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STRUCTURAL STABILITY OF LORENZ ATTRACTORS

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Dedicated to the memory of Rufus Bowen

Introduction.

E. Lorenz studied the following system of differential equations in connection with problems in hydrodynamics [5]:

$$(1) \qquad \begin{cases} \dot{x} = -10x + 10y \\ \dot{y} = 28x - y - xz \\ \dot{z} = -8/3z + xy. \end{cases}$$

The apparent dynamical behavior of the solutions of this system of differential equations has been a topic of recent interest ([1], [12]). The first author introduced a geometric description of a flow which seems to have the qualitative dynamics of the solutions of the Lorenz equations (1). This geometrically defined flow has a complicated attractor which is not topologically structurally stable in a persistent way [1]. The second author proved that there is an uncountable set of these attractors, each with a different topological type [12]. These results represent some of the striking behavior which has been discovered in an attempt to classify generic flows up to topological equivalence. This paper is an attempt to demonstrate that the Lorenz attractor is not as pathological as these results indicate. We prove that the continuous family of attractors described in [12] is indeed a complete family of attractors occurring for flows in a neighborhood

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of the geometric Lorenz flow. In particular, we construct a 2-parameter family of flows containing the geometric Lorenz flow, which has the property that any perturbation of the geometric Lorenz flow is topologically conjugate on a neighborhood of the attractor to a nearby member of the family. In this sense, the geometric Lorenz attractor is structurally stable of codimension 2. The Lorenz equations (1) are invariant under rotation of $\mathbf{R}^3$ around the z-axis by π. Within the space of flows possessing this symmetry, the geometric Lorenz flow is structurally stable of codimension one rather than two. The proofs of these results are based upon

(i) the constructions of suspensions [9] and inverse limits [11] in dynamical systems, and

(ii) recent results on the bifurcations of maps of the unit interval [2].

Henceforth, we shall work exclusively with the geometric Lorenz flow, ignoring the question of whether it accurately portrays the solutions of (1). Thus our principal result can be stated as:

Theorem. — There is an open set $\mathcal{U}$ *in the space of all vector fields in* $\mathbf{R}^3$ *and a continuous mapping* k *of* $\mathcal{U}$ *into a two-dimensional disk, such that*

(A) *each* $X \in \mathcal{U}$ *has a 2-dimensional (" Lorenz ") attractor;*

(B) X *and* $Y \in \mathcal{U}$ *are topologically conjugate by a homeomorphism close to the identity iff they have the same image under* k.

Note. — Here " near the identity " means within M of the identity for the C^0 distance. For the " only if part " M is the diameter of the holes in figure 2, or about 20 units for equation (1). For the " if part " M is the front to back thickness of figure 2, or less than 1/10 for equation (1). The underlying problem [12] is that there are conjugacies changing kneading sequences, but they are not even the identity on π_1 of figure 2.

There are several natural choices of coordinates in the two-dimensional disk. However, we do not go into this subject heavily.

We choose a pair, without proving how they are related to kneading sequences, which would seem to be a natural choice. Another natural choice are the " Parry-coordinates ", (m, c). That is, in [7] Parry proves that any f satisfying (8) is topologically conjugate to one and only one map $P = P_{c,m}$, given by:

$$P(x) = \begin{cases} m(x-c)+1 & 0 \leq x < c \\ m(x-c) & c < x \leq 1. \end{cases} \tag{2}$$

It would, of course, be nice to know how these coordinates are related to one another.

The authors would like to thank the referee for a number of helpful comments, including a simpler proof of the Lemma in Section II.

1. The model differential equations.

We begin with the topological 3-cell T shaped as in the Figure 1

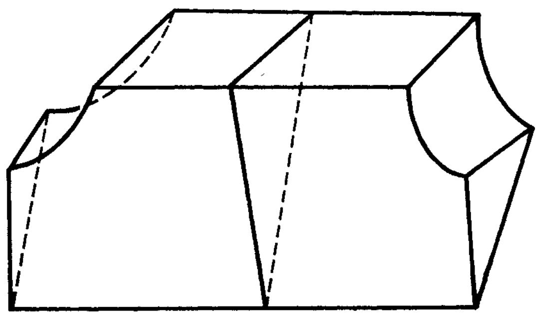

FIG. 1

except that the front and back faces are tangent along the x-axis. Thus the three " triangles " drawn, actually have cusps at their bottom " vertex ".

