TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 283, Number 2, June 1984

STRUCTURAL STABILITY OF EQUIVARIANT VECTOR FIELDS ON TWO-MANIFOLDS

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ABSTRACT. A class of vector fields on two-dimensional manifolds equivariant under the action of a compact Lie group is defined. Properties of openness, structural ability, and density are proved.

Introduction and statement of results. Let G be a compact Lie group acting smoothly on a smooth compact connected two-dimensional manifold M. In this paper we define a subset of the space $\mathfrak{X}_G^r(M)$ of C^r, $r \geqslant 1$, equivariant vector fields on M which correspond to the Morse-Smale vector fields studied by Peixoto, Palis, Smale [**17, 11, 15**] and others (see [**13**]). We prove that this subset of equivariant vector fields, which we call G-Morse-Smale vector fields, is open, and each G-Morse-Smale vector field X is equivariantly structurally stable in $\mathfrak{X}_G^r(M)$. That is, if $Y \in \mathfrak{X}_G^r(M)$ is near X then there exists an equivariant homeomorphism of M that sends trajectories of X into trajectories of Y. Also, we prove that it is dense in $\mathfrak{X}_G^r(M)$, with some exceptions; the most important exception corresponds to the fact that the density of the Morse-Smale vector fields for nonorientable 2-manifolds in the C^r topology, with $r \geqslant 2$, is still an open question (see Gutierrez [**6**]). Our results generalize those of Peixoto [**17**].

The definition of G-Morse-Smale vector fields given here is different from that given by Field [**3, 5**] in that it allows the presence of graphs in the nonwandering set. In this way, we have enough vector fields to get density, along with structural stability. However, in higher dimensions the situation is unclear since no results on structural stability have been proved so far. We believe that the concept of "modulus of stability" (Melo [**10**], Palis [**12**]) seems to be more appropriate to the equivariant framework. Examples of equivariant vector fields on 3-manifolds with modulus of stability equal to one are given in [**14**].

In fact, this phenomenom appears even locally. A G-equivariant vector field X on M defines a $G \times R$ action on M (see §1). A critical element of X is a compact $G \times R$ orbit. We require that it be normally hyperbolic. Of course, normal hyperbolicity does not imply local structural stability, but it may provide modulus of stability. We may impose an additional condition on the G-action (see §§1, 2), in order to get local

Received by the editors February 24, 1982 and, in revised form, May 24, 1983.
1980 *Mathematics Subject Classification*. Primary 58F10; Secondary 57E15.
Key words and phrases. Compact Lie group, equivariant vector field, normal hyperbolicity, stratumwise transversality, graph, structural stability, modulus of stability.
[1]Partially supported by CNPQ (Brasil).

stability (a normally hyperbolic critical element with this additional condition is called G-hyperbolic). But this condition is very restrictive, and it would be preferable to have a definition which avoids it.

Our results were announced in [**20**]. As pointed out by M. J. Field (Math. Rev. **80c**: 58012), we provide a class of examples of Ω-stable vector fields on 2-manifolds which are not Axiom A in the sense of Pugh and Shub [**19**].

Now we state the main results in a more precise way.

DEFINITION. A G-equivariant vector field on M is said to be G-Morse-Smale if:

(i) its nonwandering set consists of a finite number of critical elements and graphs, all of them G-hyperbolic;

(ii) the stable and unstable manifolds of critical elements have G-transversal intersection.

The definition of G-hyperbolic graph is in Definition 3.4. The other definitions are found in §1.

THEOREM A. *The set of G-Morse-Smale vector fields is open in* $\mathfrak{X}^r_G(M)$.

THEOREM B. *G-Morse-Smale vector fields are equivariantly structurally stable.*

THEOREM C. *Let d be the dimension of the principal G-orbits (the maximal dimension of the G-orbits). The following hold.*

(a) *Let $d = 0$. Then the set of G-Morse-Smale vector fields is dense in $\mathfrak{X}^1_G(M)$. If M/G is orientable, or otherwise the projective plane, the Klein bottle or the torus with a cross-cap, then the set of G-Morse-Smale vector fields is dense in $\mathfrak{X}^r_G(M)$, $r \geq 1$.*

(b) *Let $d = 1$. If M is not the torus then the set of G-Morse-Smale vector fields is dense in $\mathfrak{X}^r_G(M)$, $r \geq 1$. (The exceptions are described in Example* 4.1(c).)

