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STRUCTURAL CONTROLLABILITY OF MULTI-INPUT LINEAR SYSTEMS

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ABSTRACT

This paper extends the results of Lin on structural controllability of single-input linear systems to multi-input linear systems. An interesting by-product of this extension is an application of "generic analysis" to the determination of the rank of a structured matrix. The algorithm for computing "generic" rank is applicable to various problems other than the controllability problem studied here.

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1. INTRODUCTION

Lin has introduced the concept of structure to obtain more realistic models of physical processes [1]. In particular for the linear time invariant system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with $x(t) \in R^n$, $u(t) \in R^r$, assume that the elements of the matrices A and B are either fixed (zero) or indeterminate (arbitrary). The pair $(\bar{A},\bar{B})$ has the same structure as (is structurally equivalent to) the pair (A,B) of the same dimensions if for every fixed (zero) entry of the matrix $[\bar{A}\ \bar{B}]$ the corresponding entry of the matrix [A B] is also fixed (zero), and for every fixed (zero) entry of the matrix [A B], the corresponding entry of the matrix $[\bar{A}\ \bar{B}]$ is fixed (zero).

This framework is consistent with physical reality in the sense that system parameter values are never known precisely with the exception of zeros that are fixed by coordinatization (e.g., the time derivative of position is velocity) or by the absence of physical connections between certain parts of a system. Also in computing solutions to control problems, a digital computer works with "true" zeros and "fuzzy" numbers, so the assumption of imprecise knowledge of nonzero system parameters suggests the desirability of investigating properties of systems which may be considered as being determined only by structure and not by numerical values.

Controllability is important in the solution of many control problems, yet the determination of controllability indices, for example, is a particularly ill-posed computational problem as is the problem of checking the controllability of an uncontrollable system. Structural controllability [1], on the other hand, is a property that is as useful as controllability and can be determined precisely by a digital computer. The purpose of this paper is to extend Lin's result on single-input systems to multi-input systems and to propose an algorithm for determining the structural controllability of a system (A,B).

Our approach to this problem is purely algebraic whereas Lin's approach was graph theoretic. In §2 we develop the necessary results in order to determine the maximal (generic) rank of a matrix and in §3 apply these results to the determination of structural controllability. In §4 we show how an algorithm to determine rank can be used to determine structural controllability of a system (A,B). Our algorithm for determining the generic rank of a matrix makes use of the fixed-zero structure of the matrix, is called Fixed-Zero-Rank-Finder (FZRF), and is described in the Appendix.

2. STRUCTURE AND THE MAXIMAL RANK OF A MATRIX

In the following, a structured matrix A is a matrix having fixed zeros in certain locations and arbitrary entries in the remaining locations. Conceptually a structured matrix is composed of zeros and indeterminates where the nonzero entries are unrelated. A structured system (A,B) is an ordered pair of structured matrices. The systems (A,B) and $(\bar{A},\bar{B})$ are structurally equivalent if there is a one-to-one correspondence between the locations of the fixed zeros and nonzero entries of the corresponding matrices of each system.

In this section we are concerned with determining the rank of a structured matrix. In order to do this, we review briefly certain concepts from [2] and chapter 0 of [3]. Let A,B,... be structured matrices with elements in R. If N is the number of arbitrary entries in A,B,... then associated with these matrices is the parameter space R^N. Every set of N values represents a data point $p \in R^N$. Thus every system structurally equivalent to (A,B), for example, is represented by a data point p. A,B,... can be considered as matrices with entries from the ring of polynomials in N variables, $R[\lambda]$ where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ is simply a list representing the nonzero entries.

Let $\Pi(A,B,\ldots)$ be a property that may be asserted about the matrices A,B,... Most properties of interest will turn out to be true for all data points except those which lie on an algebraic hypersurface in R^N. More precisely, consider polynomials $\psi_i \in R[\lambda]$. A <u>variety</u> $V \subset R^N$ is the set of common zeros of a finite number of polynomials $\psi_1, \ldots, \psi_K$. V is proper if $V \neq R^N$ and nontrivial if $V \neq \emptyset$. A data point $p \in R^N$ is <u>typical</u> relative to V if $p \in V^C$, the complement of V. A property Π is a function from R^N to the set $\{0,1\}$ where $\Pi(p) = 0(1)$ means Π fails (holds) at p. A property Π is <u>generic</u> relative to the proper variety V if $\ker \Pi \subset V$; and Π is generic if such a V exists.

The useful implications of generic analysis follow from the fact that if $V \subset R^N$ is a proper, nontrivial variety, then V is a closed set. Therefore, if a property is generic relative to V, the property holds at any point $p' \in V^C$ and in a sufficiently small neighborhood of p'. In addition, if $p' \in V$, with V nontrivial and proper, then every neighborhood of p' contains points $p \in V^C$. All points at which a generic property fails to hold, therefore, are atypical in the sense that they lie on a hypersurface in R^N. Any such point can be suitably perturbed an arbitrarily small amount so that the property holds. Therefore, system properties which are generic can be expected to hold for practically every set of parameter values, i.e., for all <u>typical</u> data points.

As an example, consider determining the generic rank of a matrix A of order $n \times m$ consisting totally of indeterminate entries (i.e., no fixed zeros). Assume $n \leq m$ and define the property Π as follows

$$\Pi(p) = \begin{cases} 1 & \text{if the rank of } A \text{ is } n \\ 0 & \text{if the rank of } A \text{ is less than } n \end{cases}$$

where $p \in R^{n \cdot m}$. Consider the polynomial ψ in $n \cdot m$ indeterminates defined as the sum of the squares of all the possible n^{th} order minors of A. Clearly $p \in \ker \Pi$ implies $\psi(p) = 0$, so it only remains to show that the variety

$$V = \{p | \psi(p) = 0\}$$

is proper. But this is trivial since every entry of A can be chosen arbitrarily. This proves the following intuitively obvious result.

<u>Proposition 2.1</u>

<u>The maximal rank of an</u> $n \times m$ <u>matrix</u> A <u>having no specified structure is equal to</u> $\min(n,m)$.

<u>Remark</u>: From the above analysis, it is clear that <u>maximal rank</u> and <u>generic rank</u> are equivalent concepts.

