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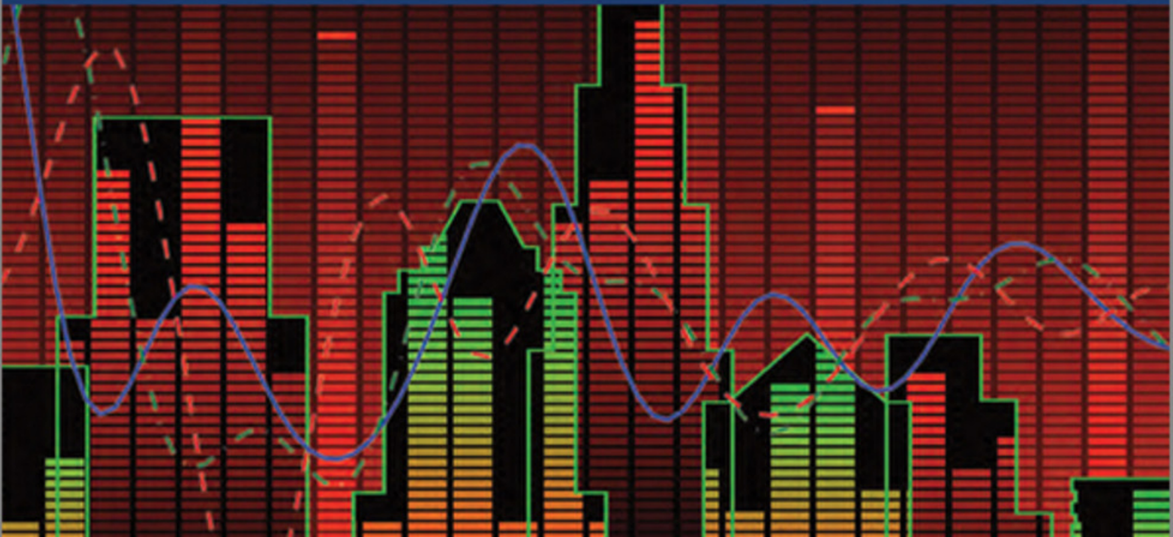
Structural Dynamic Analysis with Generalized Damping Models

Identification

Sondipon Adhikari



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To
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Sunanda Adhikari
and
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Preface

Among the various ingredients of structural dynamics, damping is one of the least understood topics. The main reason is that unlike the stiffness and inertia forces, damping forces cannot always be obtained from "first principles". The past two decades have seen significant developments in the modeling and analysis of damping in the context of engineering dynamic systems. Developments in composite materials including nanocomposites and their applications in advanced structures, such as new generation of aircrafts and large wind turbines, have led to the need for understanding damping in a better manner. Additionally, the rise of vibration energy harvesting technology using piezoelectric and electromagnetic principles further enhanced the importance of looking at damping more rigorously. The aim of this book is to systematically present the latest developments in the modeling and analysis of damping in the context of general linear dynamic systems with multiple degrees-of-freedom. The focus has been on the mathematical and computational aspects. This book will be relevant to aerospace, mechanical and civil engineering disciplines and various sub-disciplines within them. The intended readers of this book include senior undergraduate students and graduate students doing projects or doctoral research in the field of damped vibration. Researchers, professors and practicing engineers working in the field of advanced vibration will find this book useful. This book will also be useful for researchers working in the fields of aeroelasticity and hydroelasticity, where complex eigenvalue problems routinely arise due to fluid–structure interactions.

There are some excellent books which already exist in the field of damped vibration. The book by Nashif *et al.* [NAS 85] covers various material damping models and their applications in the design and analysis of dynamic systems. A valuable reference on dynamic analysis of damped structures is [SUN 95]. The book by Beards [BEA 96] takes a pedagogical approach toward structural vibration of damped systems. The handbook by Jones [JON 01] focuses on viscoelastic damping and analysis of structures with such damping models. These books represented the state of the art at the time of their publications. Since these publications, significant research works have gone into the dynamics of damped systems. The aim of this book is to cover some of these latest developments. The attention is mainly limited to theoretical and computational aspects, although some references to experimental works are given.

One of the key features of this book is the consideration of general non-viscous damping and how such general models can be seamlessly integrated into the framework of conventional structural dynamic analysis. New results are illustrated by numerical examples and, wherever possible, connections are made to well-known concepts of viscously damped systems. A related title, Structural Dynamic Analysis with Generalized Damping Models: Analysis [ADH 14], is complementary to this book, and, indeed, they could have been presented together. However, for practical reasons, it has proved more convenient to present the material separately.

The related book, Structural Dynamic Analysis with Generalized Damping Models: Analysis [ADH 14] focuses on the analysis of linear systems with general damping models. This book, Structural Dynamic Analysis with Generalized Damping Models: Identification, deals with the identification and quantification of damping. There are ten chapters and one appendix in the two volumes combined ? covering analysis and identification of dynamic systems with viscous and non-viscous damping.