(c) *Let $d = 2$. Then M is the sphere S^2, the projective plane P^2, or the torus T^2. A G-equivariant vector field on S^2 or P^2 must be null. On T^2 it is null, the rational or the irrational flow (see Example* 4.1(b)).

Proofs of Theorems A, B and C are in §§4, 5 and 6, respectively. In §1 we give general definitions and results. §2 contains a proof of local stability and §3 describes G-hyperbolic graphs. Examples are given §4.

We are grateful to J. Palis for his advice and stimulating discussions. Conversations with A. Lins, W. Melo, G. Tavares and J. Sotomayor were very helpful. We would like to thank the referee for some suggestions and corrections.

1. Generalities on equivariant vector fields. In this section, we recall some basic definitions and results on equivariant dynamical systems. The main reference is Field [**5**].

Let G be a compact Lie group acting smoothly on smooth manifolds M and N. A map f: $M \to N$ is said to be equivariant if it commutes with the group action. That is, $f(gx) = gf(x)$ for all g in G and x in M. The action of G on M induces a natural linear action on the tangent bundle TM by $gv = Dg(x)(v)$, for $v \in T_xM$, where $Dg(x)$: $T_xM \to T_{gx}M$ is the differential of the diffeomorphism g: $M \to M$ given by the action of $g \in G$. An equivariant vector field on M is an equivariant section X: $M \to TM$. Thus $X(gx) = Dg(x)X(x)$. The flow X_t induced by X on M is equivariant: for each $t \in R$, $X_t(gx) = gX_t(x)$, for all $g \in G$ and $x \in M$. So, we can speak of an action of $G \times R$ on M, namely $(g, t)x = X_t(gx)$. Similarly, if f is a G-equivariant diffeomorphism on M, we have an action of $G \times Z$ on M defined by $(g, n)x = f^n(gx)$. Let $\mathfrak{X}^r_G(M)$ and $\mathrm{Diff}^r_G(M)$ denote, respectively, the spaces of all C^r G-equivariant vector fields and diffeomorphisms on M, with the C^r topology, when M is compact.

Let $f\colon M \to M$ be a C^1 diffeomorphism. We say that an f-invariant C^1 compact submanifold V of M is normally hyperbolic for f if there exists a Df-invariant splitting $TV \oplus N^u \oplus N^u$ of T_VM into continuous subbundles such that, relative to some Riemannian matric on M.

$$\sup_{x \in V} \|Df \mid N^s_x\| < \inf_{x \in V} m(Df \mid T_xV),$$
$$\sup_{x \in V} \|Df \mid T_xV\| < \inf_{x \in V} m(Df \mid N^u_x).$$

Here, $m(A) = \inf\{\|AX\|\colon \|X\| = 1\} = \|A^{-1}\|^{-1}$ (see [**8**]).

If V is G-invariant and $f \in \mathrm{Diff}^1_G(M)$, the Riemannian metric may be taken to be equivariant and the bundles N^u and N^s are G-vector bundles over V. If V is left invariant by a C^1 flow X_t, we say that X_t is normally hyperbolic at V if for some $t \neq 0$, the diffeomorphism X_t is normally hyperbolic at V. This definition is independent of the value of t chosen.

Let $f \in \mathrm{Diff}^r_G(M)$. A critical element for f is a $G \times Z$-orbit V which is compact. Thus $V = \{\alpha, f(\alpha), \ldots, f^{p-1}(\alpha)\}$, where α is a G-orbit and p is the period of α, the smallest strictly positive integer for which $f^p(\alpha) = \alpha$. We say V is normally hyperbolic for f it is normally hyperbolic for f^p. Similarly, for $X \in \mathfrak{X}^r_G(M)$, a critical element is a $G \times R$-orbit which is compact.

There is a natural stable and unstable manifolds theory for critical elements of equivariant diffeomorphisms and vector fields. If V is a normally hyperbolic critical element for $f \in \mathrm{Diff}^r_G(M)$, there exist C^r G-invariant locally f-invariant submanifolds $W^u_{\mathrm{loc}}(V)$ and $W^s_{\mathrm{loc}}(V)$ of M, tangent at V to $TV \oplus N^u$ and $TV \oplus N^s$, respectively. Also, there exists a G-invariant neighborhood U of V such that

$$W^u_{\mathrm{loc}}(V) = \{z \in U\colon f^n(z) \in U, n \leqslant 0, \text{ and } d(f^n(z), V) \to 0 \text{ as } n \to -\infty\},$$
$$W^s_{\mathrm{loc}}(V) = \{z \in V\colon f^n(z) \in U, n \geqslant 0, \text{ and } d(f^n(z), V) \to 0 \text{ as } n \to +\infty\}.$$

