduced implies $ex = exe$. Also $xe = exe$ from (iv).

From now on N will be assumed to be unital unless otherwise specified, and all N-modules will be unitary.

Theorem 1. *A left strongly regular near-ring is left and right regular. Every principal left N-subgroup Na is generated by an idempotent, and every left N-subgroup is a two-sided ideal. Moreover $(N, +)$ is abelian and N is isomorphic to a subdirect sum of near-fields.*

Proof. The first statement comes from Proposition 1. Now in any regular near-ring a principal left N-subgroup is clearly generated by an idempotent e (see e.g. (**4**, Thm 4.2)) and in our case $eN = Ne$ by Corollary 2. Therefore $Ne = \{x \mid x(1-e) = 0\} = \mathrm{Ann}(1-e)$ because $re(1-e) = 0$ from the centrality of e and, conversely, $x(1-e) = 0$ implies $(1-e)x = 0$ so $x = ex = xe$. Now $\mathrm{Ann}(1-e) = Ne$ is a left ideal and since e is central it is also closed under right N-multiplication. Thus every principal N-subgroup, and hence every N-subgroup, is a two-sided ideal. From (**4**, Thm 4.4) we have finally that N is isomorphic to a subdirect sum of near-fields and $(N, +)$ is abelian.

This theorem has many consequences. In the first place, it follows from (**3**, 5.3) that in a strongly regular near-ring, primitive and maximal ideals coincide and the primitive radical is zero. Moreover from (**2**) every prime ideal is completely prime (P is completely prime if $ab \in P \Rightarrow a \in P$ or $b \in P$) and the prime radical is zero. In fact

Lemma 4. *If N is left strongly regular, every prime ideal is maximal.*

Proof. Let P be prime and suppose $P \not\subseteq M$ for a maximal ideal M. Let $x \in M \setminus P$. Since $0 = x - ax^2 = (1 - ax)x$ for some a, and P is completely prime, therefore $1 - ax \in P \subset M$. But $x \in M$ so $1 \in M$, a contradiction.

Another consequence of the theorem is that no d.g. near-ring can be left strongly regular unless it is a ring because of the abelian addition. In fact if $E(G)$, the endomorphism near-ring of the group G, were right strongly regular for finite G then it too would be a finite (commutative) ring by (**7**, 9.44c).

As to whether the left-right definitions are actually equivalent, we note that right strong regularity does not imply the existence of zero divisors, so perhaps there exist integral near-rings as examples. These would have the necessary condition of being reduced and are guaranteed not to be left strongly regular if not near-fields.

There are two definitions of biregularity in the literature. Write $B(N)$ for the set of central idempotents in N, and $NaN = \{\Sigma\, n_i a t_i\}$.

B_1(**10**): N is biregular if for all $a \in N \exists e$ in $B(N)$ such that $NaN = Ne$.

B_2(Betsch, (**7**, p. 94)): N is biregular if $\exists E \subset B(N)$ such that

(a) for all $e \in E$, Ne is an ideal of N,

(b) for all $n \in N$ there exists $e \in E$ with $Ne = (n)$, the principal ideal generated by n,

(c) for all $e, f \in E$, $e + f = f + e$,

(d) for all $e, f \in E$, $ef \in E$ and $e + f - ef \in E$.

https://doi.org/10.1017/S0013091500003564 Published online by Cambridge University Press

Now in (**10**) it is shown that when B_1 holds we have

(i) each NaN is an ideal,

(ii) for all $e, f \in B(N)$, $f - ef = -ef + f$

(iii) for all $e, f \in B(N)$, $e + f - ef = e - ef + f \in B(N)$.

Proposition 4. *If N is left strongly regular, it is B_1 and if N is a B_1 near-ring, it is B_2.*

Proof. By the theorem, if $n \in N$ then Nn is an ideal generated by a central idempotent e so $NnN = Nn = Ne$.

Now suppose N is a B_1 – near-ring. Let $E = B(N)$. Then $\forall e \in E$, $NeN = Ne$ is an ideal (from (i)) so (a) is true. Moreover, given x, NxN is an ideal from (i), so $(x) = NxN$ and $NxN = Ne$ for some $e \in B(N)$. Thus (b) is true. Next let $e, f \in E$. Again $Nf = NfN$ is an ideal so $e + f - e \in Nf$, that is $e + f - e = nf$ for some n. Multiplying on the right by f gives $ef + f - ef = nf = e + f - e$. Using (ii) we get $f = e + f - e$ so (c) is true. Finally, (d) follows directly from (iii).

