STRUCTURAL STABILITY OF HYBRID SYSTEMS[1]

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Abstract

We study hybrid systems from a global geometric perspective as piecewise smooth dynamical systems. Based on an earlier work, we define the notion of the hybrifold as a single piecewise smooth state space reflecting the dynamics of the original system. Structural stability for hybrid systems is introduced and analyzed in this framework. In particular, it is shown that a Zeno state is locally structurally stable and that a standard equilibrium on the boundary of a domain implies structural instability.

1 Introduction

The study of various dynamical properties of hybrid systems has been quite intensive the last decade, e.g., [20, 6, 13, 21, 8, 10, 4, 12, 9, 22]. Recently we suggested a geometric approach for the analysis of a class of hybrid systems [17, 18]. We introduced the notions of the hybrifold and hybrid flow in order to study the hybrid system as a single piecewise smooth dynamical system. This study is continued in the current paper; in particular, structural stability for hybrid systems is discussed in a geometric framework.

Roughly speaking, a smooth dynamical system is structurally stable if it is topologically equivalent to every system which is sufficiently close to it. We will generalize this idea to hybrid systems. To make the notion precise, we will specify what is meant by "close" and "topologically equivalent." Structural stability for continuous time dynamical systems (i.e., flows) was introduced by Andronov and Pontryagin [2], who proved that "most" smooth systems on a two-dimensional disk are structurally stable. The importance of this property for models of real-life systems is clear: if a system is structurally stable, its qualitative properties are insensitive to small perturbations of the system. If it is not, even a small error in measurement can produce a system which qualitatively looks entirely different from the "true" system.[1] The importance of structural stability for smooth (continuous and discrete time) dynamical systems was reemphasized in the influential work of Smale [19], Peixoto [15], and others during the 1960s. It was hoped that in some sense "most" systems were structurally stable, as in dimension two. However, this hope was soon shattered by the discovery of very large sets of non-structurally stable systems in dimensions greater than two. Despite this, robustness of its qualitative behavior is still a very important piece of information about a system.

The classical theory of structural stability of smooth dynamical systems states, for instance, that hyperbolic equilibria are locally structurally stable. This result extends directly to hybrid systems in the sense that if a hyperbolic equilibrium of a hybrid system is in the interior of a domain, then the equilibrium is locally structurally stable. A standard equilibrium on the boundary of a domain, however, is not locally structurally stable. The main contribution of the paper is to show that Zeno states, which necessarily lie on the boundary of a domain [18], are locally structurally stable. This means that it may be difficult to remove Zenoness by perturbing the system.

The outline of the paper is as follows. Hybrid systems and their executions are defined in Section 2, together with hybrifolds and hybrid flows. In Section 3 we introduce the notions of topologically equivalence and structural stability for hybrid systems, and show that the Zeno state is locally structurally stable. Some discussion on related work is given in Section 4.

2 Preliminaries

We start by giving the definition of a hybrid system and its execution. Then we describe how so called regular hybrid systems without branching can be studied on a quotient space called the hybrifold. See [17] and [18] for more details.

2.1 Hybrid system and execution

Definition 1 (Hybrid system) *An n-dimensional hybrid system is a 6-tuple* $\mathbf{H} = (Q, E, \mathcal{D}, \mathcal{X}, \mathcal{G}, \mathcal{R})$, *where*

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[1]This is not to say that all important systems are structurally stable. Take, for example, the Hamiltonian ones.

- $Q = \{1, \ldots, k\}$ *is the set of* discrete states *of* $\mathbf{H}$*, where* $k \geq 1$ *is an integer;*

- $E \subset Q \times Q$ *is the set of* edges*;*

- $\mathcal{D} = \{D_i : i \in Q\}$ *is the set of* domains *of* $\mathbf{H}$*, where* $D_i \subset \{i\} \times \mathbf{R}^n$ *for all* $i \in Q$*;*

- $\mathcal{X} = \{X_i : i \in Q\}$ *is the set of* vector fields *such that* X_i *is Lipschitz on* D_i *for all* $i \in Q$*;*

- $\mathcal{G} = \{G(e) : e \in E\}$ *is the* guards*, where for each* $e = (i, j) \in E$*,* $G(e) \subset D_i$*;*

- $\mathcal{R} = \{R_e : e \in E\}$ *is the set of* resets*, where for each* $e = (i, j) \in E$*,* R_e *is a relation*[2] *between elements of* $G(e)$ *and elements of* D_j*, i.e.,* $R_e \subset G(e) \times D_j$*.*

Given $\mathbf{H}$, the basic idea is that starting from a point in some domain D_i we flow according to X_i until (and if) we reach some guard $G(i, j)$, then switch via the reset $R_{(i,j)}$, continue flowing in D_j according to X_j and so on. The hybrid time trajectory, defined next, is interpreted as the time instants when discrete transitions from one domain to another take place.