Moreover, $W^u_{\mathrm{loc}}(V)$ and $W^s_{\mathrm{loc}}(V)$ have the structure of C^r equivariant locally trivial fibrations over V. The fiber $W^{uu}_{\mathrm{loc}}(p)$ at $p \in V$ is C^r, its dimension is the dimension of N^u_p, and is tangent to N^u_p. Similarly for $W^{ss}_{\mathrm{loc}}(p)$. Also, there exist an open neighborhood Q of f in $\mathrm{Diff}^r_G(M)$, a G-invariant neighborhood U of V, and continuous maps $F, F^u, F^s\colon Q \to \mathrm{Diff}^r_G(M)$ such that:

(a) $F(f) = F^u(f) = F^s(f) =$ identity map of M;

(b) for $f' \in Q$, $F(f')(V) = V'$ is contained in U and is a normally hyperbolic critical element of f' of the same period as V;

(c) $F^u(f')(W^{uu}_{\mathrm{loc}}(p)) = W^{uu}_{\mathrm{loc}}(F(f')(p)), p \in V, f' \in Q$.

Similarly for W^u, W^{ss} and W^s. As usual, the global stable and unstable manifolds are obtained by iterating f. For vector fields the theory is similar.

For the notion of G-transversality of stable and unstable manifolds see Bierstone [**1**] and Field [**4**]. For two-dimensional manifolds, that notion is equivalent to that one called "stratumwise transversality". This property is defined as follow. If $H = G_x$ is the isotropy subgroup of a point $x \in M$, then the conjugacy class (H) of H is called the type of the G-orbit $G(x)$. The union $M_{(H)}$ of all G-orbits of type (H) is a differentiable G-fiber bundle with the G-orbits as fibres. The manifold M is stratified by orbit type (see Bredon [**2**]). If $W \subset M$ and H is a subgroup of G, we denote $W_{(H)} = W \cap M_{(H)}$. Let W^s and W^u be stable and unstable manifolds of critical elements of an equivariant diffeomorphism or vector field. Let $x \in M$ and $H = G_x$. We say W^s is stratumwise transverse to W^u at x if $W^s_{(H)}$ is transverse to $W^u_{(H)}$ in $M_{(H)}$. That is, we require that the tangent spaces of $W^s_{(H)}$ and $W^u_{(H)}$ at x generate the tangent space of $M_{(H)}$ at x.

We remark that the normal hyperbolicity hypothesis for a critical element does not imply local structural stability. In order to assure stability, we require an additional condition, which we describe below for flows.

Let V be a normally hyperbolic critical element of $X \in \mathfrak{X}^r_G(M)$. Thus $V = (G \times R)(x)$ is a compact $G \times R$-orbit, for some $x \in M$. There are three cases to be considered:

(a) $V = R(x) =$ trajectory of X,

(b) $V = G(x)$,

(c) $V \neq R(x), G(x)$;

in this case, $G(x)$ is a global Poincaré section for the restriction of X to V. In fact, in the 2-dimensional case, V is equivariantly diffeomorphic to $G(x) \times S^1$. We recall that the isotropy group G_x is constant along a trajectory of the vector field X. Also, $N(G_x)(x)$ is the subset of points of the G-orbit $G(x)$ whose isotropy group is G_x. Here, $N(G_x) = \{g \in G\colon gG_xg^{-1} = G_x\}$ is the normalizer of G_x. It is known that $N(G_x)(x)$ is diffeomorphic to $N(G_x)/G_x$ and $G(x)$ is diffeomorphic to G/G_x. Rank $N(G_x)$ denotes the dimension of the maximal tori $N(G_x)/G_x$.

Definition. Let $V = (G \times R)(x)$ be a normally hyperbolic critical element of $X \in \mathfrak{X}^r_G(M)$. Let $H = G_x$. We say that V is G-hyperbolic if one of the following conditions is satisfied.

(a) $V = R(x) =$ trajectory of X.

(b) If $V = G(x)$ and $X|V = 0$ then rank $N(H)/H = 0$; if $V = G(x)$ and $X|V \neq 0$ then rank $N(H)/H = 1$.

(c) If $V \neq G(x)$ and $V \neq R(x)$ then rank $N(H)/H = 0$.

Remark. We discuss below the conditions of the above definition.

