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On The Structure and Control of Piecewise-Linear Systems

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Abstract

Elementary algebric topology is used in the study of the structure and optimal control of piecewise-linear and nonlinear systems.

1 Introduction

In this paper we study the behaviour of piecewise-linear systems by dissecting the ambient space into a geometric simplicial complex. This will allow the application of algebric topology to obtain a numerical procedure for the determination of the global topology of the trajectories of such systems. Algebric topology has been applied previously (for example, **Easton, 1975**) to obtain the fine structure of choatic attractors; in particular, of their Poincare sections. Here we wish to determine the global topology of trajectories, so that a structure 'with one hole in'is a limit cycle, etc.

Particular examples of piecewise-linear systems have alsow been studied in detail (a notable example is **Chua 's circuit; see Chua, Komuro and Matsumoto, 1986**), but little attention has been given, at least from control theory viewpoint, to the general theory of the these systems . We shall introduce some ideas in this paper which may be useful in the developing such a general theory. This would be an important step towards a better understanding of general nonlinear systems, and the methods proposed here follow closely the ad hoc methods of local linearization used widely in engineering.

We begin in section **2** with an outline of the results from algebric topolpgy which we shall need; the proofs of these results can be found in **Hilton and Wylie, 1965** or **Spanier, 1966.** In section **3** we shall study piecewise-linear systems (without control) by making a simplicial decomposition of $\mathbb{R}^n$ and deriving a numerical technique for the recursive evaluation of the homology groups of a simplicial approximation to a trajectory of the system. This is very similar to the standard numerical techniques (Runge-Kutta, etc.) for the numerical evaluation of the phase-plane trajectories. In section **4** we shall consider optimal control of piecewise-linear control systems and extend the linear-quadratic regulator problem to these systems. Finally, in section **5** we shall show how to apply the simplicial approximation theorem to obtain a piecewise-approximation to a continuous nonlinear systems, and indicate briefly the evaluation of the optimal control for such a system.

2 Elementary Algebric Topology

In this section we shall review the ideas of algebric topology which we use in the paper; see **Hilton and Wylie, 1965 or Spainer, 1966.** If $a^0, a^1, \ldots, a^p \in \mathbb{R}^n$ then $b \in \mathbb{R}^n$ is **dependent** on the $a's$ if there exist $\lambda_0, \ldots, \lambda_p \in \mathbb{R}$ such that

$$\sum_{i=0}^{p} \lambda_i a^i = b$$

If the vectors $a^i - a^0, i = 1, \ldots, p$ are linearly independent, then a p-simplex S_p (of dimension p) with vertices $a^0, a^1, \ldots, a^p$ is the set of points dependent on the $a's$ for which the **barycentric coordinates** $\lambda_i > 0, i = 0, \ldots, p$.

A **face** of a simplex is a subset of the simplex consisting of points with at least one $\lambda_i = 0$.

A **geometric** simplicial complex K in $\mathbb{R}^n$ is a set of simplexes of $\mathbb{R}^n$ such that if $S \in K$ then all the faces of S also belong to K and any two simplexes are disjoint or intersect in a closed face of each. If K and L are complexes a map v from the vertices of K to those of L which is such that $va^0, \ldots, va^p$ spans a simplex of L if $a^0, \ldots, a^p$ spans a simplex of K is called **a simplicial map**. For any complex K we let $\mid K \mid$ denote the union of all the simplexes of K; note that $\mid K \mid \subseteq \mathbb{R}^n$ whereas K is a set of simplexes. $\mid K \mid$ is called the **polyhedron** of K. Clearly, any simplicial map $v : K \longrightarrow L$ induces a map $\mid v \mid : \mid K \mid \longrightarrow \mid L \mid$ by linearity in each simplex.

If K is a complex we define the complex K' (first derived complex) by applying regular substitution to each simplex of K. Thus, if $S_p = (a^{i_0}, \ldots, a^{i_p})$ is a simplex of K, we define a vertex

$$b^{i_0 \ldots i_p} = \frac{1}{p+1} \sum_{k=0}^{p} a^{i_k}$$

so that $b^{i_0 \ldots i_p}$ is the **centre of mass** of S_p . (If p=0 then $b^{i_0} = a^{i_0}$ so that the vertices of K are vertices of K') . If $\sigma_1, \ldots, \sigma_m$ are subset of the set of indices $i_0, \ldots, i_p, \ldots$ which label the $b's$, then $\{b^{\sigma_1}, b^{\sigma_2}, \ldots, b^{\sigma_m}\}$ is a simplex of K' if and only if the $\sigma's$ can be renumbered so that $\sigma_1 \supset \sigma_2 \supset \ldots \supset \sigma_m$. This process can be continued to produce the complexes $K'', \ldots, K^{(r)}, \ldots$. Given a **continuous** map $f : \mid K \mid \longrightarrow \mid L \mid$, a simplicial map $v : K \longrightarrow L$ is called a **simplicial approximation** to f if for each $x \in \mid K \mid, \mid v \mid (x)$ belongs to the closed simplex of L containing f(x). An important result of combinatorial topology is

Theorem 2.1 Simplicial Approximation Theorem. *If K, L are complexes and* $f : \mid K \mid \longrightarrow \mid L \mid$ *is continuous, then there exists* $r \geq 0$ *and a simplicial approximation* $v : K^{(r)} \longrightarrow L$ *to* f. □

Let $\mid L \mid^{|K|}$ denote the set of continuous functions $f : \mid K \mid \longrightarrow \mid L \mid$ and topologize the space with the topology of uniform convergence on compact sets. Then we have

Corollary 2.1 *If K, L are complexes, the set of simplicial maps* $v : K^{(r)} \longrightarrow L^{(s)}$, $r, s \in \mathbb{N}$, *is dense in* $\mid L \mid^{|K|}$. □

We next describe the **homology groups** $H_p(k)$ which are **homotopy** invariants of the polyhedron $\mid K \mid$ of K. The set of **p-chains** $C_p(K)$ (or simply C_p) is the free abelian group generated by the p-simplexes of K. We define the **boundary homomorphism** $\partial_p : C_p \longrightarrow C_{p-1}$ on each oriented simplex $S_p = (a^0 a^1 \ldots a^p)$ by

$$\partial_p S_p = \sum_i (-1)^i (a^0 a^1 \ldots \hat{a^i} \ldots a^p),$$

and by linearity to C_p . (An **oriented** simplex is just a simplex with some fixed ordering of the vertices. Two orientations induced by two orderings are considered to be identical if the orderings differ by an even permutation).