Definition 2 (Hybrid time trajectory) *A (forward) hybrid time trajectory is a sequence (finite or infinite)* $\tau = \{I_j\}_{j=0}^N$ *of intervals such that* $I_j = [\tau_j, \tau_j']$*, for all* $j \geq 0$ *if the sequence is infinite; if* N *is finite, then* $I_j = [\tau_j, \tau_j']$ *for all* $0 \leq j \leq N-1$ *and* I_N *is either of the form* $[\tau_N, \tau_N']$ *or* $[\tau_N, \tau_N')$*. The sequences* τ_j *and* τ_j' *satisfy:* $\tau_j \leq \tau_j' = \tau_{j+1}$*, for all* j*.*

We use $\langle\tau\rangle$ to denote the set $\{0, \ldots, N(\tau)\}$ if $N(\tau)$ is finite, and $\{0, 1, 2, \ldots\}$ if $N(\tau)$ is infinite.

Definition 3 (Execution) *A (*forward*)* execution *of a hybrid system* $\mathbf{H}$ *is a triple* $\chi = (\tau, q, x)$*, where* τ *is a hybrid time trajectory,* $q : \langle\tau\rangle \to Q$ *is a map, and* $x = \{x_j : j \in \langle\tau\rangle\}$ *is* C^1 *maps such that for all* $t \in I_j$*,* $x_j : I_j \to D_{q(j)}$ *satisfies*

$$\dot{x}_j(t) = X_{q(j)}(x_j(t)).$$

Furthermore, for all $j \in \langle\tau\rangle$*, we have*

$$(q(j), q(j+1)) \in E, \quad x_j(\tau_j') \in G(q(j), q(j+1)),$$

and

$$(x_j(\tau_j'), x_{j+1}(\tau_{j+1})) \in R_{(q(j), q(j+1))}.$$

For an execution $\chi = (\tau, q, x)$, denote by $\tau_\infty(\chi)$ its *execution time*:

$$\tau_\infty(\chi) = \sum_{j=0}^{N(\tau)} (\tau_j' - \tau_j) = \lim_{j \to N(\tau)} \tau_j' - \tau_0.$$

[2] If a reset relation R_e is actually a map $G(e) \to D_j$, with $e = (i, j) \in E$, we write $y = R_e(x)$ instead of $(x, y) \in R_e$.

An execution χ is called *infinite*, if $N(\tau) = \infty$ or $\tau_\infty(\chi) = \infty$; *Zeno*, if $N(\tau) = \infty$ and $\tau_\infty(\chi) < \infty$; and *maximal* if it is not a strict prefix of any other execution. The last statement means that there exists no other execution $\chi' = (\tau', q', x')$ such that τ is a strict prefix of τ' and $x = x'$ on τ (in the sense that $x_j = x_j'$ on I_j for all $j \in \langle\tau\rangle$).

We say that an execution $\chi = (\tau, q, x)$ starts at a point $p \in D = \bigcup_j D_j$ if $p = x_0(\tau_0)$ and $\tau_0 = 0$. It *passes through* p if $p = x_j(t)$ for some $j \in \langle\tau\rangle$, $t \in I_j$, $t > \tau_0$.

A hybrid system is called *deterministic* if for every $p \in D$ there exists at most one maximal execution starting from p. It is called *non-blocking* if for every $p \in D$ there is at least one infinite execution starting from p. Necessary and sufficient conditions for a hybrid system to be deterministic and non-blocking can be found in [10]. Roughly speaking, resets have to be functions, guards have to be mutually disjoint and whenever a continuous trajectory of one of the vector fields in $\mathcal{X}$ is about to exit the domain in which it lies, it has to hit a guard.