Case (a). V is either a hyperbolic singularity or a hyperbolic closed trajectory of X.

Case (b). If rank $N(H)/H = 0$, then X is null on V. If Y is near X then Y is null on the corresponding normally hyperbolic submanifold V'. If $X|V \neq 0$ and rank $N(H)/H = 1$ then $X|V$ consists of closed trajectories. The trajectory through each $y \in V$ coincides with $N(G_y)(y)$. The same happens for Y near X.

Case (c). The condition rank $N(H)/H = 0$ implies that each trajectory of $X|V$ intersects $G(x)$ in a finite number of points and so it is closed. The same is true of all Y near X.

2. Local stability. Let V be a G-hyperbolic critical element of $X \in \mathfrak{X}^r_G(M)$. In this section, we give a proof of the structural stability of X in a neighborhood of V, by using the following equivariant version of a theorem of Palis and Takens [**16**].

(2.1) THEOREM. *Let X, $Y \in \mathfrak{X}^r_G(M)$. Let V and V' be critical elements of X and Y, respectively. Assume that h: $V \to V'$ is a G-equivariant conjugacy between $X|V$ and $Y|V'$ that can be lifted to invertible linear bundle maps h^s: $N^s \to N'^s$ and h^u: $N^u \to N'^u$. Then we can extend h to a G-equivariant conjugacy H between X and Y from a neighborhood of V to a neighborhood of V'.*

The proof of this theorem is a matter of checking that the pertinent constructions in the proof in [**16**] can be made equivariantly. By conjugacy H between X and Y we mean that H is a homeomorphism conjugating the flows X_t and Y_t: $HX_t = Y_tH$ for all $t \in R$. If we weaken the latter condition by just requiring that H sends trajectories of X into trajectories of Y then H is said to be a topological equivalence. We say X is equivariantly structurally stable in a neighborhood U of V if there exists a neighborhood Q of X in $\mathfrak{X}^r_G(M)$ such that, for each $Y \in Q$, there exists a G-equivariant topological equivalence h between X and Y, from U to $h(U)$.

(2.2) THEOREM. *Let V be a G-hyperbolic critical element of $X \in \mathfrak{X}^r_G(M)$. Then X is equivariantly structurally stable in a neighborhood of V.*

PROOF. If Y is near X, it follows from the Remark in §1 that we can get a G-equivariant topological equivalence h: $V \to V'$ between $X|V$ and $Y|V'$. By reparametrizing the vector fields, we may assume that h is a conjugacy. Since the splitting $T_{V'}M = TV' \oplus N'^u \oplus N'^s$ is close to the splitting $T_VM = TV \oplus N^u \oplus N^s$, there are liftings h^s: $N^s \to N'^s$ and h^u: $N^u \to N'^u$ of h to bundle isomorphisms. Now Theorem 2.1 applies.

The following consequence of Theorem 2.2 could be proved directly.

(2.3) COROLLARY. *$W^s(V) \cap M_{(H)} \neq \varnothing$ if and only if $W^s(V') \cap M_{(H)} \neq \varnothing$; similarly, for W^u. ($M_{(H)}$ is a stratum of G-orbits of type (H).)*

3. G-hyperbolic graphs. In this section, we define a G-hyperbolic graph and prove structural stability in its neighborhood.

Let M be a two-dimensional manifold. A graph of a vector field X on M is a closed and connected subset of M consisting of saddle points and separatrices such that:

(i) the α- and ω-limit sets of each separatrix of the graph are saddle points;

(ii) each saddle point of the graph has at least one stable and one unstable separatrix contained in the graph.

(3.1) PROPOSITION. *Let $X \in \mathfrak{X}^r_G(M)$. Let L be a graph of X such that the stable and unstable manifolds of its saddle points are G-transverse to each other. Then*:

(a) *Each saddle point in L is contained in a zero-dimensional stratum.*

(b) *If p is a saddle point in L, then $W^s(p) - p$ and $W^u(p) - p$ are contained in one-dimensional strata.*

(c) *Let p, q be saddle points in L and suppose that $W^u(p) \cap W^s(q) \neq \varnothing$. Then $W^u(G(p)) - G(p) = W^s(G(q)) - G(q)$.*

