KYUNGPOOK Math. J. 52(2012), 137-147
http://dx.doi.org/10.5666/KMJ.2012.52.2.137

Strongly Summable Double Sequence Spaces in $n-$Normed Spaces Defined by Ideal Convergence and an Orlicz Function

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Abstract. In this paper we introduce some new double sequence spaces via ideal convergence and an Orlicz function in $n-$normed spaces and examine some properties of the resulting spaces.

1. Introduction

Let X be a non-empty set, then a family of sets $I \subset 2^X$ (the class of all subsets of X) is called an ideal if and only if for each $A, B \in I$, we have $A \cup B \in I$ and for each $A \in I$ and each $B \subset A$, we have $B \in I$. A non-empty family of sets $F \subset 2^X$ is a filter on X if and only if $\varnothing \notin F$, for each $A, B \in F$, we have $A \cap B \in F$ and each $A \in F$ and each $A \subset B$, we have $B \in F$. An ideal I is called non-trivial ideal if $I \neq \varnothing$ and $X \notin I$. Clearly $I \subset 2^X$ is a non-trivial ideal if and only if $F = F(I) = \{X/A : A \in I\}$ is a filter on X. A non-trivial ideal $I \subset 2^X$ is called admissible if and only if $\{\{x\} : x \in X\} \subset I$. Further details on ideals of 2^X can be found in Kostyrko, et.al [5]. The notion was further investigated by Salat et.al [13], Tripathy and Hazarika [21, 23, 15, 29], Tripathy and Mahanta [27] and others.

The notion of double sequences has been investigated from different aspect by Tripathy and Dutta [25], Tripathy and Sarma [22, 24, 30] and many others in recent years.

By the convergence of a double sequence we mean the convergence on the Pringsheim sense that is, a double sequence $x = (x_{k,l})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > n$ Pringsheim [10]. We shall write more briefly as "$P-$*convergent*".

The double sequence $x = (x_{k,l})$ is bounded if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l. Let l_∞^2 the space of all bounded double such that

$$\|x_{k,l}\|_{(\infty,2)} = \sup_{k,l} |x_{k,l}| < \infty.$$

Received January 9, 2011; revised September 14, 2011; accepted November 24, 2011.
2010 Mathematics Subject Classification: Primary 40A99; Secondary 40A05.
Key words and phrases: P-convergent, n-normed space, Orlicz function.

Recall Krasnoselski and Rutickii [6] that an Orlicz function M is continuous, convex, nondecreasing function define for $x > 0$ such that $M(0) = 0$ and $M(x) > 0$. If convexity of Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$ then this function is called the modulus function and characterized by Nakano [9]. It was further investigated from sequence space point of view by Ruckle [12], Tripathy and Chandra [28] and many others. An Orlicz function M is said to satisfy Δ_2-*condition* for all values u, if there exists $K > 0$ such that $M(2u) \leq KM(u)$, $u \geq 0$.

Lemma 1.1. *Let M be an Orlicz function which satisfies Δ_2-condition and let $0 < \delta < 1$. Then for each $t \geq \delta$, we have $M(t) < K\delta^{-1}M(2)$ for some constant $K > 0$.*

Recently different classes of sequences have been introduced by using an Orlicz function and their algebraic and topological properties have been investigated by Et et.al. [1], Tripathy and Mahanta [18], Tripathy et.al. [20], Tripathy and Hazarika [15], Tripathy and Sarma [24, 30] and many others.

A double sequence space E is said to be solid or normal if $(\alpha_{k,l}x_{k,l}) \in E$, whenever $(x_{k,l}) \in E$ and for all double sequences $\alpha = (\alpha_{k,l})$ of scalars with $|\alpha_{k,l}| \leq 1$ for all $k, l \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and X be a real vector space of dimension d, where $n \leq d$. A real-valued function $\|., ..., .\|$ on X satisfying the following four conditions:

(i) $\|(x_1, x_2, \ldots, x_n)\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,

(ii) $\|(x_1, x_2, \ldots, x_n)\|$ is invariant under permutation,

(iii) $\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha| \|(x_1, x_2, \ldots, x_n)\|, \alpha \in \mathbb{R}$,

(iv) $\|(x_1 + x_1^{\imath}, x_2, \ldots, x_n)\| \leq \|(x_1, x_2, \ldots, x_n)\| + \|(x_1^{\imath}, x_2, \ldots, x_n)\|$

is called an $n-$norm on X, and the pair $(X, \|(., \ldots, .)\|)$ is called an $n-$normed space [3].

