# STRUCTURE AND STABILITY <br> OF A SPHERICAL SHOCK WAVE <br> IN A VAN DER WAALS GAS 

By C. C. WU and P. H. ROBERTS<br>(Departments of Physics and Mathematics, University of California, Los Angeles, California 90095, USA)

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## SUMMARY

Strong spherical shock waves are studied for an imperfect gas, here modelled by a van der Waals equation of state. Similarity solutions of Guderley type are shown to exist, in which the radius of the shock is proportional to $(-t)^{\text {a }}$, where $t$ is time measured from the moment at which the shock focuses. The exponent $\alpha$ depends on both the ratio of specific heats, $\gamma$, and on the van der Waals excluded volume, b. For small $b$, the solution resembles the Guderley solution and is well described by the Chester-Chisnell-Whitham (CCW) approximation. A new branch of solutions, which the CCW approximation fails to locate, is shown to exist for larger $b$.

The linear stability of the similarity solutions is examined directly, without the use of the CCW approximation. Normal modes grow $(\operatorname{Re}(\beta)<0)$ or decay $(\operatorname{Re}(\beta)>0)$ as $(-t)^{\alpha \beta}$, where the 'growth rate', $\beta$, is a function of $\gamma, b$ and $n$, the spherical harmonic wavenumber. No physically meaningful, discrete, spherically symmetric ( $n=0$ ) modes were found. This case was examined numerically and nonlinearly for shocks launched by a spherical 'piston'; no evidence of instability was discovered. For $n>0$, the existence of an infinite discrete spectrum of normal modes is indicated for all $\gamma$ and $b$. In every case examined, the shock is unstable if $b=0$ (the ideal gas), $\operatorname{Re}(\beta)$ for the most unstable mode being negative for all $n \geqslant 1$ but tending to zero as $n \rightarrow \infty$; it is shown that $\beta \sim i n \mathrm{~V}[(\gamma-1) /(\gamma+1)]$ as $n \rightarrow \infty$. It is found that, in general terms, stability is enhanced if either $n$ or $b$ is increased. Special attention is given to the structure and stability of solutions in the limit $\gamma \rightarrow 1$. The limit of the nearly incompressible fluid is also briefly considered.

A way of increasing the light emission from a sonoluminescing bubble is suggested by the present analysis and is described.

## 1. Introduction

The study of shocks generated by spherical and cylindrical implosions in an ideal gas has received much attention in the past decade. The prime impetus has come from plasma fusion research. Our work $(1,2)$ has, however, been motivated by the suggestion $(3,4)$ that such shocks are an essential part of the mechanism responsible for sonoluminescence, that is, the light which under certain conditions is emitted from a bubble of gas trapped in a liquid and compressed by incident spherically symmetric sound waves.