The inclusion of structure into the problem makes it possible for matrices to have less than full rank independently of parameter values. In this case, $\psi(p) = 0$ for every $p \in R^{n \cdot m}$ (in the above proof) so that V is not proper. The additional analysis required to handle this situation is developed next.

The first investigations into structural properties of matrices were by Frobenius in 1912 [4] and in a more general manner by König in 1931 [5]; see also [6],[7],[8],[9].

Let A be an $n \times m$ structured matrix. A line of the matrix A refers to either a row or a column. A set of independent entries of A is a set of nonzero entries, no two of which lie on the same line. Let M^* denote the cardinality of the largest set of independent entries of A and let M_* denote the minimum number of lines of A which contain all the nonzero entries. Then the classic result of König is

Theorem 2.1 (König)

If A is a finite rectangular matrix consisting of zero and nonzero entries, then

$$M^* = M_*$$

The relation between König's Theorem and the structural properties of the matrix A is made clear by the following Lemma. Assume $n \leq m$.

Lemma 2.1

Every term in the determinantal expansion of every t^{th} order submatrix of A, $1 \leq t \leq n$, vanishes if and only if for some k in the range

$m - t < k \leq m$, A <u>contains a zero submatrix of order</u> $(n+m-t-k+1) \times k$.

Proof

(Necessity) Assume every term in the determinantal expansion of every t^{th} order submatrix of A vanishes for some t in the range $1 \leq t \leq n$. Then since each nonzero term in such an expansion consists of the product of t independent entries of A , it follows from Theorem 2.1 that

$$M^* = M_* \leq t-1$$

Otherwise there would exist a t^{th} order minor of A having a nonzero term in its expansion. Therefore, there are r rows and s columns such that

$$r + s = M_*$$

Let $t_1(t_2)$ designate any rows (columns) such that

$$r + t_1 + s + t_2 = t - 1$$

where $t_1 \geq 0$, $t_2 \geq 0$. Then there exists a zero submatrix of A of order $n - (r+t_1) \times m - (s+t_2)$. With $k = m - (s+t_2)$ there are

$$n - (r+t_1) = n + m - (r+t_1) - m$$

$$= n + m - t + (s+t_2) + 1 - m$$

$$= n + m - t - k + 1$$

rows and k columns which define a zero submatrix of A .

(Sufficiency) By assumption there exists a zero submatrix of A of order $(n+m-t-k+1) \times k$ for some k in the range $m - t < k \leq m$. Using permutation operations*, A can be brought to the form

* A permutation matrix is a square matrix which in each row and each column has one entry unity and all others zero. Permutation operations clearly preserve structure.

$$\begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix}$$

where A_1 is of order $(n+m-t-k+1) \times (m-k)$ and A_3 is of order $(t-m+k-1) \times k$. Assume there exists a t^{th} order submatrix of A having a nonzero term in its determinantal expansion. Then there exists a set of t independent entries in A. If $m-k$ such entries are chosen from the first $m-k$ columns of the above matrix, $t-(m-k)$ remain to be selected from the remaining k columns. However, from the size of A_3, the maximum number of independent entries is $t - (m-k) - 1$ which is a contradiction.

Q.E.D.

<u>Remark</u>: By setting $m = n = t$, this Lemma reduces to the result first published by Frobenius [4],[6].

Lemma 2.1 is only sufficient to determine a bound on the rank of a matrix, since every term in the determinantal expansion of a matrix need not vanish for the determinant to be zero. The usual analysis, therefore, is not capable of taking into consideration the effect of the nonzero entries of A on the rank of A. The additional assumption that A is a structured matrix makes applicable the generic analysis discussed above and yields a solution to the maximal rank problem.

Assume A is a structured matrix with N arbitrary entries. For clarity we restate what is meant by the generic rank of A. The structured matrix A has generic rank equal to t, i.e.,

$$\text{rank } A = t \ (g)$$

if there exists a proper variety $V \subset R^N$ such that all data points $p \in R^N$ for which rank $A \neq t$ lie on V. Assume A is of order $n \times m$ with $n \leq m$.

<u>Definition</u>: A is of form (t) for some t, $1 \leq t \leq n$, if for some k in the range $m - t < k \leq m$, A contains a zero submatrix of order $(n+m-t-k+1) \times k$.

<u>Example</u>

Each of the following matrices is of form (4), but $k = 5$ for the first and $k = 4$ for the second

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ X & X & X & X & X \\ X & X & X & X & X \\ X & X & X & X & X \end{bmatrix} \qquad \begin{bmatrix} X & 0 & 0 & 0 & 0 \\ X & 0 & 0 & 0 & 0 \\ X & 0 & 0 & 0 & 0 \\ X & X & X & X & X \\ X & X & X & X & X \end{bmatrix}$$

<u>Remark</u>: If A has form (t), then clearly A has form (j) for $t < j \leq n$.

The importance of form (t) is the following:

<u>Lemma 2.2</u>

<u>For any</u> t, $1 \leq t \leq n$, rank $A < t$ <u>for every</u> $p \in R^N$ <u>if and only if</u> A <u>has form</u> (t).

Proof

(Sufficiency) If A has form (t), then from Lemma 2.1, rank $A < t$ for every $p \in R^N$.

(Necessity) Assume A does not have form (t). Then by Lemma 2.1 there exists at least one nonzero term in the determinantal expansion of at least one t^{th} order submatrix of A. Let this term be

$$a_{i_1 j_1} \cdots a_{i_t j_t}$$

Choose $a_{i_1 j_1} = \ldots = a_{i_t j_t} = 1$ and all the other entries of A to be zero. Therefore there exists $p \in R^N$ such that rank $A \geq t$.

Q.E.D.

The main result of this section is the following:

Theorem 2.2

rank $A = t$ (g)

(i) for $t = n$ if and only if A is not of form (n)

(ii) for $1 \leq t < n$ if and only if A is of form (t+1) but not of form (t).

Proof

(Necessity)

(i) By assumption, there exists $p \in R^N$ such that rank $A = n$. By Lemma 2.2, A is not of form (n).