A trivial example of $n-$normed space is $X = \mathbb{R}$ equipped with the following Euclidean $n-$norm:

$$\|(x_1, x_2, \ldots, x_n)\|_E = abs \left(\begin{vmatrix} x_{11} \ldots x_{1n} \\ \ldots \\ x_{n1} \ldots x_{nn} \end{vmatrix} \right)$$

where $x_i = (x_{i1}, \ldots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \ldots, n$.

The notion of $n-$norm from different aspect by Gahler [2], Gunawan and Mashadi [4], Misiak [8], Tripathy and Dutta [26] and many others.

2. Main results

In this section we introduce the notion of different types of $I-$convergent double sequences. This generalizes and unifies different notions of convergence for double sequences. We shall denote the ideal of $2^{\mathbb{N}\times\mathbb{N}}$ by I_2.

The notion of paranormed sequence space was studied at the initial stage by Simons [14]. Later on different classes of paranormed sequence spaces were introduced and their different properties were investigated by Maddox [7], Rath and Tripathy [11], Tripathy [17], Tripathy and Sen [16, 19] and many others.

Let I_2 be an ideal of $2^{\mathbb{N}\times\mathbb{N}}$, M be an Orlicz function, $p=(p_{k,l})$ be a bounded double sequence of strictly positive real numbers and $(X,\|(.,\dots,.)\|)$ be an $n-$normed space. Further $w\,(n-X)$ denotes $X-$valued sequence space. Now, we define the following double sequence spaces:

$$w^{I_2}\left[M,p,\|(.,\dots,.)\|\right]_o=$$
$$\left\{\begin{array}{c} x=(x_{k,l})\in w\,(n-X):\forall\varepsilon>0,\\ \left\{(n,m)\in\mathbb{N}\times\mathbb{N}:\ \frac{1}{nm}\sum_{k,l=1,1}^{m,n}\left[M\left(\left\|\left(\frac{x_{k,l}}{\rho},z_1,\dots,z_{n-1}\right)\right\|\right)\right]^{p_{k,l}}\geq\varepsilon\right\}\in I_2\\ \text{for some }\rho>0\text{ and for every }z_1,\dots,z_{n-1}\in X\end{array}\right\},$$

$$w^{I_2}\left[M,p,\|(.,\dots,.)\|\right]=$$
$$\left\{\begin{array}{c} x=(x_{k,l})\in w\,(n-X):\forall\varepsilon>0,\\ \left\{(n,m)\in\mathbb{N}\times\mathbb{N}:\ \frac{1}{nm}\sum_{k,l=1,1}^{m,n}\left[M\left(\left\|\left(\frac{x_{k,l}-L}{\rho},z_1,\dots,z_{n-1}\right)\right\|\right)\right]^{p_{k,l}}\geq\varepsilon\right\}\in I_2\\ \text{for some }\rho>0, L\in X\text{ and for every }z_1,\dots,z_{n-1}\in X\end{array}\right\},$$

$$w^{I_2}\left[M,p,\|(.,\dots,.)\|\right]_\infty=$$
$$\left\{\begin{array}{c} x=(x_{k,l})\in w\,(n-X):\exists K>0,\\ \left\{(n,m)\in\mathbb{N}\times\mathbb{N}:\ \frac{1}{nm}\sum_{k,l=1,1}^{m,n}\left[M\left(\left\|\left(\frac{x_{k,l}}{\rho},z_1,\dots,z_{n-1}\right)\right\|\right)\right]^{p_{k,l}}\geq K\right\}\in I_2\\ \text{for some }\rho>0\text{ and for every }z_1,\dots,z_{n-1}\in X\end{array}\right\}$$

and

$$w\left[M,p,\|(.,...,.)\|\right]_\infty=\left\{\begin{array}{c} x=(x_{k,l})\in w\,(n-X):\\ \exists K>0,\ \frac{1}{nm}\sum_{k,l=1,1}^{m,n}\left[M\left(\left\|\left(\frac{x_{k,l}}{\rho},z_1,...,z_{n-1}\right)\right\|\right)\right]^{p_{k,l}}\leq K\\ \text{for some }\rho>0\text{ and for every }z_1,...,z_{n-1}\in X\end{array}\right\}.$$