We can best describe T in terms of three model linear differential equations:

$$(3)\qquad \begin{cases} \dot{x} = ax \\ \dot{y} = -by, & 0<c<a<b. \\ \dot{z} = -cz \end{cases}$$

Let S, the top square of T, be given by

$$(4)\qquad z=1, \quad -1/2 \leq x, y \leq 1/2.$$

With initial conditions $(x_0, y_0, 1)$ in S, we solve

$$(5)\qquad \begin{cases} x = x_0 e^{at} \\ y = y_0 e^{-bt} \\ z = e^{-ct}. \end{cases}$$

Now the right hand " triangle " of T is taken in the plane $x=1$, so that the trajectory meets this plane when

$$(6)\qquad \begin{cases} x=1 \\ y = y_0 x_0^{b/a} = y_0 x_0^u, & 0<x_0 \leq 1/2 \\ z = x_0^{c/a} = x_0^s, & 0<s<1<u \end{cases}$$

with similar equations holding for $-1/2 \leq x_0 < 0$. We now flow in a smooth, non-zero (non-linear) way so as to take this " triangle " into a subset of S as indicated in Figure 2.

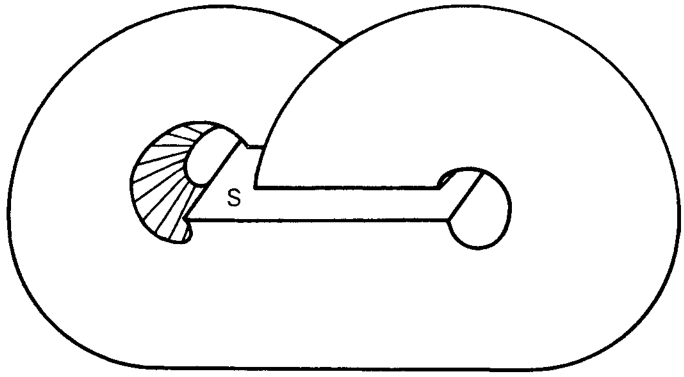

FIG. 2

There is of course a choice here. But we can clearly make this choice so that the resulting Poincaré map $F : S \to S$ has the form

$$F(x_0, y_0) = (f(x_0), H(x_0, y_0))$$

where

$$(7) \qquad \begin{cases} a) \ H(x_0, y_0) > \frac{1}{4} & \text{for} \quad x_0 > 0, \\ b) \ H(x_0, y_0) < -\frac{1}{4} & \text{for} \quad x_0 < 0, \end{cases}$$

and where f satisfies:

$$(8) \qquad \begin{cases} a) \ f(0^+) = -1/2, \quad f(0^-) = 1/2, \\ b) \ f'(x) > \sqrt{2}, \\ c) \ -1/2 < f(x) < 1/2, \end{cases}$$

b) and *c)* holding throughout the range $-1/2 \leq x \leq 1/2$. This describes a flow with a Lorenz attractor. Condition (8) above implies that f is *locally eventually onto* [12]. This means that if $J \subset [-1/2, 1/2]$ is any subinterval, then there is an $n > 0$ such that $f^n(j) = [-1/2, 1/2]$. Here we have chosen $-1/2 \leq x_0, y_0 \leq 1/2$ to simplify our differential equations. Below, we will use the more usual domain:

$$0 \leq x, \ y \leq 1.$$

2. One-Dimensional Analysis.

The Lorenz attractor is a two-dimensional set in $\mathbf{R}^3$. In [12] it is described as the inverse limit of a semiflow on a two-dimensional branched manifold. The return map of this semiflow is a discontinuous function on an interval. This function $f: \mathrm{I} \to \mathrm{I}$, as a map of the unit interval, has the following properties:

(9)
$$\begin{cases} a)\ f \text{ is locally eventually onto;} \\ b)\ f \text{ has a single discontinuity } c \text{ and is strictly increasing on } [0, c) \text{ and } (c, 1]; \\ c)\ f_-(c)=1, \quad f_+(c)=0, \quad f(0)<c<f(1); \\ d)\ f'(x)\to\infty \text{ as } x\to c \text{ (from right and left).} \end{cases}$$

Figure 3 displays the graph of f.

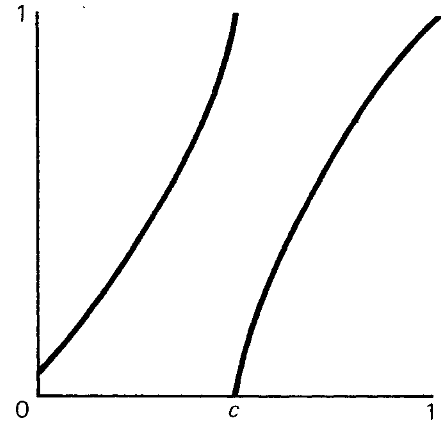

Fig. 3

This section is devoted to a study of the map f. In particular, we use the theory of symbolic dynamics to obtain two real numbers which characterize f up to topological equivalence. Recall that two maps $f: \mathrm{M}\to\mathrm{M}$ and $g: \mathrm{N}\to\mathrm{N}$ are *topologically equivalent* if there is a homeomorphism $h: \mathrm{M}\to\mathrm{N}$ so that $h\circ f = g\circ h$. The results of this section are obtained by constructing a topological space from a set of sequences with the property that the shift map on the sequence space induces a map which is almost topologically equivalent to f.