Definition. $u \in N$ is a *generator* of N if the ideal generated by u is all of N. For our left strongly regular near-rings, u is a generator if and only if u is a unit, since the ideal generated by u is Nu and $Nu = N$ implies $vu = 1$ for some v. Therefore V is in no proper maximal ideal so $Nv = N$ and $xv = 1$ for some x. Then $x = xvu = u$. Thus for left strongly regular near-rings our definition of generator is consistent with (**7**, p. 75).

Proposition 5. *If N is left strongly regular, then for all $x \in N$ there is a unit $u \in N$ such that $x = ux^2$.*

Proof. We have $x = ax^2$ for some a and $a = za^2$ for some z. Put $u = 1 + a - za$. Then $ux^2 = (1 + a - za)x^2 = x^2 + ax^2 - zax^2 = x^2 + x - zaxx = x^2 + x - za(ax^2)x = x^2 + x - ax^2x = x$. Moreover $ua = (1 + a - za)a = a + a^2 - a = a^2$. Thus if u is in a maximal ideal so is a, and then so is 1 from the way u was defined. Therefore u is a unit.

Theorem 2. *The following are equivalent:*

(a) *N is left strongly regular;*

(b) *N is regular and a subdirect sum of near-fields;*

(c) *every N-subgroup is a two sided ideal, and all ideals I satisfy $I = I^2$* (where in general $IJ = \{ij \mid i \in I, j \in J\}$).

Proof. (a)$\Leftrightarrow$(b) follows from (**4**, 4.4) and the observation that a regular near-ring with C.I. is left strongly regular.

(a)$\Rightarrow$(c) If L and K are left ideals, they are two sided by Theorem 1 so $LK \subset L \cap K$. Moreover if $x \in L \cap K$ then $x = ax^2 = ax \cdot x \in LK$ so $LK = L \cap K$. In particular $L^2 = L$.

(c)$\Rightarrow$(a) For any $x \in N$, Nx is a two-sided ideal and $(Nx)^2 = Nx$ so $x = rxsx$ for some r, s. But $xs \in Nx$ so $x = rtx^2$ for some t, as required.

Recall that a *left V-ring* is a ring which satisfies either of the equivalent (**6**)

https://doi.org/10.1017/S0013091500003564 Published online by Cambridge University Press

conditions (i) every left ideal is an intersection of maximal left ideals (ii) every simple left module is injective. A strongly regular ring is both a left and right V-ring. For near-rings we distinguish between simple modules (those having no N-submodules) and irreducible modules (those having no N-subgroups). As well, it is not clear if there exist any injective modules (see (5)).

However there are near-ring modules that appear to play the role of injective modules in some situations. M is defined to be a *Baer module* if it satisfies "Baer's criterion": For every left ideal L, every homomorphism $f: L \to M$ can be extended to N. In (5), for instance, it was shown that N is semi-simple iff every module is Baer. The next two theorems show that the Baer modules are a suitable generalisation of injective modules in the present context also. First we prove a lemma.

Lemma 5. *If N is left strongly regular and S is a multiplicatively closed subset of N not containing* 0 *then any ideal P maximal with respect to $P \cap S = \emptyset$ is* (*completely*) *prime.*

Proof. Let $ab \in P$ and suppose $a \notin P$, $b \notin P$. By Theorem 1, Na and Nb are ideals and by the maximality of $P \exists s \in S \cap (Na + P)$ and $\exists t \in S \cap (Nb + P)$. Now $(Na + P)(Nb + P) = NaNb + P$ (see e.g. (7, page 61)) and since $Na = Ne$ for some central idempotent e, $NaNb + P = Nab + P \subset P$. Therefore $st \in S \cap P$ which is a contradiction.