The structure of strong spherical shocks was analysed by Guderley (5), who studied the case of an ideal gas. He found that solutions exist of similarity form, in which the radius $R_{S}(t)$ of the shock and other variables
are proportional to $(-t)^{a}$ up to the time $t=0$ of implosion and proportional to $t^{a}$, for the same constant $\alpha$, after that moment.
In previous papers (1,2), we studied the generation of spherically symmetric shocks in bubbles of air trapped in water and compressed by incident spherically symmetric sound waves. We found that the density of the air in the bubble is increased during the compression phase of the sound wave and may approach that of water. The assumption that the air is then an ideal gas becomes suspect. We noted that, when the imperfections of the air are modelled by a simple van der Waals law, similarity solutions of Guderley type continue to exist, with of course a different exponent $\alpha$ which now depends on the van der Waals excluded volume $b$. We found that, under many conditions of excitation, an implosion would approach similarity form as the moment of collapse approaches, a result anticipated in ( 6 to 9 ) for the case of an ideal gas. Indeed, Greenspan and Nadim (9) made precisely that point in their discussion of sonoluminescence. In short, similarity solutions are ubiquitous. In the present paper, we study these solutions more systematically (section 3), and we investigate their linear stability (section 4).
The validity of the shock-wave model of sonoluminescence may well hinge on the shock remaining spherical as it implodes, right down to high compressions, small bubble radii, and high Mach numbers M. Plausibly, the level of energy concentration is limited by the stability of the shock. This provided the principal motivation for this paper. Fusion research has also inspired several theoretical and experimental studies of cylindrical and spherical implosions, and the stability of the resulting shock waves. For example, see the experimental results reported in $(\mathbf{1 0}, \mathbf{1 1})$ and the theoretical findings described in (12,13). The experimental results are equivocal. Singh et al. (10) show photographs of an imploding spherical shock after its radius has been reduced by only about 50 per cent; no instabilities are apparent, but we would argue that instabilities (which according to linear analysis grow algebraically rather than exponentially; see section 4) would not be prominent at such an early stage even if the shock were unstable. Singh et al. (10) do not specify the Mach number of the shock at that stage, but our estimate based on the information available suggests that it was modest, perhaps less than 5. Watanabe and Takayama (11), who were concerned with cylindrically symmetric implosions, also limited their experiments to $M<5$ (approximately); they report that instabilities arise. The theoretical analyses of shock stability are based on the so-called 'CCW approximation', so named after the influential contributions of Chester, Chisnell and Whitham ( $\mathbf{1 4}$ to 16). We do not use that approximation in this paper, although we do make a number of comparisons which suggest that, while the CCW approximation provides a quick and relatively accurate determination of the structure and stability of shocks in an ideal gas, it performs badly when applied to shocks in a van der Waals gas once $b$
becomes sufficiently large. Indeed, once the CCW approximation is abandoned, a branch of solutions of an entirely different structure is revealed, this being the only branch that exists when $b$ is sufficiently large. Our results show that the instabilities to which shocks are subject are progressively removed if $b$ is increased for fixed $n$ or if $n$ is increased for fixed $b$, where $n$ is the harmonic number of the perturbation. The imperfect (van der Waals) shock is apparently more stable than the corresponding shock in an ideal gas, but the most dangerous modes of instability are, in both cases, those of longest wavelength. Sonoluminescence has recently been observed in gases, such as ethane, in which the ratio of specific heats, $\gamma$, is close to unity. We pay special attention in section 5 to the structure and stability of shocks in the limit $\gamma \rightarrow 1$. Shocks in a nearly incompressible fluid are briefly discussed in Appendix A. In Appendix B, the limit $n \rightarrow \infty$ is studied for the case of an ideal gas.

## 2. Basic equations

The basic equations of this study are the Euler equations

$$
\begin{gather*}
\frac{\partial \rho}{\partial t}+\nabla \cdot(\rho \mathbf{v})=0  \tag{2.1}\\
\frac{\partial E}{\partial t}+\nabla \cdot[(E+p) \mathbf{v}]=0  \tag{2.2}\\
\frac{\partial}{\partial t}\left(\rho v_{i}\right)+\nabla_{j}\left(\rho v_{i} v_{j}\right)=-\nabla_{i} p \tag{2.3}
\end{gather*}
$$

where $\rho$ is gas density, v is fluid velocity, $p$ is pressure and $E=\rho e+\frac{1}{2} \rho v^{2}$ is the total energy density, $e$ being the internal energy density. We assume that the gas obeys a van der Waals equation of state of the form

$$
\begin{equation*}
p=\frac{\mathscr{R} T}{\mathscr{V}-b}, \quad e=c_{v} T=\frac{\mathscr{V}-b}{\gamma-1} p, \quad S=c_{\nu} \ln \left[p(\mathscr{V}-b)^{\gamma}\right]+\text { constant } \tag{2.4}
\end{equation*}
$$

where $\mathscr{V}=1 / \rho$ is specific volume, $S$ is specific entropy, $T$ is temperature, $\mathscr{R}$ is the gas constant, $c_{v}=\mathscr{R} /(\gamma-1)$ is the specific heat at constant volume, $\gamma$ is the ratio of specific heats and $b$ is the van der Waals excluded volume; $\mathscr{R}$, $\gamma(>1)$ and $b$ are constants. It follows from (2.1) to (2.4) that (2.2) may also be written as

$$
\begin{equation*}
\left(\frac{\partial}{\partial t}+\mathbf{v} \cdot \nabla\right)\left[p(\mathscr{V}-b)^{\gamma}\right]=0 \tag{2.5}
\end{equation*}
$$

that is, the specific entropy following the motion is constant.