In [ADH 14] Chapter 1 gives an introduction to the various damping models. Dynamics of viscously damped systems are discussed in Chapter 2. Chapter 3 considers dynamics of non-viscously damped single-degree-of-freedom systems in detail. Chapter 4 discusses non-viscously damped multiple degree-of-freedom systems. Linear systems with general non-viscous damping are studied in Chapter 5. Chapter 6 proposes reduced computational methods for damped systems. A method to deal with general asymmetric systems is described in the appendix.

In this book, Structural Dynamic Analysis with Generalized Damping Models: Identification, Chapter 1 describes parametric sensitivity of damped systems. Chapter 2 describes the problem of identification of viscous damping. The identification of non-viscous damping is detailed in Chapter 3. Chapter 4 gives some tools for the quantification of damping.

This book is the result of the last 15 years of research and teaching in the area of damped vibration problems. Initial chapters started taking shape when I offered a course on advanced vibration at the University of Bristol. The later chapters originated from the research work with numerous colleagues, students, collaborators and mentors. I am deeply indebted to all of them for numerous stimulating scientific discussions, exchanges of ideas and, on many occasions, direct contributions toward

the intellectual content of the book. I am grateful to my teachers Professor C. S. Manohar (Indian Institute of Science, Bangalore), Professor R. S. Langley (University of Cambridge) and, in particular, Professor J. Woodhouse (University of Cambridge), who was heavily involved with the works reported in Chapters 2–4 of this book. I am very thankful to my colleague Professor M. I. Friswell with whom I have a long-standing collaboration. Some joint works are directly related to the content of this book (Chapter 1 of this book in particular). I would also like to thank Professor D. J. Inman (University of Michigan) for various scientific discussions during his visits to Swansea. I am thankful to Professor A. Sarkar (Carleton University) and his doctoral student M. Khalil for joint research works. I am deeply grateful to Dr A. S. Phani (University of British Columbia) for various discussions related to damping identification and contributions toward Chapters 2 and 5 of [ADH 14] and Chapter 2 of this book. Particular thanks go to Dr N. Wagner (Intes GmbH, Stuttgart) for joint works on non-viscously damped systems and contributions in Chapter 4 of [ADH 14]. I am also grateful to Professor F. Papai for involving me in research works on damping identification. My former PhD students B. Pascual (contributed in Chapter 6 of [ADH 14]), J. L. du Bois and F. A. Diaz De la O deserve particular thanks for various contributions throughout their time with me and putting up with my busy schedules. I am grateful to Dr Y. Lei (University of Defense Technology, Changsha) for carrying out joint research with me on non-viscously damped continuous systems. I am grateful to Professor A. W. Lees (Swansea University), Professor N. Lieven, Professor F. Scarpa (University of Bristol), Professor D. J. Wagg (University of Sheffield), Professor S. Narayanan (Indian Institute of Technology (IIT) Madras), Professor G. Litak (Lublin University), E. Jacquelin (Université Lyon), Dr A. Palmeri (Loughborough University), Professor S. Bhattacharya (University of Surrey), Dr S. F. Ali (IIT Madras), Dr R. Chowdhury (IIT Roorkee), Dr P. Duffour (University College London), and Dr P. Higino, Dr G. Caprio and Dr A. Prado (Embraer Aircraft) for their intellectual contributions and discussions at different times. Besides the names mentioned here, I am also thankful to many colleagues, fellow researchers and students working in this field of research around the world, whose names cannot be listed here due to page limitations. The lack of explicit mentions by no means implies that their contributions are any lesser. The opinions presented in the book are entirely mine, and none of my colleagues, students, collaborators and mentors have any responsibility for any shortcomings.

I have been fortunate to receive grants from various companies, charities and government organizations including an Advanced Research Fellowship from UK Engineering and Physical Sciences Research Council (EPSRC), the Wolfson Research Merit Award from the Royal Society and the Philip Leverhulme Prize from the Leverhulme Trust. Without these findings, it would have been impossible to have conducted the works leading to this book. Finally, I want to thank my colleagues at the College of Engineering at Swansea University. Their support proved to be a key factor in materializing the idea of writing this book.

Last, but by no means least, I wish to thank my wife Sonia and my parents for their constant support, encouragement and putting up with my ever-increasing long periods of “non-engagement” with them.

