Research papers

Struct Multidisc Optim 21, 333–344 © Springer-Verlag 2001

Structural reanalysis for topological modifications – a unified approach

U. Kirsch* **and P.Y. Papalambros**

Abstract A unified approach for structural reanalysis of all types of topological modifications is presented. The modifications considered include various cases of deletion and addition of members and joints. The most challenging problem where the structural model is itself allowed to vary is presented. The two cases, where the number of degrees of freedom is decreased and increased, are considered. Various types of modified topologies are discussed, including the common conditionally unstable structures. The solution procedure is based on the combined approximations approach and involves small computational effort. Numerical examples show that accurate results are achieved for significant topological modifications. Exact solutions are obtained efficiently for modifications in a small number of members.

Key words structural analysis, approximate reanalysis, topological optimization

1 Introduction

Topological optimization of structures has been the subject of numerous studies in recent years (Bendsøe and Mota-Soares 1992; Kirsch 1989; Rozvany *et al.* 1995). This type of optimization can greatly improve the design, and potential savings are generally more significant than those resulting from fixed-topology optimization. However, optimization of the topology is more difficult due to changes in the structural model. Members and joints are deleted or added during the solution process and the reanalysis model becomes complicated. Developing reanalysis procedures for general topological modifications is particularly challenging when the number of degrees of freedom (DOF) is modified and the structural response is significantly changed.

Reanalysis methods are intended to analyze modified designs efficiently. The object is to evaluate accurately the structural response (displacements, stresses and forces) for successive modifications in the design, without solving repeatedly the complete set of modified analysis equations. These methods can be divided broadly into exact (direct closed-form) and approximate ones. In this study both exact and approximate solutions will be presented.

Several exact methods for calculating the modified response due to changes in the design have been proposed in the past. Most of these methods are based on the Sherman-Morrison-Woodbury formulae (Sherman and Morrison 1949; Woodbury 1950). Exact methods are suitable for changes in a relatively small number of members and are inefficient in cases of changes in a large proportion of the structure. Improved versions of the Sherman-Morrison-Woodbury approach have been proposed by several authors (Akgun *et al.* 1998; Kirsch 1981).

Approximate reanalysis methods are more efficient than exact methods and are usually suitable for moderate changes in the structure. In most approximations the quality of the results and the efficiency of the calculations are conflicting factors that should be considered. That is, better approximations are often achieved at the expense of more computational effort. Approximate reanalysis methods can be divided into the following classes (Barthelemy and Haftka 1993; Kirsch 1993a).

1. Global (multi-point) approximations, such as reduced basis (Noor 1994) and response surface methods (Sobieszczanski-Sobieski and Haftka 1997). These approximations are obtained by analysing the structure at a number of design points, and they are valid for the whole design space. However, global approximations may require much computational effort, particularly in problems with a large number of design variables.
2. Local (single-point) approximations, such as the first-order Taylor series expansion or the binomial series expansion about a given design point. Local approxi-

Received April 4, 2000

U. Kirsch and P.Y. Papalambros

Department of Mechanical Engineering and Applied Mechanics, University of Michigan, Ann Arbor, MI 48109, USA
e-mail: kirsch@tx.technion.ac.il

* *On leave from:* Department of Civil Engineering, Technion, Haifa 32 000, Israel

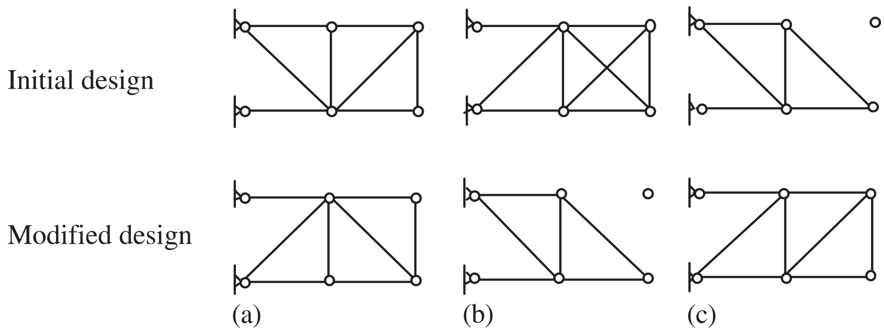

Fig. 1 Types of topological changes

mations are based on information calculated at a single point. These methods are more efficient but they are effective only in cases of small changes in the design variables. To improve the quality of the results, various means have been proposed (Schmit and Farshi 1974; Svanberg 1987).

In this study, a third class attempting to give global qualities to local approximations, is presented. The method has been used in previous studies for various problems and different types of design variables (Kirsch 1993b, 1999, 2000; Kirsch and Liu 1995, 1997). The solution procedure is based on the combined approximations (CA) approach, where the binomial series terms are used as basis vectors in a reduced basis approximation. Similar to local approximations, the calculations are based on results of a single exact analysis. Each reanalysis involves small computational effort and calculation of derivatives is not required. The method is easy to implement and can be used with general finite element programs. Accurate results are achieved for significant design modifications and exact solution is obtained in certain special cases. In previous studies reanalysis procedures have been developed for some particular cases of topological optimization (Kirsch 1993b; Kirsch and Liu 1997).

A unified approach of structural reanalysis for all types of topological modifications is presented in this study. The modifications include various cases of deletion and addition of members and joints. The most challenging problem is when the number of DOF's is decreased or increased. Various types of modified topologies are discussed, including the common case of conditionally unstable structures. Formulation of the reanalysis problem is presented in Sect. 2, solution procedures are developed in Sect. 3, and numerical results are demonstrated in Sect. 4.

2 Formulation of reanalysis for various topological modifications

In topological optimization, members are deleted or added, the structural model is itself allowed to vary during the solution process, and the reanalysis model becomes complicated. In this study, the following typical cases of topological modifications will be considered.

1. Deletion and addition of members, where the number of DOF's is unchanged (Fig. 1a). In such cases, the number of analysis equations is also unchanged and only numerical values of the coefficients of the equations are modified. It will be shown later in Sect. 3 that exact solutions can be obtained efficiently by the CA method in various cases of topological and geometrical modifications, where the number of modified members is relatively small.
2. Deletion and addition of members, and deletion of some joints, where the number of DOF's is decreased (Fig. 1b). Some common cases where the resulting structure is conditionally unstable are demonstrated. It will be shown that in the latter cases approximate reanalysis by the CA method with a reduced number of unknowns provides accurate results.
3. Deletion and addition of members, and addition of some joints, where the number of DOF's is increased (Fig. 1c). In this case it is necessary to augment the analysis model such that the new degrees of freedom are included in the modified model. A general solution procedure is presented, where an exact modified initial analysis is first efficiently carried out. The modified initial analysis is then used for reanalysis of further modifications.