If $p_{k,l}=1$ for all $k,l\in\mathbb{N}$ we obtain $w^{I_2}\left[M,p,\|.,\dots,.\|\right]=w^{I_2}\left[M,\|.,\dots,.\|\right]$, $w^{I_2}[M,p,\|.,\dots,.\|]_o=w^{I_2}[M,\|.,\dots,.\|]_o$, $w^{I_2}\left[M,p,\|(.,\dots,.)\|\right]_\infty=w^{I_2}\left[M,\|(.,\dots,.)\|\right]_\infty$ and $w\left[M,p,\|.,\dots,.\|\right]_\infty=w\left[M,\|.,\dots,.\|\right]_\infty$.

The following well-known inequality will be used in this study: If $0\leq\inf_{k,l}p_{k,l}=H_o\leq p_{k,l}\leq\sup_{k,l}=H<\infty$, $D=\max\left(1,2^{H-1}\right)$, then

$$|x_{k,l}+y_{k,l}|^{p_{k,l}}\leq D\left\{|x_{k,l}|^{p_{k,l}}+|y_{k,l}|^{p_{k,l}}\right\}$$

for all $k,l\in\mathbb{N}$ and $x_{k,l},y_{k,l}\in\mathbb{C}$. Also $|x_{k,l}|^{p_{k,l}}\leq\max\left(1,|x_{k,l}|^H\right)$ for all $x_{k,l}\in\mathbb{C}$.

Theorem 2.1. *The sets* $w^{I_2}\left[M,p,\|(.,...,.)\|\right]_o, w^{I_2}\left[M,p,\|(.,...,.)\|\right]$ *and* $w^{I_2}\left[M,p,\|(.,...,.)\|\right]_\infty$ *are linear spaces over the complex field* $\mathbb{C}$.

Proof. We will prove only for $w^{I_2}\left[M,p,\|(.,\ldots,.)\|\right]_o$ and the others can be proved similarly. Let $x,y\in w^{I_2}\left[M,p,\|(.,\ldots,.)\|\right]_o$ and $\alpha,\beta\in\mathbb{C}$. Then

$$\left\{(n,m)\in\mathbb{N}\times\mathbb{N}:\frac{1}{nm}\sum_{k,l=1,1}^{m,n}\left[M\left(\left\|\left(\frac{x_{k,l}}{\rho_1},z_1,\ldots,z_{n-1}\right)\right\|\right)\right]^{p_{k,l}}\geq\frac{\varepsilon}{2}\right\}\in I_2,\text{ for some }\rho_1>0$$

and

$$\left\{(n,m)\in\mathbb{N}\times\mathbb{N}:\frac{1}{nm}\sum_{k,l=1,1}^{m,n}\left[M\left(\left\|\left(\frac{y_{k,l}}{\rho_2},z_1,\ldots,z_{n-1}\right)\right\|\right)\right]^{p_{k,l}}\geq\frac{\varepsilon}{2}\right\}\in I_2,\text{ for some }\rho_2>0.$$

Since $\|.,...,.\|$ is a $n-$norm and M is an Orlicz function, the following inequality holds:

$$\begin{aligned}
&\frac{1}{nm}\sum_{k,l=1,1}^{m,n}\left[M\left(\left\|\left(\frac{\alpha x_{k,l}+\beta y_{k,l}}{|\alpha|\,\rho_1+|\beta|\,\rho_2},z_1,\ldots,z_{n-1}\right)\right\|\right)\right]^{p_{k,l}}\\
\leq&\frac{D}{nm}\sum_{k,l=1,1}^{m,n}\left[\frac{|\alpha|}{|\alpha|\,\rho_1+|\beta|\,\rho_2}M\left(\left\|\left(\frac{x_{k,l}}{\rho_1},z_1,\ldots,z_{n-1}\right)\right\|\right)\right]^{p_{k,l}}\\
&+\frac{D}{nm}\sum_{k,l=1,1}^{m,n}\left[\frac{|\beta|}{|\alpha|\,\rho_1+|\beta|\,\rho_2}M\left(\left\|\left(\frac{y_{k,l}}{\rho_2},z_1,\ldots,z_{n-1}\right)\right\|\right)\right]^{p_{k,l}}\\
\leq&\frac{D}{nm}\sum_{k,l=1,1}^{m,n}\left[M\left(\left\|\left(\frac{x_{k,l}}{\rho_1},z_1,\ldots,z_{n-1}\right)\right\|\right)\right]^{p_{k,l}}\\
&+\frac{D}{nm}\sum_{k,l=1,1}^{m,n}\left[M\left(\left\|\left(\frac{y_{k,l}}{\rho_2},z_1,\ldots,z_{n-1}\right)\right\|\right)\right]^{p_{k,l}}
\end{aligned}$$