Sondipon ADHIKARI
October 2013

Nomenclature

C'_{jj}	diagonal element of the modal damping matrix
$\alpha_k^{(j)}$	terms in the expansion of approximate complex modes
α_1, α_2	proportional damping constants
α_j	coefficients in Caughey series, $j = 0, 1, 2, \cdots$
$\mathbf{0}_j$	a vector of j zeros
$\mathbf{A}$	state-space system matrix
$\mathbf{a}_j$	a coefficient vector for the expansion of jth complex mode
$\boldsymbol{\alpha}$	a vector containing the constants in Caughey series
$\bar{h}(\mathrm{i}\omega)$	frequency response function of an SDOF system
$\mathbf{B}$	state-space system matrix
$\mathbf{b}_j$	a vector for the expansion of jth complex mode
$\bar{\mathbf{f}}(s)$	forcing vector in the Laplace domain
$\bar{\mathbf{f}}'(s)$	modal forcing function in the Laplace domain
$\bar{\mathbf{p}}(s)$	effective forcing vector in the Laplace domain
$\bar{\mathbf{q}}(s)$	response vector in the Laplace domain
$\bar{\mathbf{u}}(s)$	Laplace transform of the state-vector of the first-order system
$\bar{\mathbf{y}}(s)$	modal coordinates in the Laplace domain
$\bar{\mathbf{y}}_k$	Laplace transform of the internal variable $\mathbf{y}_k(t)$
$\mathbb{R}^+$	positive real line
$\mathbf{C}$	viscous damping matrix
$\mathbf{C}'$	modal damping matrix
$\mathbf{C}_0$	viscous damping matrix (with a non-viscous model)
$\mathbf{C}_k$	coefficient matrices in the exponential model for $k = 0, ..., n$, where n is the number of kernels

$\boldsymbol{\mathcal{G}}(t)$	non-viscous damping function matrix in the time domain
$\boldsymbol{\Delta}\mathbf{K}$	error in the stiffness matrix
$\boldsymbol{\Delta}\mathbf{M}$	error in the mass matrix
β	non-viscous damping factor
β_c	critical value of β for oscillatory motion, $\beta_c = \frac{1}{3\sqrt{3}}$
$\beta_i(\bullet)$	proportional damping functions (of a matrix)
$\beta_k(s)$	coefficients in the state-space modal expansion
β_{mU}	the value of β above which the frequency response function always has a maximum
$\mathbf{F}$	linear matrix pencil with time step in state-space, $\mathbf{F} = \mathbf{B} - \frac{h}{2}\mathbf{A}$
$\mathbf{F}_1, \mathbf{F}_2$	linear matrix pencils with time step in the configuration space
$\mathbf{F}_j$	regular linear matrix pencil for the jth mode
$\mathbf{f}'(t)$	forcing function in the modal coordinates
$\mathbf{f}(t)$	forcing function
$\mathbf{G}(s)$	non-viscous damping function matrix in the Laplace domain
$\mathbf{G}_0$	the matrix $\mathbf{G}(s)$ at $s \to 0$
$\mathbf{G}_\infty$	the matrix $\mathbf{G}(s)$ at $s \to \infty$
$\mathbf{H}(s)$	frequency response function matrix
$\hat{\mathbf{u}}_j$	real part of $\hat{\mathbf{z}}_j$
$\hat{\mathbf{v}}_j$	imaginary part of $\hat{\mathbf{z}}_j$
$\hat{\mathbf{z}}_j$	jth measured complex mode
$\mathbf{I}$	identity matrix
$\mathbf{K}$	stiffness matrix
$\mathbf{M}$	mass matrix
$\mathbf{O}_{ij}$	a null matrix of dimension $i \times j$
$\boldsymbol{\Omega}$	diagonal matrix containing the natural frequencies
$\mathbf{p}$	parameter vector (in Chapter 1)
$\mathbf{P}_j$	a diagonal matrix for the expansion of jth complex mode
$\boldsymbol{\phi}_j$	eigenvectors in the state-space
$\boldsymbol{\psi}_j$	left eigenvectors in the state-space
$\mathbf{q}(t)$	displacement response in the time domain
$\mathbf{q}_0$	vector of initial displacements
$\mathbf{Q}_j$	an off-diagonal matrix for the expansion of jth complex mode
$\mathbf{r}(t)$	forcing function in the state-space
$\mathbf{R}_k$	rectangular transformation matrices (in Chapter 4, [ADH 14])