Modifying the topology, the resulting structures may be classified as:

1. Stable structures (S). This is the typical case where the analysis equations are satisfied and reanalysis is straightforward.
2. Conditionally Unstable structures (CU), which are common in topological optimization problems solved for a single (or a small number of) loading condition(s). In such cases the forces in the structure satisfy equilibrium conditions for a specific loading, but the necessary relationship, which exists between the joints and the members in a stable structure, is not satisfied. That is, the structure can carry only specific loading conditions. Some analysis equations of the modified structure become zero identities and exact analysis is meaningless. It will be shown that the CA approach may provide accurate results in such cases.
3. Unstable structures (U). In such cases the structure (or part of it) is unstable for a general loading con-

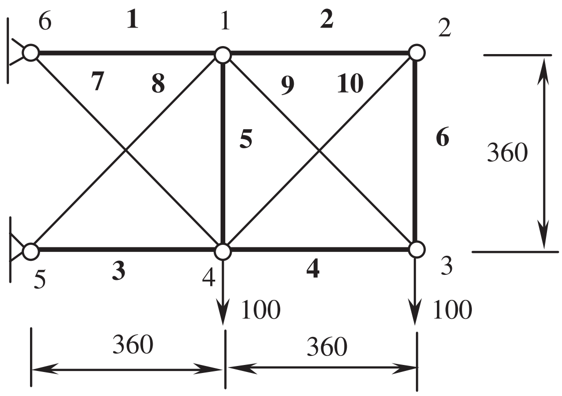

Fig. 2 Ten-bar truss

dition, the analysis equations are not satisfied, and a collapse of the structure will occur. Unstable structures are not considered in this study.

To illustrate the various cases, consider the initial ten-bar truss shown in Fig. 2. The number of unknown forces is fourteen (ten member forces and four reactions) and the number of independent equilibrium conditions is twelve. The truss is statically indeterminate, having two redundant members, thus various cases of deletion of two members could be considered to obtain a statically determinate structure. However, in many of the above cases the resulting structure might be unstable or conditionally unstable. Assuming all forty-five possible cases of deletion of two members, only twenty-nine of the resulting structures are stable (Fig. 3), whereas four structures are conditionally unstable (Fig. 4) and twelve structures are unstable (Fig. 5). Some of the conditionally unstable structures could be transformed into stable structures by deleting or adding zero force members. It is instructive to note that conditionally unstable structures could be obtained also by addition of some members and joints. In addition, various stable and conditionally unstable structures could be obtained by deletion of three or four members, as can be observed in Figs. 6 and 7.

2.1 The number of DOF's is unchanged

The basic reanalysis problem presented in this section can be stated as follows.

1. Given an initial design, the corresponding stiffness matrix $\mathbf{K}_0$ and the initial load vector $\mathbf{R}_0$, the displacements $\mathbf{r}_0$ are computed by the equilibrium equations

$$\mathbf{K}_0\mathbf{r}_0 = \mathbf{R}_0 \,. \tag{1}$$

It is assumed that the stiffness matrix $\mathbf{K}_0$ is given from the initial analysis in the decomposed form

$$\mathbf{K}_0 = \mathbf{U}_0^T\mathbf{U}_0 \,, \tag{2}$$

where $\mathbf{U}_0$ is an upper triangular matrix.

2. Assume addition or deletion of members so that the modified stiffness matrix $\mathbf{K}$ and the modified load vector $\mathbf{R}$ are given by

$$\mathbf{K} = \mathbf{K}_0 + \Delta\mathbf{K} \,, \tag{3}$$

$$\mathbf{R} = \mathbf{R}_0 + \Delta\mathbf{R} \,, \tag{4}$$

where $\Delta\mathbf{K}$ and $\Delta\mathbf{R}$ are the changes in the stiffness matrix and in the load vector, respectively, due to the change in topology.

3. The goal is to find efficient and accurate approximations of the modified displacements $\mathbf{r}$ due to various changes in the topology, without solving the complete set of modified analysis equations

$$\mathbf{K}\mathbf{r} = (\mathbf{K}_0 + \Delta\mathbf{K})\mathbf{r} = \mathbf{R}_0 + \Delta\mathbf{R} \,. \tag{5}$$

Once the displacements are evaluated, the explicit stress-displacement relations can readily determine the stresses

$$\sigma = \mathbf{S}\mathbf{r} \,, \tag{6}$$

where $\mathbf{S}$ is the stress transformation matrix. Thus the presented approximations of $\mathbf{r}$ are intended only to replace the set of implicit analysis equations (5).

The above formulation is general, it is suitable for different types of structure, and can be extended readily to include various types of topological modifications. The solution procedure is presented subsequently in Sect. 3.

2.2 The number of DOF's is decreased

This type of reanalysis problem is encountered in many topological optimization problems where some members and joints are deleted from an initial ground structure, consisting of numerous members and joints. As a result, the number of DOF's is decreased, and the number of analysis equations is changed.

The sizes of the stiffness matrix and the load vector are decreased according to number of joints deleted from the structure. The modified stiffness matrix and the modified load vector can be expressed as

$$\mathbf{K} = \mathbf{K}_0 + \Delta\mathbf{K} = \begin{bmatrix} \mathbf{K}_M & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} , \tag{7}$$

$$\mathbf{R} = \mathbf{R}_0 + \Delta\mathbf{R} = \begin{Bmatrix} \mathbf{R}_M \\ \mathbf{0} \end{Bmatrix} , \tag{8}$$

where $\mathbf{K}$ and $\mathbf{R}$ are the modified stiffness matrix and the modified load vector, respectively, of the complete set of equations, including the original degrees of freedom; and $\mathbf{K}_M$ and $\mathbf{R}_M$ are the stiffness matrix and the load vector, respectively, of the modified structure with a reduced number of degrees of freedom. Since some analysis equations become zero identities, stiffness analysis of the