PROOF. First we observe that the existence of a saddle point implies that all G-orbits have dimension zero. Let S be a slice for the action of G at p, on which the isotropy group G_p acts orthogonally (slice theorem, see Bredon [**2**]). The G-transversality condition and the G-invariance of W^s and W^u imply that G_p is the group $D_2 = \{R(0), R(\pi), A(0), A(\pi)\}$, where $R(0)$ is the identity, $R(\pi)$ the rotation through π, $A(0)$ the reflection about the x-axis, and $A(\pi)$ the reflection about the y-axis. There are four types of G-orbits on S, corresponding to the isotropy groups R_0, A_0, A_1, D_2. Here $R_0 = \{R(0)\}$, $A_0 = \{R(0), A(0)\}$, $A_1 = \{R(0), A(\pi)\}$. The stratum $S_{(D_2)} = p$ has dimension zero, and $S_{(A_0)}$ and $S_{(A_1)}$ have dimension one. Since, in S, $W^u(p) - p$ coincides with $S_{(A_1)}$ (or with $S_{(A_0)}$), and similarly for $W^s(p) - p$, (a) and (b) have been proved. For (c), let γ be a trajectory of X contained in $W^u(p) \cap W^s(q)$, so that p and q are the α- and ω-limit sets of γ: $\alpha(\gamma) = p$, $\omega(\gamma) = q$. Then, if $g \in G$ is the reflection about $W^s(p)$, $\alpha(g\gamma) = g\alpha(\gamma) = gp = p$ and $\omega(g\gamma) = g\omega(\gamma) = gq$. Hence $W^u(p) - p \subset W^s(G(q)) - G(q)$ and, by G-invariance of W^u and W^s, $W^u(G(p)) - G(p) \subset W^s(G(q)) - G(q)$. The reverse inclusion is proved in the same way.

Let G be a finite group acting on M and let $M = M_1 \cup M_2 \cup \cdots \cup M_k$ be the stratification of M by G-orbit types. Assume M_1 consists of principal G-orbits, and let $N = M_2 \cup \cdots \cup M_k$ be the nonprincipal part of M. As is well known, M_1 is open and dense in M. For each i, the quotient map $\pi_i\colon M_i \to M_i^* = M_i/G$ is a differentiable covering map, whose fibers are the G-orbits. Although each M_i^* is a differentiable manifold, $M^* = M/G$ is a topological manifold with boundary $N^* = N/G$. Let $\pi\colon M \to M^*$ denote the quotient map.

Given an equivariant vector field X on M, we define its projection $X^* = \pi(X)$ on M^* by $X^*(\pi_i(x)) = d\pi_i(x) \cdot X(x)$, for $x \in M_i$. Each M_i^* is left invariant by X^*, and $X^* \mid M_i^*$ is a differentiable vector field. Conversely, if Z is a vector field on M^* that leaves each M_i^* invariant, we can lift it to a vector field X on M, such that $\pi(X) = Z$. If X_t and X_t^* denote the flows of X and X^*, we have $X_t^* \circ \pi = \pi \circ X_t$.

Let p be a singularity for X. Then $\pi(p) = q$ is a singularity for X^*. Since λ is an eigenvalue of $DX(p)$ if and only if λ is an eigenvalue of $DX(gp)$ for each g in G, we can define the eigenvalues of X^* at q as being the eigenvalue of $DX(p)$, where $p \in M$ is such that $\pi(p) = q$. If p is a saddle point for X, q is said to be a saddle point for X^*.

We will use the following notation: $p_1 < p_2$ if there is a trajectory γ such that $\alpha(\gamma) = p_1$ and $\omega(\gamma) = p_2$. The following proposition follows immediately from Proposition 3.1.

(3.2) PROPOSITION. *Let L be a graph as in Proposition* 3.1. *Let $p_1, \ldots, p_n$ be the saddle points of L, with $p_1 < p_2 < \cdots < p_n < p_1$. Then*:

(a) *$\pi(L)$ is a graph for X^*.*

(b) *Let $q_1, \ldots, q_m$ be the saddle points of $\pi(L)$, and suppose that $q_1 < q_2 < \cdots < q_m < q_1$. If $\pi(p_1) = q_1$, then $\pi(p_2) = q_2, \ldots, \pi(p_m) = q_m$, and if $m < n$, $\pi(p_{m+1}) = q_1$, $\pi(p_{m+2}) = q_2, \ldots, \pi(p_{2m}) = q_m$, and so forth, the cycle being repeated.*

The following lemma may be proved by successive application of the following facts:

(a) By a theorem due to Hartman [7], C^2 vector fields on two manifolds are C^1 linearizable near hyperbolic singularities. Thus, if p is a saddle point of X, there is a coordinate system $\varphi\colon U \to R^2$, around p, such that $X_t(x^1, x^2) = (x^1e^{\lambda t}, x^2e^{\mu t})$. Here $(x^1, x^2) = \varphi(x)$, for $x \in U$, and λ, μ are the eigenvalues of x at p. We assume that $\varphi(p) = (0,0)$, $\varphi(x) = (x^1,0)$ for $x \in W^s(p)$, and $\varphi(x) = (0, x^2)$ for $x \in W^u(p)$.