(ii) In this case, by Lemma 2.2, A is not of form (t). Assume A is not of form (t+1) and define

$$\Pi = \begin{cases} 1 \text{ if rank } A \geq t+1 \\ 0 \text{ if rank } A \leq t \end{cases}$$

Define the polynomial $\psi \in R[\lambda]$ as the sum of squares of all minors of order greater than or equal to $t+1$. By Lemma 2.2, there exists $p \in R^N$ such that rank $A \geq t+1$ and so the variety defined by ψ is proper. Therefore

$$\text{rank } A \geq t+1 \ (g)$$

which contradicts rank $A = t$ (g) and establishes that A is of form (t+1).

(Sufficiency)

(i) Define

$$\Pi = \begin{cases} 1 \text{ if rank } A = n \\ 0 \text{ if rank } A < n \end{cases}$$

and $\psi \in R[\lambda]$ as the sum of squares of all minors of A of order n. Since A is not of form (n), by Lemma 2.2, there is a $p \in R^N$ for which

$$\text{rank } A = n$$

so

$$\text{rank } A = n \ (g) .$$

(ii) By assumption A is of form (t+1) so, by Lemma 2.2, rank $A < t+1$ for every $p \in R^N$. Define

$$\Pi = \begin{cases} 1 \text{ if rank } A = t \\ 0 \text{ if rank } A < t \end{cases}$$

and $\psi \in R[\lambda]$ as the sum of squares of all minors of A of order t. Since A is not of form (t), it follows from Lemma 2.2 that the variety defined by $\psi(p) = 0$ is proper and so

$$\text{rank } A = t \ (g)$$

Q.E.D.

The actual determination of the generic rank of a matrix involves the ability to computationally recognize the existence of form (t) for any t. This necessitates searching the given matrix for patterns of fixed zeros. Since it is possible to precisely represent and recognize a fixed zero using a digital computer, it is a straightforward task to implement an algorithm to determine generic rank. Because the algorithm depends on the ability to recognize fixed zeros, the program is given the designation Fixed-Zero-Rank-Finder (FZRF). The details of FZRF are given in the Appendix.

3. STRUCTURAL CONTROLLABILITY

In this section we apply the results developed in §2 to the problem of structural controllability first studied by Lin [1]. Consider (A,B) to be a structured system with associated parameter space R^{N+M} where N(M) is the number of non-fixed entries in A(B).

Definition: A system (A,B) is structurally controllable if there exists a system structurally equivalent to (A,B) which is controllable in the usual sense.

Proposition 3.1

The system (A,B) is structurally controllable if and only if all uncontrollable pairs structurally equivalent to (A,B) lie on a proper variety in R^{N+M}.

Proof

(Necessity) Define the polynomial $\psi \in R[\lambda]$ as the sum of squares of all maximal order minors of the controllability matrix

$$[B\ AB\ \dots\ A^{n-1}B]$$

By assumption there exists $p \in R^{N+M}$ for which the associated system is controllable. Therefore $\psi(p) = 0$ defines a proper variety.

(Sufficiency) Obvious from the definition of a proper variety. Q.E.D.

As usual, the intuitive content of this concept is that all uncontrollable systems, structurally equivalent to a structurally controllable system are atypical. Proposition 3.1 is an obvious generalization of the controllability results of Lee and Markus [10] to structured systems and characterizes structural controllability but does not give conditions for determining structural controllability.

The ability to recognize a structurally controllable system is <u>not</u> a simple application of FZRF to the controllability matrix. For example, consider the system

$$A = \begin{bmatrix} a_1 & a_5 & a_7 & a_9 \\ a_2 & a_6 & a_8 & a_{10} \\ a_3 & 0 & 0 & 0 \\ a_4 & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The controllability matrix is

$$\left[\begin{bmatrix} b_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ b_2 \\ 0 \\ 0 \end{bmatrix}, b_1\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}, b_2\begin{bmatrix} a_5 \\ a_6 \\ 0 \\ 0 \end{bmatrix}, b_1a_1\begin{bmatrix} X \\ X \\ a_3 \\ a_4 \end{bmatrix}, b_2a_5\begin{bmatrix} X \\ X \\ a_3 \\ a_4 \end{bmatrix}, X\begin{bmatrix} X \\ X \\ a_3 \\ a_4 \end{bmatrix}, X\begin{bmatrix} X \\ X \\ a_3 \\ a_4 \end{bmatrix}\right]$$

where the X's denote general nonzero terms. FZRF applied to this matrix would determine that the rank is four and hence the system is structurally controllable. Clearly the last two rows are dependent for any set of parameter values and the system is not structurally controllable. The problem with FZRF, of course, is that the nonzero entries in the above matrix are not independent so the structure inherent in (A,B) is lost in the formation of the controllability matrix. The analysis, therefore, requires a more involved investigation of matrix structural properties. In our investigation we rely on the previous work of Lin and define two special forms of the matrix [A B] as follows:

Definition: If there exists a permutation matrix P satisfying

$$P^T[A \quad B]\begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} A_1 & 0 & 0 \\ A_2 & A_3 & B_2 \end{bmatrix}$$

with A_1 of order $t \times t$, $1 \leq t < n$, then the matrix [A B] has Form I.†
The matrix [A B] has Form II if it has form (n); i.e., if [A B] contains a zero submatrix of order $(n+r-k+1) \times k$ for $r < k \leq n + r$.

The significance of Forms I and II arise from one of the main results of Lin which we now state (in the following, b is a structured $n \times 1$ matrix).

Lemma 3.1 (Lin)

The system (A,b) is structurally uncontrollable* if and only if the matrix [A b] has Form I or Form II.

We can now state our result as follows:

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* A system (A,B) is structurally uncontrollable if every system structurally equivalent to (A,B) is uncontrollable in the usual sense.

† We assume that B_2 has at least one nonzero element.

Theorem 3.1

The system (A,B) is structurally uncontrollable if and only if the matrix [A B] has Form I or Form II.

This Theorem is proved using four Lemmas concerning the structural properties of the matrices A and B . The first two concern the eigenvectors of a structured matrix. If s is an eigenvalue of A , the corresponding left eigenspace is the set of vectors x satisfying $x^T(A-sI) = 0$. Any such non-zero x is a left eigenvector of A corresponding to s .