From the above inequality we get

$$\left\{(n,m)\in\mathbb{N}\times\mathbb{N}:\ \frac{1}{nm}\sum_{k,l=1,1}^{m,n}\left[M\left(\left\|\left(\frac{\alpha x_{k,l}+\beta y_{k,l}}{|\alpha|\,\rho_1+|\beta|\,\rho_2},z_1,\ldots,z_{n-1}\right)\right\|\right)\right]^{p_{k,l}}\geq\varepsilon\right\}$$

$$\begin{aligned}
\subset&\left\{(n,m)\in\mathbb{N}\times\mathbb{N}:\ \frac{D}{nm}\sum_{k,l=1,1}^{m,n}\left[M\left(\left\|\left(\frac{x_{k,l}}{\rho_1},z_1,\ldots,z_{n-1}\right)\right\|\right)\right]^{p_{k,l}}\geq\frac{\varepsilon}{2}\right\}\\
\cup&\left\{(n,m)\in\mathbb{N}\times\mathbb{N}:\ \frac{D}{nm}\sum_{k,l=1,1}^{m,n}\left[M\left(\left\|\left(\frac{y_{k,l}}{\rho_2},z_1,\ldots,z_{n-1}\right)\right\|\right)\right]^{p_{k,l}}\geq\frac{\varepsilon}{2}\right\}.
\end{aligned}$$

Two sets on the right hand side belong to I_2 and this completes the proof. □

It is also easy verify that the space $w\left[M,p,\|(.,\ldots,.)\|\right]_{\infty}$ is also a linear space.

Theorem 2.2. *For fixed* $(n,m)\in\mathbb{N}\times\mathbb{N}$, $w\left[M,p,\|.,\ldots,.\|\right]_{\infty}$ *paranormed space with respect to the paranorm defined by*

$$g_{(n,m)}(x)=\inf\left\{\begin{array}{c}\rho^{\frac{p_{n,m}}{H}}:\left(\sup_{n,m}\frac{1}{nm}\sum_{k,l=1,1}^{m,n}\left[M\left(\left\|\left(\frac{x_{k,l}}{\rho},z_1,\ldots,z_{n-1}\right)\right\|\right)\right]^{p_{k,l}}\right)^{\frac{1}{H}}\leq 1,\\ \text{for some } \rho>0 \text{ and for every } z_1,\ldots,z_{n-1}\in X\end{array}\right\}.$$

Proof. $g_{(n,m)}(\theta)=0$ and $g_{(n,m)}(-x)=g_{(n,m)}(x)$ are easy to prove, so we omit them. Let us take $x,y\in w\left[M,p,\|(.,\ldots,.)\|\right]_{\infty}$. Let

$$A(x)=\left\{\rho>0:\sup_{n,m}\frac{1}{nm}\sum_{k,l=1,1}^{m,n}\left[M\left(\left\|\left(\frac{x_{k,l}}{\rho},z_1,\ldots,z_{n-1}\right)\right\|\right)\right]^{p_{k,l}}\leq 1,\forall z\in X\right\}$$

and

$$A(y)=\left\{\rho>0:\sup_{n,m}\frac{1}{nm}\sum_{k,l=1,1}^{m,n}\left[M\left(\left\|\left(\frac{y_{k,l}}{\rho},z_1,\ldots,z_{n-1}\right)\right\|\right)\right]^{p_{k,l}}\leq 1,\forall z\in X\right\}.$$

