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## **Structural Properties of Buy-Back Contracts for Price-Setting Newsvendors**

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### **Abstract**

This paper analyzes a buy-back contract in the Stackelberg setting of a manufacturer (leader) selling to a price sensitive, newsvendor retailer (follower). We primarily focus our analysis on a stochastic multiplicative demand form, and contrast our model with a price insensitive framework. In the latter case it is known that the manufacturer's profit function is not well-behaved, but a buy-back contract can coordinate the chain and the channel-coordinating decisions are independent of the demand distribution. We show that in our setting, though buy-backs are not able to attain coordination, the profit functions for both channel partners are unimodal under relatively mild conditions. Moreover, we identify the necessary and sufficient conditions under which: i) the manufacturer's optimal wholesale and buy-back prices are *independent* of the demand uncertainty in spite of being exposed to it, and ii) a *no-return* contract is optimal for the manufacturer, i.e., the optimal buy-back price is zero. The conditions depend only on the structure of the deterministic price-sensitive part and are satisfied by many demand functions used in the literature. The optimal performance measures for the decentralized channel, like the profit division among the partners and the degree of channel efficiency, for distribution-free buy-back contracts are also shaped by the form of the deterministic element. Moreover, these measures are equivalent to those for the corresponding optimal deterministic price-only contracts. A particular transformation technique enables us to prove that all our results hold true for any non-decreasing hazard rate distribution. However, we also demonstrate that when the price sensitive stochastic demand is of the additive form, none of our structural results remain valid anymore.

**Key Words:** Price sensitive newsvendor, Buy-back contracts, Supply chain performance.

# 1 Introduction

Recent operations management literature has focussed a great deal on developing effective contracting schemes between channel partners in a decentralized setting. Especially, there is a growing interest in analyzing the performance of realistic contracts like price-only, buy-back and consignment/revenue-sharing (refer to Wang, et al. 2004, Cachon, 2003, and Pasternack, 2002, for a detailed discussion). While these contracts might not be able to attain coordination<sup>1</sup>, their widespread use can be attributed to the simplicity and cost-effectiveness in terms of administration, compared to complex coordinating contracts. One particularly interesting feature of some of these contracting models is that they are based on the framework of price sensitive stochastic end customer demand (Yano and Gilbert, 2004). The particular demand characteristic obliges the retailers to simultaneously decide on the optimal pricing and inventory policies. This is in sharp contrast with the traditional operations literature where the underlying assumption is of stochastic, but price insensitive, end customer demand, with order quantity as the only retail decision (Pasternack 1985; Lariviere 1998; Lariviere and Porteus 2001).

The most commonly observed contract in practice is a wholesale *price only* type, which involves the manufacturer charging a per unit wholesale price for the quantity ordered by the retailer. It is well-known that coordination of a two-echelon decentralized channel in a price insensitive newsvendor framework cannot be achieved through such price-only contracts. The optimal decision and performance of the individual parties, as well as the channel efficiency (i.e., the ratio of the optimal total profit of a decentralized system to that of a centralized one) then depends crucially on the co-efficient of variation of the random demand (Lariviere and Porteus 2001). Consequently, a number of contracts have been proposed in the literature that can attain coordination in price insensitive newsvendor settings (Cachon 2003, Section 2).

One of the most popular such coordinating contracts is the *buy-back* type. In buy-back contracts, the retailer still pays the wholesale price, but is assured of some financial restitution in the form of a per unit buy-back price ( $<$  wholesale price) for returning the unsold items at the end of the season to the manufacturer. Obviously, contrary to price-only contracts, the manufacturer is then exposed to the risk of a poor demand outcome. Pasternack (1985), Lariviere (1998) and Tsay, et al. (1998) have analyzed this contract in price-taking newsvendor settings. The profit function of the manufacturer is complicated and not well-behaved under such a framework (Theorem 6, Lariviere,

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<sup>1</sup>Coordinating contracts allow a decentralized chain to attain the profit performance of a centralized system and arbitrarily allocate the gains among the channel partners.

1998). However, the risk-sharing mechanism induces the retailer to order the same as the centralized system, resulting in coordination. Depending on the values of the contract parameters, the total optimal channel profit can also be allocated arbitrarily among the partners. Moreover, the channel-coordinating parameters (wholesale and buy-back prices) are independent of the demand distribution. Buy-backs have been used extensively in sectors like publishing, fashion apparel and cosmetics (Kandel 1996; Emmons and Gilbert 1998).

In this paper, we also analyze a buy-back contract in a two-echelon Stackelberg context, but for a *price-setting* newsvendor facing stochastic and price sensitive end customer demand. The retailer (follower) then needs to decide both on the retail price and the order quantity before any uncertainty is resolved, based on the wholesale and buy-back prices offered by the manufacturer (leader). It has already been shown that a buy-back contract cannot attain coordination in this setting (Bernstein and Federgruen 2002; Cachon 2003). Hence, we do not address that issue in our paper. Rather, we concentrate on identifying some structural properties of the optimal decentralized system. In that sense we address fundamentally different issues compared to Pasternack (1985) and Lariviere (1998) in a comparatively enlarged setting. Additionally, our framework extends the performance analysis of “selling to a newsvendor under a price-only contract” model of Lariviere and Porteus (2001) by incorporating retail pricing and buy-backs. We first demonstrate the parallels and contrasts in the behavior of the optimal decision variables and profit functions in our setting to that of a price insensitive one. Subsequently, we assess the performance of the optimal buy-back contract. Specifically, we study the following issues:

1. Is there any change in the behavior of the manufacturer’s profit function as we move from a price insensitive to a price sensitive framework?
2. Is it ever optimal for the manufacturer not to offer a buy-back opportunity to the retailer when demand is price sensitive, i.e., set the optimal buy-back price to zero?
3. More generally, are there any conditions under which the manufacturer’s optimal decisions, even though exposed to uncertainty, are independent of the demand distribution?
4. If indeed there are certain conditions under which the above two properties hold true, how does the supply chain perform under those settings? For example: i) how do the optimal decision variables and profits of the decentralized system compare to that of a centralized one?, and ii) how do the optimal profits of the two channel partners in a decentralized setting compare to each other?

The analysis in this paper primarily focusses on a multiplicative demand form, with a quite general random element. Under these assumptions, we show that the profit function for the manufacturer (and the retailer), in contrast to the price insensitive setting, is well-behaved under relatively mild conditions. Moreover, we develop the necessary and sufficient conditions on the demand function under which: i) a no buy-back contract is optimal for the manufacturer, and ii) the manufacturer's optimal decisions are independent of the customer demand uncertainty. Since we obtain closed form expressions for the optimal decisions and profits, we are able to evaluate the performance of the decentralized chain under the necessary and sufficient conditions. We note that some of the optimal decisions and performances for stochastic buy-back scenarios are equivalent to optimal deterministic price-only settings. Subsequently, we study an additive demand function, the analysis of which turns out to be quite complicated. In general, none of our insights from the multiplicative demand hold true anymore.

Buy-back contracts in price sensitive settings, like ours, have previously also been analyzed in the literature. For example, Kandel (1996) notes that the resulting problem is quite complicated, and indicates that buy-backs might not be able to achieve coordination. Padmanabhan and Png (1997) study the joint effect of demand uncertainty and retail competition on manufacturer profitability in such settings. Marvel and Peck (1995) deal with both valuation uncertainty and demand (number) uncertainty simultaneously in their model. Gurnani and Sharma (2004) not only discuss how demand uncertainty affects the optimal contract, but also the issue of information asymmetry. Recently, Bernstein and Federgruen (2002) have formally shown that buy-backs cannot coordinate a newsvendor with price-dependent demand unless the manufacturer's decisions are made contingent on the retail price chosen. Nevertheless, Emmons and Gilbert (1998) demonstrate that if the wholesale price is set high enough both the retailer and the manufacturer might be better off with buy-backs than price-only contracts. However, these papers do not, in general, address the structural characterization of the manufacturer's optimal decision variable values/profit function and/or the performance of the optimal decentralized system, like we do.

Rather, our framework is more closely related to Wang, et al. (2004), Petruzzi (2004) and Granot and Yin (2004, G&Y). This paper is similar to the first two from the objective perspective, as they also analyze the supply chain performance for a particular contract in a price setting newsvendor setup. However, Wang, et al. and Petruzzi study consignment<sup>2</sup> and price-only contracts, respectively, whereas we focus on buy-backs. Contrasting our results with Wang, et al., we are able to demonstrate that the form of the deterministic part of the demand plays a more

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<sup>2</sup>In such contracts, the retailer offers to deduct a percentage from the selling price and remit the balance to the manufacturer, based on which the manufacturer decides on the end customer price and order quantity.

important role in buy-back contracts than consignments. Moreover, the results of those two and this paper jointly establish under what conditions consignment, price-only and buy-back contracts all become equivalent. On the other hand, our basic model setting is similar to that of G&Y, though their analysis is based on a more restrictive demand setting. For example, the random part of the demand is uniformly distributed in G&Y, but can be any IFR distribution in this paper (our deterministic part is also less restrictive). More importantly, our distinct contribution is to be able to synthesize the results from specific demand forms to identify a general condition under which the optimal buy-back contract exhibits some interesting structural properties. Methodologically also we differentiate our work by utilizing a transformation technique that considerably simplifies the derivation and analysis. Overall, we feel that our research generates analytical insights with greater range and applicability compared to G&Y (refer to Section 4 for more details about the contributions of this paper).