Define T as the set of points in the parameter space R^N , associated with A , as follows:

$$T = \{a | a \in R^N \text{ , rank } (A-sI) < n-1 \text{ for some } s\}$$

Note that $T \neq \phi$ since A = 0 is an element of T .

Lemma 3.2

Assume rank A = n (g) . There exists a proper variety in R^N which contains T .

Proof

Define b as an nxl vector containing no fixed zeros and consider the system (A,b) with associated parameter space R^{N+n} . If (A,b) is structurally uncontrollable, Lemma 3.1 implies that the matrix [A b] has Form II. But by Theorem 2.2, this means that rank A < n for every $a \in R^N$ so (A,b) must be structurally controllable. Define the polynomial $\psi \in R[\lambda]$ as the sum of squares of all the maximal order minors of the matrix

$$[b \quad Ab \ldots A^{n-1}b]$$

ψ defines a variety in R^{N+n} which is proper since (A,b) is structurally controllable. Fix b so that there exists $a \in R^N$ for which $\psi \neq 0$ as a polynomial in the nonzero entries of A. Clearly for every $a \in T$, the corresponding system (A,b) is uncontrollable.* Therefore, T is contained in the proper variety defined by

$$V = \{a \mid a \in R^N, \ \psi(a) = 0\}$$

Q.E.D.

This Lemma states that under the assumption rank A = n (g), all the eigenspaces associated with the eigenvalues of A are generically of dimension one. (Note that this does not rule out repeated eigenvalues, only linearly independent eigenvectors corresponding to the same eigenvalue.)

Define G as the set of points in R^N, associated with A, as follows

$$G = \{a \mid a \in R^N, \text{ and there exists at least one left eigenvector of } A \text{ having a zero component}\}$$

Note $G \neq \emptyset$ since $A = 0$ is an element of G.

From the definition of Form I, it follows that if [A B] has Form I, then A has Form I, i.e., there exists a permutation matrix P such that

$$P^T A P = \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix}$$

with A_1 of order $t \times t$, $1 \leq t < n$.

<u>Lemma 3.3</u>

<u>Assume rank</u> A = n (g) <u>and</u> A <u>does not have Form</u> I. <u>There exists a proper variety in the parameter space</u> R^N <u>which contains</u> G.

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* i.e., rank $[A-sI \quad b] < n$ for some s.

Proof

Consider the set of all structured n x 1 vectors containing at least one zero and one nonzero entry. This set is composed of 2^n-2 vectors. Let b_i, $i=1,\ldots,2^n-2$ represent these vectors. Consider the system (A,b_i) with associated parameter space R^{N+n_i} where n_i is the number of nonzero entries in b_i. If (A,b_i) is structurally uncontrollable, it can be shown that rank A = n (g) is contradicted so (A,b_i) must be structurally controllable for each $i = 1,2,\ldots,2^n-2$.

Define the polynomials ψ_i in the nonzero entries of A and b_i as the sum of squares of all n^{th} order minors of

$$[b_i \; Ab_i \; \ldots \; A^{n-1}b_i]$$

ψ_i defines a proper variety in R^{N+n_i} since (A,b_i) is structurally controllable. For each i, fix b_i so that there exists $a \in R^N$ for which $\psi_i \neq 0$ as a polynomial in the nonzero entries of A. Consider the variety $V \subset R^N$ defined by

$$V = \bigcup_{i=1}^{2^n-2} V_i$$

where

$$V_i = \{a \mid a \in R^N, \; \psi_i(a) = 0\}$$

V is proper since each V_i is proper. Also $G \subset V$. To see this, let A' be the matrix associated with the data point $a \in G$. Then there exists a left eigenvector x of A' containing a zero component. Therefore there is a b_i such that the system (A',b_i) is uncontrollable (i.e., $x^T(A'-sI) = 0$

and $x^T b_i = 0$ for some s), where the nonzero entries of b_i correspond to zero entries of x . Thus $\psi_i = 0$ which implies $a \in V$.

Q.E.D.

This Lemma states that under the assumptions rank $A = n$ (g) and A does not have Form I, all components of each eigenvector of A are nonzero generically.

<u>Definition</u>: Suppose we have two matrices with components (g_{ij}) and (h_{ij}). We say that their product is <u>identically</u> <u>zero</u> ($\equiv 0$) if

$$g_{ik} h_{kj} = 0 \quad \text{for every } i,k,j$$

For example, the product

$$[a_1 \quad 0 \quad a_2] \begin{bmatrix} -a_2 \\ a_3 \\ a_1 \end{bmatrix} = 0$$

is not identically zero even though it is zero while the product

$$[a_1 \quad a_2 \quad 0] \begin{bmatrix} 0 \\ 0 \\ a_3 \end{bmatrix} \equiv 0$$

is identically zero.

<u>Lemma 3.4</u>

<u>Assume</u> rank $[A \quad B] = n$ (g) <u>and</u> rank $A = q$ (g) , $n-m \le q < n$. <u>Then there exists a permutation operation</u> P <u>such that</u>

$$P^TAP = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1,k+1} \\ A_{21} & A_{22} & \cdots & A_{2,k+1} \\ & & \ddots & \\ A_{k+1,1} & A_{k+1,2} & \cdots & A_{k+1,k+1} \end{bmatrix} \qquad P^TB = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{k+1} \end{bmatrix} \tag{3.1}$$

<u>where</u> A_{ii} <u>is of order</u> $t_i \times t_i$, $0 \leq t_i < n$, <u>and either</u> $A_{11} = 0$ <u>in which case</u> $t_1 \leq n-q$, <u>or</u>

$$A_{11} = \begin{bmatrix} A'_{11} & 0 & \cdots & 0 \\ A'_{21} & A'_{22} & \cdots & 0 \\ & & \cdots & \\ & & & 0 \\ A'_{v1} & & & A'_{vv} \end{bmatrix} \tag{3.2}$$

<u>with</u> rank $A_{11} = t_1$ (g), $1 \leq v \leq t_1$, A'_{ii} <u>of order</u> $v_i \times v_i$, <u>and not of Form I,</u> $1 \leq k \leq q-t_1+1$ <u>and finally</u> B_{k+1} <u>has no zero rows. Also the pair</u>