We begin by establishing some notation and terminology. Sequences will be denoted by underbars: $\underline{x}=\{x_i\}_{i=0}^{\infty}$. The *kneading sequences* of f satisfying (2) are the sequences $\underline{a}$, $\underline{b}$ of 0's and 1's defined by:

(10)
$$a_i = \begin{cases} 0 & \text{if } f^i(0^+)<c \\ 1 & \text{if } f^i(0^+)\geq c \end{cases}$$
$$b_i = \begin{cases} 0 & \text{if } f^i(1^-)\leq c \\ 1 & \text{if } f^i(1^-)>c \end{cases}$$

$i=0, 1, 2, \ldots$. In particular $\underline{a}$ and $\underline{b}$ are always infinite sequences.

The (one sided) *shift space* (on two symbols) Σ is the set of sequences of 0's and 1's, indexed by the non-negative integers, with the topology it inherits from the distance defined by $d(\underline{x},\underline{y})=\sum_{i=0}^{\infty}|x_i-y_i|2^{-i}$. The *shift map* $\sigma:\Sigma\to\Sigma$ is defined by $\sigma(\underline{x})=\underline{y}$ where $y_i=x_{i+1}$. A *subshift* is the restriction of σ to a closed subset $\Gamma\subset\Sigma$ which is invariant under σ. Define an ordering on Σ by $\underline{x}<\underline{y}$ if there exists an n such that $x_i=y_i$ for $i<n$ and $x_n=0$, $y_n=1$. This agrees with the usual order of the real line when $\underline{x}$ and $\underline{y}$ are the binary expansions of distinct real numbers.

We use the kneading sequences of f to construct a subshift $\Gamma\subset\Sigma$. Define Γ by $\underline{x}\in\Gamma$ if

$$(11)\qquad \underline{a}\leq\sigma^i\underline{x}\leq\underline{b},\qquad i=0,1,2,\ldots$$

Let $I_0=[0,c]$, $I_1=[c,1]$.

Note that there are numbers γ, δ such that $\gamma<c<\delta$ and $f(\gamma)=f(\delta)=c$. We use the conventions $f^{-1}(I_0)=[0,\gamma]\cup[c,\delta]$ and $f^{-1}(I_1)=[\gamma,c]\cup[\delta,1]$, though f has a discontinuity at c. Using this convention, we define $I(x_0,\ldots,x_n)$ to be $\bigcap_{i=0}^{n}f^{-i}(I_{x_i})$ for $\underline{x}\in\Gamma$ and $n\geq 0$.

Lemma. — *If* $\underline{x}\in\Gamma$, *then* $I(x_0,\ldots,x_n)$ *is a closed line interval* J *such that* f^i *maps* J *homeomorphically into* $I(x_i)$ *for* $0\leq i\leq n$. *The interval* J *is maximal with respect to this property.*

Proof. — We proceed by induction on n. The case $n=0$ is trivial. Assume the lemma is true for n, and consider $J=I(x_0,\ldots,x_{n+1})$ for some $\underline{x}\in\Gamma$. Now $\sigma\underline{x}\in\Gamma$ so that $K=I(x_1,\ldots,x_{n+1})$ satisfies the lemma by the induction hypothesis. Note that $J=I(x_0)\cap f^{-1}(K)$. There are several cases to consider. We write $\alpha=f(0)$ and $\beta=f(1)$.

Case 1. — $K\subset[\alpha,\beta]$. Then $f(I_{x_0})\supset K$ for $x_0=0$ or 1, so that $J=I(x_0)\cap f^{-1}(K)$ is a closed interval and $f(J)=K$. The two properties of the lemma are clearly satisfied.

Case 2. — α is an interior point of K. If $x_0=1$, then $f(I_1)\supset K$ and we proceed as in Case 1. Assume $x_0=0$. Then $f(I_0)\supset[\alpha,c]$, so that $J=I_0\cap f^{-1}(K)$ has the form $[0,b']$, where f maps $[0,b']$ onto $[\alpha,b'']$ and b'' is the right end point of K. Once again the properties of the Lemma are clearly satisfied.

Case 3. — β is an interior point of K. This case is similar to case 2.

Case 4. — $K\subset[0,\alpha]$. If $x_0=1$, we proceed as in Case 1. We further assert that we cannot have $x_0=0$. Indeed, we will show that $x_0=0$ and $K\subset[0,\alpha]$ imply that $\underline{x}<\underline{a}$, contradicting the assumption that $\underline{x}\in\Gamma$.