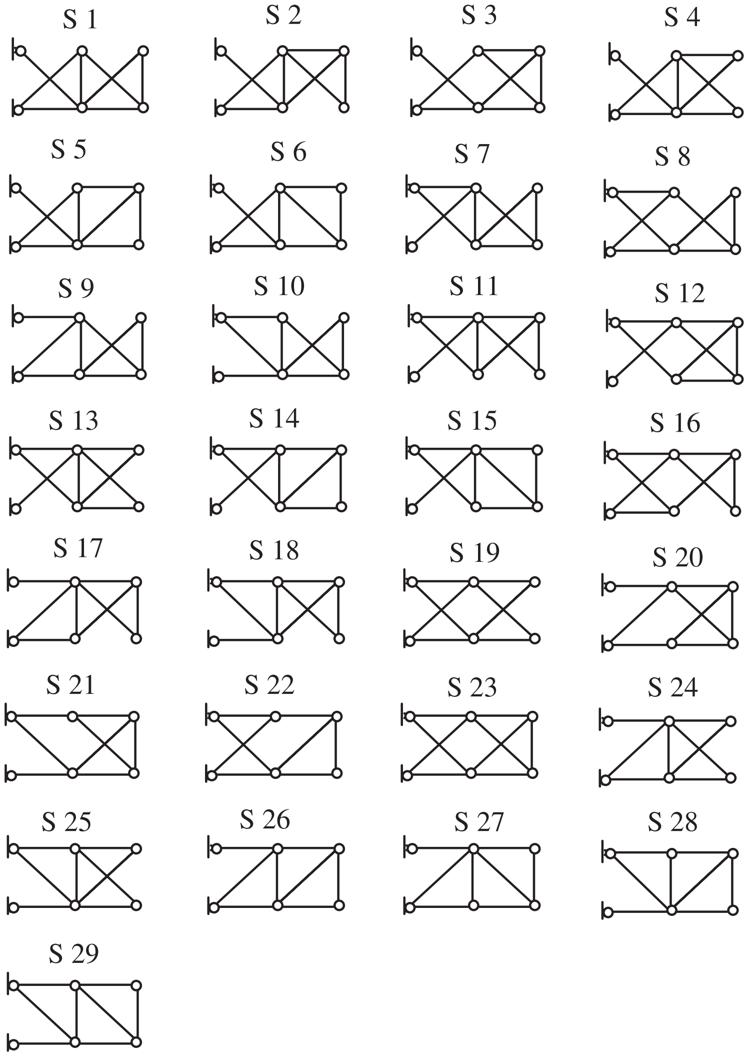

Fig. 3 Stable structures obtained by deletion of two members

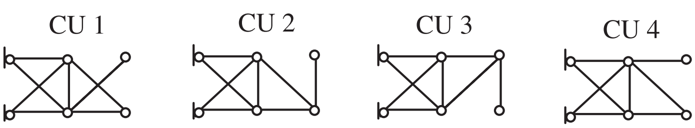

Fig. 4 Conditionally unstable structures obtained by deletion of two members

complete set of modified equations cannot be carried out. The set of modified equations (5) to be solved is reduced to

$$\mathbf{K}_M \mathbf{r} = \mathbf{R}_M \,, \tag{9}$$

where $\mathbf{r}$ is now a reduced vector of modified displacements.

It has been noted that in many cases the resulting structure represented by (9) is conditionally unstable. However, despite the reduction in the size of the stiffness matrix, the number of modified analysis equations is usually large and efficient reanalysis is still useful. Approximate reanalysis by the CA method with a reduced number of unknowns may provide accurate results. The solution procedure for such cases is discussed in Sect. 3.2 and numerical results are demonstrated in Sect. 4.

2.3 The number of DOF's is increased

In cases where some members and joints are added to the initial structure, the number of DOF is increased, the number of analysis equations is changed, and the sizes of the stiffness matrix and the load vector are increased according to the number of joints added to the structure. Let us define the augmented stiffness matrix $\mathbf{K}_A$ and the augmented load vector $\mathbf{R}_A$, with an increased number of

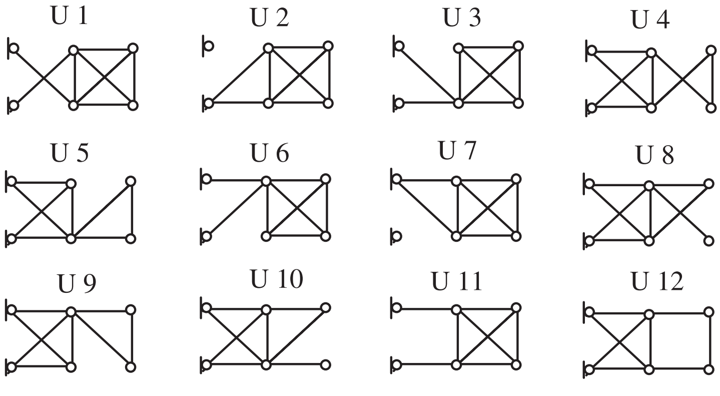

Fig. 5 Unstable structures obtained by deletion of two members

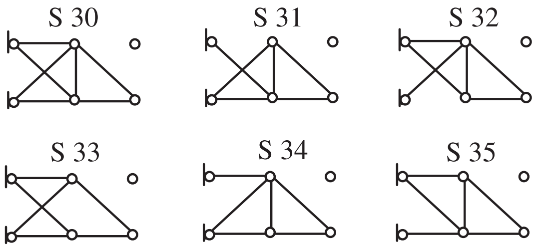

Fig. 6 Stable structures obtained by deletion of three or four members

DOF's, by

$$\mathbf{K}_A = \begin{bmatrix} \mathbf{K}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{R}_A = \begin{Bmatrix} \mathbf{R}_0 \\ \mathbf{0} \end{Bmatrix}. \tag{10}$$

Increasing the number of DOF's, the matrix of changes $\Delta\mathbf{K}$ and the vector of changes $\Delta\mathbf{R}$ can be expressed in terms of corresponding submatrices and subvectors as

$$\Delta\mathbf{K} = \begin{bmatrix} \Delta\mathbf{K}_{00} & \Delta\mathbf{K}_{0N} \\ \Delta\mathbf{K}_{N0} & \Delta\mathbf{K}_{NN} \end{bmatrix}, \quad \Delta\mathbf{R} = \begin{Bmatrix} \Delta\mathbf{R}_0 \\ \Delta\mathbf{R}_N \end{Bmatrix}, \tag{11}$$

where $\Delta\mathbf{K}_{00}$ and $\Delta\mathbf{R}_0$ are the changes in stiffness coefficients and in the loads, respectively, for the original DOF's; and $\Delta\mathbf{K}_{NN}$ and $\Delta\mathbf{R}_N$ are the changes in the stiffness coefficients and the loads, respectively, for the new DOF's.

The modified stiffness matrix and the modified load vector are given by

$$\mathbf{K} = \mathbf{K}_A + \Delta\mathbf{K}, \quad \mathbf{R} = \mathbf{R}_A + \Delta\mathbf{R}. \tag{12}$$

Thus, the new degrees of freedom are included in the set of modified equations (5).

In many cases the number of added degrees of freedom is relatively small, compared with the original number. It will be shown subsequently in Sect. 3.3 that in such cases it is possible to calculate efficiently the modified displacements.