Clearly,

$$\partial_{p-1}\partial p = 0 \qquad \text{for all p.}$$

put

$$Z_p(K)(= Z_p) = \text{kernel of } \partial_p$$
$$B_p(K)(= B_p) = \text{image of } \partial_{p+1}$$

and define

$$H_p(K) = Z_p/B_p$$

A sequence

$$\longrightarrow G_{n+1} \xrightarrow{\alpha_{n+1}} G_n \xrightarrow{\alpha_n} G_{n-1} \longrightarrow \cdots$$

of abelian groups G_i and homomorphisms α_i is **exact** if ker α_n = image α_{n+1} for each n. Such a sequence gives rise to the **(short)** exact sequences

$$0 \longrightarrow G'_n \longrightarrow G_n \longrightarrow G'_{n-1} \longrightarrow 0$$

where G'_n = ker α_n . This process is called **splicing** the sequence. Let K_1, K_2, K_3 be simplicial complexes , and suppose we have the exact sequences

$$0 \longrightarrow C_p(K_1) \xrightarrow{\alpha_p} C_p(K_2) \xrightarrow{\beta_p} C_p(K_3) \longrightarrow 0$$

for each p. Then the homomorphisms α_p, β_p induces homomorphisms

$$\alpha_{*p} : H_p(K_1) \longrightarrow H_p(K_2)$$
$$\beta_{*p} : H_p(K_2) \longrightarrow H_p(K_3)$$

It is easy to show that the function ∂_{*p} defined by

$$\partial_{*p}\{z_3\} = \{\alpha_{p-1}^{-1}\partial_p\beta_p^{-1}\}$$

is a homomorphism

$$\partial_{*p} : H_p(K_3) \longrightarrow H_{p-1}(K_1).$$

(Here, {z} denotes the image of $z \in Z_p$ in H_p under the canonical map $Z_p \longrightarrow Z_p/B_p$.) Moreover , it can be shown that the induced sequence

$$\cdots \xrightarrow{\partial_*} H_p(K_1) \xrightarrow{\alpha_*} H_p(K_2) \xrightarrow{\beta_*} H_p(K_3) \xrightarrow{\partial_*} H_{p-1}(K_1) \xrightarrow{\alpha_*} \cdots \tag{2.1}$$

is exact.

Finally, we shall derive the **Mayer-Vietoris** sequence. First, if $v : K \longrightarrow L$ is a simplicial map we define $C_p(v) : C_p(K) \longrightarrow C_p(L)$ by

$$C_p(v)((a^0, a^1, \ldots, a^p)) = (v(a^0), v(a^1), \ldots, v(a^p))$$

where the simplex on the right is replaced by 0 if the vertices are not distinct. Now if K_1 and K_2 are subcomplexes of a complex K then $K_1 \cap K_2$ and $K_1 \cup K_2$ are also subcomplexes of K and $C_p(K_1), C_p(K_2) \subseteq C_p(K)$ for each p. Moreover, $C_p(K_1 \cap K_2) = C_p(K_1) \cap C_p(K_2)$ and $C_p(K_1) + C_p(K_2) = C_p(K_1 \cup K_2)$. If $i_l : K_1 \cap K_2 \subseteq K_l$, $j_l : K_l \subseteq K_1 \cup K_2$, j=1,2, are the inclusions , then we have short exact sequences

$$0 \rightarrow C_p(K_1 \cup K_2) \xrightarrow{i_p} C_p(K_1) \oplus C_p(K_2) \xrightarrow{j_p} C_p(K_1 \cup K_2) \rightarrow 0$$

where

$$i_p(c) = (C_p(i_1)c, -C_p(i_2)c),$$

$$j_p(c_1, c_2) = C_p(j_1)c_1 + C_p(j_2)c_2$$

The **Mayer-Vietoris** sequence of the triple (K, K_1, K_2) now follows from **(2.1)**:

$$\cdots \xrightarrow{\partial_*} H_p(K_1 \cap K_2) \xrightarrow{i_*} H_p(K_1) \oplus H_p(K_2) \xrightarrow{j_*}$$
$$H_p(K_1 \cup K_2) \xrightarrow{\partial_*} H_{p-1}(K_1 \cap K_2) \xrightarrow{i_*} \cdots$$

3 Piecewise-Linear Systems

In this section we shall consider a system of linear equation of the form[1]

$$\begin{aligned}
\dot{x} &= A_1 x + b_1 \quad && x \in P_1 \\
\dot{x} &= A_2 x + b_2 \quad && x \in P_2 \\
&\vdots && \vdots \\
\dot{x} &= A_m x + b_m \quad && x \in P_m
\end{aligned} \tag{3.1}$$

where the P_i are polygonal regions in $\mathbb{R}^n$ which form a partition of $\mathbb{R}^n$, i.e.

$$\cup P_i = \mathbb{R}^n \ , \quad P_i \cap P_j = \emptyset \quad , \quad i \neq j \tag{3.2}$$

Let K denote an **(infinite)** geometric simplicial complex (**Hilton** and **Wylie**, **1965**) which is **compatible** with the polygonal dissection $\{ P_1, \ldots, P_m \}$ of $\mathbb{R}^n$. By this we mean that , for each $i \in \{1, \ldots, m\}$, there exist a subcomplex K_i of K such that $\overline{K_i} = P_i$ (where $\overline{K_i}$ is the polyhedron of K_i, i.e. the set of points in $\mathbb{R}^n$ spanned by the closed simplexes of K_i).

In order to simplify the discussion we shall make the following assumptions. For each linear subsystem

$$\dot{x_i} \ = \ A_i x + b_i \quad , \quad x \in \mathbb{R}^n$$

of **(3.1)** extended to $\mathbb{R}^n$ we shall assume that the trajectories are transversal almost everywhere (with respect to (n-1)- dimensional measure in $\mathbb{R}^n$) to the (n-1)-dimensional boundaries of all simplexes in K_i . This implies that there is no finite piece of a trajectory lying in the (n-1)-dimensional skeleton of K.

To facilitate the study of a system of the form **(3.1)** we shall first consider a single homogeneous equation

$$\dot{x} \ = \ Ax \tag{3.3}$$

[1] P_i are an open polytopes. We shall discuss the definition of the systems on their dosures later.

and write it in simplicial coordinates . To this end , let S be a closed simplex in $\mathbb{R}^n$ with vertices $v_0, v_1, \ldots, v_n$. Then any point $x \in S$ can be written uniquely in the form

$$x = \sum_{i=0}^{n} \xi_i v_i \quad \text{where} \quad \sum_{i=0}^{n} \xi_i = 1 \quad , \quad \xi_i \geq 0. \tag{3.4}$$

Thus, the solution x(t) of **(3.3)** may be written in the form

$$x(t) \;=\; \sum_{i=0}^{n} \xi_i(t) v_i \tag{3.5}$$

where

$$\sum_{i=0}^{n} \xi_i(t) = 1 \quad , \quad \xi_i \geq 0.$$

This expansion is valid for all t such that $x(t) \in S$. Note that if we extend the vector field Ax outside S then we may still write the solution x(t) uniquely in the form **(3.2)** with

$$\sum_{i=0}^{n} \xi_i(t) \;=\; 1,$$

but now we no longer have $\xi_i \geq 0$ for each i.