2.2 Hybrifold and hybrid flow

Unless specified otherwise, we will from now on assume that $\mathbf{H}$ is *regular*. In short, this means that $\mathbf{H}$ is deterministic and non-blocking and that all the building blocks of $\mathbf{H}$ are piecewise smooth. Furthermore, each continuous time orbit always exits a domain through a guard which lies on the boundary of the domain, and enters it through the image of a reset map which is a homeomorphism and takes values on the boundary of the "next" domain. For a more detailed and precise formulation of the notion of regularity, see [17] and [18].

Given $\mathbf{H}$, define a map $\Phi^{\mathbf{H}} : \Omega_0 \to D$, (where $\Omega_0 \subset \mathbf{R} \times D$ will be specified later) as follows. Let $p \in D$ be arbitrary. Because of the assumption that $\mathbf{H}$ is deterministic and non-blocking, there exists a unique infinite execution $\chi(p) = (\tau, q, x)$ starting at p. For any $0 \leq t < \tau_\infty(\chi(p))$ there exist a unique $j \in Q$ such that $t \in [\tau_j, \tau_j')$. Then define

$$\Phi^{\mathbf{H}}(t, p) = x_j(t).$$

To define $\Phi^{\mathbf{H}}(t, p)$ for negative t, set $\Phi^{\mathbf{H}}(t, p) = \Phi^{\bar{\mathbf{H}}}(-t, p)$, where $\bar{\mathbf{H}}$ is the *reverse* hybrid system [18, 17]. Let Ω_0 be the largest subset of $\mathbf{R} \times D$ on which $\Phi^{\mathbf{H}}$ is defined. It can be shown [17, 18] that Ω_0 contains a neighborhood of $\{0\} \times \operatorname{int} D$ in $\mathbf{R} \times D$. Moreover, for all $p \in D$, $\Phi^{\mathbf{H}}(0, p) = p$, and

$$\Phi^{\mathbf{H}}(t, \Phi^{\mathbf{H}}(s, p)) = \Phi^{\mathbf{H}}(t + s, p),$$

whenever both sides are defined.

The basic idea in construction of the hybrifold from a hybrid system is simple: "glue" each guard to the image of the corresponding reset via the reset map. More precisely, let $\sim$ be the equivalence relation on D generated by $p \sim \tilde{R}_e(p)$, for all $e \in E$ and $p \in \overline{G(e)}$. Here $\overline{G(e)}$ is the closure of the guard $G(e)$ and $\tilde{R}_e$ is the extended reset which coincides with R_e on $G(e)$ but is defined on a neighborhood of $\overline{G(e)}$ where it is a piecewise smooth homeomorphism onto its image. The existence of $\tilde{R}_e$ is ensured by regularity of $\mathbf{H}$ [17, 18]. Collapse each equivalence class to a point to obtain the quotient space

$$M_{\mathbf{H}} = D/\sim .$$

Definition 4 (Hybrifold) *We call $M_{\mathbf{H}} = D/\sim$ the* hybrifold *of* $\mathbf{H}$.

Denote by π the natural projection $D \to M_{\mathbf{H}}$, which assigns to each p its equivalence class $p/\sim$. Put the *quotient topology* on $M_{\mathbf{H}}$. Recall that this is the smallest topology that makes π continuous, i.e., a set $V \subset M_{\mathbf{H}}$ is open if and only if $\pi^{-1}(V)$ is open in D. Some basic properties of the hybrifold include [17, 18]: the hybrifold $M_{\mathbf{H}}$ is a topological n-manifold with boundary; both $M_{\mathbf{H}}$ and its boundary are piecewise smooth; and the restriction $\pi|_{\text{int } D} : \text{int } D \to \pi(\text{int } D)$ is a diffeomorphism. It is not difficult to see that $M_{\mathbf{H}}$ can be naturally equipped with a distance function which makes π into a piecewise isometry. The hybrifold enables us to study the dynamics of a hybrid system on a single phase space.

Definition 5 (Hybrid flow) *The* hybrid flow *of* $\mathbf{H}$, $\Psi^{\mathbf{H}} : \Omega \to M_{\mathbf{H}}$, *is given by*

$$\Psi^{\mathbf{H}}(t, \pi(p)) = \pi\Phi^{\mathbf{H}}(t, p).$$

Here $\Omega = \{(t, \pi(p)) : (t, p) \in \Omega_0\}$. In other words, orbits of $\Psi^{\mathbf{H}}$ are obtained by projecting orbits of $\Phi^{\mathbf{H}}$ by π. By the $\Phi^{\mathbf{H}}$-orbit of p we mean the collection of points $\Phi^{\mathbf{H}}(t, p)$ for all possible t (i.e., all t such that $(t, p) \in \Omega_0$).