Let $\rho_1\in A(x)$ and $\rho_2\in A(y)$. If $\rho=\rho_1+\rho_2$, then we have

$$\begin{aligned}&\sup_{n,m}\frac{1}{nm}\sum_{k,l=1,1}^{m,n}\left[M\left(\left\|\left(\frac{x_{k,l}+y_{k,l}}{\rho},z_1,\ldots,z_{n-1}\right)\right\|\right)\right]\\ \leq&\frac{\rho_1}{\rho_1+\rho_2}\sup_{n,m}\frac{1}{nm}\sum_{k,l=1,1}^{m,n}\left[M\left(\left\|\left(\frac{x_{k,l}}{\rho_1},z_1,\ldots,z_{n-1}\right)\right\|\right)\right]\\ &+\frac{\rho_2}{\rho_1+\rho_2}\sup_{n,m}\frac{1}{nm}\sum_{k,l=1,1}^{m,n}\left[M\left(\left\|\left(\frac{y_{k,l}}{\rho_1},z_1,\ldots,z_{n-1}\right)\right\|\right)\right].\end{aligned}$$

Thus

$$\sup_{n,m}\frac{1}{nm}\sum_{k,l=1,1}^{m,n}\left[M\left(\left\|\left(\frac{x_{k,l}+y_{k,l}}{\rho_1+\rho_2},z_1,\ldots,z_{n-1}\right)\right\|\right)\right]^{p_{k,l}}\leq 1$$

and

$$\begin{aligned}g_{(n,m)}(x+y)&=\inf\left\{(\rho_1+\rho_2)^{\frac{p_{n,m}}{H}}:\rho_1\in A(x)\ \textit{and}\ \rho_2\in A(y)\right\}\\ &\leq\inf\left\{(\rho_1)^{\frac{p_{n,m}}{H}}:\rho_1\in A(x)\right\}+\inf\left\{(\rho_2)^{\frac{p_{n,m}}{H}}:\rho_2\in A(y)\right\}\\ &=g_{(n,m)}(x)+g_{(n,m)}(y).\end{aligned}$$

Now, let $\lambda_{k,l}^{u} \to \lambda$, where $\lambda_{k,l}^{u}, \lambda \in \mathbb{C}$ and $g_{(n,m)}\left(x_{k,l}^{u} - x_{k,l}\right) \to 0$ as $u \to \infty$. We have to show that $g_{(n,m)}\left(\lambda_{k,l} x_{k,l}^{u} - \lambda x_{k,l}\right) \to 0$ as $u \to \infty$. Let

$$A\left(x^{u}\right)=\left\{\rho_{u}>0: \sup_{n,m} \frac{1}{nm} \sum_{k,l=1,1}^{m,n}\left[M\left(\left\|\left(\frac{x_{k,l}^{u}}{\rho_{u}}, z_{1}, \ldots, z_{n-1}\right)\right\|\right)\right]^{p_{k,l}} \leq 1, \forall z \in X\right\}$$

and

$$A(x^{u}-x)=\left\{\rho_{u}^{\imath}>0: \sup_{n,m} \frac{1}{nm} \sum_{k,l=1,1}^{m,n}\left[M\left(\left\|\left(\frac{x_{k,l}^{u}-x_{k,l}}{\rho_{u}^{\imath}}, z_{1}, \ldots, z_{n-1}\right)\right\|\right)\right]^{p_{k,l}} \leq 1, \forall z \in X\right\}.$$

If $\rho_{u} \in A\left(x^{u}\right)$ and $\rho_{u}^{\imath} \in A\left(x^{u}-x\right)$ then we observe that