Equations (2.1) to (2.4) break down at a shock front, where they are replaced by the Rankine-Hugoniot conditions which, for the present van der Waals model are, in the frame of reference moving with the shock,

$$
\begin{align*}
& \frac{\mathbf{n} \cdot \mathbf{v}_{2}}{\mathbf{n} \cdot \mathbf{v}_{1}}=\frac{\rho_{1}}{\rho_{2}}= 1-\frac{2\left(M^{2}-1\right)}{(\gamma+1) M^{2}}\left(1-b \rho_{1}\right),  \tag{2.6}\\
& \frac{p_{2}}{p_{1}}= 1+\frac{2 \gamma\left(M^{2}-1\right)}{\gamma+1},  \tag{2.7}\\
& \mathbf{n} \wedge \mathbf{v}_{2}=\mathbf{n} \wedge \mathbf{v}_{1}, \tag{2.8}
\end{align*}
$$

and for which equation (2.5) is replaced by $S_{2} \geqslant S_{1}$ or, equivalently, $p_{2}\left(\mathscr{V}_{2}-b\right)^{\gamma} \geqslant p_{1}\left(\mathscr{V}_{1}-b\right)^{\gamma}$. Here the suffices ${ }_{1}$ and ${ }_{2}$ denote values immediately ahead of, and immediately behind, the shock front, $n$ is the unit normal to the front, and
where

$$
\begin{gather*}
M=\mathbf{n} \cdot \mathbf{v}_{1} / a_{1},  \tag{2.9}\\
a=\sqrt{ }[\gamma p / \rho(1-b \rho)], \tag{2.10}
\end{gather*}
$$

is the Mach number for the flow ahead of the shock, $a$ being the speed of sound.

Following the lines of much previous research, we formulate the theory in terms of $a^{2}$ rather than $p$. We therefore use (2.10) to replace (2.2), (2.3) and (2.7) by

$$
\begin{gather*}
\left(\frac{\partial}{\partial t}+\mathbf{v} \cdot \nabla\right) a^{2}=-\frac{\gamma-1+2 b \rho}{1-b \rho} a^{2} \nabla \cdot \mathbf{v}  \tag{2.11}\\
\left(\frac{\partial}{\partial t}+\mathbf{v} \cdot \nabla\right) \mathbf{v}=-\frac{1}{\gamma}(1-b \rho) \nabla a^{2}-\frac{a^{2}}{\gamma \rho}(1-2 b \rho) \nabla \rho  \tag{2.12}\\
\frac{a_{2}^{2}}{a_{1}^{2}}=\left[1-\frac{2\left(M^{2}-1\right)}{(\gamma+1) M^{2}}\left(1-b \rho_{1}\right)\right]^{2}\left[1+\frac{2 \gamma\left(M^{2}-1\right)}{\gamma+1}\right] /\left[1-\frac{2\left(M^{2}-1\right)}{(\gamma+1) M^{2}}\right] . \tag{2.13}
\end{gather*}
$$

In what follows, we shall examine the structure (section 3) and stability (section 4) of a strong spherical shock. The structure has a similarity form similar to that discovered by Guderley (5) for the ideal gas. Guderley supposed that a strong spherical shock imploded into a stagnant medium in which

$$
\begin{equation*}
\mathbf{v}=0, \quad \rho=\rho_{0}, \quad p=p_{0}, \quad a=a_{0} \tag{2.14}
\end{equation*}
$$

and reached the origin, $O$, at time $t=0$. Subsequently $(t>0)$, the shock was reflected at the origin and moved outwards. Guderley showed that the equations admit a similarity solution in which the radius of the shock is

$$
R_{S}(t)= \begin{cases}A_{i}(-t)^{\alpha} & \text { for } t<0  \tag{2.15}\\ A_{o} t^{\alpha} & \text { for } t>0\end{cases}
$$