## 1.1 Model Framework

The basic modelling framework is a manufacturer selling to a price-setting newsvendor retailer in a decentralized Stackelberg setting. The retail demand function  $X(p, \epsilon)$  comprises of two elements: a riskless part which is a deterministic decreasing function of the retail price  $p$ ,  $d(p)$ , and a non-negative random variable  $\epsilon$  having support on  $(L, U)$  with mean  $\mu(> 0)$ , density  $f(u)$  and distribution  $F(u)$ . Any unmet demand is lost. The buy-back contractual agreement between the two parties consists of the manufacturer offering to charge a per unit wholesale price  $w$  for the quantity ordered and pay a per unit buy-back price  $b$  for return of any leftover inventory at the end of the selling season, to the retailer. Based on these contract parameters, the retailer then decides on its optimal retail price  $p$  and ordering quantity  $y$  before the start of the selling season. Note that  $w$  and  $b$  cannot explicitly depend on  $p$ ; furthermore,  $p > w > c > 0$  and  $w > b \geq 0$  for realistic solutions. Suppose that there is no salvage cost or value for the leftover stock. Building on the above framework, we address the issues discussed before for a multiplicative  $X(p, \epsilon) = d(p)\epsilon$  (Wang, et al. 2004, and references therein) in Section 2. Subsequently, in Section 3, we study whether the results are valid for another widely-used  $X(p, \epsilon)$  form:  $X(p, \epsilon) = d(p) + \epsilon$ .

Before proceeding to the detailed analysis, we present one assumption and one lemma about  $\epsilon$  which are used throughout the paper (we do not repeat them subsequently).<sup>3</sup>

**Assumption 1** Let  $r(z) = \frac{f(z)}{1-F(z)}$  (hazard rate). We assume that  $r(z)$  is non-decreasing for  $z \in (L, U)$ .

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<sup>3</sup>Note that  $L(\geq 0)$  can be 0, and  $U$  can be infinity.

As indicated in Petruzzi and Dada (1999), the above *non-decreasing hazard rate* property (represented throughout as IFR) of  $\epsilon$  is a mild requirement which is satisfied by almost all theoretical distributions used in the operations management literature including Uniform, Gamma with shape parameter  $\geq 1$ , Beta with both parameters  $\geq 1$ , Weibull with shape parameter  $\geq 1$ , Normal, Exponential, Left-truncated (at 0) Normal, and Left-truncated (at 0) Logistic. Furthermore, we can show that (all proofs are provided in the Appendix):

**Lemma 1** *Let  $\Theta(z) = \int_z^{+\infty} (u - z)f(u)du$  and  $V(z) = \frac{\mu - \Theta(z)}{\int_z^{\infty} uf(u)du}$  for any  $z \in (L, U)$ . Under Assumption 1, we have  $V(z)$  is decreasing for  $z \in (L, U)$ .  $\square$*

## 2 Analysis for Multiplicative Demand, $X(p, \epsilon) = d(p)\epsilon$

In this section, we first characterize the optimal profit functions and the optimal decision variable values for both parties of the decentralized system, as well as that of the centralized one, when the demand is multiplicative, i.e.,  $X(p, \epsilon) = d(p)\epsilon$ . Later on we also assess the optimal performance for the decentralized system. Without loss of generality, let the mean of the random variable,  $\mu$ , be 1 for multiplicative demand.<sup>4</sup> We further assume that the demand function  $d(p)$  exhibits the following properties:

**Assumption 2** *The elasticity  $\eta(p) = p \frac{d'(p)}{d(p)} < 0$  of the demand function is non-increasing in  $p \in (0, +\infty)$ . Furthermore,  $\frac{p}{\eta(p)}$  is monotone and concave, while  $p(1 + \frac{1}{\eta(p)})$  is increasing on  $(0, +\infty)$ .*

Normally, we would indeed expect the percent change in demand due to a percent change in price to be equal or higher at higher prices (i.e., absolute value of price-elasticity to be non-decreasing in  $p$ ). In fact, a large family of  $d(p)$ s used in the literature satisfies the above assumption. Examples include,  $d(p) = ap^{-k}, a > 0, k > 1$ ,  $d(p) = (a - kp)^\gamma, a, k, \gamma > 0$ ,  $d(p) = ae^{-kp}, a, k > 0$ ,  $d(p) = ak^{-p}, a > 0, k > 1$ ,  $d(p) = a - p^k, a > 0, k > 1$ ,  $d(p) = a - e^{kp}, a > 0, k > 0$  and  $d(p) = a - k^p, a > 0, k > 1$ . Hence, Assumption 2 is quite broad in terms of its scope and applicability.

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<sup>4</sup>All our results hold true for  $\mu \neq 1$ , with some minor modifications of expressions.

## 2.1 Characterization of the Profit Functions

For any given buy back contract  $(w, b)$  offered by the manufacturer, the retailer needs to determine its optimal  $y$  and  $p$ . The retailer's profit is given by

$$\Gamma(p, y) = pd(p)[1 - \Theta(\frac{y}{d(p)})] + b\Lambda(\frac{y}{d(p)})d(p) - wy, \quad (1)$$

where  $\Lambda(z) = \int_0^z (z - u)f(u)du$  and  $\Theta(z) = \int_z^{+\infty} (u - z)f(u)du$ . In the above expression, the first term represents the consumer revenue, the second term denotes the buy-back revenue, and  $wy$  represents the purchase cost. It is possible to characterize  $\Gamma(p, y)$  making use of the following expressions for  $\Lambda(z)$  and  $\Theta(z)$ :

$$\Lambda(z) = zF(z) - \int_0^z uf(u)du \quad \text{and} \quad \Theta(z) = 1 - \int_0^z uf(u)du - z[1 - F(z)]. \quad (2)$$

Taking the partial derivatives of the profit function with respect to  $p$  and  $y$ , we get

$$\begin{aligned} \frac{\partial \Gamma(p, y)}{\partial p} &= d(p) \int_0^{\frac{y}{d(p)}} uf(u)du \{V(\frac{y}{d(p)}) + (p - b)\frac{d'}{d}\}, \\ \frac{\partial \Gamma(p, y)}{\partial y} &= p[1 - F(\frac{y}{d(p)})] + bF(\frac{y}{d(p)}) - w. \end{aligned}$$

**Theorem 1** *Under Assumptions 1 and 2, for any given ordering quantity  $y(\geq 0)$ , the retailer's expected total profit function  $\Gamma(p, y)$  is unimodal in  $p$ . Hence, there exists a unique retail price  $p(y)$ , solution of  $\frac{\partial \Gamma(p, y)}{\partial p} = 0$ , which maximizes  $\Gamma(p, y)$ , and  $p(y)$  is non-increasing. Furthermore,  $\Gamma(p(y), y)$  is concave.  $\square$*

So for any given  $(w, b)$ , the corresponding optimal  $(p, y)$  should satisfy the first order conditions  $\frac{\partial \Gamma(p, y)}{\partial p} = 0$  and  $\frac{\partial \Gamma(p, y)}{\partial y} = 0$ . These two equations can be simplified to express  $(w, b)$  in terms of optimal  $(p, y)$  as follows:

$$\begin{pmatrix} w \\ b \end{pmatrix} = p \begin{pmatrix} 1 + \frac{1}{\eta(p)} F(\frac{y}{d(p)}) V(\frac{y}{d(p)}) \\ 1 + \frac{1}{\eta(p)} V(\frac{y}{d(p)}) \end{pmatrix}. \quad (3)$$

From the manufacturer's perspective, the goal is to maximize its expected total profit  $\pi(w, b) = (w - c)y - bd(p)\Lambda(\frac{y}{d(p)})$  by optimally selecting the wholesale price  $w$  and the buy-back price  $b$ , keeping in mind the retailer's optimal actions. Researchers normally strive to solve the manufacturer's maximization problem directly in terms of  $w$  and  $b$  (Granot and Yin 2004). This involves using  $\frac{\partial \Gamma(p, y)}{\partial p} = 0$  and  $\frac{\partial \Gamma(p, y)}{\partial y} = 0$  to express the optimal  $p$  and  $y$  in terms of  $w$  and  $b$ , substituting them in  $\pi$ , and then optimizing over  $w$  and  $b$ . The process is quite cumbersome, and oftentimes results in a messy analysis. In order to get around this problem, we decided to use (3) to transform

the manufacturer's maximization problem in terms of  $p$  and  $y$ . Substitution of (3) in  $\pi(w, b)$  and simplification yields the following tractable form of  $\pi$ .

$$\pi(p, y) = pd(p)[1 - \Theta(\frac{y}{d(p)})](1 + \frac{1}{\eta}) - cy. \quad (4)$$

Taking partial derivatives with respect to  $p$  and  $y$ , we get the following first order conditions for  $\pi(p, y)$ .

$$\frac{\partial \pi(p, y)}{\partial p} = d(p) \int_0^{\frac{y}{d(p)}} uf(u)du \{ [1 + (\frac{p}{\eta(p)})'] V(\frac{y}{d(p)}) + (1 + \eta(p)) \}, \quad (5)$$

$$\frac{\partial \pi(p, y)}{\partial y} = p[1 - F(\frac{y}{d(p)})](1 + \frac{1}{\eta(p)}) - c. \quad (6)$$

**Theorem 2** *Under Assumptions 1 and 2, for any given  $y(\geq 0)$  there exists a unique  $P(y)$ , solution of  $\frac{d\pi(p, y)}{dp} = 0$ , which maximizes  $\pi(p, y)$ . Furthermore,  $\pi(P(y), y)$  is concave,  $y \in (0, +\infty)$ . Hence, there exists a unique  $(p^D, y^D)$  which maximizes  $\pi(p, y)$ .  $\square$*

Note that the determinant of the hessian of the manufacturer's profit function is always negative when demand is price insensitive, implying that it is not well-behaved (Lariviere 1998). However, when the demand is price-dependent and the overall demand function is of the multiplicative form, the profit function exhibits a "nice" unimodal structure for almost all usual randomness distributions and a sufficiently large family of deterministic demand functions. Clearly, price sensitivity of demand plays a major role in determining the characteristics of the manufacturer's profit function. The above property, contrary to price insensitive settings, allows us to identify the optimal contract through standard optimization approaches as we will see in the next section.