$$\begin{aligned}
& M\left(\left\|\left(\frac{\lambda_{k,l}^{u} x_{k,l}^{u}-\lambda x_{k,l}}{\rho_{u}\left|\lambda_{k,l}^{u}-\lambda\right|+\rho_{u}^{\imath}|\lambda|}, z_{1}, \ldots, z_{n-1}\right)\right\|\right) \\
\leq & M\left(\left\|\left(\frac{\lambda_{k,l}^{u} x_{k,l}^{u}-\lambda x_{k,l}^{u}}{\rho_{u}\left|\lambda_{k,l}^{u}-\lambda\right|+\rho_{u}^{\imath}|\lambda|}, z_{1}, \ldots, z_{n-1}\right)\right\|\right. \\
& \left.+\left\|\left(\frac{\lambda x_{k,l}^{u}-\lambda x_{k,l}}{\rho_{u}\left|\lambda_{k,l}^{u}-\lambda\right|+\rho_{u}^{\imath}|\lambda|}, z_{1}, \ldots, z_{n-1}\right)\right\|\right) \\
\leq & \frac{\rho_{u}\left|\lambda_{k,l}^{u}-\lambda\right|}{\rho_{u}\left|\lambda_{k,l}^{u}-\lambda\right|+\rho_{u}^{\imath}|\lambda|} M\left(\left\|\left(\frac{x_{k,l}^{u}}{\rho_{u}}, z_{1}, \ldots, z_{n-1}\right)\right\|\right) \\
& +\frac{\rho_{u}^{\imath}|\lambda|}{\rho_{u}\left|\lambda_{k,l}^{u}-\lambda\right|+\rho_{u}^{\imath}|\lambda|} M\left(\left\|\left(\frac{x_{k,l}^{u}-x_{k,l}}{\rho_{u}^{\imath}}, z_{1}, \ldots, z_{n-1}\right)\right\|\right).
\end{aligned}$$

From this inequality, it follows that

$$\left[M\left(\left\|\frac{\lambda_{k,l}^{u} x_{k,l}^{u}-\lambda x_{k,l}}{\rho_{u}\left|\lambda_{k,l}^{u}-\lambda\right|+\rho_{u}^{\imath}|\lambda|}, z\right\|\right)\right]^{p_{k,l}} \leq 1$$

and consequently

$$\begin{aligned}
& g_{(n,m)}\left(\lambda_{k,l}^{u}x_{k,l}^{u}-\lambda x_{k,l}\right) \\
&= \inf\left\{\left(\rho_u\left|\lambda_{k,l}^{u}-\lambda\right|+\rho_u^{\imath}\left|\lambda\right|\right)^{\frac{p_{n,m}}{H}} : \rho_u\in A\left(x^u\right) \text{ and } \rho_u^{\imath}\in A\left(x^u-x\right)\right\} \\
&\le \left(\left|\lambda_{k,l}^{u}-\lambda\right|\right)^{\frac{p_{n,m}}{H}} \inf\left\{\left(\rho_u\right)^{\frac{p_{n,m}}{H}} : \rho_u\in A\left(x^u\right)\right\} \\
&\quad + \left(\left|\lambda\right|\right)^{\frac{p_{n,m}}{H}} \inf\left\{\left(\rho_u^{\imath}\right)^{\frac{p_{n,m}}{H}} : \rho_u^{\imath}\in A\left(x^u-x\right)\right\} \\
&\le \max\left\{\left|\lambda\right|,\left(\left|\lambda\right|\right)^{\frac{p_{n,m}}{H}}\right\} g_{(n,m)}\left(x_{k,l}^{u}-x_{k,l}\right).
\end{aligned}$$

Hence by our assumption the right hand side tends to zero as $u\to\infty$. This completes the proof. □

Corollary 2.3. *It can be noted that* $g=\inf_{n,m\in\mathbb{N}} g_{(n,m)}$ *also gives a paranorm on the above sequence spaces. However if one consider the sequence space* $w[M,p,\|(.,\ldots,.)\|]_{\infty}$ *which is larger space than the space* $w^{I_2}[M,p,\|.,...,.\|]_{\infty}$ *the construction of the paranorm is not clear and we leave it as an open problem. However it should be noted that for a fixed* $G\in I_2$*, the space*

$$w_G\left[M,p,\|(.,\ldots,.)\|\right]_{\infty} = \left\{\left\{\begin{array}{c} x=(x_{k,l})\in w\left(n-X\right):\exists K>0, \\ (n,m)\in\mathbb{N}\times\mathbb{N}: \\ \sup_{(n,m)\in\mathbb{N}\times\mathbb{N}/F}\ \frac{1}{nm}\sum_{k,l=1,1}^{m,n}\left[M\left(\left\|\left(\frac{x_{k,l}}{\rho},z_1,\ldots,z_{n-1}\right)\right\|\right)\right]^{p_{k,l}} \ge K \\ for\ some\ \rho>0\ and\ for\ every\ z_1,\ldots,z_{n-1}\in X \end{array}\right\}\in I_2\right\}$$

which is subspace of the space $w^{I_2}[M,p,\|(.,\ldots,.)\|]_{\infty}$ *is a paranormed space with the paranorms* $g_{(n,m)}$ *for* $(n,m)\notin G$ *and* $g_G=\inf_{(n,m)\in\mathbb{N}\times\mathbb{N}/G} g_{(n,m)}$.