In general, $\Psi^{\mathbf{H}}(t, \pi(p))$ may not be a single point. Therefore, assume that $\mathbf{H}$ is *without branching* which, roughly speaking, means the following: if C is the equivalence class of some point lying on the boundary of a domain, then there exists at most one $p \in C$ which can be reached from its corresponding domain by a trajectory of the corresponding vector field, and at most one $q \in C$ whose trajectory enters the corresponding domain. If the hybrid system satisfies this property, then it can be shown that $\Psi^{\mathbf{H}}(t, \pi(p))$ is indeed a single point, for all $(t, p) \in \Omega$. For details, please see [17].

Next we establish some basic properties of the hybrid flow for a regular hybrid system without branching.

For each $t \in \mathbf{R}$ and $x \in M_{\mathbf{H}}$, let

$$M(t) = \{y \in M_{\mathbf{H}} : \Psi^{\mathbf{H}}(t, y) \text{ is defined}\},$$

and

$$J(x) = \{s \in \mathbf{R} : \Psi^{\mathbf{H}}(s, x) \text{ is defined}\}.$$

Observe that if $x = \pi(p)$, then $J(x) \cap [0, \infty) = [0, \tau_\infty(x))$, where $\tau_\infty(x)$ is the execution time of $\chi(p)$, the unique execution of $\mathbf{H}$ starting at p. Also, for $t > 0$, $M(t)$ contains all points $x = \pi(p)$ such that $\tau_\infty(x) > t$.

If $M(t)$ is not empty, denote by $\Psi^{\mathbf{H}}_t : M(t) \to M_{\mathbf{H}}$ the *time t map of* $\Psi^{\mathbf{H}}$ defined by $\Psi^{\mathbf{H}}_t(x) = \Psi^{\mathbf{H}}(t, x)$. It is possible to prove the following results [17, 18]: if each vector field X in $\mathcal{X}$ is smooth (in addition to being globally Lipschitz), then for each $x \in M_{\mathbf{H}}$ the map $t \mapsto \Psi^{\mathbf{H}}_t(x)$ is continuous and smooth except at countably many points in $J(x)$; each map $\Psi^{\mathbf{H}}_t$ is injective; whenever both sides are defined $\Psi^{\mathbf{H}}_t\Psi^{\mathbf{H}}_s(x) = \Psi^{\mathbf{H}}_{t+s}(x)$; and there is an open and dense subset of Ω on which $\Psi^{\mathbf{H}}$ is smooth. Observe that the local flow of a smooth vector field satisfies these properties.

A point $y \in M_{\mathbf{H}}$ is called an *ω-limit point of $x \in M_{\mathbf{H}}$* if $y = \lim_{m\to\infty} \Psi^{\mathbf{H}}_{t_m}(x)$, for some sequence $t_m \to \tau_\infty(x)$. The set of all ω-limit points of x is called the *ω-limit set of x* and is denoted by $\omega(x)$.

Definition 6 (Zeno state) *A point $z \in M_{\mathbf{H}}$ is called a* Zeno state *for x if $z \in \omega(x)$ and $\tau_\infty(x) < \infty$.*

We will also refer to points in $\pi^{-1}(z)$ as *Zeno states* in $\mathbf{H}$. If the execution starting from $x \in M_{\mathbf{H}}$ is Zeno, then $\omega(x)$ consists of exactly one Zeno state for x and

$$\omega(x) \subset \bigcap_{e \in E_\infty(x)} \pi(\overline{G(e)}),$$

see [17, 18]. Here $E_\infty(x) \subset E$ denotes the set of discrete transitions which occurs infinitely many times in the execution starting from x.

The following example illustrates Zeno.