3 Reanalysis by the CA approach

Most reanalysis methods developed in the past are suitable for the relatively simple case where the number of DOF's (or analysis equations) is unchanged. The reanalysis approach presented in this section is suitable for problems where the number of DOF's and the sizes of $\mathbf{K}$, $\mathbf{r}$ and $\mathbf{R}$ are changed. In the procedure presented below, the computed terms of the binomial series expansion are used as high quality basis vectors in reduced basis approximations. The unknown coefficients of the reduced basis expression can be determined by solving a reduced set of the analysis equations. The efficiency and accuracy are further improved by introducing an uncoupled set of basis vectors, using a Gram-Schmidt orthonormalization.

3.1 The number of DOF's is unchanged

Evaluation of modified displacements by the CA method for cases where the number of DOF's is unchanged is briefly described in this section. A detailed discussion of the solution process is given elsewhere (Kirsch 1999).

Given the initial stiffness matrix $\mathbf{K}_0$ in the decomposed form of (2) and the initial displacements $\mathbf{r}_0$, calculation of the modified displacements $\mathbf{r}$ for any assumed changes $\Delta\mathbf{K}$, $\Delta\mathbf{R}$, in the stiffness matrix and in the load vector, involves the following steps.

1. The modified stiffness matrix $\mathbf{K}$ and the modified load vector $\mathbf{R}$ are first introduced. Since $\mathbf{K}_0$ and $\mathbf{R}_0$ are given, this step involves only introduction of $\Delta\mathbf{K}$ and $\Delta\mathbf{R}$.
2. The basis vectors $\mathbf{r}_i$ are calculated by the following recurrence relation:

$$\mathbf{r}_i = -\mathbf{K}_0^{-1}\Delta\mathbf{K}\mathbf{r}_{i-1} = -\mathbf{B}\mathbf{r}_{i-1}, \quad i = 2, 3, \ldots, s,$$
$$\mathbf{r}_1 = \mathbf{K}_0^{-1}\mathbf{R}, \tag{13}$$

where s is the number of vectors considered (it is assumed that $s \ll$ number of DOF's) and the matrix $\mathbf{B}$

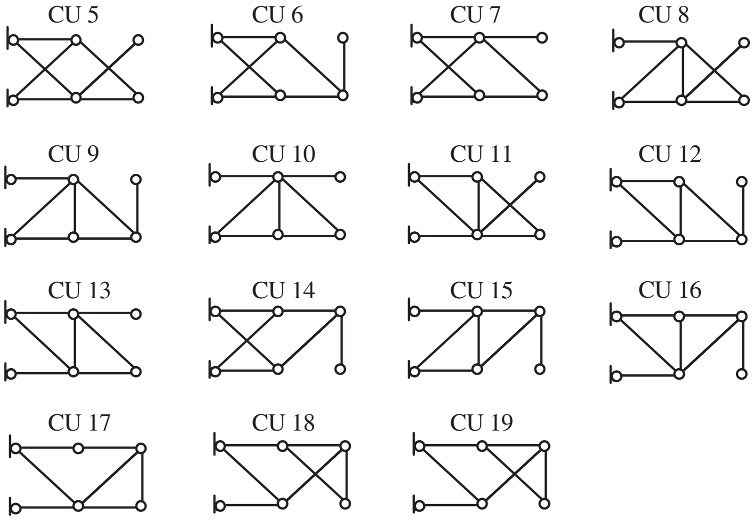

Fig. 7 Conditionally unstable structures obtained by deletion of three or four members

is defined by

$$\mathbf{B} \equiv \mathbf{K}_0^{-1} \Delta\mathbf{K} . \tag{14}$$

The matrix of the basis vectors $\mathbf{r}_B$ is defined by

$$\mathbf{r}_B = \{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_s\} . \tag{15}$$

In cases where $\Delta\mathbf{R} = \mathbf{0}$, the first basis vector is simply $\mathbf{r}_1 = \mathbf{r}_0$. Calculation of the basis vectors by (13) involves only forward and backward substitutions in cases where $\mathbf{K}_0$ is available in the form of (2) from the initial analysis of the structure. For example, assuming that $\mathbf{r}_1$ is given, the vector $\mathbf{r}_2$ is calculated by

$$\mathbf{K}_0\mathbf{r}_2 = -\Delta\mathbf{K}\mathbf{r}_1 . \tag{16}$$

We solve first for the vector of unknowns $\mathbf{t}$ by the forward substitution

$$\mathbf{U}_0^T\mathbf{t} = -\Delta\mathbf{K}\mathbf{r}_1 . \tag{17}$$

The vector $\mathbf{r}_2$ is then calculated by the backward substitution

$$\mathbf{U}_0\mathbf{r}_2 = \mathbf{t} . \tag{18}$$

Similarly, $\mathbf{r}_3$ is calculated by

$$\mathbf{K}_0\mathbf{r}_3 = -\Delta\mathbf{K}\mathbf{r}_2 . \tag{19}$$

3. The reduced matrix $\mathbf{K}_R$ and the reduced vector $\mathbf{R}_R$ are calculated by

$$\mathbf{K}_R = \mathbf{r}_B^T\mathbf{K}\mathbf{r}_B , \quad \mathbf{R}_R = \mathbf{r}_B^T\mathbf{R} . \tag{20}$$

4. The vector of unknown coefficients $\mathbf{y}$ is calculated by solving the set of $(s \times s)$ equations

$$\mathbf{K}_R\mathbf{y} = \mathbf{R}_R . \tag{21}$$

5. The modified displacements $\mathbf{r}$ are evaluated by

$$\mathbf{r} = y_1\mathbf{r}_1 + y_2\mathbf{r}_2 + \ldots + y_s\mathbf{r}_s = \mathbf{r}_B\mathbf{y} , \tag{22}$$

where $\mathbf{y}$ is a vector of s coefficients to be determined.

To improve the efficiency and accuracy of the approximations, the reduced set of simultaneous equations (21) can be transformed into an uncoupled form. An uncoupled set of new basis vectors $\mathbf{V}_i$ $(i = 1, 2, \dots, s)$ is introduced using a Gram-Schmidt orthonormalization (Kirsch 1999; Leu and Huang 1998). The new vectors are determined by the original ones $\mathbf{r}_i$ from

$$\mathbf{V}_i = \left|\mathbf{r}_1^T\mathbf{K}\mathbf{r}_i\right|^{-1/2} \mathbf{r}_1 , \tag{23}$$

$$\overline{\mathbf{V}}_i = \mathbf{r}_i - \sum_{j=1}^{i-1} \left(\mathbf{r}_i^T\mathbf{K}\mathbf{V}_j\right) \mathbf{V}_j ,$$
$$\mathbf{V}_i = \left|\overline{\mathbf{V}}_i^T\mathbf{K}\overline{\mathbf{V}}_i\right|^{-1/2} \overline{\mathbf{V}}_i , \quad i = 2, \dots, s , \tag{24}$$

where $\overline{\mathbf{V}}_i$ and $\mathbf{V}_i$ are the i-th non-normalized and normalized vectors, respectively. Defining the matrix $\mathbf{V}_B$ of new basis vectors and the vector $\mathbf{z}$ of new coefficients, the reduced system (20) becomes uncoupled and the final displacements are given by the explicit expression

$$\mathbf{r} = \mathbf{V}_B\mathbf{z} = \mathbf{V}_B \left(\mathbf{V}_B^T\mathbf{R}\right) . \tag{25}$$