In a centralized system there is a single decision maker for both the retailer and the manufacturer, and there are no financial transactions between the two parties. The channel as a whole has to decide on the following: the optimal retail price  $p^C$ , and the optimal ordering quantity  $y^C$ . The expected total profit of the centralized system can be expressed as  $\Pi(p, y) = pd(p)[1 - \Theta(\frac{y}{d(p)})] - cy$ , which is similar to the retailer's profit function in (1) with  $b = 0$  and  $w = c$ . Hence, following Theorem 1 we have (refer also to Petruzzi and Dada 1999):

**Theorem 3** *Under Assumptions 1 and 2, for any given ordering quantity  $y(\geq 0)$ , the centralized system's expected total profit function  $\Pi(p, y)$  is unimodal in  $p$ . Hence, there exists a unique retail price  $\tilde{p}(y)$ , solution of  $\frac{\partial \Pi(p, y)}{\partial p} = 0$ , which maximizes  $\Pi(p, y)$ , and  $\tilde{p}(y)$  is non-increasing. Furthermore,  $\Pi(\tilde{p}(y), y)$  is concave.  $\square$*



## 2.2 Optimal Decision Variable Values

Based on Theorems 2 and 3 we can determine the following optimal decision variable values:

**Corollary 1** *The optimal decisions for the centralized system are to charge a retail price  $p^C$ , and order a quantity  $y^C$ , where  $(p^C, y^C)$  can be obtained from the simultaneous solutions of  $\frac{\partial \Pi(p, y)}{\partial p} = 0$  and  $\frac{\partial \Pi(p, y)}{\partial y} = 0$ .*

*On the other hand, the optimal contract from the manufacturer's perspective and the optimal retail response in the decentralized system are given by:*

- *Retailer: Charge a retail price  $p^D$  to the customers, and order  $y^D$  from the manufacturer, where  $y^D$  is the solution to  $\frac{d\pi(P(y), y)}{dy} = 0$ , and  $p^D = P(y^D)$ .*
- *Manufacturer: Offer the contract  $(w^D, b^D)$  to the retailer, where the optimal wholesale price  $w^D$  and the optimal buy-back price  $b^D$  can be determined by substituting  $(p^D, y^D)$  in (3).□*

In what follows, we study the behavior of the optimal contract  $(w^D, b^D)$ . As indicated before, according to Pasternack (1985) and Lariviere (1998), the channel-coordinating contract is independent of the demand distribution when demand is price insensitive. Moreover, Kandel (1996), Marvel and Peck (1995), Granot and Yin (2004) and Gurnani and Sharma (2004) provide examples of the optimality of a no buy-back contract, i.e.,  $b^D = 0$ , from the manufacturer's viewpoint. We are interested in identifying a general condition under which the distribution-free nature of the optimal contract and/or the optimality of no buy-back contracts are valid for our multiplicative price sensitive framework. This issue has not been analyzed before in the literature. Assuming that the optimal solutions are interior points, we can indeed show that:

**Theorem 4** *The manufacturer's optimal contract parameters are independent of the demand distribution iff the demand elasticity  $\eta(p) = \frac{p}{A+Bp}$ , where  $A$  and  $B$  are constants, i.e.,  $\frac{d(p)}{d'(p)}$  is linear in terms of  $p$ .*

*Moreover, a no buy-back contract is optimal from the manufacturer's viewpoint iff  $\eta(p) = \text{constant}$ , i.e.,  $A = 0$ . In other words,  $b^D = 0$  iff  $d(p)$  is iso-elastic.□*

Both results are somewhat counter-intuitive. Traditionally, buy-backs are supposed to render the manufacturer more vulnerable to demand uncertainty, but improve its profitability, when compared to price-only contracts. The

above result shows that (only) demand functions of the form  $d = \text{constant}[(p + \frac{A}{B})^{\frac{1}{B}}]$  or  $d = \text{constant}[e^{\frac{p}{A}}]$  result in distribution-free optimal buy-back contracts. In other words, for such demand functions the manufacturer can set the optimal contract without any knowledge about the distribution of  $\epsilon$ . Note that the optimal price-only contracts in those scenarios might still be dependent on the distribution of  $\epsilon$  (refer to Section 2.4). Obviously, the optimal contract parameters for any  $d(p)$  for which  $\frac{d(p)}{d'(p)}$  is not linear in  $p$  will depend on the demand distribution (e.g.,  $d(p) = a - k \ln(p)$ ,  $a, k > 0$ ). Though the condition in the theorem seems restrictive, many demand functions used in the literature for studying joint pricing-inventory decisions indeed are special cases of the two demand forms. Examples include  $d(p) = ap^{-k}$ ,  $a > 0, k > 1$ ,  $d(p) = (a - kp)^\gamma$ ,  $a, k, \gamma > 0^5$ ,  $d(p) = ae^{-kp}$ ,  $a, k > 0$ ,  $d(p) = ak^{-p}$ ,  $a > 0, k > 1$  (Gilbert and Emmons, 1998; Petruzzi and Dada, 1999, 2002; Wang, et al. 2004; Petruzzi 2004; Granot and Yin 2004). It seems that  $\eta(p) = \frac{p}{A+Bp}$  is the common thread which connects the demand functions in the literature. However, only when  $d = \text{constant}[(p)^{\frac{1}{B}}]$ , i.e., iso-elastic  $d(p) = ap^{-k}$ ,  $a > 0, k > 1$ , it makes sense for the manufacturer not to offer any buy-back opportunities to the retailer, and the optimal buy-back contract becomes equivalent to a price-only contract (and even to an optimal consignment contract; refer to Petruzzi, 2004, and Section 4). It is worthwhile to point out that when demand is multiplicative, the usual assumption about  $d(p)$  is indeed iso-elastic (Petruzzi and Dada 1999; Wang, et al. 2004).

Since  $\eta(p) = \frac{p}{A+Bp}$  or equivalently  $\frac{d(p)}{d'(p)} = A + Bp < 0$  is not too restrictive, we would assume it to hold true for our subsequent analysis of the multiplicative demand function (rest of Section 2). In order to satisfy Assumption 2 we need  $A \leq 0, B + 1 > 0$ . Depending on the values of  $A$  and  $B$  we can determine the following closed form expressions for the optimal decision variables/contract parameters based on Corollary 1.

**Proposition 1** For  $\eta(p) = \frac{p}{A+Bp}$ ,  $A \leq 0, B + 1 > 0$ , the optimal decision variable/contract parameter values are given by the expressions in Table 1, where  $Z_1 \in (L, U)$  is the unique solution of  $[1 - F(z)] + \frac{c}{A}(\frac{1}{V(z)} + B) = 0$  and  $Z_2 \in (L, U)$  is the unique solution of  $1 + BV(z) = 0$ .  $\square$

The optimal values for the most commonly used demand functions in the literature can be easily deduced by substituting for  $A$  and  $B$  in Table 1. For example:

- If the demand function is iso-elastic, i.e.,  $d(p) = ak^{-p}$ ,  $a > 0, k > 1$ , then  $A = 0$  and  $B = -\frac{1}{k} < 0$  ( $1+B > 0$ ).
- If  $d(p)$  is exponential, i.e.,  $d(p) = ae^{-kp}$ ,  $a, k > 0$ , then  $A = -\frac{1}{k} < 0$  and  $B = 0$ .

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<sup>5</sup>In the literature  $\gamma$  is usually 1, i.e., linear  $d(p)$ ; however, our demand form includes any  $\gamma > 0$ .

$\eta = \frac{p}{A+Bp}$		Retail Price	Retail Order Quantity	Wholesale Price, $w^D$	Buy-back Price, $b^D$
$A < 0, 1 + B > 0$	<b>Cent</b>	$p^C = \frac{-AV(Z_1)}{(1+B)}$	$y^C = d(p^C)Z_1$		
	<b>Decent</b>	$p^D = \frac{-A\{1+(1+B)V(Z_1)\}}{(1+B)(1+B V(Z_1))}$	$y^D = d(p^D)Z_1$	$\frac{(c-A)}{(1+B)} > 0$	$\frac{-A}{(1+B)} > 0, < w^D$
$A = 0, B < 0, 1 + B > 0$	<b>Cent</b>	$p^C = \frac{c}{(1-F(Z_2))}$	$y^C = d(p^C)Z_2$		
	<b>Decent</b>	$p^D = \frac{c}{(1+B)(1-F(Z_2))}$	$y^D = d(p^D)Z_2$	$\frac{c}{(1+B)} > 0$	0

**Table 1: Optimal Decision Variable Values for  $X(p, \varepsilon) = d(p)\varepsilon$  and  $\eta(p) = p/(A + Bp)$**

- If  $d(p) = (a - kp)^\gamma$ ,  $a, k, \gamma > 0$  ( $\gamma = 1$  implies linear demand), then  $A = -\frac{a}{k\gamma} < 0$  and  $B = \frac{1}{\gamma} > 0$ .