Example 1 (Water tank system [1, 8, 18, 17])
Consider the hybrid system $WT = (Q, E, \mathcal{D}, \mathcal{X}, \mathcal{G}, \mathcal{R})$, where $Q = \{1, 2\}$, $E = \{(1, 2), (2, 1)\}$, $D_i = \{i\} \times [0, \infty) \times [0, \infty)$, $i = 1, 2$,

$$X_1 = (w - v_1, -v_2)^T, \quad X_2 = (-v_1, w - v_2)^T,$$

$$G(1, 2) = \{(1; x_1, x_2) \in D_1 : x_2 = \ell_2\},$$

$$G(2, 1) = \{(2; x_1, x_2) \in D_2 : x_1 = \ell_1\},$$

and

$$R_{(1,2)}(1; x_1, \ell_2) = (2; x_1, \ell_2),$$

$$R_{(2,1)}(2; \ell_1, x_2) = (1; \ell_1, x_2).$$

The water tank interpretation is as follows. For $i \in Q$, x_i denotes the volume of water in tank i and v_i the constant flow of water out of tank i. The desired minimum volume ℓ_i of water in tank i is to be achieved by dedicating the constant inflow w exclusively to one tank at a time. The control strategy is to switch the inflow to the first tank whenever $x_1 = \ell_1$ and to the second tank whenever $x_2 = \ell_2$ (assuming that the initial volumes are greater than ℓ_1 and ℓ_2, respectively). Assume $\max(v_1, v_2) < w < v_1 + v_2$ and, for simplicity, $\ell_1 = \ell_2 = 0$. Then, every infinite execution of WT is Zeno. Moreover, the "origin" of M_{WT} is a Zeno state.

3 Structural Stability of Hybrid Systems

Intuitively speaking, a smooth dynamical system is structurally stable if it is topologically equivalent to every system which is sufficiently close to it. We will generalize this idea to hybrid systems.

Definition 7 (Topological equivalence) *Two regular hybrid systems without branching* $\mathbf{H}$ *and* $\mathbf{H}'$ *are called* topologically equivalent *if their hybrid flows are topologically equivalent. That is, there exists a homeomorphism* $h : M_{\mathbf{H}} \to M_{\mathbf{H}'}$ *carrying orbits of the hybrid flow of* $\mathbf{H}$ *to those of* $\mathbf{H}'$*, preserving the orientation, but not necessarily preserving the time.*

In other words, $\mathbf{H}$ and $\mathbf{H}'$ are topologically equivalent if their dynamics are qualitatively the same.

We now define C^r-topology on the space of regular hybrid systems without branching. Relative to this topology we say that $\mathbf{H}'$ is close to $\mathbf{H}$ if they have exactly the same states, discrete transitions, domains and guards; further, vector fields and reset maps of $\mathbf{H}'$ should be close to the corresponding vector fields and reset maps of $\mathbf{H}$, and the corresponding reset maps should have exactly the same images. Let us make this more precise.

For $r \geq 1$ and two C^r-manifolds M and N, denote by $C^r(M, N)$ the space of all C^r maps from M to N equipped with the C^r-topology. So $f, g \in C^r(M, N)$ are close if $T_x^k f$ is close to $T_x^k g$, for all $x \in M$ and $0 \leq k \leq r$, where $T_x^k f$ denotes the k^{th} derivative (or tangent map) of f at x. For more details, see [7].

Let $\mathcal{H}^r$ denote the set of all regular hybrid systems without branching $\mathbf{H} = (Q, E, \mathcal{D}, \mathcal{X}, \mathcal{G}, \mathcal{R})$ with all components of class C^r. That is, all vector fields in $\mathcal{X}$, guards in $\mathcal{G}$, and reset maps in $\mathcal{R}$ are of class C^r. We define a C^r topology on $\mathcal{H}^r$ by specifying a collection of sets which we declare to be its basis. Here is how we do that.

Let $\mathbf{H} = (Q, E, \mathcal{D}, \mathcal{X}, \mathcal{G}, \mathcal{R}) \in \mathcal{H}^r$. For each $i \in Q$, choose a neighborhood $\mathcal{U}_i$ of X_i in $C^r(D_i, TD_i)$. For each $e \in E$, choose a neighborhood $\mathcal{U}_e$ of R_e in $C^r(G(e), \operatorname{im} R_e)$. Let $\mathcal{U}$ be the set of all hybrid systems $\mathbf{H}' = (Q', E', \mathcal{D}', \mathcal{X}', \mathcal{G}', \mathcal{R}') \in \mathcal{H}^r$ such that

$$Q' = Q, \ \ E' = E, \ \ \mathcal{D}' = \mathcal{D}, \ \ \mathcal{G}' = \mathcal{G},$$

and for every $i \in Q$ and $e \in E$,

$$X_i' \in \mathcal{U}_i, \quad \text{and} \quad R_e' \in \mathcal{U}_e.$$

Using all possible such sets $\mathcal{U}$ as a basis, we generate a topology $\mathcal{T}^r$ on $\mathcal{H}^r$.