Suppose $K=[d,e]$ with $0\leq d\leq e\leq\alpha$ and $x_0=0$. Then, for some $k\leq n$, $f^k(K)\subset I_0$ and $f^k(e)=c$. Otherwise, there would exist an $\varepsilon>0$ such that $f^i(e+\varepsilon)\in I_{x_{i+1}}$

for $0 \leq i \leq n$, contradicting the maximality of K. Pick the smallest $\ell \leq k$ such that $f^{\ell}(e)=c$ or $f^{\ell}(\alpha)=c$. Then for $0 \leq i < \ell$, we have $f^i(e) \subset \operatorname{int}(I_{x_{i+1}})$. Since $e \leq \alpha$, we have either:

($4a$) for all $0 \leq i < \ell$, $f^i(e)$ and $f^i(\alpha)$ are on the same side of c and $f^{\ell}(e) \leq f^{\ell}(\alpha)$, or:

($4b$) there is a smallest integer $0 \leq \ell_1 < \ell$ such that $f^{\ell_1}(e) < c < f^{\ell_1}(\alpha)$.

Since $\alpha = f(0)$, case ($4a$) gives $x_i = a_i$ for $i \leq \ell$. Also, since K is a non-trivial interval and $f^i | \mathrm{K}$ preserves orientation for $0 \leq i \leq n$, we have $f^{\ell}(\mathrm{K}) \subset \mathrm{I}_0$. This gives $x_{\ell+1}=0$. Since $f^{\ell}(\alpha) \geq c$, we have $a_{\ell+1}=1$, so $\underline{x} < \underline{a}$ as required.

On the other hand, case ($4b$) clearly gives $x_i = a_i$ for $i < \ell_1$, $x_{\ell_1+1}=0$, and $a_{\ell_1+1}=1$ which again gives $\underline{x} < \underline{a}$.

Case 5. — $\mathrm{K} \subset [\beta, 1]$. This case is similar to Case 4. The Lemma is proved.

Define now a map $\varphi : \Gamma \to \mathrm{I}$ by $\varphi(\underline{x}) = \bigcap_{i=0}^{\infty} f^{-i}(\mathrm{I}_{x_i})$. The above Lemma together with the expansiveness of f imply that φ is well defined. We prove φ is onto. First $\underline{a}, \underline{b} \in \Gamma$ and $\varphi(\underline{a})=0$, $\varphi(\underline{b})=1$. Now let $x \in \mathrm{I}$ and assume $0<x<1$. Then define $\underline{x} \in \Sigma$ by $x_i = j$ if $f^i(x^+) \in \mathrm{I}_j$. This means there is an interval (x, δ) such that $f^i((x, \delta)) \subset \operatorname{Int} \mathrm{I}_j$. Now $\varphi(\underline{x}) = x$ so it suffices to show $\underline{x} \in \Gamma$. But $0 < x^+$ so that $f^{i+1}(0^+) < f^{i+1}(x^+)$ as long as $f^i(0^+)$ and $f^i(x^+)$ are on the same side of c, $i = 0, 1, 2, \ldots$ Let n be the biggest such i. Then:

$$a_0 \ldots a_n = x_0 \ldots x_n \quad \text{but} \quad a_{n+1} = 0 < 1 = x_{n+1},$$

so that $\underline{a} < \underline{x}$. Similarly $\underline{x} < \underline{b}$.

The map φ is well defined and onto. Moreover, φ is $1-1$ except on the set of sequences mapped to $\{f^{-i}(x)\}_{i \geq 0}$ by φ. On this set, φ is $2-1$ with:

$$\varphi(x_1, \ldots, x_n, 1, 0, \underline{a}) = \varphi(x_1, \ldots, x_n, 0, 1, \underline{b}).$$

This establishes that $f : \mathrm{I} \to \mathrm{I}$ is topologically equivalent to the map induced by $\sigma : \Gamma \to \Gamma$ on the quotient space of Γ defined by identifying the sequences $\{x_1, \ldots, x_n, 0, 1, \underline{a}\}$ and $\{x_1, \ldots, x_n, 1, 0, \underline{b}\}$. The map induced by σ is not well defined at the point of the quotient corresponding to $\{0, 1, \underline{a}\}$ and $\{1, 0, \underline{b}\}$. This is its only discontinuity.

If $\underline{x}$ is a sequence of 0's and 1's, denote by $\underline{x}'$ the sequence obtained from $\underline{x}$ by changing all of the terms of the sequence. With this notation we state the main result of this section.

Theorem. — *Let* $f_1, f_2 : \mathrm{I} \to \mathrm{I}$ *be two maps of the unit interval satisfying properties* (2). *Then* f_1 *and* f_2 *are topologically equivalent if and only if the kneading sequences of* f_1 *and* f_2 *satisfy* $\{\underline{a}_1, \underline{b}_1\} = \{\underline{a}_2, \underline{b}_2\}$, $\underline{a}_1 = \underline{a}_2$, $\underline{b}_1 = \underline{b}_2$ *or* $\underline{a}_1 = \underline{b}_2'$, $\underline{b}_1 = \underline{a}_2'$.