(b) Let γ be a trajectory of X such that $\alpha(\gamma) = p_1$, $\omega(\gamma) = p_2$ are saddle points of X. Let $\varphi_i\colon U_i \to R^2$ be coordinate systems around p_i, $i = 1, 2$, as in (a). Let $S_1 \subset U_1$ be a cross-section through a point of γ such that, for some $T > 0$, $X_T(S_1) = S_2$ is contained in U_2. Assume that the points in S_1 and S_2 have coordinates $(x^1, 1)$ and $(1, y^2)$, respectively. Let $g\colon S_1 \to S_2$ be the map $(1, y^2) = g(x^1, 1) = X_T(x^1, 1)$. Then $y^2 = k(x^1, 1)x^1$, where $k\colon S_1 \to R$ is a continuous function bounded away from zero.

(3.3) LEMMA. *Let L be a graph with saddle points $p_1,\ldots,p_n$. Let λ_i, μ_i, $\lambda_i < 0 < \mu_i$, be the eigenvalues of X at p_i. Let φ_i be coordinate systems around p_i, as above. Through each separatrix of L consider cross-sections as above. Assume that an arc of the trajectory starting from a point $(x, 1) \in S_1$ intersects each cross-section once, and intersects S_1 once again in the point $(y, 1)$. Then*

$$|y| = K(x, 1)\,|x|^{|\lambda_1\cdots\lambda_n|/\mu_1\cdots\mu_n}$$

where $K\colon S_1 \to R$ is a continuous function bounded away from zero.

Lemma 3.3 motivates the following definition (inspired by Sotomayor [**22**]).

(3.4) DEFINITION. Let L be a graph of $X \in \mathfrak{X}^r_G(M)$. Let $q_1,\ldots,q_m$ be the saddle points of $\pi(L)$, and let λ_i, μ_i, $\lambda_i < 0 < \mu_i$, be the eigenvalues of X^* at q_i, $i = 1,\ldots,m$. We say L is a G-hyperbolic graph if:

(a) the separatrices of the saddle points of L have G-transversal intersections,

(b) $|\lambda_1,\ldots,\lambda_m| \neq \mu_1\cdots\mu_m$.

As an immediate consequence of Lemma 3.3 we have

(3.5) COROLLARY. *Let L be a G-hyperbolic graph and let λ_i, μ_i $(i = 1,\ldots,m)$ be as in* 3.4. *Then L is an attractor if*

$$|\lambda_1\cdots\lambda_m| > \mu_1\cdots\mu_m,$$

and is a repulsor if

$$|\lambda_1\cdots\lambda_m| < \mu_1\cdots\mu_m.$$

REMARK. Corollary 3.5 is true for vector fields of class C^1 [**22**].

The following proposition says that G-hyperbolic graphs are persistent by small equivariant perturbations of the vector field.

(3.6) PROPOSITION. *Let L be a G-hyperbolic graph of the vector field $X \in \mathfrak{X}^r_G(M)$, $r \geqslant 1$. There exists a neighborhood Q of X in $\mathfrak{X}^r_G(M)$ such that if $Y \in Q$ then L is a G-hyperbolic graph for Y. Moreover, if L is an attractor (repulsor) for X then L is an attractor (repulsor) for Y.*

PROOF. The proposition follows from Proposition 3.1 and Corollary 3.5.

Now we state the main result of this section.

(3.7) PROPOSITION. *Let L be a G-hyperbolic graph of a vector field $X \in \mathfrak{X}_G^r(M)$, $r \geqslant 1$. Then X is structurally stable in a neighborhood of L.*

PROOF. Suppose that L is an attractor (similar argument applies to repulsor). Let Q be a neighborhood of X, as in Proposition 3.6, and $Y \in Q$. Let $X^* = \pi(X)$ and $Y^* = \pi(Y)$. Then $\pi(L)$ is contained in the boundary N^* of M^*, and is an attractor graph for X^* and Y^*. In a collar neighborhood of $\pi(L)$ we define a topological equivalence h between X^* and Y^* via arc length. Now, via π, we can lift h to a G-equivariant topological equivalence between X and Y, defined on a G-invariant neighborhood of L.