In our first lemma we show how to invert the relation (3.4) in order to express the barycentric coordinates ξ_i in terms of x.

Lemma 3.1 *If the n+1 vectors $v_0, v_1, \ldots, v_n \in \mathbb{R}^n$ span a nondegenerate simplex in $\mathbb{R}^n$, then any $x \in \mathbb{R}^n$ may be written uniquely in the form*

$$x = \sum_{i=0}^{n} \xi_i v_i \quad , \quad \sum_{i=0}^{n} \xi_i = 1 \tag{3.6}$$

and the barycentric coordinates ξ_i of x are given by

$$\xi = V^{-1} \overline{x} \tag{3.7}$$

where

$$\xi = (\xi_0, \ldots, \xi_n)^T \quad , \quad V = (\overline{v}_0 \; \overline{v}_1 \; \ldots \overline{v}_n)$$

and

$$\overline{v}_i = \begin{pmatrix} v_i \\ 1 \end{pmatrix} \quad , \quad \overline{x} = \begin{pmatrix} x \\ 1 \end{pmatrix}$$

Proof. We have already seen that the expansion (**3.6**) holds and so it remains to prove (**3.7**).

First note that the $(n+1) \times (n+1)$ matrix V is nonsingular. This can be proved as follows. The vectors $v_o, \ldots, v_n$ are linearly dependent and any n of them are linearly independent since they span a nondegenerate simplex in $\mathbb{R}^n$. Hence we may write

$$v_0 = \sum_{i=1}^{n} \alpha_i v_i \tag{3.8}$$

for some $\alpha_i's$, not all zero. Since $\{v_1, \ldots, v_n\}$ forms a linearly independent set, so does $\{\overline{v}_1, \ldots, \overline{v}_n\}$. If V is singular then we can write

$$\overline{v_0} = \sum_{i=1}^{n} \beta_i \overline{v}_i$$

i.e.

$$v_0 = \sum_{i=1}^{n} \beta_i v_i \quad , \quad 1 = \sum_{i=1}^{n} \beta_i$$

Since the expression (**3.8**) is unique, we must have $\alpha_i = \beta_i, 1 \leq i \leq n$. Hence,

$$v_0 = \sum_{i=1}^{n} \alpha_i v_i \quad with \quad \sum_{i=1}^{n} \alpha_i = 1$$

This contradicts the uniqueness of expansions (**3.6**), since we also have

$$v_0 = \sum_{1=0}^{n} \xi_i v_i \quad , \quad \xi_0 = 1 \quad , \quad \xi_i = 0 \quad , \quad i \geq 1$$

Hence V is vertible and (**3.7**) now follow directly from (**3.6**) . □ We can now express a linear system of the form (**3.3**) in terms of barycentric coordinates. In fact, we have

Theorem 3.1 *Consider the linear system*

$$\dot{x} \; = \; Ax \quad , \quad x(0) = x_0 \in \mathbb{R}^n \tag{3.9}$$

and let $v_0, v_1, \ldots, v_n$ span a nondegenerate simplex in $\mathbb{R}^n$. If $\alpha_0, \ldots, \alpha_n$ are the barycentric coordinates of the origin $0 \in \mathbb{R}^n$ and $\xi_0(t), \ldots, \xi_n(t)$ are the *barycentric coordinates of x(t), i.e.*

$$0 = \sum_{i=0}^{n} \alpha_i v_i \quad , \quad \sum_{i=0}^{n} \alpha_i = 1,$$

$$x(t) \;=\; \sum_{i=0}^{n} \xi_i(t) v_i \quad , \quad \sum_{i=0}^{n} \xi_i(t) = 1,$$

then the system **(3.9)** *takes the form*

$$\dot{\xi} \;=\; \overline{A}\xi - \alpha \quad , \quad \xi_0 = V^{-1}\overline{x}_0$$

in barycentric coordinates, where

$$\xi = (\xi_0, \ldots, \xi_n)^T \quad , \quad \alpha = (\alpha_0, \ldots, \alpha_n)^T,$$

$$\overline{A} \;=\; V^{-1} \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} V$$

Proof. From **(3.9)** we obtain the system

$$\begin{pmatrix} \dot{x} \\ 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix}$$

defined on $\mathbb{R}^{n+1}$. Thus

$$V^{-1} \begin{pmatrix} \dot{x} \\ 1 \end{pmatrix} \;=\; V^{-1} \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} V V^{-1} \begin{pmatrix} x \\ 1 \end{pmatrix}$$

$$= \overline{A}\xi$$

since

$$V^{-1} \begin{pmatrix} x \\ 1 \end{pmatrix} \;=\; V^{-1}\overline{x} \;=\; \xi, \text{ by (3.7).}$$

Now,

$$x = \sum_{i=0}^{n} \xi_i v_i \quad , \quad \sum_{i=0}^{n} \xi_i = 1,$$

and so

$$\dot{x} = \sum_{i=0}^{n} \dot{\xi}_i v_i \quad , \quad \sum_{i=0}^{n} \dot{\xi}_i = 0 \tag{3.10}$$

However,

$$V^{-1} \begin{pmatrix} \dot{x} \\ 1 \end{pmatrix}$$

is the barycentric coordinate vector of $\dot{x}$ and so it equal η where

$$\dot{x} = \sum_{i=0}^{n} \eta_i v_i \quad , \quad \sum_{i=0}^{n} \eta_i = 1 \tag{3.11}$$

Subtracting **(3.10)** from **(3.11)** gives

$$0 = \sum_{i=0}^{n} (\eta_i - \dot{\xi}_i) v_i \quad , \quad \sum_{i=0}^{n} (\eta_i - \dot{\xi}_i) = 1$$

and by uniqueness of barycentric coordinates, $\alpha_i = \eta_i - \dot{\xi}_i$. The result now follows. □

Remark. Of course,

$$\alpha = V^{-1}\overline{0} = V^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Corollary 3.1 *The nonhomogeneous system*

$$\dot{x} = Ax + b$$

has the expression

$$\dot{\xi} = \overline{A}\xi - \zeta \tag{3.12}$$

where ζ is the barycentric coordinate vector of $-b$.