We only present the detailed expressions for the last case in Table 2 where  $Z_1 \in (L, U)$  is the unique solution of  $[1 - F(z)] - \frac{ck\gamma}{a}(\frac{1}{V(z)} + \frac{1}{\gamma}) = 0$  (since most of the literature deals with  $\gamma = 1$ ).

		Retail Price	Retail Order Quantity	Wholesale Price, $w^D$	Buy-back Price, $b^D$
$d(p) = (a - kp)^\gamma$	<b>Cent</b>	$\frac{a}{k\gamma} \frac{V(Z_1)}{(1 + \frac{1}{\gamma}V(Z_1))}$	$a \frac{Z_1}{(1 + \frac{1}{\gamma}V(Z_1))}$		
	<b>Decent</b>	$\frac{a}{k(\gamma+1)} \frac{\gamma + (\gamma+1)V(Z_1)}{\gamma + V(Z_1)}$	$a\gamma^2 \frac{Z_1}{(\gamma + V(Z_1))(\gamma+1)}$	$\frac{1}{(\gamma+1)}(\frac{a}{k} + \gamma c)$	$\frac{a}{k} \frac{1}{(\gamma+1)}$

**Table 2: Optimal Decision Variable Values for  $X(p, \varepsilon) = d(p)\varepsilon$  and  $d(p) = (a - kp)^\gamma$ ,  $a, k, \gamma > 0$**

### 2.3 Performance of the Optimal Contract

In this section we compare the optimal decision variables and profits of the decentralized system to that of the centralized one for a multiplicative model. Furthermore, we also compare the optimal profits of the two channel partners in the decentralized system. Let  $\Pi^C = \Pi(p^C, y^C)$  and  $\Pi^D = \pi^D + \Gamma^D = \pi(p^D, y^D) + \Gamma(p^D, y^D)$  be the optimal expected total profits in the centralized and the decentralized system, respectively. In addition, assume  $\pi^D$  and  $\Gamma^D$  to represent the optimal profits for the manufacturer and the retailer, respectively, in the decentralized scenario. Then we can show that:

**Proposition 2** For  $\eta(p) = \frac{p}{A+Bp}$ ,  $A \leq 0, B + 1 > 0$ , the optimal decision variable/profit comparisons are given by the expressions in Table 3.<sup>6</sup>  $\square$

$\eta = \frac{p}{A+Bp}$	$\frac{p^C}{p^D}$	$\frac{y^C}{y^D}$	$\frac{\Pi^C}{\Pi^D}$	$\frac{\pi^D}{\Gamma^D}$
$A \leq 0, 1+B > 0, B \neq 0$	$\frac{(1+B)V(Z_1)}{(1+B)V(Z_1)+1} < 1$	$(1+B)^{\frac{1}{B}} > 1$	$\frac{(1+B)^{1+\frac{1}{B}}}{2+B} \in (1, \frac{e}{2})$	$1+B$
$A < 0, B = 0$	$\frac{V(Z_1)}{V(Z_1)+1} < 1$	$e > 1$	$\frac{e}{2} > 1$	$1$

**Table 3: Comparisons of Optimal Decision Variable Values and Profits for  $X(p, \varepsilon) = d(p)\varepsilon$  and  $\eta(p) = p/(A + Bp)$**

There are a number of points worth noting from the above table. First, under the condition  $\eta(p) = \frac{p}{A+Bp}$ ,  $A \leq 0, B + 1 > 0$ , the decentralized system is clearly not coordinated by the optimal buy-back contract. Second, the popular performance measures of the decentralized system,  $\frac{\Pi^C}{\Pi^D}$  and  $\frac{\pi^D}{\Gamma^D}$ , are independent of the distribution of  $\varepsilon$ . Rather, the performances are influenced solely by  $B$ , the slope of the line representing  $\frac{d}{d'}$ . In fact, depending on whether  $B$  is positive or negative, either of the channel partners might be more profitable. Specifically note that:

**Corollary 2** Since  $B \in (-1, \infty)$ , the ratio  $\frac{\Pi^C}{\Pi^D}$  lies in the range  $(1, \frac{e}{2}]$  and is unimodal. It initially increases from 1 ( $\frac{\Pi^C}{\Pi^D}$  tends towards 1 as  $B$  tends towards -1), attains its maximum as  $B$  tends towards 0 (limit  $\frac{\Pi^C}{\Pi^D}$ , as  $B \rightarrow 0 = \frac{e}{2} \approx 1.359$ ) and then decreases asymptotically approaching 1 as  $B$  tends towards  $\infty$ .

On the other hand,  $\frac{\pi^D}{\Gamma^D}$  lies in the range  $(0, \infty)$  and is linearly increasing in  $B$ .  $\square$

Clearly, when  $\frac{d}{d'}$  is decreasing ( $-1 < B < 0$ ) like iso-elastic  $d(p) = ap^{-k}$ ,  $k > 1$ , the retailer's profit is more than the manufacturer. For a highly decreasing slope ( $B \approx -1$ ), which translates to lower  $k$ , the channel does not lose much from decentralization, but almost all of it is captured by the retailer. As  $B$  increases (or equivalently  $k$  increases), the decentralized channel becomes less efficient but the manufacturer's relative performance improves. When  $B \approx 0$ , i.e.,  $k \approx \infty$ , the channel tends to lose the most from decentralization, but the two partners share the optimal decentralized profit almost equally. In particular, when  $\frac{d}{d'}$  is horizontal, i.e.,  $B = 0$ , which is the exponential demand function, the two partners make exactly equal profits and the decentralized channel is the

<sup>6</sup>When  $A = 0, B < 0$  and  $(1+B) > 0$ , then  $1 + BV(z) = 0$ , which implies  $\frac{p^C}{p^D} = 1 + B$ , as evident from Table 1.

most inefficient. The optimal centralized profit in that case is  $\frac{\epsilon}{2} \approx 1.359$  times more than the total optimal profit of the decentralized system. As soon as  $\frac{d}{d'}$  is increasing, i.e.,  $B > 0$ , like  $d(p) = (a - kp)^\gamma, a, k, \gamma > 0$ , the manufacturer becomes more profitable compared to the retailer. For small  $B$  values (high  $\gamma$ ) the decentralized channel is still quite inefficient. The higher the slope of  $\frac{d}{d'}$  (higher  $B$  or equivalently lower  $\gamma$ ), the more is the profit allocation skewed towards the manufacturer, but the decentralized system as a whole starts performing better. For  $B \approx \infty$ , i.e.,  $\gamma \approx 0$ , the total decentralized channel is almost as profitable as the centralized one, but nearly the entire profit now comes from the manufacturer. Obviously, we can once again substitute the different values of  $B$  to evaluate the performance of the widely-used deterministic demand functions.

## 2.4 Illustrative Numerical Examples

Our main goal in this section is to show that many of the distribution-free properties of the optimal buy-back contracts do not hold true for an optimal price-only contract under our setting. In the interest of space, we just focus on a specific demand form for illustrative numerical examples. Suppose that  $X(p, \epsilon) = d(p)\epsilon$ , where  $d(p) = 500 - 20p$  and  $\epsilon \sim U(A, B)$ . Moreover,  $c = 1$ , and  $w^W, p^W$  and  $y^W$  are the optimal wholesale price, retail price and ordering quantity for a price-only contract, respectively.

The optimal prices for both channel partners are presented in Fig. 1. The figure clearly indicates that  $w^D$  and  $b^D$  are independent of the distribution, while  $w^W$  is not so. However, when the demand function is almost deterministic (the variance decreases over the horizontal axis, while the mean remains constant), the retailer will seldom use the buy-back option even when offered, and the buy-back and price-only contracts become similar. Hence, when variance tends towards 0,  $w^W \rightarrow w^D$  and  $p^W \rightarrow p^D$ . Also, we noted that  $y^C > y^D > y^W$ , and  $(y^D - y^W)$  decreases as variance decreases. Even the optimal performances of the decentralized chain for a price-only contract ( $\frac{\Pi^C}{\Pi^W}$  and  $\frac{\pi^W}{\Gamma^W}$ , where the  $W$  superscript is for the price-only contract setting) are dependent on the distribution of  $\epsilon$ . So are the changes in the optimal profits for the two parties due to inclusion of buy-backs in a price-only contract ( $\frac{\pi^D}{\pi^W}, \frac{\Gamma^D}{\Gamma^W}$ ). We noted similar behavior when  $\epsilon$  and/or  $d(p)$  is exponential. Obviously, when  $d(p)$  is iso-elastic, buy-back contracts are equivalent to price-only ones since  $b^D = 0$  (Section 2.2).