The displacements, calculated by (25), can be expressed as an additively separable quadratic function of the basis vectors $\mathbf{V}_i$ by

$$\mathbf{r} = \sum_{i=1}^{s} \mathbf{V}_i \left(\mathbf{V}_i^T\mathbf{R}\right) . \tag{26}$$

For any assumed number of basis vectors, identical results are obtained by considering either the original set of basis vectors or the new set of uncoupled basis vectors. One advantage in using the new vectors is that all expressions for evaluating the displacements are explicit functions of the original basis vectors. Calculation of any new basis vector $\mathbf{V}_i$ results in an additional term of the displacements expression (26) that is a function of the original vectors $\mathbf{r}_j$ $(j = 1, 2, \ldots, i)$. Consequently, additional vectors can be considered without modifying the calculations that were carried out already.

The efficiency and accuracy of the results achieved by the CA method have been demonstrated in previous studies. The CA approximations provide accurate solutions even in cases where the series of the original basis vectors (the binomial series) diverges. In many problems a small number of basis vectors (2–3 vectors) is sufficient to obtain adequate accuracy. Several criteria for evaluating the errors involved in the approximations have been presented. These errors can be reduced by considering additional basis vectors.

3.1.1
Exact solutions

Exact solutions are obtained by the CA in the general case where an added basis vector is a linear combination of the previous vectors. In most problems the CA approach does not provide exact results. However, the vectors determined by the binomial series are almost linearly dependent (Kirsch and Papalambros 2001). Therefore, "nearly exact" solutions are often achieved with a small number of basis vectors. Exact solutions can be obtained efficiently by the CA method also in cases of low rank modifications. In such cases, exact solutions achieved by the CA method and the Sherman-Morrison-Woodbury formulae are equivalent (Akgun *et al.* 1998). As a typical example consider the case of changes in m truss members (deletion or addition of members). The exact solution is achieved by the CA method if one basis vector is introduced for each changed member by

$$\mathbf{r}_i = \mathbf{K}_0^{-1} \Delta\mathbf{K}_i \mathbf{r}_0 \,, \quad i = 1, \ldots, m \,, \tag{27}$$

where $\Delta\mathbf{K}_i$ is the contribution of the i-th member to $\Delta\mathbf{K}$. If some of the basis vectors are linearly dependent, the exact solution is achieved for a smaller number of vectors. The exact solution is given in this case by (Kirsch and Liu 1995)

$$\mathbf{r} = \mathbf{r}_0 + \sum_{i=1}^{m} y_i \mathbf{r}_i \,, \tag{28}$$

where $\mathbf{r}_0$ is the vector of initial displacements. This procedure is efficient in cases where the number of changed members is much smaller than the number of degrees of freedom.

It is instructive to note that exact solutions can be achieved efficiently also for geometrical changes by viewing these changes as corresponding topological modifications. For example, modifying the coordinates of a single joint, it is possible to obtain the exact solution for the new design by viewing the change in the geometry as two simultaneous changes in the topology. That is, all members connected to the joint are deleted, and new members are added at the modified location (Kirsch and Liu 1997).

3.2
The number of DOF's is decreased

In cases when some joints are deleted from the structure, the sizes of the resulting stiffness matrix and load vector are decreased accordingly. Since the number of DOF's is decreased, the number of analysis equations is changed and complete exact analysis can be employed for the reduced set (9). If the resulting structure is conditionally unstable, the modified stiffness matrix is singular and exact analysis cannot be carried out. However, approximate reanalysis by the CA method with a reduced number of unknowns may provide accurate results. Assuming that the number of basis vectors considered is smaller than the number of degrees of freedom of the modified structure, the solution procedure described above in Sect. 3.1 can be used without any modification. This is the case even when the modified set of equations is of the form of (7)–(9) with some zero identities, as will be shown by the numerical examples in Sect. 4.2.

It is instructive to note that the reduced stiffness matrix used by the CA method is usually not singular even in cases when the modified stiffness matrix is singular. The reduced matrix is singular when the basis vectors are linearly dependent.

3.3
The number of DOF's is increased

Adding some joints to the structure, the number of DOF's is increased. Some procedures for using the CA method in such cases have been proposed in the past (Chen *et al.* 1998; Kirsch and Liu 1997). The procedure presented is more general and provides more accurate results. Increasing the number of DOF's, it is necessary first to establish a Modified Initial Analysis (MIA), such that the new degrees of freedom are included in the analysis model. Considering the augmented stiffness matrix and load vector (10), the MIA model can be selected such that reanalysis will be convenient. Once the MIA is established, it is then possible to analyse modified structures conveniently due to addition or deletion of members, keeping the number of degrees of freedom unchanged.

Considering the formulation of Sect. 2.3, the MIA is established as follows. The matrix of changes in the stiffness $\Delta\mathbf{K}$ is expressed first as a sum of the two matrices $\Delta\mathbf{K}_0$ and $\Delta\mathbf{K}_N$ by

$$\Delta\mathbf{K} = \Delta\mathbf{K}_0 + \Delta\mathbf{K}_N \,. \tag{29}$$

The above matrices are defined in such a way that the modified initial analysis will be easy to carry out. The modified initial stiffness matrix $\mathbf{K}_M$ is expressed as

$$\mathbf{K}_M = \mathbf{K}_A + \Delta\mathbf{K}_0 \,. \tag{30}$$

Matrices $\Delta\mathbf{K}_0$ and $\Delta\mathbf{K}_N$ are defined as

$$\Delta\mathbf{K}_0 = \alpha \begin{bmatrix} \mathbf{0} & \Delta\mathbf{K}_{0N} \\ \Delta\mathbf{K}_{N0} & \Delta\mathbf{K}_{NN} \end{bmatrix},$$

$$\Delta\mathbf{K}_N = \begin{bmatrix} \Delta\mathbf{K}_{00} & (1-\alpha)\Delta\mathbf{K}_{0N} \\ (1-\alpha)\Delta\mathbf{K}_{N0} & (1-\alpha)\Delta\mathbf{K}_{NN} \end{bmatrix}, \tag{31}$$

where α is a scalar multiplier to be selected ($0 < \alpha \leq 1$). Substituting the expressions of $\mathbf{K}_A$ and $\Delta\mathbf{K}_0$ [(10) and (31), respectively] into (30) yields

$$\mathbf{K}_M = \begin{bmatrix} \mathbf{K}_0 & \alpha\Delta\mathbf{K}_{0N} \\ \alpha\Delta\mathbf{K}_{N0} & \alpha\Delta\mathbf{K}_{NN} \end{bmatrix}. \tag{32}$$