The following result is an immediate consequence of the definition.

Proposition 1 *If* $\mathbf{H_1}$ *is close to* $\mathbf{H_2}$ *relative to* $\mathcal{T}^r$*, then* $M_{\mathbf{H_1}} = M_{\mathbf{H_2}}$.

Definition 8 (Structural stability) *A hybrid system* $\mathbf{H} \in \mathcal{H}^r$ *is said to be* C^r-structurally stable *if it has a neighborhood* $\mathcal{U}$ *in* $\mathcal{H}^r$ *such that every element of* $\mathcal{U}$ *is topologically equivalent to* $\mathbf{H}$.

We claim that the water tank system is structurally stable.

Example 1 (Cont'd)
To see that WT is C^r-structurally stable, for any $r \geq 1$, denote by θ the angular variable in the polar coordinate system in $\mathbf{R}^2$ (see Fig. 1) and observe that for $i = 1, 2$,

$$0 < m_i^- \leq X_i\theta \leq m_i^+ < \infty$$

on $D_i - \{(0,0)\}$. Here $X_i\theta$ denotes the derivative of θ along X_i, i.e., $X_i\theta = d\theta(X_i)$. Let $\mathcal{U}$ be the set of all

$$\mathbf{H}' = (Q, E, \mathcal{D}, \mathcal{X}', \mathcal{G}, \mathcal{R}') \in \mathcal{H}^r$$

such that for all $i \in Q$, $X_i'(0,0) \neq \mathbf{0}$,

$$0 < m_i^{-'} < X_i'\theta < m_i^{+'}$$

on $D_i - \{(0,0)\}$, $\operatorname{im} R_e' = \operatorname{im} R_e$, for all $e \in E$, and

$$\|R_e' - R_e\| < \epsilon_e,$$

where $m_i^{-'} < m_i^-$, $m_i^{+'} > m_i^+$, and ϵ_e will be specified later.

For $e \in E$, consider the map $P_e : \overline{G(e)} \to \overline{G(e)}$ defined as the first-return map for $\overline{G(e)}$ of the hybrid flow of $\mathbf{H}$. It is of class C^r and its Lipschitz constant at $(0,0)$ is less than or equal to

$$\eta = \nu_1\mu_1\nu_2\mu_2,$$

where ν_1, ν_2 are Lipschitz constants of $R_{(1,2)}, R_{(2,1)}$, respectively, at $(0,0)$, and μ_1, μ_2 are the norms of the derivatives at $(0,0)$ of the maps $h_1 : \overline{\operatorname{im} R_{(2,1)}} \to \overline{G(1,2)}$ and $h_2 : \overline{\operatorname{im} R_{(1,2)}} \to \overline{G(2,1)}$, defined by flowing from the image of a reset to the guard along the corresponding vector field (cf., [11]).

Let $P_e' : \overline{G(e)} \to \overline{G(e)}$ be the corresponding first-return map for $\mathbf{H}'$. That it is well defined is guaranteed by the above condition on X_1' and X_2' (this is not difficult to check). The map P_e' is C^r and, as above, its Lipschitz constant at $(0,0)$ is less than or equal to

$$\eta' = \nu_1'\mu_1'\nu_2'\mu_2',$$

where $\nu_1', \nu_2', \mu_1', \mu_2'$ are the corresponding numbers for $\mathbf{H}'$ (i.e., ν_1', ν_2' are Lipschitz constants of $R_{(1,2)}', R_{(2,1)}'$, respectively, at $(0,0)$, etc.)