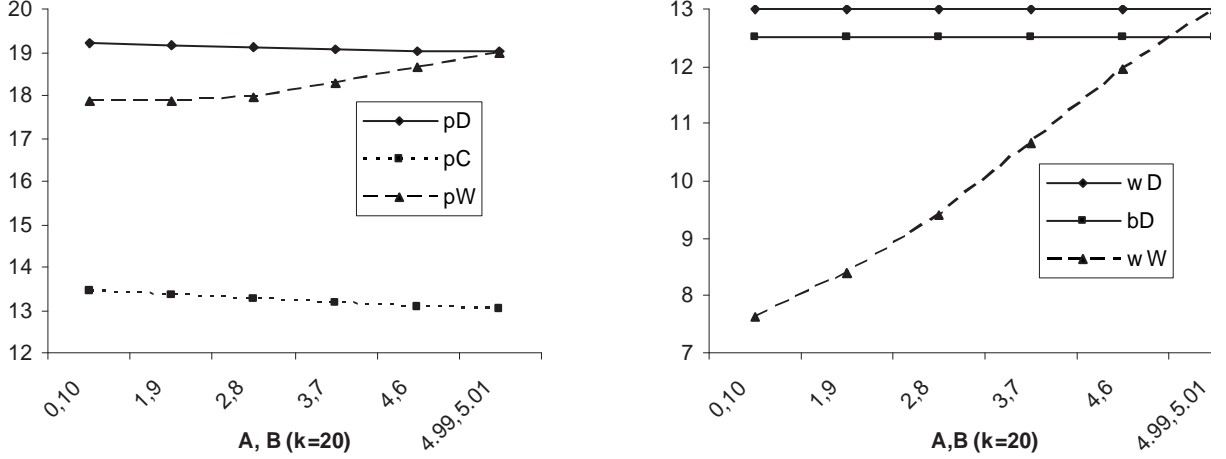


Figure 1: Optimal Prices for Buy-back & Price-only Contracts,  $X(p, \epsilon) = (500 - 20p)\epsilon$ ,  $\epsilon \sim U(A, B)$

### 3 Analysis for Additive Demand, $X = (a - kp) + \epsilon$

We now shift the focus of our analysis from multiplicative to additive price sensitive stochastic demand of the form  $X = d(p) + \epsilon$ , where  $d(p) = a - kp$  ( $a > 0, k > 0$ ) (Petruzzi and Dada 1999). We assume the mean of  $\epsilon$ ,  $\mu$ , to be positive, but not necessarily equal to 1. The retailer's expected total profit can then be written as

$$\Gamma(p, y) = p[y - \Lambda(y - d)] + b\Lambda(y - d) - wy, \quad (7)$$

where  $(w, b)$  is the contract offered by the manufacturer based on which the retailer decides on the optimal price  $p$  and the optimal order quantity  $y$ . Chen et al. (2004) have shown that there exists a unique  $(p, y)$  which maximizes  $\Gamma(p, y)$  for any IFR  $\epsilon$ . This  $(p, y)$  must satisfy the first-order conditions  $\frac{\partial \Gamma(p, y)}{\partial p} = 0$  and  $\frac{\partial \Gamma(p, y)}{\partial y} = 0$ , i.e.,

$$y - \Lambda(y - d) - kpF(y - d) + kbF(y - d) = 0 \text{ and } p[1 - F(y - d)] + bF(y - d) - w = 0.$$

We can then express the buy-back contract  $(w, b)$  in terms of optimal  $(p, y)$  as (like in Section 2):

$$\begin{pmatrix} w \\ b \end{pmatrix} = \begin{pmatrix} p - \frac{y - \Lambda(y - d)}{k} \\ p - \frac{y - \Lambda(y - d)}{kF(y - d)} \end{pmatrix}. \quad (8)$$

Substitution of the above expressions in the manufacturer's expected profit function  $\pi(w, b) = [w(p, y) - c]y - b(p, y)\Lambda(y - d)$  results in  $\pi(p, y) = [y - \Lambda(y - d)][p - \frac{yF(y - d) - \Lambda(y - d)}{kF(y - d)}] - cy$ . Let  $z = y - d(p)$  be the riskless leftovers at the end of the period. The manufacturer's expected profit can then be rewritten as

$$\pi(p, z) = [d(p) + z - \Lambda(z)][p - \frac{d}{k} - \frac{zF(z) - \Lambda(z)}{kF(z)}] - c[d(p) + z]. \quad (9)$$

The overall analysis of the above profit function turns out to be considerably more complicated than multiplicative demand. We can only conclude the following:

**Theorem 5** *Under the assumption  $\epsilon$  is IFR (Assumption 1), for any given  $z$ , there exists a unique  $p(z)$  which maximizes  $\pi(p, z)$ . Moreover, if  $F(u)$  is uniform on  $(0, B)$ ,  $\pi(p(z), z)$  is unimodal in  $z$ .*

*The retailer's optimal decisions are  $(p^D, y^D)$ , where  $y^D = z^D + d(p^D)$  and  $(p^D, z^D)$  are obtained from simultaneously solving the first order conditions of  $\pi(p, z)$ . On the other hand, the manufacturer's optimal action is to offer the buy-back contract  $(w^D, b^D)$ , where  $w^D = w(p^D, z^D)$  and  $b^D = b(p^D, z^D)$  based on the transformation in (8).  $\square$*

In fact, the manufacturer's optimal contract is now a function of the distribution of  $\epsilon$ , even when  $\epsilon \sim U(A, B)$ . Since the profit function of the centralized system is similar to that of the retailer (substituting  $w = c$  and  $b = 0$ ), based on Chen et al. (2004) we can determine the unique optimal decisions,  $p^C$  and  $y^C$ . However, when we compare the optimal decisions and profits of the centralized and decentralized systems, none of the interesting insights of the multiplicative demand hold true anymore (Granot and Yin, 2004, also indicate likewise).

### Illustrative Numerical Examples

Fig. 2 uses  $X(p, \epsilon) = (500 - 20p) + \epsilon$ ,  $\epsilon \sim U(A, B)$  and  $c = 1$ . The notation for the price-only contract remains the same as in Section 2.4. Clearly, the optimal buy-back price is positive and  $w^D$ ,  $b^D$  and  $w^W$  are now all dependent on  $A$  and  $B$ . Though we do not show,  $\frac{\pi^C}{\Gamma^D}$  and  $\frac{\pi^D}{\Gamma^D}$  are also functions of the stochasticity of the demand function. It seems that the manufacturer can retain its "leadership" in terms of profit ( $\frac{\pi^D}{\Gamma^D} > 1$ ). Moreover,  $p^D > p^W > p^C$  and  $w^W < w^D$  ( $y^C > y^D > y^W$  also holds true). Obviously, as the variance tends to zero, the optimal buy-back and price-only contracts start becoming similar.

## 4 Concluding Discussion

In this paper, we have investigated a buy-back contract for a serial two-echelon decentralized supply chain. The newsvendor retailer in our setting faces price sensitive stochastic demand, while the manufacturer is the Stackelberg leader offering the contract. Our analysis is able to shed light on a number of structural properties of the optimal contract. Specifically, we are able to identify the major role that the form of the deterministic part of the demand

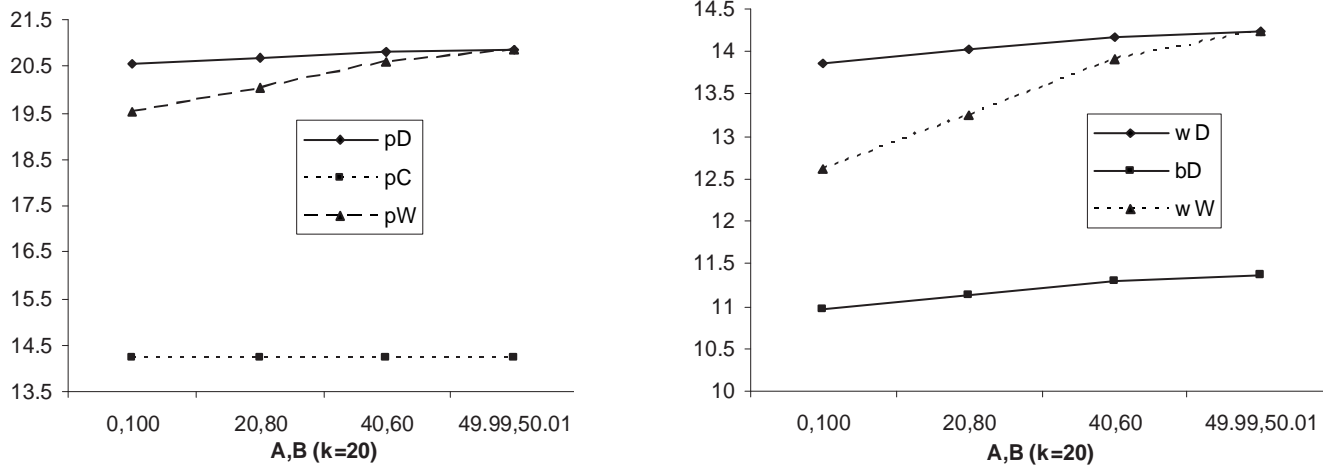


Figure 2: **Optimal Prices for Buy-back/Price-only Contracts**,  $X(p, \epsilon) = (500 - 20p) + \epsilon$ ,  $\epsilon \sim U(A, B)$

plays in shaping some of the properties. All our results have been developed for a comprehensive family of uncertainty distributions satisfying the mild condition of non-decreasing hazard rate.