$\mathbf{R}_k$	residue matrix associated with pole s_k
$\mathbf{S}$	a diagonal matrix containing eigenvalues s_j
$\mathbf{T}$	a temporary matrix, $\mathbf{T} = \sqrt{\mathbf{M}^{-1}\mathbf{K}}$ (Chapter 2)
$\mathbf{T}_k$	Moore-Penrose generalized inverse of $\mathbf{R}_k$
$\mathbf{T}_k$	a transformation matrix for the optimal normalization of the kth complex mode
$\mathbf{\Theta}$	normalization matrix
$\mathbf{u}(t)$	the state-vector of the first-order system
$\mathbf{u}_0$	vector of initial conditions in the state-space
$\mathbf{u}_j$	displacement at the time step j
$\mathbf{v}(t)$	velocity vector $\mathbf{v}(t) = \dot{\mathbf{q}}(t)$
$\mathbf{v}_j$	a vector of the j-modal derivative in Nelson's methods (in Chapter 1)
$\mathbf{v}_j$	velocity at the time step j
$\boldsymbol{\varepsilon}_j$	error vector associated with jth complex mode
$\boldsymbol{\varphi}_k(s)$	eigenvectors of the dynamic stiffness matrix
$\mathbf{W}$	coefficient matrix associated with the constants in Caughey series
$\mathbf{X}$	matrix containing the undamped normal modes $\mathbf{x}_j$
$\mathbf{x}_j$	undamped eigenvectors, $j = 1, 2, \cdots, N$
$\mathbf{y}(t)$	modal coordinate vector (in Chapter 2, [ADH 14])
$\mathbf{y}_k(t)$	vector of internal variables, $k = 1, 2, \cdots, n$
$\mathbf{y}_{k,j}$	internal variable $\mathbf{y}_k$ at the time step j
$\mathbf{Z}$	matrix containing the complex eigenvectors $\mathbf{z}_j$
$\mathbf{z}_j$	complex eigenvectors in the configuration space
$\boldsymbol{\zeta}$	diagonal matrix containing the modal damping factors
$\boldsymbol{\zeta}_v$	a vector containing the modal damping factors
χ	merit function of a complex mode for optimal normalization
χ_R, χ_I	merit functions for real and imaginary parts of a complex mode
Δ	perturbation in the real eigenvalues
δ	perturbation in complex conjugate eigenvalues
$\dot{q}_0$	initial velocity (SDOF systems)
ϵ	small error
η	ratio between the real and imaginary parts of a complex mode
$\mathcal{F}$	dissipation function
γ	non-dimensional characteristic time constant
γ_j	complex mode normalization constant

γ_R, γ_I	weights for the normalization of the real and imaginary parts of a complex mode
$\hat{\theta}(\omega)$	frequency-dependent estimated characteristic time constant
$\hat{\theta}_j$	estimated characteristic time constant for jth mode
$\hat{t}$	an arbitrary independent time variable
κ_j	real part of the complex optimal normalization constant for the jth mode
λ	complex eigenvalue corresponding to the oscillating mode (in Chapter 3, [ADH 14])
λ_j	complex frequencies MDOF systems
$\mathcal{M}_r$	moment of the damping function
$\mathcal{D}$	dissipation energy
$\mathcal{G}(t)$	non-viscous damping kernel function in an SDOF system
$\mathcal{T}$	kinetic energy
$\mathcal{U}$	potential energy
μ	relaxation parameter
μ_k	relaxation parameters associated with coefficient matrix $\mathbf{C}_k$ in the exponential non-viscous damping model
ν	real eigenvalue corresponding to the overdamped mode
$\nu_k(s)$	eigenvalues of the dynamic stiffness matrix
ω	driving frequency
ω_d	damped natural frequency of SDOF systems
ω_j	undamped natural frequencies of MDOF systems, $j = 1, 2, \cdots, N$
ω_n	undamped natural frequency of SDOF systems
$\omega_{\max}$	frequency corresponding to the maximum amplitude of the response function
ω_{d_j}	damped natural frequency of MDOF systems
ρ	mass density
i	unit imaginary number, $\mathrm{i} = \sqrt{-1}$
τ	dummy time variable
θ_j	characteristic time constant for jth non-viscous model
$\tilde{\mathbf{f}}(t)$	forcing function in the modal domain
$\tilde{\omega}$	normalized frequency ω/ω_n
ς_j	imaginary part of the complex optimal normalization constant for the jth mode
ϑ	phase angle of the response of SDOF systems

ϑ_j	phase angle of the modal response
ψ	a trail complex eigenvector (in Chapter 2, [ADH 14])
$\widehat{\mathbf{A}}$	asymmetric state-space system matrix
$\widehat{\mathbf{C}}$	fitted damping matrix
$\widehat{f}(\omega_j)$	fitted generalized proportional damping function (in Chapter 2)
$\widetilde{\mathbf{A}}$	state-space system matrix for rank-deficient systems
$\widetilde{\mathbf{B}}$	state-space system matrix for rank-deficient systems
$\widetilde{\mathbf{i}}_r$	integration of the forcing function in the state-space for rank-deficient systems
$\widetilde{\mathbf{i}}_r$	integration of the forcing function in the state-space
$\widetilde{\mathbf{\Phi}}$	matrix containing the state-space eigenvectors for rank-deficient systems
$\widetilde{\boldsymbol{\phi}}_j$	eigenvectors in the state-space for rank-deficient systems
$\widetilde{\mathbf{r}}(t)$	forcing function in the state-space for rank-deficient systems
$\widetilde{\mathbf{u}}(t)$	the state vector for rank-deficient systems
$\widetilde{\mathbf{y}}_k(t)$	vector of internal variables for rank-deficient systems, $k = 1, 2, \cdots, n$
$\widetilde{\mathbf{y}}_{k,j}$	internal variable $\mathbf{y}_k$ at the time step j for rank-deficient systems
$\widetilde{\boldsymbol{y}}_{k_j}$	jth eigenvector corresponding to the kth the internal variable for rank-deficient systems
ξ	a function of ζ defined in equation [3.132] (Chapter 3, [ADH 14])
ζ	viscous damping factor
ζ_c	critical value of ζ for oscillatory motion, $\zeta_c = \frac{4}{3\sqrt{3}}$
ζ_j	modal damping factors
ζ_L	lower critical damping factor
ζ_n	equivalent viscous damping factor
ζ_U	upper critical damping factor
ζ_{mL}	the value of ζ below which the frequency response function always has a maximum
a_k, b_k	non-viscous damping parameters in the exponential model
B	response amplitude of SDOF systems
B_j	modal response amplitude
c	viscous damping constant of an SDOF system
c_k	coefficients of exponential damping in an SDOF system
c_{cr}	critical damping factor
d_j	a constant of the j-modal derivative in Nelson's methods