The rationale of this selection is that, once the decomposed form (2) is available, factorization of the modified initial stiffness matrix $\mathbf{K}_M$ by

$$\mathbf{K}_M = \mathbf{U}_M^T \mathbf{U}_M \tag{33}$$

is straightforward. Specifically, matrix $\mathbf{U}_M$ can be expressed as

$$\mathbf{U}_M = \begin{bmatrix} \mathbf{U}_0 & \mathbf{U}_{0N} \\ \mathbf{0} & \mathbf{U}_{NN} \end{bmatrix}, \tag{34}$$

where the elements of matrix $\mathbf{U}_0$ are already given. That is, the rows and columns corresponding to the original degrees of freedom are unchanged and only rows and columns corresponding to the new degrees of freedom are calculated. In general the number of added joints is small, and the factorization (33) involves small computational effort.

As to the selected value of α it can be observed that $\alpha = 1$ yields

$$\mathbf{K}_M = \begin{bmatrix} \mathbf{K}_0 & \Delta\mathbf{K}_{0N} \\ \Delta\mathbf{K}_{N0} & \Delta\mathbf{K}_{NN} \end{bmatrix}. \tag{35}$$

One drawback of this selection is that matrix $\mathbf{K}_M$ is not necessarily positive definite and the factorization (33) might not be suitable. Still, it would be possible to use the symmetric factorization

$$\mathbf{K}_M = \mathbf{L}_M \mathbf{D}_M \mathbf{L}_M^T \,, \tag{36}$$

where $\mathbf{L}_M$ is a lower triangular matrix and $\mathbf{D}_M$ is a diagonal matrix. However, in this case $\mathbf{K}_M$ does not represent a real structure and experience has shown that the accuracy of the approximations deteriorates.

In the procedure presented subsequently, this difficulty is overcome by selecting a small α value such that matrix $\mathbf{K}_M$ (32) is a good approximation of the matrix $\mathbf{K}_A + \alpha\Delta\mathbf{K}$ [since $\alpha\Delta\mathbf{K}_{00} \ll \mathbf{K}_0$, see (10), (11)]. Considering the above definitions, the solution procedure involves the following two stages.

1. The modified initial analysis (MIA) is established. Assuming a small α value, matrix $\mathbf{K}_M$ is introduced and factorized. Since the decomposed form (2) is available, this operation involves a small computational effort. The modified initial displacements $\mathbf{r}_M$ are then calculated by

$$\mathbf{K}_M \mathbf{r}_M = \mathbf{R} \,. \tag{37}$$

Given the form (33), this calculation involves only forward and backward substitutions.

2. Once $\mathbf{r}_M$ has been determined, the displacements due to the remaining change in the stiffness matrix $\Delta\mathbf{K}_N$ are calculated. Specifically, the modified equations to be solved are

$$\mathbf{K}\mathbf{r} = (\mathbf{K}_M + \Delta\mathbf{K}_N)\mathbf{r} = \mathbf{R} \,, \tag{38}$$

where $\Delta\mathbf{K}_N$ is defined by (31). This can be done by the CA procedure described in Sect. 3.1 for the simple case where the number of DOF's is unchanged, with $\mathbf{r}_M$, $\mathbf{K}_M$, $\mathbf{U}_M$ and $\mathbf{R}$ replacing $\mathbf{r}_0$, $\mathbf{K}_0$, $\mathbf{U}_0$ and $\mathbf{R}_0$, respectively, as initial values.

4 Numerical examples

The approach presented in this study is general and suitable for all types of structure. For simplicity of presentation small-scale truss example structures will be considered. In all examples, cross-sectional areas of unity have been assumed.

4.1 The number of DOF's is unchanged

To illustrate numerical results for conditionally unstable structures, consider the classical ten-bar truss shown in Fig. 2, subjected to a single loading condition of two concentrated loads. The modulus of elasticity is 30 000 and the eight unknowns are the horizontal (to the right) and the vertical (downward) displacements in joints 1, 2, 3 and 4, respectively. The following cases of deletion of members have been solved:

1. Deletion of members $2+6$, giving the conditionally unstable structure CU 1 (Fig. 4).
2. Deletion of members $4+9$, giving the conditionally unstable structure CU 3 (Fig. 4).

3. Deletion of members 5 + 8 + 9, giving the conditionally unstable structure CU 17 (Fig. 7).
4. Deletion of members 4 + 5 + 8 + 9, giving the conditionally unstable structure CU 19 (Fig. 7).

The exact solutions have been achieved for all the above cases with only three basis vectors, as summarized in Table 1.

To illustrate results for structures with a larger number of members, consider the fifty-bar truss shown in Fig. 8a, subjected to a single concentrated load. The modulus of elasticity is 10 000 and the forty unknowns are the horizontal (to the right) and the vertical (downward) displacements at joints 2 through 21, respectively. Deleting ten diagonal members, the modified design is shown in Fig. 8b. Despite the relatively large number of deleted members, the exact solution shown in Table 2 has been achieved with only three basis vectors.

To illustrate results for deletion of many members and conditionally unstable structures, consider the 19-bar tower truss shown in Fig. 9, subjected to a single loading condition of two concentrated loads. The modulus of elasticity is 10 000 and the 12 unknowns are the horizontal (to the right) and the vertical (upward) displacements at joints 2, 3, 4, 6, 7, and 8, respectively. The following cases have been solved.

1. Deletion of 6 members to obtain the topology shown in Fig. 10a.
2. Deletion of 7 members to obtain the topology shown in Fig. 10b.
3. Deletion of 9 members to obtain the topology shown in Fig. 10c.