Proof. As before we have

$$\dot{\xi} = \overline{A}\xi - \alpha + V^{-1} \begin{pmatrix} b \\ 0 \end{pmatrix}$$

and since

$$\alpha = V^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

the results follows. □

Now return to the system **(3.1)** and let $S_1, S_2, S_3, \ldots$ denote the open n-dimentional simplexes in K .Simplicial coordinates in S_i will be denoted by ξ_i and the linear equation which represent system **(3.1)** in S_i will be denoted by

$$\dot{\xi}_i = \Gamma_i \xi_i - \zeta_i \qquad [i] \qquad (3.13)$$

where $\Gamma_i = \overline{A}_k$ for some $k \in \{1, \ldots, m\}$. Since these equations specify the system on open simplexes we must define the behaviour of the system on the simplexes of dimension < n. For simplexes S_i, S_j with a boundary simplex in common there are two possibilities. If this boundary simplex lies entirely in a single polytope P_k then we merely extend **(3.13)** by continuity. If, on the other hand, it lies in the boundary of a pair of polytopes P_k and P_l then we shall assume that some arbitrary but well-defined choice has been made for one of the systems

$$\dot{x} = A_k x + b_k$$

or

$$\dot{x} = A_l x + b_l$$

and uniquely extend **(3.13)** onto the boundary according to this choice. (Of course, the system will in general be discontinuous).

For any given initial condition $x_0 \in \mathbb{R}^n$ the solution of the system **(3.13)** will specify a unique sequence of simplexes $S_{i_1}, S_{i_2}, \ldots$ through which to solution passes (in that order).

(It is possible that the solution is not defined for all time. Consider , for example the system

$$\left.\begin{aligned} \dot{x}_1 &= x_1 + 1 \\ \dot{x}_2 &= 0 \end{aligned}\right\} \qquad x \in \{(x_1, x_2) : x_1 \leq 0\}$$

$$\left.\begin{aligned} \dot{x}_1 &= x - 1 \\ \dot{x}_2 &= 0 \end{aligned}\right\} \qquad x \in \{(x_1, x_2) : x_1 > 0\}$$

Taking simplexes which have a boundary in common on the x_2 axis we see that the solution cannot be extended beyond the x_2 axis.) For any simplex S_k of this sequence the solution trajectory satisfies the equation **(3.13k)** i.e.

$$\dot{\xi}_k = \Gamma_k \xi_k - \zeta_k$$

Assuming that Γ_k is diagonalizable (a simplex extention to the Jordan form follows easily) we can find an invertible (complex) matrix N_k such that

$$\xi_k(t) = N_k^{-1} e^{\Lambda_k t} N_k \xi_0 - \int_0^t N_k^{-1} e^{\Lambda_k (t-s)} N_k \zeta_k ds$$

where Λ_k is the diagonal matrix of eigenvalues of Γ_k . Thus,

$$\xi_k(t) = N_k^{-1} e^{\Lambda_k t} N_k \xi_0 - N_k^{-1} \Lambda_k^{-1} (I - e^{\Lambda_k t}) N_k \zeta_k$$

Hence we can write the i^{th} component $(1 \leq i \leq n+1)$ of $\xi_k(t)$ in the form

$$(\xi_k(t))_i = \sum_{j=1}^{n+1} \alpha_{ij}^k e^{\lambda_j t} + \beta_i^k \tag{3.14}$$

where $\{\lambda_j\}$ is the set of eigenvalues of Γ_k, and α_{ij}^k depends on λ_j and ξ_0 and β_i^k depends on λ_j and ζ .

In order to determine the sequence $S_{i_1}, S_{i_2}, \ldots$ of simplexes through which a given trajectory passes suppose that the trajectory has passed through the simplexes $S_{i_1}, S_{i_2}, \ldots, S_{i_k}$ and is on the boundary of S_{i_k} at the point x_{i_k} . Then we have

Lemma 3.2 *The trajectory will enter simplex S_k if*

$$\sum_{j=1}^{n+1} \alpha_{ij}^k (\xi_0) A_j > 0 \tag{3.15}$$

for all i for which ξ_{0i} is zero where ξ_0 is the vector of S_k -simplicial coordinates of x_{i_k} .

Proof. The expression for the simplicial coordinates of the trajectory is given by **(3.14)** . If ξ_{i0} =0 then the trajectory can only enter this simplex if ξ_i increases (where ξ is the simplicial coordinate vector of S_k). Since $\xi(t)$ is differentiable in S_k the results follows directly from (3.14). □

Of course, if the trajectory enters the interior of simplex S_k then it will leave again (if at all) at the smallest time T for which there exists i such that

$$\sum_{j=1}^{n+1} \alpha_{ij}^k e^{\lambda_j T} + \beta_i^k = 0 \tag{3.16}$$

and

$$\sum_{j=1}^{n+1} \lambda_j \alpha_{ij}^k e^{\lambda_j T} < 0 \tag{3.17}$$

The expressions **(3.15)-(3.17)** enable us to determine numerically the sequence $S_{i_1}, S_{i_2}, \ldots$ of simplexes through which a given trajectory passes. Renumbering the simplexes, we can assume that a given trajectory passes through the sequence of simplexes $S_1, S_2, \ldots$. Let K_k denote the simplicial complex generated by the simplexes $S_1, \ldots, S_k$. We shall be interested in the topology of K_k as reflected in its homology groups $H_p(K_k)$. These are , of course, only homotopy invariants but are useful in determining the 'number of holes 'in a global structure , so that they will distinguish between limit cycles, periodic doubling phenomena and possibly even the global behaviour of 'strange attractors '.

Just as we solve a nonlinear differential equation numerically by evaluating the current phase space coordinates recursively from the previous ones, we shall now outline a method for the recursive numerical evaluation of the homology groups $H_p(K_k)$. For a sufficiently fine simplicial decomposition of $\mathbb{R}^n$, this will provide an **approximation** to the topology of the trajectories in much the same way as numerical integration gives an approximation to the phase plane portraits. This method will not, of course, provide the **fine** topology of strange attractors, but will give the topology of some open neighbourhood of the attractors.