Theorem 3. *N is left strongly regular if and only if every left N-subgroup is an ideal which is an intersection of maximal* (*left*) *ideals.*

Proof. ($\Rightarrow$) From Theorem 1, it suffices to show that every ideal I is an intersection of maximal ideals. If $b \notin I$ then since $Nb = Ne$ for some central idempotent e, let M be an ideal containing I and maximal with respect to the property $b \notin M$ (i.e. $e \notin M$). But since $\{e\}$ is multiplicatively closed, does not contain 0, and is disjoint from M, M is a prime ideal by Lemma 5. Hence by Lemma 4, M is maximal. Thus we have shown that if $b \notin I$, there is a maximal ideal M containing I with $b \notin M$, as required.

($\Leftarrow$) Since Nx^2 is an intersection of maximal ideals, if $x \neq ax^2$ for all a, there should exist a maximal ideal M such that $x^2 \in M$, $x \notin M$. However this contradicts the fact that, when left N-subgroups are ideals, a maximal ideal is completely prime.

Recall (5) that the N-module I is *loosely injective* if whenever $0 \to I \to A \xrightarrow{f} B \to 0$ is a short exact sequence, there is a map $g: B \to A$ such that $g(B)$ is a submodule of A and $fg = 1_B$. The sequence is then said to *split*.

Theorem 4. *Consider the following conditions on N*:

(a) *Every simple module is loosely injective*;

(b) *Every left ideal is an intersection of maximal left ideals*;

(c) *Every simple module is Baer.*

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

Proof. (a) $\Rightarrow$ (b) If M is an N-module with $0 \neq x \in M$, choose Y to be maximal among submodules of M with $x \notin Y$. Let $D = \cap\{\text{proper submodules } B \supsetneq Y\}$ if any

https://doi.org/10.1017/S0013091500003564 Published online by Cambridge University Press

exist. The D/Y is a simple module and there is an exact sequence $0 \to D/Y \to M/Y \to M/D \to 0$ which splits since D/Y is loosely injective. Therefore (**5**, Lemma 2.1) $M/Y \simeq D/Y \oplus K/Y$ for some submodule K, $Y \subseteq K \subseteq M$. Now $x \in D$ so $x \notin K$ which contradicts the maximality of K. Therefore $D = \emptyset$ and Y is maximal. Thus we have shown that for all $x \in M$ there is a maximal submodule Y with $x \notin Y$, so the intersection of all maximal submodules is (0). In particular if $M = N/I$ for some left ideal I, $I = \cap\{\text{maximal left ideals} \supseteq I\}$.

(b)$\Rightarrow$(c) Consider the diagram

$$\begin{array}{c} S \\ \alpha \uparrow \\ 0 \to I \to N \end{array}$$

where S is simple and I is a left ideal of N. Let $J = \operatorname{Ker} \alpha$ and let M be a maximal left ideal such that $M \supset J$, $M \not\supset I$ (which exists by (b)). Then $I \not\supset M \cap I \supseteq J$ but since $S = I/J$ is simple $M \cap I = J$ so $S \simeq I/J \simeq I/M\,I \simeq M + I/M \simeq N/M$. Therefore we define $\bar{\alpha} : N \to S = N/M$ to be the canonical quotient map and this completes the diagram.

Corollary. *If N is left strongly regular, every simple module is Baer.*

Theorem 5. *If N is regular, has* C.N. *and every maximal N-subgroup is closed under right N-multiplication, then every irreducible N-module is Baer.*

Remark. If a regular near-ring with C.N. is strongly regular, then of course Theorem 5 is unnecessary since the corollary is stronger. For *rings*, C.N.$\Rightarrow$C.I. and so a regular ring with C.N. is strongly regular. However it is not known if either statement is true for near-rings.

The theorem requires a lemma, and both proofs are adapted from (**9**).

Lemma 6. *If every maximal left N-subgroup of N is closed under right N-multiplication, and M is an irreducible N-module, then for all x, y ∈ M,* $\operatorname{Ann} x = \operatorname{Ann} y$.

Proof. As noted earlier, $\operatorname{Ann} x$ is a left ideal. Since M is irreducible, $M = Nx$ and $N/\operatorname{Ann} x \simeq Nx \simeq M \simeq N/\operatorname{Ann} y$ so both $\operatorname{Ann} x$ and $\operatorname{Ann} y$ are maximal left ideals. By hypothesis, they are ideals. If $a \in \operatorname{Ann} x$ then since $y = nx$, $ay = anx$. Since $\operatorname{Ann} x$ is a two-sided ideal, $an \in \operatorname{Ann} x$ so $ay = 0$ i.e. $a \in \operatorname{Ann} y$. By symmetry $\operatorname{Ann} x = \operatorname{Ann} y$.