Table 1 Ten-bar truss, various cases of deletion of members

Deleted members	Displacement 1	2	3	4	5	6	7	8
2 + 6	2.40	5.80	*	*	−3.60	15.18	−2.40	5.80
4 + 9	2.11	4.67	3.30	13.62	*	14.81	−1.35	5.57
5 + 8 + 9	*	*	2.40	19.76	−3.60	20.96	−3.60	10.38
4 + 5 + 8 + 9	1.20	*	2.40	19.76	*	20.96	−3.60	10.38

* irrelevant results

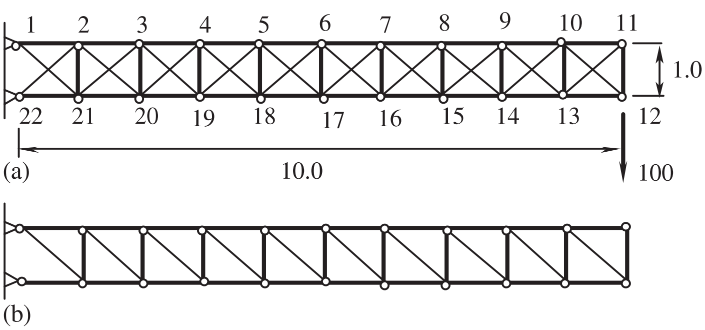

Fig. 8 (a) Fifty-bar truss initial topology, (b) modified topology

Table 2 Fifty-bar truss, deletion of members

DOG	Displacement	DOG	Displacement	DOG	Displacement	DOG	Displacement
1	0.09	11	0.39	21	−0.55	31	−0.40
2	0.14	12	3.13	22	7.07	32	2.28
3	0.17	13	0.42	23	−0.54	33	−0.34
4	0.46	14	4.05	24	6.04	34	1.54
5	0.24	15	0.44	25	−0.52	35	−0.27
6	0.93	16	5.03	26	5.02	36	0.92
7	0.30	17	0.45	27	−0.49	37	−0.19
8	1.55	18	6.05	28	4.04	38	0.45
9	0.35	19	0.45	29	−0.45	39	−0.10
10	2.29	20	7.07	30	3.12	40	0.13

Table 3 19-bar tower truss, deletion of members

Displacement number	Case 1 CA	Case 1 exact	Case 2 CA	Case 2 exact	Case 3 CA	Case 3 exact
1	0.49	0.48	0.78	0.76	0.71	0.70
2	0.27	0.26	0.21	0.21	0.32	0.32
3	1.57	1.58	2.07	2.09	*	*
4	0.43	0.43	0.33	0.32	*	*
5	2.88	2.88	3.61	3.62	3.53	3.53
6	0.48	0.48	0.35	0.32	0.54	0.53
7	0.50	0.50	0.72	0.70	*	*
8	−0.27	−0.27	−0.32	−0.32	*	*
9	1.54	1.54	2.00	2.03	1.97	1.98
10	−0.42	−0.43	−0.51	−0.53	−0.42	−0.43
11	2.93	2.93	3.64	3.65	3.62	3.62
12	−0.48	−0.48	−0.63	−0.64	−0.42	−0.43

* irrelevant results

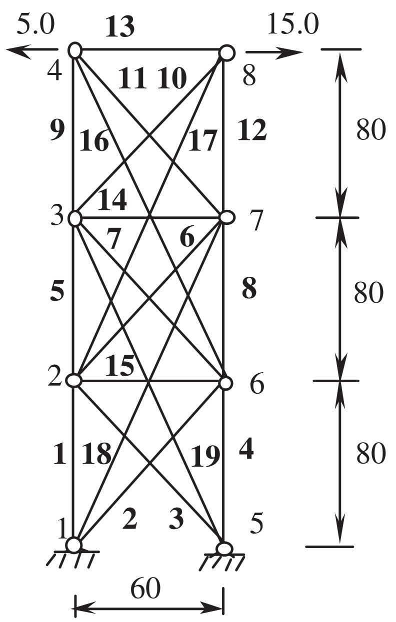

Fig. 9 19-bar tower truss

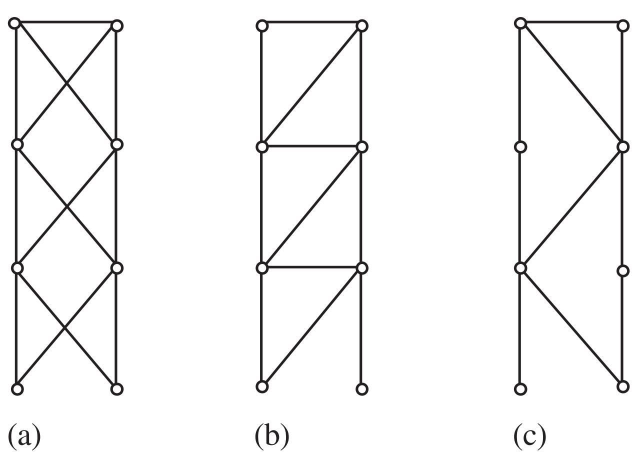

Fig. 10 Modified topologies, 19-bar truss

Considering three basis vectors, the results given in Table 3 indicate the accuracy of the approximations achieved by the CA method.

4.2 The number of DOF's is decreased

To illustrate numerical results for cases where the number of DOF's is decreased, consider again the initial ten-bar truss shown in Fig. 2. The following cases of elimination of members and joints have been solved (see the modified topologies in Fig. 6).

1. Deletion of members $2+6+10$ and joint 2, to obtain the modified topology S30.
2. Deletion of members $2+5+6+10$ and joint 2, to obtain the modified topology S33.
3. Deletion of members $2+6+7+10$ and joint 2, to obtain the modified topology S34.
4. Deletion of members $2+6+8+10$ and joint 2, to obtain the modified topology S35.

The exact solutions, summarized in Table 4, have been achieved for all the above cases with only three basis vectors.

4.3 The number of DOF's is increased

To illustrate reanalysis for the case of addition of members and joints, consider the initial six-bar truss shown in Fig. 11a. The six unknowns are the horizontal (to the right) and the vertical (downward) displacements in joints 1, 2 and 3, respectively. The initial displacement vector and decomposed stiffness matrix are

$$\mathbf{r}_0 = \{1.20, 11.59, -4.80, 20.98, -3.60, 10.39\},$$

Table 4 Ten-bar truss, various cases of deletion of members and joints

Deleted members	Displacements 1	2	3	4	5	6	7	8
2+6+10	2.40	−5.79	*	*	−3.60	−15.18	−2.40	−5.79
2+5+6+10	2.40	−5.79	*	*	−3.60	−15.18	−2.40	−5.79
2+6+7+10	3.60	−10.37	*	*	−2.40	−19.77	−1.20	−11.57
2+6+8+10	1.20	−11.57	*	*	−4.80	−20.96	−3.60	−10.37

* irrelevant results

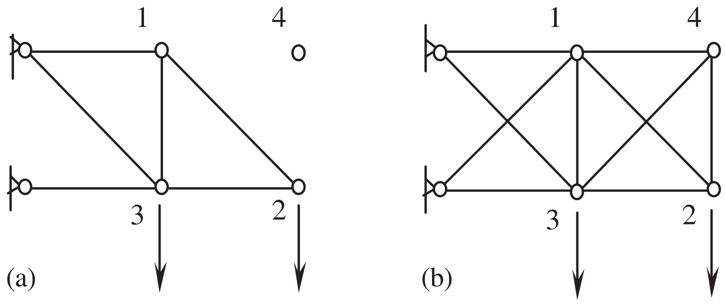

Fig. 11 (a) Six-bar truss, initial topology, (b) modified topology

$$\mathbf{U}_0 = \begin{bmatrix} 10.62 & 2.77 & -2.77 & -2.77 & 0 & 0 \\ & 10.25 & -2.12 & -2.12 & 0 & -8.13 \\ & & 10.03 & 1.72 & -8.31 & -1.72 \\ & & & 3.78 & 3.78 & -3.78 \\ & & & & 10.62 & 2.77 \\ & & & & & 4.67 \end{bmatrix}.$$