We focus most of our analysis on a multiplicative demand form. In this particular framework our main results are as follows:

- We prove that the profit functions for both parties are unimodal for quite a general form of the riskless (deterministic) part of the demand function,  $d(p)$ . In fact, the price-dependent demand induces the manufacturer's profit function to be quite well-behaved compared to price insensitive settings. Moreover, note the transformation technique of (3). This enables us to considerably streamline the overall analysis of the paper by focussing on the “inverse” problem of the manufacturer's profit function. Our experience suggests that this technique can be a powerful analytical tool in solving similar multi-echelon supply chain problems, especially where both echelons have the same number of decision variables (refer also to Lariviere and Porteus, 2001).
- The manufacturer's optimal contract decisions are independent of the uncertainty of customer demand iff the elasticity of the demand function is of the form  $\frac{p}{A+Bp}$ , where  $A(\leq 0)$  and  $B(B+1 > 0)$  are constants. Most of the widely-used demand functions in the literature satisfy the condition, a common trait that has not been reported before. However, counter-examples are also not rare, e.g.,  $d(p) = a - k \ln(p)$ ,  $a, k > 0$ . The advantage of the distribution-free property is the fact that a manufacturer can design an optimal contract



for multiple independent retailers based only on the form of their riskless demand functions, and without any knowledge of their market uncertainty. However, note that if the manufacturer is going to offer the contract based on this information, it means that it is acceptable to the particular manufacturer to enter into a contract without exactly knowing what the optimal profit would be, since the manufacturer's profit is still dependent on the extent of uncertainty (Lariviere 1998; Tsay, et al. 1998). Even when buy-back contracts are distribution-free, the optimal wholesale price in relatively simpler price-only contracts might be dependent on the demand distribution. Hence, quite interestingly, buy-backs simultaneously expose and shield the manufacturer from the effects of demand uncertainty.

- There is only one riskless demand function, iso-elastic, under which the value of the distribution-free optimal buy-back price is zero. That is, only for an iso-elastic  $d(p)$  the manufacturer would not offer any buy-back opportunities to the retailer even when allowed to do so and buy-back contracts become equivalent to price-only ones. Petruzzi (2004) has shown that iso-elastic riskless demand is also the only scenario when optimal price-only contracts are equivalent to optimal consignment contracts of Wang, et al. (2004), i.e., the two channel partners make the same profit from both contracts and the price charged to the consumers are the same. Hence, if (and only if) the riskless demand in a multiplicative price sensitive setting is of the iso-elastic form, the three contracts - price-only, buy-back and consignment - are equivalent. Under that condition managers of both parties are indifferent to the type of contract in use.
- As explained in Section 2.3, the optimal performances ( $\frac{\Pi^C}{\Pi^D}$  and  $\frac{\pi^D}{\pi^C}$ ) of the distribution-free optimal buy-back contracts are also distribution-free. They are driven only by the form of the riskless demand function, especially the slope of  $\frac{d}{p}$ , i.e.,  $B \in (-1, \infty)$ . The retailer is better off when the slope is decreasing, while the manufacturer is better off for an increasing slope (they make exactly equal profit when  $B = 0$ ). On the other hand, the total decentralized channel makes almost the same profit as the centralized one for extreme values of  $B$ , i.e.,  $\approx -1$  or  $\approx \infty$ , but is most inefficient for a horizontal  $\frac{d}{p}$ , i.e.,  $B = 0$ . Note that for price-only contracts the stochasticity of the demand function normally plays a more active role in the optimal performances. Moreover, as per Wang, et al. (2004), the performance of the optimal consignment contract is not affected much by the value of  $B$ . For example, in that case the retailer is always better off compared to the manufacturer, irrespective of the value of  $B$ . Hence, the form of the riskless demand function might be more relevant in buy-back contract settings than consignments.

- It is worthwhile to point out that many of the distribution-free properties mentioned above can be determined by solving a *deterministic* wholesale price only contract. Specifically:

**Theorem 6** *If  $\eta(p) = \frac{p}{A+Bp}$  ( $A \leq 0, B + 1 > 0$ ), and  $X(p, \epsilon) = d(p)\epsilon$  (IFR  $\epsilon$ ), the optimal wholesale price, and the optimal ratio of the profits between the manufacturer and the retailer as well as that between the centralized and the total decentralized system for the buy-back contract are equivalent to those for the corresponding optimal price-only contract for a deterministic model.*

*In particular, the optimal wholesale price for the buy-back contract is exactly the same as the optimal retail price for the deterministic price-only contract.  $\square$*

But the optimal wholesale prices for price-only contracts are not equivalent in price sensitive and insensitive settings even if  $\eta(p) = \frac{p}{A+Bp}$  (except when  $A = 0$ ).

Three remarks are in order here. First, *all our results hold true for any IFR  $\epsilon$* . In fact, while we assume that  $\epsilon$  is IFR, this particular property is only used for proving the unimodality of the profit functions, not the insights. Hence most of our results will hold true for multiplicative demand even if  $\epsilon$  is not IFR, as long as there are unique optimal decision variables for both parties and the riskless demand function is “well-behaved”. Second, while operations literature tends to thrive on analyzing the effects of uncertainty, for certain stochastic buy-back models the deterministic price-sensitive part might be the driver of many of the insights. Hence, manufacturers need to understand (maybe empirically) the form of the riskless part under these circumstances. It has implications in terms of what type of contract to offer, the role of uncertainty, as well as the profit allocation and the channel efficiency. Third, we would like to reiterate our contributions compared to Granot and Yin (2004, G&Y). G&Y’s model setting and objective are indeed similar to ours. However, their analysis is based on specific forms of  $d(p)$  - iso-elastic, linear and exponential - and  $\epsilon \sim U(0, 2)$ . Evidently, our analysis is considerably more general. In addition, our distinguishing contributions are: i) characterizing the profit functions for both channel partners for a quite general  $d(p)$  and any IFR  $\epsilon$ , ii) developing the necessary and sufficient conditions under which the optimal buy-back contract demonstrate certain specific structural properties (e.g., distribution-free or no buy-back), iii) identifying the parameter ( $B$ ) which drives the performance of the decentralized chain, iv) equivalence result in Theorem 6 for stochastic buy-back contracts and deterministic price-only contracts under certain forms of  $d(p)$  and any IFR  $\epsilon$ , and v) the transformation technique which greatly simplifies the analysis.

## 4.1 Future Research Opportunities

The most important extension to our modelling framework would be to investigate the performance of an optimal buy-back contract when the random component of the demand is an additive term to the deterministic part. We have shown that none of the major insights from the multiplicative structure remain valid under that scenario. Obviously, demand uncertainty then becomes an integral component of the optimal decisions for both parties. As Petruzzi and Dada (1999) have pointed out, any change in price affects the two most common measures of uncertainty, variance and coefficient of variation, differently for additive and multiplicative demand functions. This naturally entails some structural differences in the behavior of the two models, which need to be studied. It might also be argued that analysis of buy-backs makes greater sense when the risk-profiles (averseness) of the parties are taken into consideration, rather than in risk-neutral frameworks. Similarly, it is important to understand whether our results remain valid in presence of competition and/or for multi-period models.

## Appendix

### Proof of Lemma 1:

We need to show that  $V'(z) < 0$ , i.e.,  $(1 - F) \int_0^z uf(u)du - (1 - \Theta)zf < 0$ . By (2), the above inequality is equivalent to  $[1 - zr(z)] \int_0^z uf(u)du - z^2f < 0$ . Since  $[1 - zr(z)] \int_0^z uf(u)du - z^2f$  at  $z = L$  is  $-L^2f(L) (< 0)$ , in order to prove this inequality, it is sufficient to show that the first derivative of the left hand side is non-positive, i.e.,  $-(r + zr') \int_0^z uf(u)du - zf - z^2(1 - F)r' \leq 0$ . This is obviously true.  $\square$

### Proofs of Theorems 1 and 3:

We first show that for any given  $y (\geq 0)$ , there is a unique  $p(y)$  which maximizes  $\Gamma(p, y)$ . Taking partial derivative with respect to  $p$  and using (2), we get

$$\frac{\partial \Gamma(p, y)}{\partial p} = d(p) \int_0^{\frac{y}{d(p)}} uf(u)du \left\{ V\left(\frac{y}{d(p)}\right) + (p - b) \frac{d'}{d} \right\}.$$

It is easy to check that if  $\frac{y}{d(p)} \leq L$ ,  $\Gamma(p, y)$  is increasing with respect to  $p$ . Hence, without loss of generality, we assume that  $\frac{y}{d(p)} > L$  in the following analysis. From Assumption 2, we know that  $\frac{d}{d'}$  is monotone, hence  $\frac{d'}{d}$  is also monotone. In the following, we consider two cases.