Remarkably, the constant exponent $\alpha$ is the same for the outgoing and ingoing shocks, although the constant-amplitude factors $A_{i}$ and $A_{o}$ are different. The constants $\alpha$ and $A_{o} / A_{i}$ are numerically determined. The constant $A_{i}$ is arbitrary as far as the similarity solution is concerned but when an implosion is initiated, by for example the 'piston' described in section 3, the solution is not at first of similarity form, but it approaches similarity form as the moment of shock focusing approaches, and it does so for only one value of $A_{i}$, determined by the way that the implosion was set off.

Other variables also scaled with distance $r=|\mathbf{x}|$ from the origin in the same way as in (2.15). In preparation for the study of these shocks, we shall introduce a stretched coordinate system defined by

$$
\mathbf{x}= \begin{cases}A_{t}(-t)^{\alpha} \xi & \text { for } t<0  \tag{2.16}\\ A_{o} t^{\alpha} \xi & \text { for } t>0\end{cases}
$$

and use $\boldsymbol{\xi}$ as an independent variable in place of $\mathbf{x}$. More precisely, we shall write

$$
\begin{equation*}
\rho=\rho_{0} G(\xi, t), \quad a^{2}=\left(\frac{\alpha r}{t}\right)^{2} Z(\xi, t), \quad \mathbf{v}=\frac{\alpha r}{t} \mathbf{V}(\xi, t) \tag{2.17}
\end{equation*}
$$

The change of variables (2.16) and (2.17) leads to new forms for (2.1), (2.11) and (2.12), namely

$$
\begin{gather*}
\frac{t}{\alpha} \frac{\partial G}{\partial t}=\xi \cdot \nabla G-\nabla \cdot(\xi G \mathbf{V})  \tag{2.18}\\
\frac{t}{\alpha} \frac{\partial Z}{\partial t}=\xi \cdot \nabla Z+\frac{2 Z}{\alpha}-\frac{1}{\xi} \mathbf{V} \cdot \nabla\left(\xi^{2} Z\right)-\frac{\gamma-1+25 G}{1-5 G} Z \nabla \cdot(\xi \mathbf{V})  \tag{2.19}\\
\frac{t}{\alpha} \frac{\partial \mathbf{V}}{\partial t}=\xi \cdot \nabla \mathbf{V}+\frac{\mathbf{V}}{\alpha}-\mathbf{V} \cdot \nabla(\xi \mathbf{V}) \\
-\frac{1}{\gamma}\left[(1-5 G) \frac{1}{\xi} \nabla\left(\xi^{2} Z\right)+(1-25 G) \frac{Z \xi}{G} \nabla G\right] \tag{2.20}
\end{gather*}
$$

where $\bar{b}=b \rho_{0}, \xi=|\xi|$ and now $\nabla_{i}=\partial / \partial \xi_{1}$. We suppose that the shock position is given by

$$
\begin{equation*}
f(\xi, t)=0 \tag{2.21}
\end{equation*}
$$

so that $\mathbf{n}=\nabla f /|\nabla f|$. The rate of change of $f$ following the shock is zero, so that the velocity of the shock along its normal,

$$
\begin{equation*}
u=\frac{\alpha R_{S}}{t} U \tag{2.22}
\end{equation*}
$$

is given by

$$
\begin{equation*}
\frac{t}{\alpha} \frac{\partial f}{\partial t}+(\xi U \mathbf{n}-\xi) \cdot \mathbf{n}|\nabla f|=0 \tag{2.23}
\end{equation*}
$$