Theorem 2.4. *Let* M, M_1 *and* M_2 *be Orlicz functions. Then we have*

(*i*) $w^{I_2}[M_1,p,\|(.,\ldots,.)\|]_o \subset w^{I_2}[M\circ M_1,p,\|(.,\ldots,.)\|]_o$ *provided that* $p=(p_{k,l})$ *is such that* $H_o>0$.

(*ii*) $w^{I_2}[M_1,p,\|(.,\ldots,.)\|]_o\cap w^{I_2}[M_2,p,\|(.,\ldots,.)\|]_o\subset w^{I_2}[M_1+M_2,p,\|(.,\ldots,.)\|]_o$.

Proof. **(i).** Let $\inf_{k,l} p_{k,l}=H_o$. For given $\varepsilon>0$, we first choose $\varepsilon_o>0$ such that $\max\left\{\varepsilon_o^H,\varepsilon_o^{H_o}\right\}<\varepsilon$. Now using the continuity of M, choose $0<\delta<1$ such that $0<t<\delta$ implies $M(t)<\varepsilon_o$. Let $x\in w^{I_2}[M_1,p,\|(.,\ldots,.)\|]_o$. Now from the definition of the space $w^{I_2}[M_1,p,\|(.,\ldots,.)\|]_o$, for some $\rho>0$

$$A(\delta)=\left\{(n,m)\in\mathbb{N}\times\mathbb{N}:\frac{1}{nm}\sum_{k,l=1,1}^{m,n}\left[M_1\left(\left\|\left(\frac{x_{k,l}}{\rho},z_1,\ldots,z_{n-1}\right)\right\|\right)\right]^{p_{k,l}}\ge\delta^H\right\}\in I_2.$$

Thus if $(n,m) \notin A(\delta)$ then

$$\begin{aligned}
&\frac{1}{nm}\sum_{k,l=1,1}^{m,n}\left[M_1\left(\left\|\left(\frac{x_{k,l}}{\rho},z_1,\ldots,z_{n-1}\right)\right\|\right)\right]^{p_{k,l}} < \delta^H \\
&\Rightarrow \sum_{k,l=1,1}^{m,n}\left[M_1\left(\left\|\left(\frac{x_{k,l}}{\rho},z_1,\ldots,z_{n-1}\right)\right\|\right)\right]^{p_{k,l}} < nm\delta^H, \\
&\Rightarrow \left[M_1\left(\left\|\left(\frac{x_{k,l}}{\rho},z_1,\ldots,z_{n-1}\right)\right\|\right)\right]^{p_{k,l}} < \delta^H \text{ for all } k,l=1,2,\ldots, \\
&\Rightarrow M_1\left(\left\|\left(\frac{x_{k,l}}{\rho},z_1,\ldots,z_{n-1}\right)\right\|\right) < \delta \text{ for all } k,l=1,2,\ldots.
\end{aligned}$$

Hence from above inequality and using continuity of M, we must have

$$M\left(M_1\left(\left\|\left(\frac{x_{k,l}}{\rho},z_1,\ldots,z_{n-1}\right)\right\|\right)\right) < \varepsilon_o \text{ for all } k,l=1,2,\ldots$$

which consequently implies that

$$\begin{aligned}
&\sum_{k,l=1,1}^{m,n}\left[M\left(M_1\left(\left\|\left(\frac{x_{k,l}}{\rho},z_1,\ldots,z_{n-1}\right)\right\|\right)\right)\right]^{p_{k,l}} < nm\max\left\{\varepsilon_o^H,\varepsilon_o^{H_o}\right\} < nm\varepsilon, \\
&\Rightarrow \frac{1}{nm}\sum_{k,l=1,1}^{m,n}\left[M\left(M_1\left(\left\|\left(\frac{x_{k,l}}{\rho},z_1,\ldots,z_{n-1}\right)\right\|\right)\right)\right]^{p_{k,l}} < \varepsilon.
\end{aligned}$$

This shows that

$$\left\{(n,m)\in\mathbb{N}\times\mathbb{N} : \frac{1}{nm}\sum_{k,l=1,1}^{m,n}\left[M\left(M_1\left(\left\|\left(\frac{x_{k,l}}{\rho},z_1,\ldots,z_{n-1}\right)\right\|\right)\right)\right]^{p_{k,l}} \geq \varepsilon\right\} \subset A(\delta)$$

and so belongs to I_2. This completes the proof.