4. Examples and proof of Theorem A. In the rest of this paper M is a compact connected boundaryless two-dimensional manifold. We denote by d the dimension of principal G-orbits. Before we start the proof, we give some examples.

(4.1) EXAMPLES. (a) Let $p\colon R^3 - \{0\} \to S^2$ be defined by $p(x) = x/|x|$, where $|x|$ is the Euclidean norm of x. If X' is a linear vector field on R^3, we define a vector field X on the sphere S^2 via p: $X(x) = Dp(x) \cdot X'(x)$, for $x \in S^2$. If X' is equivariant then X is equivariant for the restriction to S^2 of the action of G. For instance, if X' is the vector field defined by the diagonal matrix diag$[a, a, b]$, $a > b$, and G is SO(2) acting as rotation around the z-axis, then X is the equivariant vector field which has the north and south poles as sources and the equator as an attractor G-orbit of singularities. All the other trajectories of X start at the poles and end in the equator, along the meridians. If x is a point in the equator, we have $N(G_x)/G_x =$ SO(2). So $G(x)$ is not G-hyperbolic for X. Adding to X a small equivariant vector field tangent to the equator and zero outside of a neighborhood of the equator, we obtain a G-Morse-Smale vector field with the equator as a closed trajectory. However, if we have $G = \mathrm{O}(2)$ instead of SO(2), the same vector field X is G-Morse-Smale, since $N(G_x)/G_x = D_2/A_0$ (see notation in proof of 3.1) has dimension zero and so the equator is G-hyperbolic. We cannot perturb X to make the equator a closed trajectory.

(b) Assume that G acts transitively on M (that is, M is a G-orbit). Then M is S^2, P^2, or T^2. Let X be a G-equivariant vector field on M. So the whole of M is a critical element for X. Of course, it is not normally hyperbolic. If M is S^2 or P^2 then X is null. If M is T^2 then all of the trajectories of X are (i) singularities or (ii) closed trajectories or (iii) trajectories dense on T^2. In fact, $\overline{R(x)}$ (the closure of a trajectory of X) has the structure of a compact connected abelian Lie group, so is a torus (see Field [**5**, §2.B1]). For S^2 and P^2, X is structurally stable. For T^2, it depends on $k = \operatorname{rank} N(G_x)/G_x$. If $k = 0$, then X must be null and is structurally stable. If $k = 1$, and $X \neq 0$, then X consists of closed trajectories and is structurally stable. If $k = 2$, X is never structurally stable. In fact, X has modulus of stability equal to one (the parameter being rotation number). This gives a complete description in the case of transitive action.

(c) Assume $d = 1$. If X is a G-equivariant vector field on M such that $M = (G \times R)(x)$ for $x \in M$, then $M = T^2$. In fact, $M = G(x) \times S^1 = S^1 \times S^1$. If $k = 0$, the trajectories of X are closed and X is structurally stable. If $k = 1$ then X has modulus of stability equal to one.

(d) Let $G = Z_2 \times Z_2 \times Z_2$ acting on $S^2 \subset R^3$ by $(a, b, c)(x, y, z) = (ax, by, cz)$, $(a, b, c) \in G$, $(x, y, z) \in S^2$. Let X be a G-Morse-Smale vector field on S^2 described as follows. The boundary of the first octant of S^2 is a graph L for X, with saddle points $p_1 = (1, 0, 0), p_2 = (0, 1, 0), p_3 = (0, 0, 1)$ and separatrices γ_i, $i = 1, 2, 3$, such that $\alpha(\gamma_i) = p_i$, $\omega(\gamma_i) = p_{i+1}$, for $i = 1, 2$, and $\alpha(\gamma_3) = p_3$, $\omega(\gamma_3) = p_1$. In the interior of the first octant there is a source, and all regular trajectories have that source as their α-limit set and L as their ω-limit set. The other octants are reflected copies of the first one.

(4.2) PROOF OF THEOREM A. If $d = 2$, the set of G-Morse-Smale vector fields is empty (see Example 4.1(b)). Let $d \leqslant 1$. Let X be a G-Morse-Smale vector field on M. Then X is not as in example 4.1(c). So, a filtration (see [**12**]) for X and for all $Y \in \mathfrak{X}_G^r(M)$ near X can be constructed observing the following:

(i) The sources and sinks of X are G-orbits of singularities, G-orbits of closed trajectories, or G-hyperbolic graphs, all of them persistent by small equivariant perturbations;

(ii) when there are saddle connections between two G-orbits of saddle points, they are contained in strata $M_{(H)}$ of dimension one, by the G-transversality property. Hence they are persistent and the G-transversality is preserved by small equivariant perturbations.