The method we propose consists of applying the **Mayer-Vietoris** sequence to K_k and K_{k+1} to obtain the change in topology from K_k to K_{k+1} . Of course at any given stage we could compute the topology of K_k directly , since an effective procedure for this exists **(Hilton, Wylie, 1965)**. The use of the **Mayer-Vietoris** sequence simplifies the recursive computation since we are adding a simplex (or rather the geomatric complex of a simplex) each time. Recall, then, that the **Mayer-Vietoris** sequence is

$$\cdots \xrightarrow{\partial_*} H_q(K_k \cap S_{k+1}) \xrightarrow{i_*} H_q(K_k) \oplus H_q(S_{k+1}) \xrightarrow{j_*} \\ H_q(K_k \cup S_{k+1}) \xrightarrow{\partial_*} H_{q-1}(K_k \cap S_{k+1}) \xrightarrow{i_*} \cdots$$

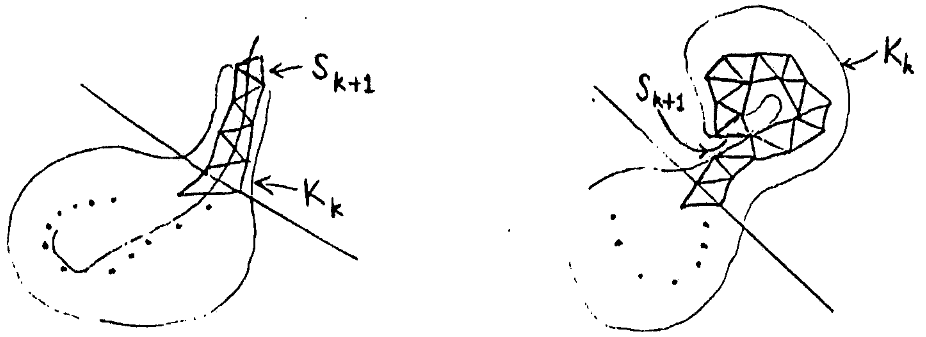
However, for any simplex S_{k+1}

$$\begin{aligned} H_q(S_{k+1}) = 0 \quad &, \quad q > 0 \\ H_0(S_{k+1}) = J \end{aligned} \tag{3.18}$$

Hence, the **Mayer-Vietoris** sequence reduces to the exact sequence

$$H_n(K_k \cap S_{k+1}) \xrightarrow{i_*} H_n(K_k) \xrightarrow{j_*} H_n(K_{k+1}) \xrightarrow{\partial_*} H_{n-1}(K_k \cap S_{k+1}) \\ \xrightarrow{i_*} \cdots \xrightarrow{\partial_*} H_0(K_k \cap S_{k+1}) \xrightarrow{i_*} H_0(K_k) \oplus J \xrightarrow{j_*} H_0(K_{k+1}) \tag{3.19}$$

We shall consider two typical case for K_{k+1};namely, (a). when K_{k+1} is such that $K_k \cap S_{k+1}$ is the complex of an (n-1)-dimentional simplex and (b). when $K_k \cap S_{k+1}$ is the union of the complex of an (n-1)-dimentional simplex and its opposite vertex. (see **fig. 3.1.**)



(a) (b)

Fig. 3.1 Typical situations for K_{k+1}

In case (a) , since S_{k+1} is contractible we expect that the homology groups of K_{k+1} are the same as those of K_k . This is clear, of course, from the fact that the homology groups are homotopy invariants. It also follows directly from **(3.18)** and **(3.19)**, since $H_p(K_k \cap S_{k+1}) = 0$ if $p > 0$ and $H_0(K_k \cap S_{k+1}) = J$. In case (b) we have, from **(3.19)**,

$$0 \xrightarrow{i_*} H_j(K_k) \xrightarrow{j_*} H_j(K_{k+1}) \xrightarrow{\partial_*} 0$$

for $j > 1$ and so

$$H_j(K_k) \cong H_j(K_{k+1}) \quad for \quad j > 1$$

Consider the case $j = 1$. Then we have the exact sequence

$$0 \xrightarrow{i_*} H_1(K_k) \xrightarrow{j_*} H_1(K_{k+1}) \xrightarrow{\partial_*} H_0(K_k \cap S_{k+1}) \xrightarrow{i_*} H_0(K_k) \oplus J \xrightarrow{j_*} H_0(K_{k+1}) \qquad (3.20)$$

However, since the complex $K_k (k \geq 1)$ is connected (being derived from a connected trajectory), we have

$$H_0(K_k) = H_0(K_{k+1}) = J$$

and so **(3.20)** reduces to the sequence

$$0 \xrightarrow{i_*} H_1(K_k) \xrightarrow{j_*} H_1(K_{k+1}) \xrightarrow{\partial_*} J \oplus J \xrightarrow{i_*} J \oplus J \xrightarrow{j_*} J$$

It follows by splicing the sequence that

$$H_1(K_{k+1}) \cong H_1(K_k) \oplus J$$

These simple cases illustrate the application of the sequence **(3.19)** to obtain recursively the homology groups of K_{k+1} . Similar ideas apply in more general cases.

4 Quadratic Control of Piecewise-Linear systems

We shall now discuss the following optimal control problem: consider the system of equations

$$\dot{x} = A_i x + B_i u \quad , \quad x \in P_i \quad , \quad i \in \{1, \ldots, m\} \tag{4.1}$$

where P_i is again a polygonal dissection of $\mathbb{R}^n$. Then we wish to minimize the quadratic cost function

$$J(u) = x^T(t_f) F x(t_f) + \int_0^{t_f} (x^T(t) Q x(t) + u^T(t) R u(t)) dt \tag{4.2}$$

subject to the constraints **(4.1)**. We shall assume that each pair (A_i, B_i) is controllable. Suppose we know the optimal control $u^*(t)$, $0 \leq t \leq t_f$ which drives the state from x_0 to x_1 . We shall assume for the present that the optimal trajectory $x^*(t)$ is transversal to the boundary of any polytope P_i which it intersects (or remains in the same polytope P_i if it is tangent to ∂P_i). Then we have

Lemma 4.1 *There exists a set of times*

$$\Delta = \{t_1, t_2, \ldots, t_h\} \tag{4.3}$$

(possibly empty) with

$$t_f > t_1 > t_2 > \ldots t_h > 0$$

and a sequence $(P_{i_1}, P_{i_2}, \ldots)$ *of polytopes (with* $P_{i_j} \in \{P_1, \ldots, P_m\}$*) such that the optimal trajectory* $x^*(t)$ *belongs to* P_{i_j} *on the interval* (t_{j-1}, t_j) *, where* $t_0 = t_f$*.*

Proof. If $x_1 \in \overline{P}_k$ then we set $i_1 = k$. Suppose first that $x_1 \in p_k$. Consider the linear-quadratic problem

$$\dot{x} = A_k x + B_k u \quad , \quad x \in \mathbb{R}^n \tag{4.4}$$

together with cost function **(4.2)**. (Note that we are considering the extension of the linear system **(4.1)** to the whole of $\mathbb{R}^n$.) The solution of this optimization is of course, well-known and is given by

$$\overline{u}(t) = -R^{-1}B_k^T P_k(t)x(t) \quad ,$$

where P satisfies the Riccati equation

$$\begin{gathered} \dot{P}_k(t) = -P_k(t)A_k - A_k^T P_k(t) - Q + P_k(t)B_k R^{-1} B_k^T P_k(t) \quad , \\ P(t_f) = F \quad . \end{gathered} \tag{4.5}$$