Proof. — Denote by $\Gamma_{\underline{a},\underline{b}}$ the subshift constructed from the kneading sequences of a map f. If the kneading sequences of f_1 and f_2 agree, then both f_1 and f_2 are topologically equivalent to the map induced by σ on the same quotient of $\Gamma_{\underline{a},\underline{b}}$. Note that $\underline{a}$ and $\underline{b}$ determine which sequence pairs of the form $(x_1, \ldots, x_n, 1, 0, \underline{a})$ and $(x_1, \ldots, x_n, 0, 1, \underline{b})$ occur in $\Gamma_{\underline{a},\underline{b}}$. If the second set of equalities is satisfied and $g: I \to I$ is the map $g(x) = 1 - x$, then f_1 and $g \circ f_2 \circ g$ have the same kneading sequences. Therefore, f_1 and $g \circ f_2 \circ g$ are topologically equivalent. Since $g = g^{-1}$, f_1 and f_2 are also topologically equivalent. Therefore, f_1 and f_2 are topologically equivalent if one of the sets of conditions on the kneading sequences are satisfied.

On the other hand, if f_1 and f_2 satisfying (2) are topologically equivalent, the topological equivalence h must map the discontinuity of f_1 to the discontinuity of f_2. Moreover, h must map the endpoints of I to themselves. If h is orientation preserving, this implies that the kneading sequences of f_1 and f_2 agree. If h is orientation reversing, this implies that $\underline{a}_1 = \underline{b}_2'$ and $\underline{b}_1 = \underline{a}_2'$, proving the theorem.

As a final remark, we note that if $f: I \to I$ satisfies the equation $f(1-x) = 1 - f(x)$, then its kneading sequences satisfy $\underline{a}' = \underline{b}$. Therefore, one binary sequence characterizes the topological equivalence class of a map satisfying this identity.

3. Inverse Limits; The Poincaré map F.

The next topic we consider is the construction of an almost everywhere $1-1$ map of the unit square which reflects the dynamic properties of the map $f: I \to I$ studied in Section II. This will be the Poincaré map of our example. Denote the unit square in $\mathbf{R}^2$ by S. Given a map $f: I \to I$ satisfying properties (9), define a map $F: S \to S$ with the following properties:

(12)
- *a)* $F(x,y) = (f(x), g(x,y))$ for some smooth function g.
- *b)* F is $1-1$ on the complement of $\{c\} \times I$, where c is the discontinuity of F.
- *c)* $\dfrac{\partial g}{\partial y} < \dfrac{1}{2}$.
- *d)* As $x \to c^{\pm}$ from the right or left, the functions $f | x \times I$ tend uniformly to a constant function, $b^{\pm}$.

These properties yield the picture of the image of F shown in Figure 4:

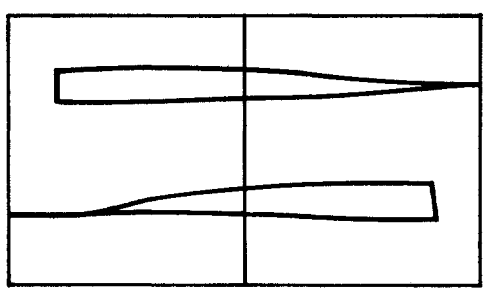

FIG. 4

The intersection of the iterated images of S by F constitutes a complicated 1-dimensional set Λ which is the ω-limit of almost all points of S. Here we study the topology of Λ and the dynamics of $F|_\Lambda$. We prove that $F|_\Lambda$ is almost the "inverse limit" of the sequence of "maps" $I \overset{f}{\leftarrow} I \overset{f}{\leftarrow} I \overset{f}{\leftarrow} \ldots$ (Recall that f is poorly defined at c.)

Let us briefly describe the construction of the inverse limit for well defined maps. If $f: X \to X$ is a map on a metric space, then the *inverse limit* of f is defined on the space $\hat{X}$ consisting of sequences $\underline{x} \subset X$ such that $f(x_{i+1}) = x_i$. The map $\hat{f}: \hat{X} \to \hat{X}$ is defined by $\hat{f}(\underline{x}) = \underline{y}$ with $y_i = f(x_i)$. If X is a space of one-sided sequences and f is the shift map, then $\hat{f}$ is the shift map on a space of two-sided sequences. A point of $\hat{X}$ consists of a point of X together with a history for the point.

There is a general technique for realizing the inverse limit of a map $f: X \to X$ on a "good" space X as an invariant set for a map which is $1-1$. The technique consists of embedding X in a larger space so that X has a tubular neighborhood and so that f can be approximated by an embedding which extends to a fiber preserving map of the tubular neighborhood. This fiber preserving map of the tubular neighborhood is required to be a contraction on each fiber and to project to f by the projection map of the tubular neighborhood. Then the map g has an invariant set to which its restriction is the inverse limit of $f: X \to X$.