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & A_{12} & & \cdots & & A_{1,k+1} \\ 0 & 0 & A_{23} & \cdots & & A_{2,k+1} \\ & & & \cdots & & \\ & & & \cdots & & \\ & & & & A_{k-1,k} & A_{k-1,k+1} \\ & & & & 0 & A_{k,k+1} \\ 0 & & & \cdots & & 0 \end{bmatrix} \qquad \bar{B} = P^TB \tag{3.3}$$

<u>with</u>

$$\bar{A}_{11} = \begin{bmatrix} A'_{11} & 0 & \cdots & 0 \\ 0 & A'_{22} & & 0 \\ & & \cdots & \\ & & 0 & A'_{vv} \end{bmatrix}$$

satisfies

$$\text{rank } \bar{A} = q \ (g) \tag{3.4}$$

and

$$\text{rank } [\bar{A} \quad \bar{B}] = n \quad (g) \tag{3.5}$$

Proof

Since rank $A = q$ (g) and rank $[A \quad B] = n$ (g) there are $n-q$ vectors x_i which satisfy

$$x_i^T A = 0$$

and for which no vector $x \in \{x_1,\ldots,x_{n-q}\}$ satisfies

$$x^T B \equiv 0$$

Define the $nx(n-q)$ matrix $X = (x_1 \ldots x_{n-q})$ and the index set $\beta \in \{1,\ldots,n\}$ designating all nonzero rows of B. Then the rows of X designated by β contain $n-q$ nonzero independent entries, for if not, there exists an $(n-q)x1$ vector y satisfying

$$(Xy)^T B \equiv 0$$

Let $\beta' \subset \beta$ designate any set of $n-q$ rows of X containing $n-q$ independent entries. Define the permutation operation P such that the last $n-q$ rows of $P^T X$ contain the $n-q$ nonzero independent entries of the rows designated by β'. Then

$$P^T A P = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \qquad P^T B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{3.6}$$

where rank $[A_1 \;\; A_2] = q$ (g), for if not, rank $A < q$ for every data point in R^N. Note that B_2 has no zero rows since $\beta' \subset \beta$. Set $A_3, A_4 = 0$. Then the vectors e_j (where the j^{th} entry is the only nonzero entry of e_j) satisfy

$$(0 \;\; e_j^T)\begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} \equiv 0 \quad \text{and} \quad (0 \;\; e_j^T)\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \not\equiv 0$$

for $1 \leq j \leq n-q$. Because of this, it is possible to choose values for the nonzero entries of A_1, A_2, B_1 and B_2 such that

$$\text{rank}\begin{bmatrix} A_1 & A_2 & B_1 \\ 0 & 0 & B_2 \end{bmatrix} = n$$

and therefore

$$\text{rank}\begin{bmatrix} A_1 & A_2 & B_1 \\ 0 & 0 & B_2 \end{bmatrix} = n \text{ (g)}$$

The permutation operation required to yield the forms of (3.1) and (3.3) depends on the properties of A_1. Three cases are possible.

First, if rank $A_1 = q$ (g) and A_1 is not of Form I, (3.6) is of the form (3.1) with $A_1 = A_{11}$ and $k = v = 1$. Also with

$$\bar{A} = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \bar{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

rank $\bar{A} = q$ (g) and rank $[\bar{A} \;\; \bar{B}] = n$ (g) as shown above.

In the second case where rank $A_1 = q$ (g) and A_1 is of Form I, there exists a permutation operation of the form

$$\begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \tag{3.7}$$

such that

$$P^T A_1 P = \begin{bmatrix} A'_{11} & 0 & & \\ A'_{21} & A'_{22} & 0 & \\ & & \cdots & \\ A'_{v1} & & & A'_{vv} \end{bmatrix}$$

where $1 \leq v \leq t_1$ and A'_{ii} is not of Form I. This follows from repeated application of the permutation operation defining Form I to each resulting block diagonal matrix until (3.2) is obtained. That is, since A_1 is of Form I, there exists a permutation operation which puts A_1 in lower block triangular form. If either of the resulting block diagonal matrices is of Form I they can be reduced to lower block triangular form. Continue this process until all block diagonal matrices are not of Form I. Thus (3.1) results with $A_{11} = P^T A_1 P$, $k = 1$ and $1 \leq v \leq t$. Also, since rank $A_1 = q$ (g), no A'_{ii} has any fixed zero eigenvalues. This implies rank $A'_{ii} = v_i$ (g), $1 \leq i \leq v$. Thus with $A_3 P = 0$, $A_4 = 0$ and

$$\bar{A}_{11} = \begin{bmatrix} A'_{11} & \cdots & 0 \\ 0 & & A'_{vv} \end{bmatrix}$$

rank $\bar{A} = q$ (g) and rank $[\bar{A} \;\; \bar{B}] = n$ (g) follow as above where

$$\bar{A} = \begin{bmatrix} P^T A_1 P & P^T A_2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \bar{B} = \begin{bmatrix} P^T B_1 \\ B_2 \end{bmatrix}$$

Consider finally the case where rank $A_1 = q'$ (g) where $q' < q$. There exist permutation operations such that

$$\begin{bmatrix} P^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} A_1' & A_2' & A_{21} \\ A_3' & A_4' & A_{22} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} P^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} B_1' \\ B_2' \\ B_2 \end{bmatrix}$$

with rank $[A_1' \quad A_2'] = q'$ (g) . Set $A_3', A_4' = 0$. Then

$$\text{rank} \begin{bmatrix} A_1' & A_2' & A_{12} \\ 0 & 0 & A_{22} \\ 0 & 0 & 0 \end{bmatrix} = q \text{ (g)}$$

and

$$\text{rank} \begin{bmatrix} A_1' & A_2' & A_{21} & B_1' \\ 0 & 0 & A_{22} & B_2' \\ 0 & 0 & 0 & B_2 \end{bmatrix} = n \text{ (g)}$$

by the same argument as before.