The optimal trajectory is then the following solution of the the equation

$$\begin{gathered} \dot{x}(t) = A_k x(t) - B_k R^{-1} B_k^T P_k(t) x(t) \ , \\ x(t_f) = x_1 \end{gathered} \tag{4.6}$$

Solving the system **(4.5)** and **(4.6)** backwards in time from t_f will then determine the optimal trajectory $\overline{x}(t)$. Let $t_1 < t_f$ be the largest time at which $x(t_1) \in \partial P_k$ and for any $\overline{t} < t_1$,$x(\overline{t}) \in P_{i_2}$,$i_2 \neq k$. It may be , of course, that $t_1 < 0$, in which case the set Δ is empty and the optimal solution to **(4.1)**,**(4.2)** with final state x_1 is just the classical solution . Let $x_2 = x(t_1)$. If x_1 is on the boundary of a polytope , then we can use the same argument , since we have assumed that the optimal trajectory does not stay on the boundary of a polytope. In this case we would consider all the extended linear problems **(4.2)** and **(4.4)** for which x_1 is on the boundary of the corresponding polytope. Then the optimal solution must belong to the interior of some polytope, say p_k again, for some finite time $t_1 < t < t_f$ and $x_2 = x(t_1) \in \partial P_k$. Of course, in either case, we have $x^*(t) = \overline{x}_1(t)$, $u^*(t) = \overline{u}(t), t \in (t_1, t_f)$. The argument is now repeated with t_1 replaced by t_f and x_2 replaced by x_1 .This time we write the cost functional in the form

$$\begin{aligned} J(u) = x^T(t_f)Fx(t_f) \quad &+ \int_{t_1}^{t_f} \left(x^T(t)Qx(t) + u^T(t)Ru(t)\right) dt \\ &+ \int_0^{t_1} \left(x^T(t)Qx(t) + u^T(t)Ru(t)\right) dt \end{aligned}$$

$$= x_1^T F x_1 \quad + \int_{t_1}^{t_f} \left(x^{*T}(t)Qx^*(t) + u^{*T}(t)Ru^*(t)\right) dt$$
$$+ \int_0^{t_1} \left(x^T(t)Qx(t) + u^T(t)Ru(t)\right) dt$$

$$= x_2^T P(t_1)x_2 + \int_0^{t_1} \left(x^T(t)Qx(t) + u^T(t)Ru(t)\right) dt$$

since, as is well known, $x_2^T P(t_1)x_2$ is the optimal cost of the control on the interval (t_1, t_f). The argument of the first part follows directly with F replaced by $P(t_1)$. □

Remark. The number of swiches h of course, depends on x_1 .

In the above discussion we have assumed that no finite piece of an optimal trajectory remains on the boundary of any polytope. This will clearly be the case when the optimal trajectories of the extended linear systems are all transversal to the boundaries in of any polytope , in such a way that the optimal trajectories of linear systems which meet on a boundary **point in the same direction** (Fig.4.1.).

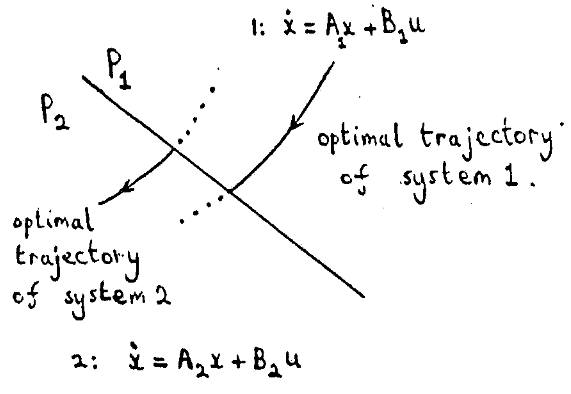


Fig. 4.1. Optimal trajectories at a boundary.

If the optimal trajectories both point away from a boundary as in **Fig.4.2** then the same argument as in **lemma 4.1** holds since we are tracing the solutions backwards in time. However, only those final states which trace back to this boundary in exactly time t_f can lie on optimal trajectories, and not those which trace back to the boundary in a time $< t_f$.

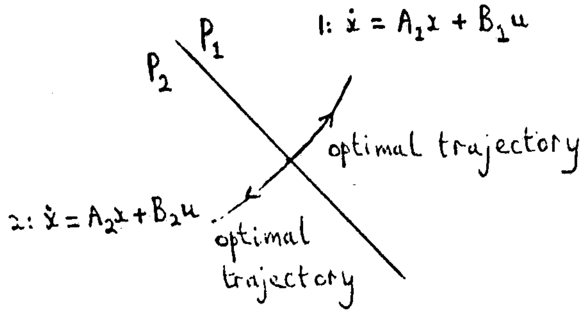


Fig. 4.2 Optimal trajectories which diverge at a boundary.

The most interesting situation occurs when optimal trajectories converge from both sides of a boundary, as in **Fig.4.3.**

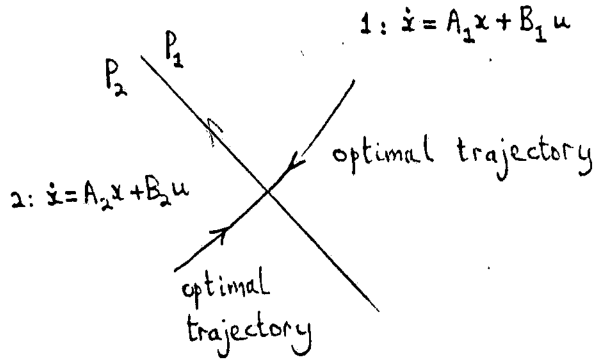


Fig. 4.3. Optimal trajectories which converge to a boundary.

The general conditions for a sliding mode to occur then exist. Consider an optimal trajectory which reaches $\partial P_1 \cap \partial P_2$ at some (minimal) time $\bar{t} < t_f$. Then the trajectory will continue on a sliding mode until it reaches a point at which the optimal trajectories are transversal to the boundary of a polytope and then continue as described in lemma 4.1. Of course, the trajectory may remain on the sliding mode for $t \in \{\bar{t}, t_f\}$. Moreover, it may reach a point on $\partial(\partial P_i)$ for some i and then move along a lower-dimentional sliding mode.

Summing up, we have proved.