For the balance of this section, we regard F as defined at $c \times I$ with two values: $F(c^\pm \times I) = b^\pm$; similarly, $f(c^\pm) = 0, 1$.

Apart from the discontinuities, the map F has all of the properties necessary for containing the inverse limit of f. We regard the unit square as a tubular neighborhood of a horizontal interval with vertical fibers and projection map π. Then F is a fiber preserving map which contracts the fibers and projects onto f. The only novelty in our situation lies in the discontinuities or ambiguities of f and F. Let us therefore examine the set $\Lambda = \bigcap_{i=0}^{\infty} F^i(S)$, paying particular attention to the treatment of the discontinuities. Since the map F fibers over the map f (*i.e.* $\pi \circ F = f \circ \pi$) each orbit of F lies over an orbit of f. Labelling the two halves of S by S_0, S_1 as shown in Figure 2, and the closure of the images of S_0 and S_1 by A and B, it is clear that the specification of the projection $\pi(x)$ determines whether $f^i(x)$ lies in A or B for $i > 0$.

We assert that $F|_\Lambda$ is topologically equivalent to a certain quotient space of the inverse limit of f. The *pinched inverse limit* P of f is the quotient space of $\hat{I}$ by the equivalence relation which identifies two sequences $\underline{x}$ and $\underline{y}$ if there is an i for which $x_i = y_i = 0$ or $x_i = y_i = 1$. We assert that the map $\psi: \Lambda \to P$ defined by $\psi(x) = \underline{x}$, $x_i = \pi(F^{-i}(x))$ is a topological equivalence between $F|_\Lambda$ and the map $\hat{f}$ induces on P. Note that ψ is well defined for those points of Λ which do not project onto points in the f-orbit of $c \in I$ since F is $1-1$ on the complement of $\pi^{-1}(c)$. For points of Λ which do project onto points in the f-orbit of c, there is an ambiguity, but this ambiguity disappears when one passes from $\hat{I}$ to P. It is clear that $\psi \circ F = \hat{f} \circ \psi$. Thus, to prove that ψ is a topological equivalence, we need only establish that it is a homeomorphism.

First, we assert ψ is $1-1$. If $\psi(x)=\psi(y)$, then $\pi(x)=\pi(y)$. Hence x and y lie on the same vertical segment in S. Now F^{-1} expands vertical segments by a factor of at least 2. Therefore distinct points x, y on a vertical segment cannot have the property that $F^{-i+1}x$ and $F^{-i+1}y$ lie on the same vertical segment in S for i sufficiently large. We conclude that $\psi(x)=\psi(y)$ implies $x=y$ and ψ is $1-1$. Next we argue that ψ is onto. If $\underline{x}\in P$, we need to find a point x in $\pi^{-1}(x_1)$ so that $\psi(x)=\underline{x}$. Let $\underline{y}$ be the sequence of 0's and 1's determined by $y_i=j$ if $x_i \in I_j$. Then the point x is identified as $(\bigcap_{i>0} F^i(S_{y_i}))\cap \pi^{-1}(x_0)$. The argument that this set is nonempty is very much like the argument that the map φ defined in Section II is well defined, so we do not repeat it here. The map ψ is a topological equivalence from $F|_\Lambda$ to the map induced by $\hat{f}$ on P. Since $\hat{f}$ and P are determined up to topological equivalence by the kneading sequences of f, so is $F|_\Lambda$.

Before proceeding further with the construction of a Lorenz attractor from the information contained in the kneading sequences of f, we examine the stability properties of F.

Proposition. — *Let* $F: S\to S$ *be a map satisfying properties* (12). *Assume that* f *is piecewise smooth and* $f'>\sqrt{2}$. *Let* G *have the form* $I'\circ F'$ *where* F' *satisfies* (12) *and is near* F *and* I' *is a* C^1 *perturbation of the identity. Then there is a piecewise continuous map* g *such that:*

1) g *satisfies* (9);
2) g *is* C^0 *near* f;
3) G *has an invariant set topologically equivalent to the pinched inverse limit of* g.

The key step of the proof revolves around the persistence of the " strong stable foliation " for F (compare [4]).

Lemma. — *Let* F *and* G *be as in the preceding proposition. Then* S *has a* G*-invariant, contracting partition into continuous curves which intersect each horizontal line in a single point.*

Proof. — We make the observation that the set D of iterated inverse images of the discontinuity of $F: S\to S$ is dense in S. Indeed, D is a set of vertical segments projecting onto the set of iterated inverse images of f. Since f is locally eventually onto, this set is dense in I.