Assume addition of a joint and four members to obtain the ten-bar truss shown in Fig. 11b. Reanalysis has been carried out in the following two stages:

1. The Modified Initial Analysis (MIA) is carried out. Selecting $\alpha = 0.001$, the initial decomposed stiffness matrix $\mathbf{U}_M$ is given by (34), where $\mathbf{U}_0$ is already given. Thus, it is necessary to calculate only the sub-matrices $\mathbf{U}_{0N}$, $\mathbf{U}_{NN}$

$$\mathbf{U}_{0N} = \begin{bmatrix} -0.0078 & 0 \\ -0.0021 & 0 \\ -0.0017 & 0 \\ -0.0038 & -0.0220 \\ -0.0028 & 0.0106 \\ 0.0080 & -0.0305 \end{bmatrix},$$

$$\mathbf{U}_{NN} = \begin{bmatrix} 0.3356 & -0.0872 \\ & 0.3220 \end{bmatrix}.$$

For the given $\mathbf{U}_M$, calculation of the modified initial displacement vector by (37) involves only forward and backward substitutions. The result is

$$\mathbf{r}_M = \{1.22, 11.74, -4.87, 21.28, -3.65, 10.52, 2.44, 20.06\}.$$

Due to the small change in stiffness, the displacements of the original degrees of freedom have changed slightly.

2. The displacements due to the remaining change in the stiffness matrix $\Delta\mathbf{K}_N$ are calculated. Employing the CA procedure described in Sect. 3.1, with $\mathbf{r}_M$, $\mathbf{K}_M$, $\mathbf{U}_M$ and $\mathbf{R}$ replacing $\mathbf{r}_0$, $\mathbf{K}_0$, $\mathbf{U}_0$ and $\mathbf{R}_0$, respectively, as initial values, the exact solution achieved with only three basis vectors is

$$\mathbf{r} = \{2.34, 5.58, -3.17, 13.13, -2.46, 6.01, 2.82, 12.65\}.$$

5 Conclusions

Most structural reanalysis methods developed in the past are suitable for the relatively simple case where the number of DOF's is unchanged. The reanalysis approach presented in this study is suitable for problems where the number of DOF's, and the sizes of the stiffness matrix and the load vector are significantly changed. A unified approach for reanalysis of all types of topological modifications has been presented. The modifications considered include both deletion and addition of members and joints. In cases when the number of DOF's is increased, it is necessary to establish first a modified initial analysis, such that the new degrees of freedom are included in the analysis model. A general solution procedure was presented, where an exact modified initial solution is calculated efficiently. The modified solution is then used for reanalysis of further topological modifications.

Numerical examples show that accurate approximations are achieved for significant topological modifications. In cases where the modified structure is conditionally unstable, approximate reanalysis by the CA method, with a reduced number of unknowns, still provides accurate results. Some cases where exact solution is obtained are demonstrated.

Acknowledgements This work has been partially supported by the Automotive Research Center under the sponsorship of the US Army TACOM, and by the General Motors Satellite Research Laboratory at the University of Michigan, Ann Arbor. The support is gratefully acknowledged.

References

Akgun, M.A.; Garcelon, J.H.; Haftka, R.T. 1998: Fast exact linear and nonlinear structural reanalysis and the Sherman-Morrison-Woodbury formulas. *NASA/ISSMO 1st Internet Conf. on Approximations and Fast Reanalysis in Engineering Optimization*

Barthelemy, J-F.M.; Haftka, R. T. 1993: Approximation concepts for optimum structural design – a review. *Struct. Optim.* **5**, 129–144

Bendsøe, M.P.; Mota-Soares C.A. (eds.) 1992 *Proc. NATO ARW on Topology Design of Structures* (held in Sesimbra, Portugal)

Chen, S.; Huang, C.; Liu, Z. 1998: Structural approximate reanalysis for topological modifications by finite element systems. *AIAA J.* **36**, 1760–1762

Kirsch, U. 1981: *Optimum structural design*. New York: McGraw Hill

Kirsch, U. 1989: Optimal topologies of structures (feature article). *Appl. Mech. Rev.* **42**, 223–239

Kirsch, U. 1993a: *Structural optimizations, fundamentals and applications*. Berlin, Heidelberg, New York: Springer

Kirsch, U. 1993b: Efficient reanalysis for topological optimization. *Struct. Optim.* **6**, 143–150

Kirsch, U. 1999: Efficient-accurate approximations for structural optimization. *AIAA J.* **37**, 1663–1669

Kirsch, U. 2000: Combined approximations – a general approach for structural optimization. *Struct. Multidisc. Optim.* **20**, 97–106

Kirsch, U.; Liu, S. 1995: Exact structural reanalysis by a first-order reduced basis approach. *Struct. Optim.* **10**, 153–158

Kirsch, U.; Liu, S. 1997: Structural reanalysis for general layout modifications. *AIAA J.* **35**, 382–388

Kirsch, U.; Papalambros, P.Y. 2001: Exact and accurate solutions in the approximate reanalysis of structures. *AIAA* (to be published)

Leu, L-J.; Huang, C-W. 1998: A reduced basis method for geometric nonlinear analysis of structures. *IASS J.* **39**, 71–75

Noor, A.K. 1994: Recent advances and applications of reduction methods. *Appl. Mech. Rev.* **47**, 125–146

Rozvany, G.I.N.; Bendsøe, M.P.; Kirsch, U. 1995: Layout optimization of structures (feature article). *Appl. Mech. Rev.* **48**, 41–118

Schmit, L.A.; Farshi, B. 1974: Some approximation concepts for structural synthesis. *AIAA J.* **11**, 489–494

Sherman, J.; Morrison, W.J. 1949: Adjustment of an inverse matrix corresponding to changes in the elements of a given column or a given row of the original matrix. *Ann. Math. Statist.* **20**, 621

Sobieszczanski-Sobieski, J.; Haftka, R.T. 1997: Multidisciplinary aerospace design optomization: survey and recent developments. *Struct. Optim.* **14**, 1–23

Svanberg, K. 1987: The method of moving asymptotes – a new method for structural optimization. *Int. J. Num. Meth. Engrg.* **24**, 359–373

Woodbury, M. 1950: Inverting modified matrices. *Memorandum Report 42*, Princeton University, Princeton, NJ: Statistical research group