**Case 1:**  $\frac{d'}{d}$  is non-decreasing.

From Lemma 1 we know that  $V(z)$  is non-increasing for  $z \in (L, +\infty)$ . We can see that  $(p-b)\frac{d'}{d} = p\frac{d'}{d} - b\frac{d'}{d}$  is non-increasing as  $p\frac{d'}{d}$  is non-increasing by Assumption 2. Hence, for any given  $y$ , there exists a unique  $p(y)$  which maximizes  $\Gamma(p, y)$ . As  $p(y)$  satisfies

$$V\left(\frac{y}{d(p)}\right) + p\frac{d'}{d} - b\frac{d'}{d} = 0, \quad (10)$$

taking derivative with respect to  $y$  on both sides, we get  $\{V'\left(\frac{y}{d(p)}\right)\left(-\frac{yd'}{d(p)^2}\right) + (p\frac{d'}{d})' - b(\frac{d'}{d})'\}p' + V'\left(\frac{y}{d(p)}\right)\frac{1}{d(p)} = 0$ . Hence,  $p' \leq 0$ , *i.e.*,  $p(y)$  is non-increasing on  $(0, +\infty)$ . Rewriting the above equation about  $p'$ , we get  $\frac{y}{d(p(y))}$  is non-decreasing on  $(0, +\infty)$ , *i.e.*,  $\frac{1}{d} - \frac{d'y}{d^2}p' \geq 0$ .

Taking the derivatives of  $\Gamma(p(y), y)$  with respect to  $y$ , we get  $\frac{d\Gamma(p(y), y)}{dy} = p[1 - F(\frac{y}{d})] + bF(\frac{y}{d}) - w$  and  $\frac{d^2\Gamma(p(y), y)}{dy^2} = p'[1 - F(\frac{y}{d})] - f(\frac{y}{d})(p-b)\{\frac{1}{d} - \frac{d'y}{d^2}p'\} < 0$ . The last inequality results from  $\frac{1}{d} - \frac{d'y}{d^2}p' \geq 0$  and  $p(y) \geq b$  by (10).

**Case 2:**  $\frac{d'}{d}$  is non-increasing: Note that both  $V(z)$  and  $\frac{d'}{d}$  are non-increasing. The remaining proof of this case is almost identical to the first case, and hence is not shown.

The proof of the centralized system is exactly the same as above with  $b = 0$  and  $w = c$ .  $\square$

### Proof of Theorem 2:

As  $\frac{\partial\pi(p, y)}{\partial p} = d(p) \int_0^{\frac{y}{d(p)}} uf(u)du\{[1 + (\frac{d}{d'})']V(\frac{y}{d(p)}) + 1 + p\frac{d'}{d}\}$ , under Assumption 2, we can see that  $[1 + (\frac{d}{d'})']V(\frac{y}{d(p)}) + 1 + p\frac{d'}{d}$  is decreasing with respect to  $p$ . Hence, for any given  $y$  there exists a unique  $P(y)$  which maximizes  $\pi(p, y)$ . As in Theorem 1, we can show that  $P(y)$  is non-increasing and  $\frac{y}{d(P(y))}$  is non-decreasing on  $(0, +\infty)$ . As  $\frac{d\pi(P(y), y)}{dy} = [1 - F(\frac{y}{d(p)})](P + \frac{d}{d'}) - c$ , under Assumption 2, it is clear that  $\frac{d^2\pi(P(y), y)}{dy^2} \leq 0$ .  $\square$

### Proof of Theorem 4:

From the first order conditions of  $\pi(p, y)$ , *i.e.*,  $\frac{\partial\pi(p, y)}{\partial p} = 0$  and  $\frac{\partial\pi(p, y)}{\partial y} = 0$ , we get

$$V = -\frac{1 + p^D\frac{d'}{d}}{1 + (\frac{d}{d'})'} \quad \text{and} \quad F = 1 - \frac{c}{p^D + \frac{d}{d'}}.$$

Note that at optimality both  $V$  and  $F$  are functions of  $p^D$ . By (3),  $b^D$  is constant iff  $p^D + \frac{d}{d'}V$  is constant. Substituting the above expression of  $V$ , we get  $b^D = p^D - \frac{\frac{d}{d'} + p^D}{1 + (\frac{d}{d'})'}$ . Clearly,  $b^D$  is a constant iff the first derivative of the above function is zero, *i.e.*,

$$\left(\frac{d}{d'}\right)''(p^D + \frac{d}{d'}) = 0. \quad (11)$$

Similarly,  $w^D$  is a constant iff the following condition is satisfied.

$$\left(\frac{d}{d'}\right)''(p^D + \frac{d}{d'} - c) = 0. \quad (12)$$

From the above discussion, the sufficient condition is obvious. Now we show the necessary part. As  $p^D$  is a function of the distribution  $F$  (this can be easily shown to be true using uniform distribution) and not a constant,  $b^D$  is distribution-free iff (11) is satisfied, and  $w^D$  distribution-free iff (12) is satisfied. If (11) is satisfied, we claim that  $(\frac{d}{d'})'' = 0$ . Suppose that  $(\frac{d}{d'})''$  is non-zero on some open interval, then we get  $p + \frac{d}{d'} = 0$  on this interval. This is a contradiction. Hence, we get  $\frac{d}{d'}$  is linear in terms of  $p$ . Similarly, if (12) is satisfied, we also get  $\frac{d}{d'}$  is linear in terms of  $p$ .

The optimality of no buy-back, i.e.,  $b^D = 0$ , is a special case of the above. Note that  $b^D = p^D - \frac{\frac{d}{d'} + p^D}{1 + (\frac{d}{d'})'}$ . Since  $\frac{d}{d'} = A + Bp^D$ ,  $b^D = \frac{-A}{1+B}$ . Hence,  $b^D = 0$  only when  $A = 0$ .  $\square$

### Proof of Proposition 1:

First we focus on the case  $A, B \neq 0$ . It is obvious that the optimal  $(p^D, y^D)$  should satisfy the first order conditions (FOCs)  $\frac{\partial \pi(p, y)}{\partial p} = 0$  and  $\frac{\partial \pi(p, y)}{\partial y} = 0$ . When  $\frac{d(p)}{d'(p)} = A + Bp < 0, A \leq 0, B + 1 > 0$ , the FOCs ((5) and (6)) result in

$$[1 + B]V\left(\frac{y}{d}\right) + 1 + \frac{p}{A + Bp} = 0 \quad \text{and} \quad (1 - F\left(\frac{y}{d}\right))[(B + 1)p + A] - c = 0. \quad (13)$$

From the first equation in (13), we get  $p = -\frac{A}{(1+B)} \frac{(1+B)V\left(\frac{y}{d}\right) + 1}{(1 + BV\left(\frac{y}{d}\right))}$ . Substituting this into the second equation in (13), we get  $[1 - F\left(\frac{y}{d}\right)] + \frac{c}{A} [B + \frac{1}{V\left(\frac{y}{d}\right)}] = 0$ . As  $V(z)$  is non-increasing, it is then easy to show that the above equation has an unique solution  $Z_1$  on  $(L, U)$ . Hence,  $\frac{y^D}{d(p^D)} = Z_1$  and  $p^D = -\frac{A}{(1+B)} \frac{(1+B)V(Z_1) + 1}{(1 + BV(Z_1))}$ . Obviously,  $y^D = d(p^D)Z_1$ .

Note that the transformation in (3) now simplifies to

$$\begin{pmatrix} w \\ b \end{pmatrix} = \begin{pmatrix} p + (A + Bp)F\left(\frac{y}{d(p)}\right)V\left(\frac{y}{d(p)}\right) \\ p + (A + Bp)V\left(\frac{y}{d(p)}\right) \end{pmatrix}. \quad (14)$$

Substituting  $p^D$  into the second equation in (13), we get  $F(Z_1) = 1 + \frac{c}{A} \frac{1 + BV(Z_1)}{V(Z_1)}$ . Then, substituting the expressions of both  $p^D$  and  $F(Z_1)$  into the first equation in (14), we get the expression for  $w^D$  in Table 1. On the other hand, based on the first equation in (13) and the second equation in (14), we get the expression for  $b^D$ . The expressions for  $p^C$  and  $y^C$  can be deduced similarly. However, the first order conditions would be

$$V\left(\frac{y}{d}\right) + \frac{p}{A + Bp} = 0 \quad \text{and} \quad (1 - F\left(\frac{y}{d}\right))p - c = 0, \quad (15)$$

though the optimal  $Z_1$  is still the solution of  $[1 - F(z)] + \frac{c}{A} [B + \frac{1}{V(z)}] = 0$ .