We assert that the set D' of iterated inverse images of the discontinuity of G also consists of a family of almost vertical segments and is dense in S. This is proved by examining the derivatives of G. The derivative of F has the form $\begin{pmatrix} a(x) & 0 \\ b(x) & d(x) \end{pmatrix}$ where $a>\sqrt{2}$, $d<1/2$ and b is small compared to d. Therefore DG has the form $\begin{pmatrix} a'(x) & c'(x) \\ b'(x) & d'(x) \end{pmatrix}$ with $a'>\sqrt{2}$, $d'<1/2$, b' is small, and c' is very small. This implies that DG (hence DG^{-1}) always has a nearly vertical eigenvector. If A is an angular sector in $\mathbf{R}^2$ containing all of these nearly vertical eigenvectors, then each DG^{-1} will map the sector A into itself. In particular, if the discontinuity set of G has tangent vectors which always lie in A, then the iterated inverse images of the discontinuity set of G will be smooth curves whose tangent vectors lie in A. We still want to prove that D′ is dense. For this purpose, it suffices to prove that any open set eventually extends across S under iteration by G until it intersects the discontinuity set of G. It is clear that this must happen since G is uniformly expanding in nearly horizontal directions. Indeed, the argument used above for G^{-1} yields an angular sector containing the horizontal direction which is mapped into itself by DG. All vectors inside the angular sector are stretched by almost $\sqrt{2}$. This implies that any open horizontal segment in S must intersect the set D′.

Therefore, the components of D′ are nearly vertical segments extending across S. Moreover, the components of the complement of D′ are each the intersection of vertical strips extending across S whose width tends to 0. We conclude that the components of D′ and the components of the complement of D′ form a family of invariant, almost vertical segments of S which are uniformly contracted by G. The lemma is proved.

Proof of Proposition. — Define a projection map $\pi' : S \to I$ whose fibers are leaves of the invariant foliation constructed above. Define $g : I \to I$ to be $\pi \circ G \circ (\pi')^{-1}$. It is clear that g satisfies (9). Since G is near F and the leaves of the invariant " foliation " of G are nearly vertical, g will be near f. We can now repeat the proof that $F|_\Lambda$ is topologically equivalent to the pinched inverse limit of f to prove that G restricted to its invariant set is topologically equivalent to the pinched inverse limit of g. The proof for F only used properties of the invariant vertical foliation which are valid for the invariant foliation of G. We leave the details to the reader.

4. Perturbations of the model flows.

Now suppose Φ is one of our model flows with vectorfield X and Φ' is the flow of a C^1-small perturbation X′ of X. Then Φ' has a hyperbolic singular point $\mathcal{O}'$ near $\mathcal{O}$ with eigenvalues near those of Φ. That Φ' is also conjugate to its linear part *via* a C^1-conjugacy [10] is true if there is no inappropriate dependence of eigenvalues. As the last is an open, dense condition, we may assume it. Furthermore the conjugacy is only valid near $\mathcal{O}'$. But away from the singular point, only a finite amount of time is involved, so that no problem arises. Thus Φ' has a return map G of the form

$$G = I' \circ F'$$

where F′ satisfies (5) and I′ is C^1 near the identity. Note that this is the form we used in the previous section.

Thus we have shown that any small perturbation of a flow $\Phi_{a,b}$ in our family yields certain data:

(15)
1) A Poincaré map G on a solid square S.
2) A nearly vertical foliation $\mathscr{U}$ of S left invariant by G.
3) *Via* (2), G induces a map $g: I \to I$ having kneading sequences $\underline{a}', \underline{b}'$.

Finally, the pair $\underline{a}'$, $\underline{b}'$ depends continuously upon Φ_{ab}.

Now suppose we are given two such sets of data Φ_i, S_i, G_i, $\mathscr{U}_i$, g_i, $i = 1, 2$, satisfying (15) and having the same kneading sequences $(\underline{a}, \underline{b})$. We want to show that Φ_1 and Φ_2 are topologically conjugate on a neighborhood of their attractors. In section II above, we constructed a topological conjugacy h from g_1 to g_2.

To proceed, enlarge the squares S_i to S_i^+ by adding thin rectangles at each side of S_i (see Figure 5).