Continue this reduction by considering next the generic rank of A_1' . Eventually a matrix $\hat{A}$ is considered where either $\hat{A} = 0$ or rank $(\hat{A}) = t_1$ (g) where $\hat{A}$ is of order $t_1 x t_1$, $1 \leq t_1 < n$. If $\hat{A} = 0$, the form (3.3) is obtained with $\bar{A}_{11} = 0$. From the form of the reduction , rank $\bar{A} = q$ (g) . Thus there exist q independent entries in $\bar{A}$. Therefore, $t_1 \leq n-q$. Replacing the zero entries of $\bar{A}$ with the original indeterminates yields (3.1) with $A_{11} = 0$.

If $\hat{A} \neq 0$, but not of Form I, then (3.3) results with $\bar{A}_{11} = \hat{A}$ and $v = 1$. Again (3.4) and (3.5) follow as above.

If $\hat{A} \neq 0$ and $\hat{A}$ is of Form I , the permutation operation of (3.7) yields (3.2). Since rank $\hat{A} = t_1$ (g) the form $\bar{A}_{11}$ of (3.3) is obtained as before. Again (3.4) and (3.5) follow since rank $\bar{A}_{11} = t_1$ (g).

Q.E.D.

Lemma 3.5

Assume rank [A B] = n (g) and the system (A,B) is structurally uncontrollable. Then the matrix [A B] has Form I.

Proof

There are three cases to consider depending on the properties of A .

First, assume rank A = n (g) and A is not of Form I. Then it can be shown that (A,b) must be structurally controllable, where b is any column of B . But this contradicts the assumption of structural uncontrollability of (A,B) and therefore cannot occur.

The second case is when rank A = n (g) and A has Form I. Then there exists a permutation P which puts [A B] in the form

$$[P^T A P P^T B] = \begin{bmatrix} A_{11} & 0 & 0 & \cdots & 0 & B_1 \\ A_{21} & A_{22} & 0 & & \vdots & B_2 \\ A_{31} & A_{32} & A_{33} & & \vdots & B_3 \\ \vdots & & & \ddots & 0 & \vdots \\ A_{v1} & \cdots & \cdots & \cdots & A_{vv} & B_v \end{bmatrix}$$

where the A_{ii} have full generic rank and are not of Form I. Furthermore, it is possible to choose parameter values for the A_{ii} , say $\widetilde{A}_{ii}$, such that $\sigma(\widetilde{A}_{ii}) \cap \sigma(\widetilde{A}_{jj}) = \emptyset$ for $i \neq j$, and it follows from Lemmas 3.2 and 3.3 that the left eigenvectors of $\widetilde{A}$ are unique up to multiplicity and of the form

$$x_i^T = (\; y_i^T \quad \tilde{x}_i^T \; 0 \; \ldots \; 0) \tag{3.8}$$

where $\tilde{x}_i$ is a left eigenvector of $\widetilde{A}_{ii}$ having no zero components and $\widetilde{A}$ is the permuted A matrix with A_{ii} replaced by $\widetilde{A}_{ii}$.

Since (A,B) is structurally uncontrollable, there is a left eigenvector of the form (3.8) which satisfies

$$x_i^T[P^T B] \equiv 0$$

If i is the first value of the index for which this is true, it is easy to show that $B_i = 0$ and if $i > 1$ the system

$$\begin{bmatrix} A_{11} & 0 & 0 \; . \; . \; . & 0 \\ A_{21} & A_{22} & 0 \; . \; . \; . & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & & & 0 \\ A_{i-1,1} & \cdots & \cdots & A_{i-1,i-1} \end{bmatrix} \begin{bmatrix} B_1 \\ \vdots \\ \vdots \\ \vdots \\ B_{i-1} \end{bmatrix}$$

is structurally controllable and , $[A_{i1} \; \ldots \; A_{i,i-1}] = 0$. Therefore, the permutation operation which exchanges $\widetilde{A}_{11}$ and $\widetilde{A}_{ii}$ yields a system in Form I .

The third case is when rank $A = q$ (g) with $q < n$. By Lemma 3.4 there exists a permutation operation P such that A and B have the forms given by (3.1) and (3.2). The following proof that [A B] must be of Form I is accomplished by choosing a particular set of parameter values and showing that if certain other parameters are not zero then (A,B) is structurally controllable.

From Lemma 3.4 we replace the form (3.1) by (3.3). If $\bar{A}_{11} = 0$, all the eigenvalues of $\bar{A}$ are zero. Then since

$$\text{rank}[\bar{A}\ \bar{B}] = n \ (g)$$

it is possible to choose parameter values $\bar{A}'$ and $\bar{B}'$ such that $(\bar{A}',\bar{B}')$ is a controllable system. This contradicts the assumption that (A,B) is structurally uncontrollable. Therefore assume $\bar{A}_{11} \neq 0$ and has the form (3.2) which is the same form as $\tilde{A}$ in the second case of this proof. Clearly we have set certain parameters to zero in $\bar{A}$ and fixed the diagonal blocks in $\bar{A}_{11}$. Since rank $\bar{A}_{11} = t_1$ (g) and $\bar{A}_{11}$ has the form (3.2), it follows from Lemma 3.2 and Lemma 3.3 that the left eigenvectors of $\bar{A}'_{11}$ are unique up to multiplicity and of the form (3.8), where $\bar{A}'_{11}$ denotes the parameter point chosen. Fix the remaining entries of $\bar{A}$, yielding $\bar{A}'$, satisfying rank $\bar{A}' = q$. Then for any eigenvalue s of $\bar{A}'_{11}$

$$\text{rank}(\bar{A}'-sI) = n-1$$

Therefore the left eigenvectors of $\bar{A}'$ associated with eigenvalues of $\bar{A}'_{11}$ are unique up to multiplicity. Consider the polynomial in the nonzero entries of $\bar{B}$ defined by

$$\psi = \prod_{i=1}^{j} \sum_{k=1}^{r} (x_i^T b_k)^2$$

where x_i is a left eigenvector of $\bar{A}'$, j is the number of eigenvectors of $\bar{A}'$ and b_k is the k^{th} column of $\bar{B}$. Assume no x_i, $1 \leq i \leq j$ satisfies