The case  $B = 0$  follows directly from the above though we need to be a bit careful about the the expression of  $d$ , which results in exponential demand function. However, when  $A = 0$  the proof is somewhat different. The first order conditions are then

$$(1 + B)(1 + BV(\frac{y}{d})) = 0 \quad \text{and} \quad (1 - F(\frac{y}{d}))p(1 + B) - c = 0. \quad (16)$$

Since we know that  $(1 + B) > 0$  so from the first equation we get that  $Z_2$  is the solution to  $1 + BV(z) = 0$ . The expression for  $p^D$  comes directly from the second equation by substituting  $Z_2$ . The expressions for  $y^D$ ,  $w^D$  and  $b^D$  then follows (e.g., since  $b^D = p + BpV$  and  $BV = -1$ ,  $b^D = 0$ ). In the centralized case also the optimal  $Z_2$  is the solution to  $1 + BV(z) = 0$  and  $p^C$  can be directly obtained by substituting  $Z_2$  in  $\frac{\partial \Pi(p, y)}{\partial y} = 0$ , i.e.,  $p(1 - F(Z_2)) - c = 0$ . Obviously,  $y^C = d(p^C)Z_2$ .  $\square$

### Proof of Proposition 2:

All the ratios for  $\frac{p^C}{p^D}$  follow from Table 1. As far as  $\frac{y^C}{y^D}$  is concerned, first let us focus on the case when  $A, B \neq 0$ . We know that in that case the general demand function can be written in the form  $d(p) = \text{constant}[(p + \frac{A}{B})^{\frac{1}{B}}]$ . So we focus on the term  $(p + \frac{A}{B})$ . Substituting the expressions for  $p^C$  and  $p^D$  from Table 1 we can easily show that:

$$p^D + \frac{A}{B} = \frac{A}{B(1 + B)(1 + BV(Z_1))} \quad \text{and} \quad p^C + \frac{A}{B} = \frac{A}{B(1 + BV(Z_1))}. \quad (17)$$

This implies that  $\frac{y^C}{y^D} = (1 + B)^{\frac{1}{B}}$ . As far as the comparison between the profit functions of the channel partners are concerned, let us substitute  $w^D$  and  $b^D$  from Table 1 in the retailer's profit function (1) and simplify using the relations in (2). Then we get (recall that  $\frac{d}{d} = A + Bp$ ),  $\Gamma(p^D, y^D) = \frac{1}{1+B}\pi(p^D, y^D)$ , where  $\Gamma(p, y)$  and  $\pi(p, y)$  are given by the expressions in (1) and (4), respectively. Hence,  $\frac{\pi^D}{\Gamma^D} = 1 + B$ .

Now we compare  $\Pi^C$  and  $\Pi^D$ . Note that  $\frac{y^D}{d(p^D)} = \frac{y^C}{d(p^C)} = Z_1$ ; hence,  $\frac{\Pi^C}{\Pi^D} = \frac{p^C d(p^C)[1 - \Theta(Z_1)] - cy^C}{p^D d(p^D)[1 - \Theta(Z_1)] - cy^D} = \frac{y^C}{y^D} \frac{(\frac{p^C}{Z_1}[1 - \Theta(Z_1)] - c)}{(\frac{p^D}{Z_1}[1 - \Theta(Z_1)] - c)}$ . By substituting the expressions for  $p^C$ ,  $p^D$  and  $\frac{y^C}{y^D}$  into the above ratio expression (note that from the first order conditions we know that  $\frac{1 + BV(Z_1)}{AV(Z_1)} = -\frac{1 - F(Z_1)}{c}$ ), we get

$$\frac{\Pi^C}{\Pi^D} = (1 + B)^{\frac{1}{B}} \frac{\frac{1 - \Theta(Z_1)}{[1 - F(Z_1)]Z_1} - 1}{\frac{[1 - \Theta(Z_1)][1 + (1 + B)V(Z_1)]}{[1 - F(Z_1)]Z_1(1 + B)V(Z_1)} - 1}.$$

Further simplifications using the relations in (2) result in

$$\frac{\Pi^C}{\Pi^D} = (1 + B)^{\frac{1}{B}} \frac{\int_0^{Z_1} uf(u)du}{\int_0^{Z_1} uf(u)du + \frac{1 - \Theta(Z_1)}{(1 + B)V(Z_1)}}.$$

Now from the definition of  $V(z)$  in Lemma 1, we know that  $\frac{1 - \Theta(Z_1)}{V(Z_1)} = \int_0^{Z_1} uf(u)du$ . Hence,  $\frac{\Pi^C}{\Pi^D} = \frac{(1 + B)^{1 + \frac{1}{B}}}{2 + B}$ .

The proof for  $A = 0, 1 + B > 0, B < 0$  is exactly the same as above except that the first order condition is  $1 + BV(Z_2) = 0$  and we should take proper note of the optimal prices from Table 1. When  $A < 0, B = 0$ , the demand function would be of the form  $d(p) = \text{constant}[e^{\frac{p}{A}}]$ . The optimal retail prices,  $p^C$  and  $p^D$ , from Table 1 then result in  $\frac{y^C}{y^D} = e$ . We can then use the same technique to prove the expressions for  $\frac{\pi^D}{\Gamma^D}$  and  $\frac{\Pi^C}{\Pi^D}$ .  $\square$

**Proof of Theorem 5:**

Note that  $z - \Lambda = \mu - \Theta$ ,  $zF - \Lambda = \int_0^z uf(u)du$  and  $\mu - \Theta = \int_0^z uf(u)du + z[1 - F]$ . Let  $T(z) = \frac{\int_0^z uf(u)du}{F(z)}$ . By using the above expressions, we get (9). For any given  $z$ , taking the partial derivatives of  $\pi(p, z)$  with respect to  $p$ , we get  $\frac{\partial \pi(p, z)}{\partial p} = -k[2p - \frac{a}{k} - \frac{T(z)}{k}] + 2[\mu - \Theta(z) + d(p)] + kc$  and  $\frac{\partial^2 \pi(p, z)}{\partial p^2} = -4k$ . Thus, for any given  $z$  there exists a unique  $p(z)$  which maximizes  $\pi(p, z)$ , where

$$p(z) = \frac{1}{4k} [(3a + kc) + 2(\mu - \Theta) + T(z)] \text{ and } p' = \frac{1}{4k} [2(1 - F) + T']. \quad (18)$$

Differentiating the manufacturer's expected profit  $\pi(p(z), z)$  and substituting the above  $p(z)$ , we get  $\frac{d\pi(p(z), z)}{dz} = \frac{1}{4k} [(a - kc) + 2(\mu - \Theta) - T][2(1 - F) - T'] - cF$ . If  $\frac{d\pi(p(z), z)}{dz} = 0$ ,

$$(a - kc) + 2(\mu - \Theta) - T = \frac{4kcF}{2(1 - F) - T'}. \quad (19)$$

Thus,  $\frac{d^2 \pi(p(z), z)}{dz^2} \Big|_{\{\frac{d\pi(p(z), z)}{dz} = 0\}} = \frac{1}{4k[2(1 - F) - T']} \{ [2(1 - F) - T']^3 - 8kcf - 4kcFT'' + 4kcfT' \}$ . In particular, if  $F$  is uniform on  $(0, B)$ ,

$$\frac{d^2 \pi(p(z), z)}{dz^2} \Big|_{\{\frac{d\pi(p(z), z)}{dz} = 0\}} = \frac{1}{4k(\frac{3}{2} - \frac{2}{B}z)} \{ [\frac{3}{2} - \frac{2}{B}z]^3 - \frac{6kc}{B} \}. \quad (20)$$

From (19), we get  $2(\mu - \Theta) - T \leq \frac{4kcF}{2(1 - F) - T'}$ . If  $F$  is uniform on  $(0, B)$ , this equation simplifies as  $(\frac{3}{2} - \frac{2}{B}z)^2 \leq \frac{4kc}{B}$ . Substituting this into (20) and noting that  $\frac{3}{2} - \frac{2}{B}z > 0$  from (19), we get  $\frac{d^2 \pi(p(z), z)}{dz^2} \Big|_{\{\frac{d\pi(p(z), z)}{dz} = 0\}} \leq \frac{1}{4k(\frac{3}{2} - \frac{2}{B}z)} [-\frac{8kc}{B}z] < 0$ .  $\square$

**Proof of Theorem 6:**

In a deterministic setting with price-only contracts the profit function for the retailer, for a given wholesale price  $w$ , can be written as  $\Gamma(p) = (p - w)d(p)$ . For  $\frac{d}{dp} = A + Bp, A \leq 0, B + 1 > 0$ , we can show that  $\frac{d\Gamma}{dp} = 0$  has a unique solution and the optimal retail price (for a given  $w$ ) is given by  $p(w) = \frac{w - A}{B + 1}$  or  $w(p) = A + (B + 1)p$ . Replacing it into the manufacturer's profit function, we have  $\pi(p) = (w(p) - c)d(p)$ . Differentiating this with respect to  $p$  and equating to zero we can show that the optimal retail price is given by  $p^* = \frac{c - A(B + 2)}{(B + 1)^2}$ . Replacing it in  $w(p)$ , we obtain the optimal wholesale price for a price-only contract in a deterministic setting as  $w^* = \frac{c - A}{B + 1}$ ,

same as  $w^D$ . The ratio between the retailer's and the manufacturer's optimal profit in a deterministic price-only contract setting is equivalent to  $\frac{w^*-c}{p^*-w^*}$ . Substituting the values of  $p^*$  and  $w^*$ , and simplifying we get the ratio to be  $(1+B)$ . We can also show that the centralized system in the deterministic price-only case results in the unique optimal retail price of  $\tilde{p}^* = \frac{c-A}{B+1} = w^* = w^D$ . The comparison between the optimal profits of the centralized and decentralized system in the deterministic price-only case is given by  $\frac{(\tilde{p}^*-c)d(\tilde{p}^*)}{(p^*-c)d(p^*)}$ . Note that when  $A < 0, B+1 > 0$ ,  $d = \text{constant}[(p + \frac{A}{B})^{\frac{1}{B}}]$ . Substituting the values of  $\tilde{p}^*$  and  $p^*$  and simplifying we get the ratio as  $(1+B)^{\frac{1}{B}} \frac{1+B}{2+B}$ .  $\square$

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