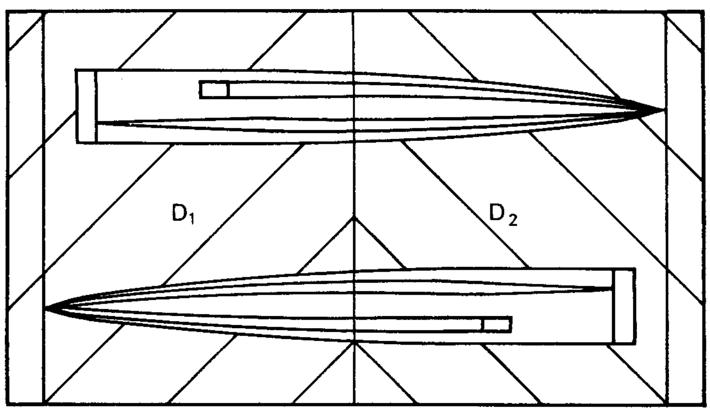

FIG. 5

The foliations on the new rectangles are induced from the foliations inside, as $G_i(S_i^+) \subset S_i$. In the figure we have also indicated a " fundamental domain " $D_i = S_i^+ - \text{Int}\, G_i(S_i^+)$ and its image $G_i(D_i)$. We follow the familiar practice of beginning by defining a conjugacy $H: D_1 \to D_2$. First, H is defined on the outside boundary ∂S_1^+ of S_1^+, so as to agree with h. That is:

(16)
$$\begin{array}{ccc} D_1 & \xrightarrow{H} & D_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ I_1^+ & \xrightarrow[h]{} & I_2^+ \end{array}$$

commutes, so far as defined. Here I_i^+ is a lengthening of the interval I_i, caused by the enlargement of the squares, S_i. Of course, G_i induces an extension of g_i to I_i^+ which we also call g_i. Finally, h is extended to I_i^+ by $h(x) = g_2^{-1} \circ h \circ g_1(x)$, where each map on the right side of the equation is first restricted to one "half" of I_1 or I_2 at a time (see the Figure 4 again).

This definition of H on ∂S_1^+ (the outer boundary of D_i) induces a definition on $G_1(\partial S_1^+)$ by:

$$H \mid G_1(\partial S_1^+) = G_0 \circ H \circ G_1^{-1}, \text{ on } G_1(\partial S_1^+).$$

Now, H is defined on ∂D_1^+, it agrees with h as far as defined, and we proceed to "fill in" the definition of H on the interior of D_1. Here we come to the basic fact that there are two types of Lorenz attractors: right-handed and left-handed ones. This depends upon whether G_i (right side of S_i) is in front or in back of G_i (left side of S_i). But this is a gross part of the flow, and since we have consistently chosen right-handed-attractors (like those of the Lorenz' equations) we are dealing with two right-handed ones here.

Thus we can extend H to the interior of D_1 so that the diagram (16) still commutes, where it is defined. Note that we have used the fact that h is a conjugacy quite heavily.

Now the process is quite standard ([6], [11]). Define, inductively $H(x)$, for $x \in G_1^i(D_1)$, by $H(x) = G_2^{-1} \circ H \circ G_1(x)$, with care that one takes the correct inverse of G_2, among at most two. This is no problem, as h is already a conjugacy. It is interesting to note (but not necessary to the argument) that one is using the fact that g_1 and g_2 have the same i-th term of their kneading sequences, to perform the i-th step of this definition. This process converges and extends to a map on all of S_1, because the connected components C of $V \cap G(D_i)$, $V \in \mathcal{U}_i$, have lengths which converge to 0 (exponentially, in fact) as $i \to \infty$. The diagram (16) remains commutative since each of these components has a single point as image under π_i.

At this point we have shown that if two Lorenz attractors (both right-handed) yield one-dimensional maps f and g with the same kneading sequences, then there are cross sections D_Φ and $D_{\Phi'}$ whose return maps F and G are topologically conjugate. The flows Φ and Φ' in a neighborhood of their attractors are "suspensions" of F and G relative to 3-cells homeomorphic to the 3-cell T described in Section I. Our theorem will be proved once we prove that any two such suspensions of topologically conjugate maps are topologically equivalent.

The argument here uses the arclength of segments of trajectories. In the 3-cell T depicted in Figure 1, insert two triangles W_1 and W_2 whose boundaries are piecewise linear triangles with a vertex at the origin and opposite edges along the left and right boundaries of S, respectively.

We shall define a topological equivalence from Φ to Φ' using W_1 and W_2 and analogous surfaces W_1' and W_2' transverse to the flow Φ'. We begin by defining the topological equivalence to be the conjugacy on the upper surface S which we constructed earlier in this section. Extend this conjugacy to each segment of a trajectory leaving S and joining it to W_1 or W_2 by ratio of arc length. This means that if the map sends a curve γ parametrized by arc length to γ' parametrized by arc length, then it sends $\gamma(t)$ to $\gamma'(\rho t)$ where ρ is the ratio of the length of γ to the length of γ'. Similarly, extend the map thus far defined to segments of trajectories which leave W_1 or W_2 and return to S by ratio of arc length. The map defined in this way will be a topological equivalence because we began with a conjugacy of return maps. Note that the continuity of the map follows from the fact that the length of trajectories approaching a hyperbolic singular point approaches the sum of the lengths of the limiting trajectories in the stable and unstable manifolds.

This completes the proof of the theorem.

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