E	Young's modulus
$f(t)$	forcing function (SDOF systems)
$f_d(t)$	non-viscous damping force
$G(\mathrm{i}\omega)$	non-dimensional frequency response function
$G(s)$	non-viscous damping kernel function in the Laplace domain (SDOF systems)
$g_{(i)}$	scalar damping functions, $i = 1, 2, \cdots$
h	constant time step
$h(t)$	impulse response function of SDOF systems
$h(t)$	impulse response function
I_k	non-proportionally indices, $k1 = 1, 2, 3, 4$
k	spring stiffness of an SDOF system
L	length of the rod
l_e	length of an element
m	dimension of the state-space for non-viscously damped MDOF systems
m	mass of an SDOF system
N	number of degrees of freedom
n	number of exponential kernels
n_d	number of divisions in the time axis
p	any element in the parameter vector **p** (in Chapter 1)
$q(t)$	displacement in the time domain
q_0	initial displacement (SDOF systems)
Q_{nc_k}	non-conservative forces
$R(\mathbf{x})$	Rayleigh quotient for a trail vector **x**
R_1, R_2, R_3	three new Rayleigh quotients
r_j	normalized eigenvalues of non-viscously damped SDOF systems (in Chapter 3, [ADH 14])
r_k	rank of $\mathbf{C}_k$ matrices
s	Laplace domain parameter
s_j	eigenvalues of dynamic systems
t	time
T_n	natural time period of an undamped SDOF system
T_{min}	minimum time period for the system
$varrho_j$	complex optimal normalization constant for the jth mode
x	normalized frequency-squared, $x = \omega^2/\omega_n^2$ (in Chapter 3, [ADH 14])

y_j	modal coordinates (in Chapter 3, [ADH 14])
$\bar{f}(s)$	forcing function in the Laplace domain
$\bar{q}(s)$	displacement in the Laplace domain
$\hat{\mathbf{U}}$	matrix containing $\hat{\mathbf{u}}_j$
$\hat{\mathbf{V}}$	matrix containing $\hat{\mathbf{v}}_j$
$\mathbf{\Phi}$	matrix containing the eigenvectors $\boldsymbol{\phi}_j$
$\dot{\mathbf{q}}_0$	vector of initial velocities
$\mathcal{F}_i(\bullet, \bullet)$	non-viscous proportional damping functions (of a matrix)
$\boldsymbol{Y}_k$	a matrix of internal eigenvectors
$\boldsymbol{y}_{kj}$	jth eigenvector corresponding to the kth the internal variable
PSD	power spectral density
$\mathbf{0}$	a vector of zeros
$\mathcal{L}$	Lagrangian (in Chapter 3, [ADH 14])
$\delta(t)$	Dirac-delta function
δ_{jk}	Kroneker-delta function
$\Gamma(\bullet)$	gamma function
γ	Lagrange multiplier (in Chapter 3, [ADH 14])
$(\bullet)^*$	complex conjugate of $(\bullet)$
$(\bullet)^T$	matrix transpose
$(\bullet)^{-1}$	matrix inverse
$(\bullet)^{-T}$	matrix inverse transpose
$(\bullet)^H$	Hermitian transpose of $(\bullet)$
$(\bullet)_e$	elastic modes
$(\bullet)_{nv}$	non-viscous modes
$(\dot{\bullet})$	derivative with respect to time
$\mathbb{C}$	space of complex numbers
$\mathbb{R}$	space of real numbers
$\perp$	orthogonal to
$\mathcal{L}(\bullet)$	Laplace transform operator
$\mathcal{L}^{-1}(\bullet)$	inverse Laplace transform operator
$\det(\bullet)$	determinant of $(\bullet)$
$\mathrm{diag}\,[\bullet]$	a diagonal matrix
$\forall$	for all
$\Im(\bullet)$	imaginary part of $(\bullet)$
$\in$	belongs to