Proof of Theorem 5. If I is a left ideal, M is an irreducible N-module and $f : I \to M$ an N-homomorphism, then $f(a) \neq 0$ for some $a \in I$ and by the regularity of N, $Na = Ne$ for some $e = e^2$. Then (**7**, 1.13) $N = Ne \oplus \operatorname{Ann} e$ and $I \supset Ne$ so $I = I \cap (Ne \oplus \operatorname{Ann} e) = Ne \oplus (I \cap \operatorname{Ann} e)$. Let $f(e) = a$. Then since $Na = Ne$, $a = ne$ so $f(a) = nu$ and $u \neq 0$ since $f(a) \neq 0$. Because $e = e^2$, $0 \neq u = f(e) = eu$ so $e \notin \operatorname{Ann} u$. We now show f is zero on $I \cap \operatorname{Ann} e$. This will allow us to define $g : N \to M$ by $g(n) = g(ne + n - ne) = f(ne) = nf(e)$ so g is an N-homomorphism extending f. Now if $x = n - ne \in I \cap \operatorname{Ann} e$ then $ef(x) = f(ex) = f(e(n - ne))$. But by Lemma 3 (ii), $(n - ne)e = 0$ implies $e(n - ne) = 0$. Thus $ef(x) = 0$. If $f(x)$ were non-zero this would say

https://doi.org/10.1017/S0013091500003564 Published online by Cambridge University Press

$e \in \text{Ann}\, f(x)$. But we saw $e \notin \text{Ann}\, u$ and this contradicts Lemma 6. Therefore $f(x) = 0$ as required.

4. Weakly regular near-rings

Ring theorists use the term "weakly regular" in the following way: for all $x, \exists a \neq 0$ with $a = axa$. This is equivalent to: every left ideal contains a non-zero idempotent. Then every regular ring is weakly regular. In an analogous manner we have

Lemma 7. *For a near-ring N (it needn't be unital or zero symmetric) the following are equivalent:*

(a) *For all $x \in N \exists a \neq 0$ in N with $a = axa$;*

(b) *Every left N-subgroup contains a non-zero idempotent.*

Moreover a zero-symmetric regular near-ring has these properties.

Proof. (a)$\Rightarrow$(b) If J is a left N-subgroup and $x \in J$, then $a = axa \neq 0 \Rightarrow 0 \neq ax$ is an idempotent in J.

(b)$\Rightarrow$(a) Given $x \in N$, the left N-subgroup Nx contains an idempotent $bx \neq 0$. Since $(bx)^3 = bx$ we have $bxb = (bxb)x(bxb)$ and so choose $a = bxb$.

Now if N is regular then for all $x \exists b$ with $x = xbx$. Put $a = bxb$ so $axa = bxbxbxb = a$. If $a = 0$ then $bxb = 0$ so $bx = bxbx = 0$. If N is zero-symmetric, then $x = xbx = 0$, a contradiction.

Rather than use the equivalent conditions of the lemma to define weak regularity, we shall follow Plasser (**8**, 3.46) and define a near-ring N to be *weakly regular* if $\forall x \in N \exists$ idempotent $x^0 \in N$ such that

(i) $xx^0 = x^0x = x$

(ii) If $e^2 = e$ and $ex = xe = x$ then $x^0e = x^0$

(iii) $(xy)^0 = x^0y^0$

It follows from (iii) that such a near-ring is reduced so it is not true that a regular near-ring is weakly regular. However we shall justify the terminology by proving that a left strongly regular near-ring is weakly regular. We continue to work in the zero-symmetric unital case, so if N is left strongly regular then for all $x \in N \exists\ a \in N$ with $x = xax = ax^2 = x^2a$. Taking $x^0 = ax$ we have (i) directly and moreover if $e^2 = e$ and $ex = xe = x$ then $x^0e = axe = ax = x^0$ so (ii) holds. It remains to show that (iii) is true, and to do this we borrow a ring theoretic definition from (**8**).

Definition. A near-ring N is called an *associate near-ring* if N is a subdirect sum of integral near-rings, and, for all $x \in N \exists$ an idempotent x^0 such that $xx^0 = x^0x = x$ and $x + 1 - x^0$ is not a zero divisor. Recall that we are using "integral near-ring" to mean there are no zero divisors. In a subdirect sum of integral near-rings, $ab = 0 \Rightarrow ba = 0$ so there is no need to distinguish between left and right zero divisors.