Theorem 4.1 *Let t_f be fixed and consider the optimal problem:*

minimize $J(u) = x^T(t_f)Fx(t_F) + \int_0^{t_f} \left(x^T(t)Qx(t) + u^T(t)Ru(t)\right) dt$

(with $R > 0$), subject to the piecewise-linear dynamics

$$\dot{x} = A_i x + B_i u \quad , \quad x \in P_i \quad , i \in \{1, \ldots, m\}.$$

(We assume some well-defined extension of the dynamics to $\cup_i \partial P_i$; for example,

$$\begin{aligned}
\dot{x} &= A_1 x + B_1 u \quad , x \in \overline{P}_1 \\
\dot{x} &= A_2 x + B_2 u \quad , x \in P_2 \cup (\partial P_2 \backslash \partial P_1) \\
\dot{x} &= A_3 x + B_3 u \quad , x \in P_3 \cup (\partial P_3 \backslash (\partial P_2 \cup \partial P_1)) \\
&\cdots \\
\dot{x} &= A_m x + B_m u \quad , x \in P_m \cup (\partial P_m \backslash (\cup_{i=1}^{m-1} \partial P_i)). \ )
\end{aligned}$$

Then, for any $x_0 \in \mathbb{R}^n$ there exist an optimal trajectory $x(t; x_0)$and a sequence of times $\{t_1, t_2, \ldots\}$ such that $0 \leq t_i < t_f$ such that, for $t \in (t_i, t_{i-1})(i = 0, 1, 2, \ldots; t_0 = t_f), x(t; x_0)$ belongs to an open polytope P_{k_i} for some k_i or is a sliding mode in ∂P_{k_i} . Moreover , for fixed t_f , there is an invertible function

$$E : \mathbb{R}^n \longrightarrow \Xi \subseteq \mathbb{R}^n$$

given by $E(x_0) = x(t_f; x_0)$. □

Remark. As we have seen above, Ξ may be a proper subset of $\mathbb{R}^n$.

In order to determine the times t_i for each $x_1 = x(t_f, x_0)$ in Ξ we must first discuss the motion on the sliding mode and obtain the optimal cost of such a trajectory. For simplicity we shall consider the case of a scalar control u.

Suppose that a sliding mode occurs on the n-1 dimentional boundary of the system

$$\dot{x} = A_1 x + B_1 u \tag{4.7}$$

and suppose that the hyperplane is given by the equation

$$d^T x = \alpha \tag{4.8}$$

Writting the equation **(4.7)** in phase-canonical form, we have

$$\dot{y_1} = \bar{A}_1 y_1 + bu$$

where

$$\bar{A}_1 = \begin{pmatrix} 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ v_1 & v_2 & \dots & \dots & \dots & v_n \end{pmatrix}$$

$$b = (0 \dots\dots 0 \;\; 1)^T,$$

for some v_i, w_j, where

$$y_1 = C_1 x$$

By (4.8), we have

$$d^T C_1^{-1} y_1 = \alpha \tag{4.9}$$

Let $y_1 = (\eta_1, \dots, \eta_n)$. Then we have

$$\begin{aligned} \dot{\eta_i} &= \eta_{i+1} \quad , \quad 1 \le i \le n-1 \\ \dot{\eta_n} &= \textstyle\sum_{j=1}^n v_j \eta_j + u \end{aligned} \tag{4.10}$$

From (4.9) we have

$$\sum_{j=1}^{n} \delta_j \eta_j = \alpha$$

for some numbers δ_j . Without loss of generality, we shall assume that $\delta_n \neq 0$ and normalize δ_n to 1. Then

$$\eta_n = \alpha - \sum_{j=1}^{n-1} \delta_j \eta_j \tag{4.11}$$

and so by **(4.10)** the equation of motion on the sliding mode is

$$\dot{\eta}_i = \eta_{i+1} \quad , \quad 1 \leq i \leq n-2$$
$$\dot{\eta}_{n-1} = \alpha - \sum_{j=1}^{n-1} \delta_j \eta_j .$$

This can be written in the form

$$\dot{\overline{\eta}} = \mathcal{M}\overline{\eta} + g$$

for an appropriate matrix $\mathcal{M}$ and vector g where $\overline{\eta} = (\eta_1, \ldots, \eta_{n-1})$. Thus if a sliding mode occurs during the time interval (t_i, t_{i-1}) , we have

$$\overline{\eta}(t) = e^{-\mathcal{M}(t_{i-1}-t)}\overline{\eta}(t_{i-1}) - \int_t^{t_{i-1}} e^{-\mathcal{M}(s-t)} g ds \tag{4.12}$$

Since η_n is given by **(4.11)**, this specifies y_1 on the sliding mode and also u, since by **(4.10)**,

$$u_s \triangleq u \,|_{\text{sliding mode}} = \dot{\eta}_n - \sum_{j=1}^{n} v_j \eta_j \tag{4.13}$$

Finally, we have

$$\begin{aligned} x_s \triangleq x \,|_{\text{sliding mode}} &= C_1^{-1} y_1 \\ &= C_1^{-1}(\overline{\eta}, \eta_n) \end{aligned} \tag{4.14}$$

If the trajectory lies on a lower dimentional hyperspace then we can use similar reasoning to determine the equation of motion on the subspace . For example, if the motion takes place on a subspace of dimention (n-2) given by a pair of equations

$$d_1^T C_1^{-1} y_1 = \alpha_1$$

$$d_2^T C_1^{-1} y_1 = \alpha_2$$

then we can assume that we can solve these equations for (η_{n-1}, η_n) in terms of $(\eta_1, \ldots, \eta_{n-2})$, say

$$\begin{pmatrix} \eta_{n-1} \\ \eta_n \end{pmatrix} = \begin{pmatrix} \alpha_1' \\ \alpha_2' \end{pmatrix} - \begin{pmatrix} \sum_{j=1}^{n-2} \delta_j^1 \eta_j \\ \sum_{j=1}^{n-2} \delta_j^2 \eta_j \end{pmatrix}$$

for some numbers α_k', δ_j^k , k=1,2 , j=1, ... ,n-2. Then, as before we have

$$\begin{aligned} \dot{\eta}_i &= \eta_{i+1} \quad , \quad 1 \leq i \leq n-3 \\ \dot{\eta}_{n-2} &= \alpha_1' - \textstyle\sum_{j=1}^{n-2} \delta_j' \eta_j \end{aligned} \tag{4.15}$$

and proceed as before.

We now have the main ingredients necessary to solve the piecewise-linear -quadratic control problem. We must solve backwards in time from the final state $x(t_f)$ either a normal Riccati equation **(4.15)** together with the dynamics **(4.6)** or a sliding mode equation of the form **(4.15)** over certain intervals of time. In the ordinary linear- quadratic problem we must store values for the Riccati matrix P(t) at discrete times between 0 and t_f . The main drawback with the piecewise-linear-quadratic problem is that the swiching times and the Riccati matrix $P_k(t)$ depend on the final state and so we must store these values for a large number of final states, perhaps at the vertices of some sufficient by fine triangulation of a bounded subject of $\mathbb{R}^n$.