(ii) Let $x \in w^{I_2}[M_1,p,\|(.,\ldots,.)\|]_o \cap w^{I_2}[M_2,p,\|(.,\ldots,.)\|]_o$. Then the fact that

$$\begin{aligned}
&\frac{1}{nm}\left[(M_1+M_2)\left(\left\|\left(\frac{x_{k,l}}{\rho},z_1,\ldots,z_{n-1}\right)\right\|\right)\right]^{p_{k,l}} \\
&\leq \frac{D}{nm}\left[M_1\left(\left\|\left(\frac{x_{k,l}}{\rho},z_1,\ldots,z_{n-1}\right)\right\|\right)\right]^{p_{k,l}} + \frac{D}{nm}\left[M_2\left(\left\|\left(\frac{x_{k,l}}{\rho},z_1,\ldots,z_{n-1}\right)\right\|\right)\right]^{p_{k,l}}
\end{aligned}$$

gives us the result. □

Theorem 2.5. *(i) If* $0 < \inf_{k,l} p_{k,l} = H_o \leq p_{k,l} < 1$, *then* $w^{I_2}[M,p,\|(.,\ldots,.)\|]_o \subset w^{I_2}[M,\|(.,\ldots,.)\|]_o$.

(ii) If $1 \leq p_{k,l} \leq \sup_{k,l} p_{k,l} = H < \infty$, *then* $w^{I_2}[M, \|(.,\ldots,.)\|]_o \subset w^{I_2}[M, p, \|(.,\ldots,.)\|]_o$.

(iii) If $0 < p_{k,l} < q_{k,l} < \infty$ *and* $\frac{q_{k,l}}{p_{k,l}}$ *is bounded, then* $w^{I_2}[M, p, \|.,\ldots,.\|]_o \subset w^{I_2}[M, q, \|(.,\ldots,.)\|]_o$.

Proof. The proof is standard, so we omit it. □

Theorem 2.6. *The sequence spaces* $w^{I_2}[M, p, \|(.,\ldots,.)\|]_o$, $w^{I_2}[M, p, \|(.,\ldots,.)\|]$, $w^{I_2}[M, p, \|(.,\ldots,.)\|]_\infty$ *and* $w[M, p, \|.,\ldots,.\|]_\infty$ *are solid.*

Proof. We give the proof for only $w^{I_2}[M, p, \|(.,\ldots,.)\|]_o$. The others can be proved similarly. Let $x \in w^{I_2}[M_1, p, \|(.,\ldots,.)\|]_o$ and $\alpha = (\alpha_{k,l})$ be a double sequence of scalars such that $|\alpha_{k,l}| \leq 1$ for all $k, l \in \mathbb{N}$. Then we have

$$\left\{(n,m) \in \mathbb{N}\times\mathbb{N} : \frac{1}{nm}\sum_{k,l=1,1}^{m,n}\left[M\left(\left\|\left(\frac{\alpha_{k,l}x_{k,l}}{\rho}, z_1,\ldots,z_{n-1}\right)\right\|\right)\right]^{p_{k,l}} \leq \varepsilon\right\}$$
$$\subset \left\{(n,m) \in \mathbb{N}\times\mathbb{N} : \frac{T}{nm}\sum_{k,l=1,1}^{m,n}\left[M\left(\left\|\left(\frac{x_{k,l}}{\rho}, z_1,\ldots,z_{n-1}\right)\right\|\right)\right]^{p_{k,l}} \leq \varepsilon\right\} \in I_2,$$

where $T = \max_{k,l}\left\{1, |\alpha_{k,l}|^H\right\}$. Hence $\alpha x \in w^{I_2}[M_1, p, \|(.,\ldots,.)\|]_o$ for all double sequences $\alpha = (\alpha_{k,l})$ with $|\alpha_{k,l}| \leq 1$ for all $k,l \in \mathbb{N}$ whenever $x \in w^{I_2}[M_1, p, \|(.,\ldots,.)\|]_o$. □

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