$\notin$	does not belong to
$\otimes$	Kronecker product
$\overline{(\bullet)}$	Laplace transform of $(\bullet)$
$\Re(\bullet)$	real part of $(\bullet)$
vec	vector operation of a matrix
$O(\bullet)$	in the order of
ADF	anelastic displacement field model
$\mathrm{adj}(\bullet)$	adjoint matrix of $(\bullet)$
GHM	Golla, Hughes and McTavish model
MDOF	multiple-degree-of-freedom
SDOF	single-degree-of-freedom

Chapter 1

Parametric Sensitivity of Damped Systems

Changes of the eigenvalues and eigenvectors of a linear vibrating system due to changes in system parameters are of wide practical interest. Motivation for this kind of study arises, on the one hand, from the need to come up with effective structural designs without performing repeated dynamic analysis, and, on the other hand, from the desire to visualize the changes in the dynamic response with respect to system parameters. Furthermore, this kind of sensitivity analysis of eigenvalues and eigenvectors has an important role to play in the area of fault detection of structures and modal updating methods. Sensitivity of eigenvalues and eigenvectors is useful in the study of bladed disks of turbomachinery where blade masses and stiffness are nearly the same, or deliberately somewhat altered (mistuned), and one investigates the modal sensitivities due to this slight alteration. Eigensolution derivatives also constitute a central role in the analysis of stochastically perturbed dynamical systems. Possibly, the earliest work on the sensitivity of the eigenvalues was carried out by Rayleigh [RAY 77]. In his classic monograph, he derived the changes in natural frequencies due to small changes in system parameters. Fox and Kapoor [FOX 68] have given exact expressions for the sensitivity of eigenvalues and eigenvectors with respect to any design variables. Their results were obtained in terms of changes in the system property matrices and the eigensolutions of the structure in its current state, and have been used extensively in a wide range of application areas of structural dynamics. Nelson [NEL 76] proposed an efficient method to calculate an eigenvector derivative, which requires only the eigenvalue and eigenvector under consideration. A comprehensive review of research on this kind of sensitivity analysis can be obtained in Adelman and Haftka [ADE 86]. A brief review of some of the existing methods for calculating sensitivity of the eigenvalues and eigenvectors is given in section 1.6 (Chapter 1, [ADH 14]).

The aim of this chapter is to consider parametric sensitivity of the eigensolutions of damped systems. We first start with undamped systems in section 1.1. Parametric sensitivity of viscously damped systems is discussed in section 1.2. In section 1.3, we discuss the sensitivity of eigensolutions of general non-viscously damped systems. In section 1.4, a summary of the techniques introduced in this chapter is provided.

1.1. Parametric sensitivity of undamped systems

The eigenvalue problem of undamped or proportionally damped systems can be expressed by

$$\mathbf{K}(\mathbf{p})\mathbf{x}_j = \lambda_j \mathbf{M}(\mathbf{p})\mathbf{x}_j \qquad [1.1]$$

where λ_j and $\mathbf{x}_j$ are the eigenvalues and the eigenvectors of the dynamic system. $\mathbf{M}(\mathbf{p}) : \mathbb{R}^m \mapsto \mathbb{R}^{N \times N}$ and $\mathbf{K}(\mathbf{p}) : \mathbb{R}^m \mapsto \mathbb{R}^{N \times N}$, the mass and stiffness matrices, are assumed to be smooth, continuous and differentiable functions of a parameter vector $\mathbf{p} \in \mathbb{R}^m$. Note that $\lambda_j = \omega_j^2$ where ω_j is the jth undamped natural frequency. The vector $\mathbf{p}$ may consist of material properties, e.g. mass density, Poisson's ratio and Young's modulus; or geometric properties, e.g. length, thickness and boundary conditions. The eigenvalues and eigenvectors are smooth differentiable functions of the parameter vector $\mathbf{p}$.

1.1.1. *Sensitivity of the eigenvalues*

We rewrite the eigenvalue equation as

$$[\mathbf{K} - \lambda_j \mathbf{M}]\, \mathbf{x}_j = \mathbf{0} \qquad [1.2]$$

$$\text{or} \quad \mathbf{x}_j^T \left[\mathbf{K} - \lambda_j \mathbf{M}\right]. \qquad [1.3]$$

The functional dependence of $\mathbf{p}$ is removed for notational convenience. Differentiating the eigenvalue equation [1.2] with respect to the element p of the parameter vector we have

$$\left[\frac{\partial \mathbf{K}}{\partial p} - \frac{\partial \lambda_j}{\partial p}\mathbf{M} - \lambda_j \frac{\partial \mathbf{M}}{\partial p}\right]\mathbf{x}_j + [\mathbf{K} - \lambda_j \mathbf{M}]\frac{\partial \mathbf{x}_j}{\partial p} = \mathbf{0}. \qquad [1.4]$$