$$x_i^T\bar{B} \equiv 0 .$$

Then, since the x_i associated with the eigenvalues of $\bar{A}'_{11}$ are unique up to multiplicity ψ is a nonconstant polynomial in the nonzero entries of $\bar{B}$. Therefore the nonzero entries of $\bar{B}$ can be chosen, yielding $\bar{B}'$, such that $\psi \neq 0$. This implies $(\bar{A}',\bar{B}')$ is controllable. Since this contradicts the assumption that (A,B) is structurally uncontrollable, it follows that some $x = x_i$, $1 \leq i \leq j$ must satisfy

$$x^T\bar{B} \equiv 0$$

Note since B_{k+1} has no zero rows, the last n-q components of x must be zero. The following q x q submatrix of $(\bar{A}'-sI)$, see (3.3), satisfies

$$\text{rank}\begin{bmatrix} \bar{A}'_{11}-sI & A_{12} & & \cdots & & A_{1k} \\ 0 & -sI & A_{23} & & & \\ & & & \cdots & & \\ & & & & -sI & A_{k-1,k} \\ & & & & & -sI \end{bmatrix} = q-1 \qquad (3.9)$$

for s an eigenvalue of $\bar{A}'_{11}$. Therefore x is independent of the entries in

$$\begin{bmatrix} A_{1,k+1} \\ \cdot \\ \cdot \\ \cdot \\ A_{k,k+1} \end{bmatrix} \qquad (3.10)$$

in the following sense. Fix $\bar{A}$ as above except for the entries of (3.10). Choose these entries, yielding an $\bar{A}''$, such that rank $\bar{A}'' = q$ and $\bar{A}' \neq \bar{A}''$. From the above analysis there exists a left eigenvector of $\bar{A}''$ satisfying

$$\bar{x}^T\bar{B} \equiv 0$$

But since the last n-q components of $\bar{x}$ must be zero and since (3.9) is true, it follows that

$$\bar{x} = x \text{ (mod a constant)} \tag{3.11}$$

Therefore x is an eigenvector for any choice of (3.10) subject to the resulting matrix having rank q. Consider (3.10) as a submatrix of zeros and indeterminates in the matrix $\bar{A}'$. Define the polynomial φ in the nonzero entries of (3.10) as

$$\varphi = \sum_{i=1}^{n-q} (x^T a_i)^2 \cdot \delta$$

where a_i is the i^{th} column of (3.10) and δ is the sum of the squares of all q^{th} order minors of $\bar{A}'$. Assume there exist nonzero rows of (3.10) corresponding to nonzero components of x. This means that

$$\sum_{i=1}^{n-q} (x^T a_i)^2$$

is a nonconstant polynomial in the nonzero entries of (3.10). Also since rank $(\bar{A}') = q$, δ is a nonconstant polynomial in these entries and it follows therefore that φ is, too. So numerical values of the nonzero entries of (3.10) can be chosen, yielding an $\bar{A}''$, such that $\varphi \neq 0$, which implies $\delta \neq 0$ and therefore rank $\bar{A}'' = q$. From the above analysis this implies x, of (3.11), is a left eigenvector of $\bar{A}''$ associated with an eigenvalue of $\bar{A}'_{11}$. But $\varphi \neq 0$ also implies

$$x^T(\bar{A}''-sI) \neq 0$$

which means x is not a left eigenvector of $\bar{A}''$. This contradiction establishes the fact that the rows of (3.10) corresponding to nonzero entries of x must be composed entirely of fixed zero entries. Then, since (3.10) is chosen such that $\delta \neq 0$,

$$\text{rank}\,(A_{k,k+1}) = t_k\,(g)$$

and therefore the last $t_k + (n-q)$ components of x must be zero. But this implies x is independent of the nonzero entries of

$$(A_{1k}^T, \ldots, A_{k-1,k}^T)^T$$

since

$$\text{rank}\begin{bmatrix} \bar{A}'_{11}-sI & A_{12} & \cdots & A_{1k-1} \\ 0 & -sI & \cdots & \\ & & \cdots & \\ & & -sI & A_{k-2,k-1} \\ & & & -sI \end{bmatrix} = q-t_k-1$$

Continuing the above analysis x is eventually obtained as a vector of the form $(x'^T 0 \ldots 0)$ where x' has the structure of (3.8) and every row of the matrix

$$(A_{12} \quad A_{13} \quad \cdots \quad A_{1,k+1})$$

corresponding to nonzero entries of x' is composed entirely of fixed zero entries. Now as in the second case of this proof, the permutation operation which exchanges $\tilde{A}_{11}$ and $\tilde{A}_{ii}$ yields a system in Form I. Clearly all the indeterminates which were set to zero can now be replaced without changing the fact that [A B] has Form I.

Q.E.D.

Proof of Theorem 3.1

(Sufficiency) If the matrix [A B] has Form I , it is obvious that every data point $p \in R^{N+M}$ is uncontrollable in the usual sense. If [A B] has Form II, then by Lemma 2.2

$$\text{rank } [A \quad B] < n$$

for every $p \in R^{N+M}$. Therefore, (A,B) is uncontrollable for every data point.

(Necessity) Consider the case where $\text{rank } [A \quad B] < n$ for every $p \in R^{N+M}$. Then, by Lemma 2.2, the matrix $[A \quad B]$ must have Form II and the Theorem is valid for this case. Therefore assume $\text{rank } [A \quad B] = n$ for some $p \in R^{N+M}$. Then $\text{rank } [A \quad B] = n\ (g)$ and from Lemma 3.5 $[A \quad B]$ must be of Form I.

Q.E.D.

Theorem 3.1 represents the complete solution to the structural controllability problem. It only remains to consider a computational method of checking the conditions of the Theorem for a given system structure.

4. COMPUTATIONAL CONSIDERATIONS

A computer algorithm to determine structural controllability using the results of Theorem 3.1 must be capable of recognizing Form I and Form II. However, it is possible to determine structural controllability using FZRF, thus avoiding Form I.