One way around the problem of large storage requirments in piecewise- linear regulator design is to use receding horizon control.(see **Banks, 1986**). For any controllable linear system

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad x(0) = x_0$$

we can choose the suboptimal control

$$u(t) = -R^{-1} B^T W^{-1}(T) x(t)$$

where

$$W(T) = \int_0^T exp(-A\tau) B R^{-1} B^T exp(-A^T \tau) d\tau$$

The horizen time T is then chosen to depend on the state x, in such a way that we obtain a stabilizing control. One such choice is known to be given by (**Shaw, 1979**)

$$V(x,T) = F(T) \tag{4.16}$$

where F is monotonically decreasing in T and

$$V(x,T) = x^T W^{-1}(T)x \tag{4.17}$$

Thus, if we consider the piecewise-linear system

$$\dot{x} = A_i x + B_i u \quad , \quad x \in P_i \quad , i \in \{1, \dots, m\}$$

again we can consider the control law

$$u_i(t) = -R^{-1} B_i^T W_i^{-1}(T) x(t) \quad , \quad x(t) \in P_i \tag{4.18}$$

where

$$W_i(T) = \int_0^T exp(-A_i \tau) B_i R^{-1} B_i^T \; exp(-A_i^T \tau) d\tau$$

of course, for the same reason as before we may have trajectories which form sliding modes on the boundaries of the polytope P_i .These can be determined in the same way as above. The horizon time may be chosen in accordance with (**4.16**) and (**4.17**) in each polytope. Then we have

Theorem 4.2 *If the motion along any sliding mode is stable then the control law* (**4.18**) *with horizon time which satisfies* (**4.16**) *and* (**4.17**) *is stabilizing.*

Proof. *Since the control law* (**4.18**) *is stabilizing for each linear system the result follows from the the assumption on the sliding mode dynamics.* □

5 Application to Nonlinear Systems

In this section we shall consider the nonlinear analytic system

$$\dot{x} = f(x, u) \tag{5.1}$$

We wish to obtain a piecewise-linear approximation to **(5.1)** for simplicial decomposition of $\mathbb{R}^n$ .The application of the simplicial approximation theorem to $f(x,u)$ is not entirely straightforward since we would obtain such an approximation for each u and the resulting system would only be linear in x. to overcome this difficulty we consider the **extended system** (see **Banks, 1988.**)

$$\left.\begin{array}{c} \dot{x} = f(x,u) \\ \dot{u} = v \end{array}\right\} \text{ or } \dot{y} = \overline{f}(y) + \begin{pmatrix} 0 \\ I \end{pmatrix} v \tag{5.2}$$

where $y = (x^T, u^T)^T, \overline{f} = (f^T, 0)^T$ and work with this system rather than **(5.1)**. We shall assume that the system **(5.2)** is controllable. Note that this is a stronger condition than the cotrollability of **(5.1)**. One main result is that there is a controllable piecewise-linear approximation to **(5.2)** which is linear in each simplex.

Theorem 5.1 *Given a controllable system of the form* **(5.2)**, *any* $\varepsilon > 0$ *and any compact set* $| C | \subseteq \mathbb{R}^{n+m}$ *,there exists a piecewise-linear system*

$$\dot{y} = \overline{f}_{pl}(y) + \begin{pmatrix} 0 \\ I \end{pmatrix} v \quad , \quad y \in \mathbb{R}^{n+m} \tag{5.3}$$

which is controllable and satisfies

$$sup_{y \in C} \quad \| \, | \overline{f}_{pl}(y) | - \overline{f}(y) \quad \| < \varepsilon \tag{5.4}$$

Proof. Since **(5.2)** is controllable, it is accessible **(Sussmann, Jurdjevic, 1972)** and hence the lie algebra of vector fields $\mathcal{L}(F)$ generated by the vector fields

$$F = \left\{ \overline{f}(.) + \begin{pmatrix} 0 \\ I \end{pmatrix} v \quad : \quad v \in \mathbb{R}^m \right\}$$

has dimension (n+m). Assume without loss of generality that $| C |$ is a compact polyhedron with geometric complex C and let $| D |$ denote some convex polyhedron with complex D containing $\overline{f}(\mid C \mid)$. Then, by the corollary to the simplicial approximation theorem (corollary 2.1), there exist a simplicial map

$$\overline{f}_{pl} : C^{(r)} \longrightarrow D^{(s)}$$

for some r,s such that **(5.4)** holds. Since the dimention of $\mathcal{L}(F)$ is n we can choose $\varepsilon > 0$ small enough so that $\mathcal{L}(F_{pl})$, where

$$F_{pl} = \left\{ \overline{f}_{pl}(.) + \begin{pmatrix} 0 \\ I \end{pmatrix} v \quad : \quad v \in \mathbb{R}^m \right\}$$

has dimension n. However, for linear systems controllability is equivalent to accessibillity and so the result follows from the fact that each linear system which is obtained by extending the linear map

$$\overline{f}_{pl} \mid_s : S \longrightarrow D^{(s)}$$

where S is a simplex of $C^{(r)}$, to $\mathbb{R}^{n+m}$ is controllable. □

The piecewise-linear system **(5.3)** can be written in simplicial coordinates in the form

$$\dot{\xi}_i = \overline{A}_i \xi_i + \overline{B}_i \begin{pmatrix} 0 \\ I \end{pmatrix} v \tag{5.5}$$

for some matrix $\overline{B}_i$ where ξ_i is the simplicial coordinates vector of $(x^T, u^T)^T$.

Suppose again that we wish to study the nonlinear system **(5.1)** with the standard quadratic cost

$$J = x^T(t_f)Fx(t_f) + \int_0^{t_f} (x^T Qx(t) + u^T(t)Ru(t))dt \tag{5.6}$$

$$= y^T(t_f)F_1 y(t_f) + \int_0^{t_f} y^T(t)Q_1 y(t)dt$$

where

$$y^T = (x^T, u^T) \quad , F_1 = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix} \quad , Q_1 = \begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix}$$

In order to obtain a nonsingular control we shall augment the cost (5.6) to include v ; i.e. we define

$$J_v = y^T(t_f)F_1y(t_f) + \int_0^{t_f} \left(y^T(t)Q_1y(t) + v^T(t)R'v(t)\right) dt \qquad (5.7)$$

for some invertible matrix R' . (This has the effect of weighting the derivative of u, and can be removed by considering the limit $R' \longrightarrow 0$). We can now apply the method of **section 4** by transformig the cost function J_v into simplicial coordinates.

6 Conclusions

In this paper we have presented a framework within which one can study piecewise-linear systems by making use of certain results from combinatorial topology . We have deliberately chosen the homology theory of geometric complexes because of its computational simplicity. Other homology theories would not lead to such a direct computational algorithm as the one given in **section 3**, based on the **Mayer-Vietoris** sequence. We emphasize that the ideas given here are merely a starting point for a complete homological study of nonlinear systems and we shall persue these methods in future papers.

7 References

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