Premultiplying by $\mathbf{x}_j^T$, we have

$$\mathbf{x}_j^T\left[\frac{\partial \mathbf{K}}{\partial p} - \frac{\partial \lambda_j}{\partial p}\mathbf{M} - \lambda_j \frac{\partial \mathbf{M}}{\partial p}\right]\mathbf{x}_j + \mathbf{x}_j^T[\mathbf{K} - \lambda_j \mathbf{M}]\frac{\partial \mathbf{x}_j}{\partial p} = \mathbf{0}. \qquad [1.5]$$

Using the identity in [1.3], we have

$$\mathbf{x}_j^T \left[\frac{\partial \mathbf{K}}{\partial p} - \frac{\partial \lambda_j}{\partial p}\mathbf{M} - \lambda_j \frac{\partial \mathbf{M}}{\partial p} \right] \mathbf{x}_j = \mathbf{0} \qquad [1.6]$$

$$\text{or} \quad \frac{\partial \lambda_j}{\partial p} = \frac{\mathbf{x}_j^T \left[\frac{\partial \mathbf{K}}{\partial p} - \lambda_j \frac{\partial \mathbf{M}}{\partial p} \right] \mathbf{x}_j}{\mathbf{x}_j^T \mathbf{M} \mathbf{x}_j}. \qquad [1.7]$$

Note that when the modes are mass normalized, $\mathbf{x}_j^T \mathbf{M} \mathbf{x}_j = 1$. Equation [1.7] shows that the derivative of a given eigenvalue depends only on eigensolutions corresponding to that particular eigenvalue. Next, we show that this fact is not true when we consider the derivative of the eigenvectors.

1.1.2. *Sensitivity of the eigenvectors*

Different methods have been developed to calculate the derivatives of the eigenvectors. One way to express the derivative of an eigenvector is by a linear combination of all the eigenvectors

$$\frac{\partial \mathbf{x}_j}{\partial p} = \sum_{r=1}^{N} \alpha_{jr} \mathbf{x}_r. \qquad [1.8]$$

This can always be done as $\mathbf{x}_r, r = 1, 2, \cdots, N$ forms a complete basis. It is necessary to find expressions for the constant α_{jr} for all $r = 1, 2, \cdots N$. Substituting this in equation [1.4], we have

$$\left[\frac{\partial \mathbf{K}}{\partial p} - \frac{\partial \lambda_j}{\partial p}\mathbf{M} - \lambda_j \frac{\partial \mathbf{M}}{\partial p} \right] \mathbf{x}_j + \sum_{r=1}^{N} \left[\mathbf{K} - \lambda_j \mathbf{M} \right] \alpha_{jr} \mathbf{x}_r = \mathbf{0}. \qquad [1.9]$$

Premultiplying by $\mathbf{x}_k^T$, we have

$$\mathbf{x}_k^T \left[\frac{\partial \mathbf{K}}{\partial p} - \frac{\partial \lambda_j}{\partial p}\mathbf{M} - \lambda_j \frac{\partial \mathbf{M}}{\partial p} \right] \mathbf{x}_j + \sum_{r=1}^{N} \mathbf{x}_k^T \left[\mathbf{K} - \lambda_j \mathbf{M} \right] \alpha_{jr} \mathbf{x}_r = 0 \qquad [1.10]$$

We consider $r = k$ and the orthogonality of the eigenvectors

$$\mathbf{x}_k^T \mathbf{K} \mathbf{x}_r = \lambda_k \delta_{kr} \quad \text{and} \quad \mathbf{x}_k^T \mathbf{M} \mathbf{x}_r = \delta_{kr}. \qquad [1.11]$$

Using these, we have

$$\mathbf{x}_k^T \left[\frac{\partial \mathbf{K}}{\partial p} - \lambda_j \frac{\partial \mathbf{M}}{\partial p} \right] \mathbf{x}_j + (\lambda_k - \lambda_j)\, \alpha_{jik} = 0. \qquad [1.12]$$

From this, we obtain

$$\alpha_{jik} = -\frac{\mathbf{x}_k^T \left[\frac{\partial \mathbf{K}}{\partial p} - \lambda_j \frac{\partial \mathbf{M}}{\partial p} \right] \mathbf{x}_j}{\lambda_k - \lambda_j}, \quad \forall\, k \neq j. \qquad [1.13]$$

To obtain the jth term α_{jj}, we differentiate the mass orthogonality relationship in [1.11] as

$$\frac{\partial(\mathbf{x}_j^T \mathbf{M} \mathbf{x}_j)}{\partial p} = 0 \quad \text{or} \quad \frac{\partial \mathbf{x}_j^T}{\partial p} \mathbf{M} \mathbf{x}_j + \mathbf{x}_j^T \frac{\partial \mathbf{M}}{\partial p} \mathbf{x}_j + \mathbf{x}_j^T \mathbf{M} \frac{\partial \mathbf{x}_j}{\partial p} = 0. \qquad [1.14]$$