We know that $\eta < 1$. It is not difficult to see that we can choose $m_i^{-'}, m_i^{+'}, \epsilon_e$ to make η' sufficiently close to η so that $\eta' < 1$. Then for every $e \in E$ and $p \in \overline{G(e)}$, $(P_e')^k(p) \to (0,0)$, as $k \to \infty$, exponentially fast, in fact, as $(\eta')^k$. Thus every execution of every $\mathbf{H}' \in \mathcal{U}$ converges to $(0,0)$. Therefore, every $\mathbf{H}' \in \mathcal{U}$ is topologically equivalent to WT, so WT is structurally stable.

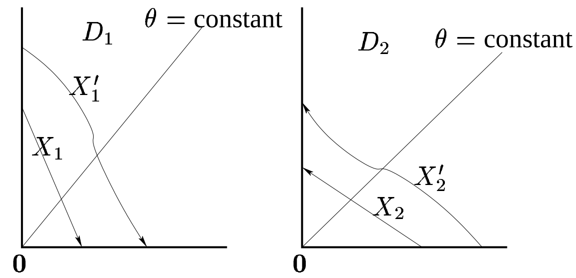

Figure 1: The water tank example.

Note that in concluding this it does not matter whether $(0, 0)$ is a Zeno state for $\mathbf{H}'$. However, we now show that every execution of $\mathbf{H}'$ is indeed Zeno.

Let $e = (i, j) \in E$ and consider a real-valued function τ_e defined as follows: for $p \in \text{im } R_e$, let $\tau_e(p)$ be the amount of time it takes the X_j'-trajectory of p to first reach the boundary of D_j (that is, to reach $G(j, i)$). Since $X_j'(0, 0) \neq \mathbf{0}$, the Implicit Function Theorem guarantees that τ_e is a smooth function. Let $T_e(p)$ be the first-return time of $p \in \text{im } R_e$. That is, $T_e(p) = \tau_{(i,j)}(p) + \tau_{(j,i)}(q)$, where q is the first intersection of the X_j'-orbit with $G(j, i)$. Then T_e is a smooth function and let L be its Lipschitz constant on some neighborhood of $\mathbf{0}$. Clearly, $T_e(\mathbf{0}) = 0$. Therefore,

$$\begin{aligned} T_e((P_e')^k(p)) &= |T_e((P_e')^k(p)) - T_e(\mathbf{0})| \\ &\leq L|(P_e')^k(p) - \mathbf{0}| \\ &\leq L(\eta')^k. \end{aligned}$$

Therefore,

$$\tau_\infty(p) = \sum_{k=0}^{\infty} T_e((P_e')^k(p)) \leq L \sum_{k=0}^{\infty} (\eta')^k < \infty,$$

for all $p \in \text{im } R_e$. We will soon see that this turns out not to be a coincidence.

3.1 Structural stability of hybrid equilibria

Recall that an *equilibrium* of a hybrid system $\mathbf{H}$ is a point $x \in M_{\mathbf{H}}$ which does not move under the hybrid flow, i.e., $\Psi^{\mathbf{H}}(t, x) = x$, for all $t \in J(x)$. Hence, a point in the interior of the domain for which the vector field vanishes is an equilibrium of the hybrid system. Equilibria also include Zeno states which make no time progress.

Definition 9 (Local structural stability) *For a hybrid system $\mathbf{H} \in \mathcal{H}^r$, an equilibrium $x \in M_{\mathbf{H}}$ is called* locally structurally stable *if there exists a neighborhood $\mathcal{U}$ of $\mathbf{H}$ in $\mathcal{H}^r$ such that every $\mathbf{H}' \in \mathcal{U}$ is* locally topologically equivalent *to $\mathbf{H}$ at x. That is, there exists an equilibrium x' of $\mathbf{H}'$ and neighborhoods U, U' of x, x' in $M_{\mathbf{H}} = M_{\mathbf{H}'}$, respectively, such that the restriction of the hybrid flow of $\mathbf{H}'$ to U' is topologically equivalent to the restriction of the hybrid flow of $\mathbf{H}$ to U.*

We now state the main result of the paper.

Theorem 1 *Suppose that $\mathbf{H} \in \mathcal{H}^r$ and let $x \in M_{\mathbf{H}}$. Suppose*

(a) for some i, $x \in \pi(\text{int } D_i)$ is a hyperbolic equilibrium for X_i,

or

(b) x is a Zeno sink, i.e., there is a neighborhood W of x in $M_{\mathbf{H}}$ such that x is the Zeno state for every execution starting in W.

Then x is locally structurally stable.