<u>Definition</u>: The extended controllability matrix of a system is the $n^2 \times n(n+r-1)$ matrix $\bar{R}$ defined by

$$\begin{bmatrix} B & I & 0 & \cdots & & & \cdot & \cdot \\ 0 & -A & B & I & 0 & \cdot & & \cdot & \cdot \\ 0 & 0 & -A & B & I & 0 & & \cdot & \cdot \\ & & \cdots & & \cdot\cdot & & \\ & & \cdots & & \cdot\cdot & & \\ 0 & \cdot & \cdot & & -A & B & I & 0 \\ 0 & \cdot & \cdot & & & & -A & B \end{bmatrix} \qquad (4.1)$$

Lemma 4.1 (Rosenbrock, pp 72-73, [12])

Consider $p \in R^{N+M}$. For this data point the associated system (A,B) is controllable if and only if

$$\text{rank } \bar{R} = n^2$$

Theorem 4.1

The following conditions on the system (A,B) are equivalent:

1) (A,B) is structurally uncontrollable.

2) The matrix [A B] is of Form I or Form II.

3) The matrix $\bar{R}$ is of form (n^2).

Proof

The equivalence of Statements 1) and 2) has already been proved.

Assume Statement 2) is true. If the matrix [A B] has Form I, then there exists a permutation operation such that this matrix has the form

$$\begin{bmatrix} A_1 & 0 & 0 \\ A_2 & A_3 & B_2 \end{bmatrix} \tag{4.2}$$

where A_1 is of order $w \times w$, $1 \leq w < n$. (4.2) implies $\bar{R}$ contains nw rows and $n(n+r-1)-w(n-1)$ columns defining a zero submatrix. However, $\bar{R}$ requires only $w(n-1) + 1$ such rows to be of form (n^2). Since $w \geq 1$, $nw \geq w(n-1) + 1$ and it follows that $\bar{R}$ is of form (n^2).

If [A B] is of Form II, then the matrix obtained by adding $n(n+r-1)-(n-r)$ columns of zeros is still of Form II. Since this matrix is simply the last n rows of $\bar{R}$, the desired result follows.

Assume Statement 3) is true. Then rank $\bar{R} < n^2$ for every $p \in R^{N+M}$

from Lemma 2.1. Thus from Lemma 4.1 the system (A,B) is uncontrollable for every $p \in R^{N+M}$.

Q.E.D.

Theorem 4.1 allows FZRF to be used on the extended controllability matrix to determine structural controllability. This relatively simple procedure is the major accomplishment of this paper.

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APPENDIX

Let A be an $n \times m$ structured matrix with $n \leq m$. The following algorithm determines the largest integer t for which there does not exist a k, $m - t < k \leq m$ such that A contains a zero submatrix of order $(n+m-t-k+1) \times k$; i.e., A is of form (j), for $j > t$, but not of form (t). Then by Theorem 2.2, the generic rank of A is t.

Step 1: Set the integer variable v to 1 and $A(0) = A$.

Step 2: Search $A(0)$ for any rows composed entirely of zeros. Assume there exists $i(0)$ such rows. Form the matrix $A(1)$ by removing these $i(0)$ zero rows from $A(0)$.

Step 3: Identify a row in $A(v)$ having the least number of nonzero entries and all rows structurally equivalent to this row. Let $j(v)$ be the number of nonzero entries and $i(v)$ the number of such structurally equivalent rows.

Step 4: Using row and column exchanges, put $A(v)$ into the form

$$\begin{bmatrix} R_k & 0 \\ A_1 & A_2 \end{bmatrix} \tag{A.1}$$

where R_k is of order $i(v) \times j(v)$ and contains the nonzero entries of rows identified in Step 3.

Step 5: Set $v = v+1$.

Step 6: If A_2 is of a nontrivial order, i.e., having one or more rows and columns, let $A(v) = A_2$ and go to Step 3. Otherwise go to Step 7.

Step 7: The A matrix is now of the form

$$\begin{bmatrix} 0 & \cdots\cdots\cdots\cdots & & 0 \\ R_1 & 0 & \cdots\cdots\cdots & 0 \\ X & R_2 & 0 \cdots\cdots & 0 \\ X & X & R_3 \cdots\cdots & 0 \\ X & \cdots & \cdots & \\ X & \cdots & R_w & \end{bmatrix} \qquad (A.2)$$

where R_v is of order $i(v) \times j(v)$ with $v = 1,\ldots,w$ and $w \leq n$. Then the generic rank of A is obtained from

$$t = n - \max_{0\leq q\leq w} \left[\sum_{s=0}^{q} [i(s)-j(s)],0\right]^*$$

with $j(0)=0$. This follows since

$$\max_{0\leq q\leq w} \left[\sum_{s=0}^{q} [i(s)-j(s)],0\right]$$

is the maximum number of generically dependent rows in A . Otherwise, from Theorem 2.2, for some $j < t$ there exists a k, $m - j < k \leq m$ defining a zero submatrix of order $(n+m-j-k+1) \times k$. But from the method of the above algorithm and since $j < t$, this implies there exist zero entries in some R_q of (A.2) or there exist additional zero rows of A . Since neither of these possibilities occur, rank $A = t$ (g) .

Step 8: End of algorithm.

Consider the following example of the application of the above algorithm.

---

* An equivalent expression is $t = \sum_{q=1}^{w} \min(i(q),j(q))$.

Assume Step 7 has been reached and A has the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X & X & 0 & 0 & 0 & 0 & 0 & 0 \\ X & X & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & X & X & 0 & 0 & 0 & 0 & 0 \\ X & 0 & X & X & X & X & X & 0 \end{bmatrix}$$

where $n = 6$, $m = 8$, the X's represent nonzero entries and

$$R_1 = X \ , \ i(1) = j(1) = 1 \ ;$$

$$R_2 = \begin{bmatrix} X \\ \\ X \end{bmatrix}, \ i(2) = 2, \ j(2) = 1 \ ;$$

$$R_3 = [X] \ , \ i(3) = j(3) = 1 \ ;$$

$$R_4 = [X \quad X \quad X \quad X] \ , \ i(4) = 1 \ , \ j(4) = 4 \ ;$$

and $i(0) = 1$.

Then with $\mathbf{W} = 4$ and $j(0) = 0$

$$\max_{0 \leq q \leq 4} \left[\sum_{s=0}^{q} [i(s) - j(s)], 0 \right] = \max(1,1,2,2,-1) = 2 \ .$$

Therefore, rank $A = 4$ (g) .