Considering the symmetry of the mass matrix and using the expansion of the eigenvector derivative, we have

$$\mathbf{x}_j^T \frac{\partial \mathbf{M}}{\partial p} \mathbf{x}_j + 2\mathbf{x}_j^T \mathbf{M} \frac{\partial \mathbf{x}_j}{\partial p} = 0 \quad \text{or} \quad \sum_{r=1}^{N} 2\mathbf{x}_j^T \mathbf{M} \alpha_{jr} \mathbf{x}_r = -\mathbf{x}_j^T \frac{\partial \mathbf{M}}{\partial p} \mathbf{x}_j. \qquad [1.15]$$

Utilizing the othonormality of the mode shapes, we have

$$\alpha_{jj} = -\frac{1}{2} \mathbf{x}_j^T \frac{\partial \mathbf{M}}{\partial p} \mathbf{x}_j. \qquad [1.16]$$

The complete eigenvector derivative is therefore given by

$$\frac{\partial \mathbf{x}_j}{\partial p} = -\frac{1}{2} \left(\mathbf{x}_j^T \frac{\partial \mathbf{M}}{\partial p} \mathbf{x}_j \right) \mathbf{x}_j + \sum_{k=1 \neq j}^{N} \frac{\mathbf{x}_k^T \left[\frac{\partial \mathbf{K}}{\partial p} - \lambda_j \frac{\partial \mathbf{M}}{\partial p} \right] \mathbf{x}_j}{\lambda_j - \lambda_k} \mathbf{x}_k. \qquad [1.17]$$

From equation [1.17], it can be observed that when two eigenvalues are close, the modal sensitivity will be higher as the denominator of the right-hand term will be very small. Unlike the derivative of the eigenvalues given in [1.7], the derivative of an eigenvector requires all the other eigensolutions. This can be computationally demanding for large systems. The method proposed by Nelson [NEL 76] can address this problem. We will discuss Nelson's method in the context of damped systems in the following sections.

1.2. Parametric sensitivity of viscously damped systems

The analytical method in the preceding section is for undamped systems. For damped systems, unless the system is proportionally damped (see section 2.4, Chapter 2 of [ADH 14]), the mode shapes of the system will not coincide with the undamped mode shapes. In the presence of general non-proportional viscous damping, the equation of motion in the modal coordinates will be coupled through the off-diagonal terms of the modal damping matrix, and the mode shapes and natural frequencies of the structure will, in general, be complex. The solution procedures for such non-proportionally damped systems follow mainly two routes: the state-space method and approximate methods in the configuration space, as discussed in Chapters 2 and 3 [ADH 14]. The state-space method (see [NEW 89, GÉR 97], for example) although exact in nature, requires significant numerical effort for obtaining the eigensolutions as the size of the problem doubles. Moreover, this method also lacks some of the intuitive simplicity of traditional modal analysis. For these reasons, there has been considerable research effort in analyzing non-proportionally damped structures in the configuration space. Most of these methods either seek an optimal decoupling of the equation of motion or simply neglect the off-diagonal terms of the modal damping matrix. It may be noted that following such methodologies, the mode shapes of the structure will still be real. The accuracy of these methods, other than the light damping assumption, depends upon various factors, for example frequency separation between the modes and driving frequency (see [PAR 92a, GAW 97] and the references therein for discussions on these topics). A convenient way to avoid the problems that arise due to the use of real normal modes is to incorporate complex modes in the analysis. Apart from the mathematical consistency, by conducting experimental modal analysis, we also often identify complex modes: as Sestieri and Ibrahim [SES 94] have put it "... it is ironic that the real modes are in fact not real at all, in that in practice they do not exist, while complex modes are those practically identifiable from experimental tests. This implies that real modes are pure abstraction, in contrast with complex modes that are, therefore, the only reality!" But surprisingly, most of the current application areas of structural dynamics, which utilize the eigensolution derivatives, e.g. modal updating, damage detection, design optimization and stochastic finite element methods, do not use complex modes in the analysis but rely on the real undamped modes only. This is partly because of the problem of considering an appropriate damping model in the structure and partly because of the unavailability of complex eigensolution sensitivities. Although, there have been considerable research efforts toward damping models, sensitivity of complex eigenvalues and eigenvectors with respect to system parameters appears to have received less attention.

In this section, we determine the sensitivity of complex natural frequencies and mode shapes with respect to some set of design variables in non-proportionally damped discrete linear systems. It is assumed that the system does not possess repeated eigenvalues. In section 2.5 (Chapter 2, [ADH 14]), the mathematical