The rest of the argument is standard and is omitted.

5. Proof of Theorem B. Let X be a G-Morse-Smale vector field on M. If $d = 2$, there is nothing to prove. Let $d = 1$. Then X is not as in Example 4.1(c). The critical elements are attractors or repulsors. It is easy to construct a G-equivariant topological equivalence between X and Y, for Y near X. Now let $d = 0$. The construction of a topological equivalence between X and Y consists of a globalization of constructions that we describe below.

(a) Let p be an isolated attractor (or repulsor) singularity for X and let p' be the corresponding one for Y. Let d be a neighborhood of p and p' whose boundary S is transversal to X and Y. We assume that S is invariant by G_p and that if $(gD) \cap D \neq \varnothing$ $(g \in G)$ then $g \in G_p$ (slice theorem, Bredon [**21**]). If h: $S \to S$ is a G_p-equivariant homeomorphism, we extend h to D via conjugation $(h = Y_t h X_{-t})$, and $h(p) = p'$. Now we equivariantly extend h to $G(D)$.

(b) If L is a graph for X, we take a segment S transversal to X and Y through a point of the graph. If h is defined on a fundamental domain contained in S, we extend h to a neighborhood of L as in the proof of Proposition 3.7.

(c) Let γ be a closed trajectory for X and let γ' be the corresponding one for Y. Let S be a section transversal to X and Y, which is invariant by X and Y (in the sense that $X_t(S) \subset S$, for some $t \in R$). In the usual way, if h: $S \to S$ is a homeomorphism sending $\gamma \cap S$ into $\gamma' \cap S$, we extend h to a neighborhood of γ via conjugation. Then we extend h equivariantly to a neighborhood of $G(\gamma)$.

(d) From each saddle point we construct a transversal segment S. If we have a separatrix between two saddle points, we extend, in an equivariant way, a homeomorphism defined on such a segment S to a neighborhood of the separatrix via arc length, by using those segments as references to measure the arcs. If the separatrix goes to an attractor (or repulsor) we use as a reference to measure the arcs, the circle or segment constructed in (a), (b) or (c).

Now, to define the homeomorphism $h\colon M \to M$, we start by defining it in a segment S, as constructed in (d), for a saddle point that does not belong to a graph. If there is no such saddle point, S could be any section as in (a), (b) or (c). The rest of the construction is standard, and is omitted.

6. Proof of Theorem C. When $d = 2$ there is no G-Morse-Smale vector field on M. However, a complete description of the G-equivariant vector fields on M is given in Example 4.1(b).

Let $d = 1$. If X is as in Example 4.1(c), then X cannot be approximated by a G-Morse-Smale vector field. This class of vector fields, whose description is given in 4.1(c), provides the only exception for nondensity, in the case $d = 1$.

Now assume that X is not as in 4.1(c). Since there are no saddle points, we do not have to worry about transversality conditions. It is easy to perturb the critical elements of X into G-hyperbolic attractors and repulsors. We omit details.

Let $d = 0$. Let $X \in \mathfrak{X}^r_G(M)$. Let $X_1 \in \mathfrak{X}^r_G(M)$ be a vector field near X, in the C^r topology, whose singularities are G-hyperbolic. We also may assume that the graphs and closed trajectories of X_1 lying on N (the nonprincipal part of M) are G-hyperbolic. Let $Y_1 = \pi(X_1)$ be the projection of X_1 on M/G. By theorems of Peixoto [**17**], Markley [**9**] and Gutierrez [**6**] we can approximate Y_1 by a vector field Y_2, in the C^r topology, which satisfies the properties of a Morse-Smale vector field on M_1/G (M_1 is the principal part of M) and coincides with Y_1 on N/G. This can be accomplished because the required perturbations can be made outside of a collar neighborhood of N/G. The cited theorems take care of the cases when M/G is orientable, the projective plane, the Klein bottle and the torus with a cross-cap. Otherwise, we use the "Closing Lemma", Pugh [**18**], which applies just for the C^1 topology. It is easy to see that $\pi^{-1}(Y_2)$ is a G-Morse-Smale vector field near X in the C^r or C^1 topology, depending on the case.

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