Proposition 6. *N is an associate near-ring iff N is a subdirect sum of integral near-rings in which for all $a = (a_i) \in N$, $a^0 = (a_i^0) \in N$ where $a_i^0 = 1$ if $a_i \neq 0$ and $a_i^0 = 0$ if $a_i = 0$.*

Proof. ($\Rightarrow$) An idempotent in N must be $e = (e_i)$ where $e_i = 0$ or 1. Therefore

https://doi.org/10.1017/S0013091500003564 Published online by Cambridge University Press

given $a \in N$ we know there exists an idempotent a^0 such that $a^0a = aa^0 = a$. Therefore $a_i^0 = 1$ when $a_i \neq 0$. Now suppose $a_i^0 = 1$ for some i with $a_i = 0$. Then $x = a + 1 - a^0$ has a component $x_i = 0$. Since $x^0 \in N$, $0 \neq 1 - x^0 \in N$ and $(1 - x^0)x = x - x = 0$ so x is a zero divisor. This is a contradiction.

($\Leftarrow$) By the definition of a^0 we have $aa^0 = a^0a = a$. Also if $x = a + 1 - a^0$ then $x_i \neq 0$ for all i so x is not a zero divisor.

Theorem 6. *A strongly regular near-ring is an associate near-ring and an associate near-ring is weakly regular.*

Proof. By Theorem 1, if N is strongly regular it is a subdirect sum of near-fields which are integral near-rings. Since for all $x \exists a$ with $x = xax = ax^2$, we take $x^0 = ax$. Then $x + 1 - x^0 = x + 1 - ax$ is not a zero divisor for if $(x + 1 - ax)y = 0$ then $(x + 1 - ax)xy = 0$ by I.F.P. so $x^2y + xy - ax^2y = 0$ or $x^2y = 0$. Then $0 = ax^2y = xy$ and so $0 = (x + 1 - ax)y = xy + y - axy = y$.

Now suppose N is an associate near-ring so we have $\forall x$ an x^0 with $xx^0 = x^0x = x$. If $e^2 = e$ and $ex = xe = x$, then $e = (e_i)$ where each $e_i = 0$ or 1. Therefore e_i must be 0 precisely when $x_i = 0$ and so $e = x^0$ and $x^0e = x^0$. Finally $(xy)^0 = x^0y^0$ follows from the description of x^0 in Proposition 6.

Acknowledgements. The author acknowledges helpful correspondence with Dr. A. Oswald, who proved parts of Theorem 1 and Propositions 3 and 4 independently. This research was supported by a grant from the National Research Council of Canada.

REFERENCES

(1) W. B. ADAMS, Near integral domains on nonabelian groups, *Monatsh. Math.* **81** (1976), 177–184.

(2) H. BELL, Near-rings in which each element is a power of itself, *Bull. Austral. Math. Soc.* **2** (1970), 363–368.

(3) M. JOHNSON, Radicals of regular near-rings, *Monatsh. Math.* **80** (1975), 331–341.

(4) S. LIGH, On regular near-rings, *Math. Japon*, **15** (1970), 7–13.

(5) G. MASON, Injective and projective near-ring modules, *Compositio Math.* **33** (1976), 43–54.

(6) G. MICHLER and O. VILLAMAYOR, On rings whose simple modules are injective, *J. Algebra* **25** (1973), 185–201.

(7) G. PILZ, Near-rings (North-Holland, Amsterdam, 1977).

(8) K. PLASSER, *Subdirekte Darstellung von Ringen und Fastringen mit Booleschen Eigenschaften* (Diplomarbeit, Univ. Linz, Austria, 1974).

(9) B. SARATH and K. VARADARAJAN, Injectivity of certain classes of modules, *J. Pure Appl. Algebra* **5** (1974), 293–305.

(10) G. SZETO, On sheaf representation of a biregular near-ring, *Canad. Math. Bull.* **20** (1977), 495–500.

UNIVERSITY OF NEW BRUNSWICK
FREDERICTON, N.B.
CANADA

https://doi.org/10.1017/S0013091500003564 Published online by Cambridge University Press