Proof. (a) Follows from the classical theory of dynamical systems; see, for example, Theorem 4.11 in [14].

(b) The proof is completely analogous to that for WT. The only difficulty is to find an analog of the function θ which was readily available in WT because of the special structure of domains and guards. However, in general, we can still do it *locally*: for every domain D_i with vector field X_i we can always find a function f_i defined in some neighborhood U_i of p_i in D_i, where $\pi(p_i) = x$ and $p_i \in D_i$, such that relative to X_i on U_i, f_i has properties analogous to those of θ relative to the vector fields in WT. One way to do this is the following. Let A (representing an "entry set" into a domain, i.e., the image of some reset) and B (representing an "exit set" from a domain, i.e., a guard) are two smooth hypersurfaces in $\mathbf{R}^n$ which meet transversely and let $p \in A \cap B$ (representing a Zeno state). Assume X is a smooth vector field on $\mathbf{R}^n$ transverse to both A and B which in particular means that it has no equilibria. That this is a fair assumption was shown in [17]: vector fields in a hybrid system never vanish at its Zeno state. Let U be a sufficiently small neighborhood of p in $\mathbf{R}^n$. Then there exists a diffeomorphism $\Phi : U \to \mathbf{R}^n$ such that $\Phi(p) = \mathbf{0}$, $\Phi(A) = \{x : x_n = 0\}$, and $\Phi(B) = \{x : x_{n-1} = 0\}$. (This is an exercise in elementary differential geometry.) Let $\Theta(x) = \arctan(x_n/x_{n-1})$ and set $f = \Theta \circ \Phi^{-1}$. Then f is a smooth function on $U \setminus (A \cap B)$, $f = 0$ on $U \cap A \setminus (A \cap B)$, and $f = \pi/2$ on $U \cap B \setminus (A \cap B)$. Furthermore, by the Flow Box theorem [14], we can choose U so small that the flow of X on U looks approximately like a collection of straight lines from A to B. Then we also get $0 < m_- \leq Xf \leq m_+ < \infty$ on U, as desired. ■

Note that in Theorem 1 the only equilibria considered on the boundary of domains are Zeno states. However, we will soon see without difficulty that a *standard equilibrium* on the boundary of a domain implies structural instability. Recall [17] that a standard equilibrium of $\mathbf{H}$ is a point $x \in \mathbf{H}$ at which all relevant vector fields vanish, i.e., if $\pi^{-1}(x) = \{p_1, \ldots, p_l\}$ where $p_j \in D_j$, then $X_j(p_j) = \mathbf{0}$.

Proposition 2 *Let $\mathbf{H} \in \mathcal{H}^r$ and suppose that $\mathbf{H}$ has a standard equilibrium in $\pi\left(\bigcup_j \partial D_j\right)$. Then $\mathbf{H}$ is structurally unstable.*

Proof. A small perturbation of the vector fields will cause the removal of the equilibrium from the boundary. In other words, if x is a standard equilibrium and $\pi^{-1}(x) = \{p_1, \ldots, p_l\}$ where $p_j \in D_j$, then it is not difficult to perturb each vector field X_j to X'_j so that X'_j has an equilibrium p'_j in the interior of D_j and no equilibria on the boundary of D_j near p_j. The new hybrid system is clearly not topologically equivalent to the original one. ■

4 Conclusions

Structural stability for hybrid systems was introduced in the paper. It is an important property, because a structural stable system is robust to modeling errors. A Zeno state is an equilibrium of a hybrid system, which arises from the interaction of the continuous and the discrete dynamics. It was shown that Zeno states are locally structurally stable. This means that Zeno may be difficult to remove by perturbing the system. Similarly, in analogy to the classical case, hyperbolic hybrid closed orbits [16] can be shown to be locally structurally stable as well, as will be presented elsewhere.

Many issues on structural stability of hybrid systems remain to be studied. For example, an important question is: in dimension two, do structurally stable systems form a "large" set (in analogy with the case of smooth systems)? An affirmative answer, in a slightly different context, was given by [3]. Namely, even if "sliding" in the sense of Filippov [5] is allowed, then the generic piecewise smooth vector field on a smooth, orientable, boundaryless, compact surface is structurally stable. Moreover, structural stability is completely characterized by a set of four conditions, which we do not discuss here.

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