The shock conditions (2.6), (2.8) and (2.13) now give, in the laboratory frame,

$$
\begin{align*}
& \frac{G_{2}}{G_{1}}=\frac{(\gamma+1) M^{2}}{(\gamma+1) M^{2}-2\left(M^{2}-1\right)\left(1-5 G_{1}\right)},  \tag{2.24}\\
& \frac{Z_{2}}{Z_{1}}=\left[1-\frac{2\left(M^{2}-1\right)}{(\gamma+1) M^{2}}\left(1-b G_{1}\right)\right]^{2}\left[1+\frac{2 \gamma\left(M^{2}-1\right)}{\gamma+1}\right] /\left[1-\frac{2\left(M^{2}-1\right)}{(\gamma+1) M^{2}}\right] \text {, }  \tag{2.25}\\
& \text { n. }\left(\mathbf{V}_{2}-\mathbf{V}_{1}\right)=\frac{2\left(M^{2}-1\right)}{(\gamma+1) M}\left(1-\sigma G_{1}\right) Z_{1}^{\frac{1}{2}},  \tag{2.26}\\
& \mathbf{n} \wedge \mathbf{V}_{2}=\mathbf{n} \wedge \mathbf{V}_{1} \text {, } \tag{2.27}
\end{align*}
$$

where by (2.9)

$$
\begin{equation*}
M^{2}=\left(U-\mathbf{n} \cdot \mathbf{V}_{1}\right)^{2} / Z_{1} \tag{2.28}
\end{equation*}
$$

and all of (2.24) to (2.28) are evaluated at the front (2.21).

## 3. The unperturbed shock

We now examine the possibility that, by suitable choice of $\alpha$, equations (2.1), (2.11) and (2.12) admit spherically symmetric similarity solutions of the form described in section 2 . In the $\boldsymbol{\xi}$-frame, these depend only on $\xi$ and are independent of $t ; \mathbf{V}$ and $\mathbf{U}$ have only radial components $V$ and $U$. The shock front (2.21) is given by

$$
\begin{equation*}
\xi=\xi_{S}=1, \quad U=1 \tag{3.1}
\end{equation*}
$$

the latter following from (2.15). The governing equations (2.18) to (2.20) imply that

$$
\begin{gather*}
(1-V) \frac{\xi}{G} \frac{d G}{d \xi}=\xi \frac{d V}{d \xi}+3 V  \tag{3.2}\\
(1-V) \xi \frac{d Z}{d \xi}+2\left(\frac{1}{\alpha}-V\right) Z=\frac{\gamma-1+25 G}{1-\tilde{b} G} Z\left(\xi \frac{d V}{d \xi}+3 V\right)  \tag{3.3}\\
(1-V) \xi \frac{d V}{d \xi}+\left(\frac{1}{\alpha}-V\right) V=\frac{1-5 G}{\gamma-1+2 \overline{5} G}\left[\xi \frac{d Z}{d \xi}+2 Z+\kappa(1-2 \bar{b} G) \frac{Z}{1-V}\right] \tag{3.4}
\end{gather*}
$$

where $\kappa=2(1-\alpha) / \alpha \gamma$. Equations (3.2) and (3.3) can be integrated once, to give $L=$ constant, where $L$ is the Larraza invariant. $\dagger$

$$
\begin{equation*}
L=\xi^{2} Z\left[\xi^{3} G(1-V)\right]^{\gamma \kappa / 3} \frac{(1-\sigma G)^{\gamma+1}}{G^{\gamma-1}} . \tag{3.5}
\end{equation*}
$$

Conservation of $L$ is related to conservation of $S$ and therefore $L$, like $S$, is not conserved across the shock front. The initial conditions are determined by the state ahead of the implosion, where $G_{1}=1$ and $V_{1}=0$. For a strong shock ( $M \rightarrow \infty$ ), (2.28) gives $Z_{1}=1 / M^{2}$ and (2.24) to (2.27) become

$$
\begin{equation*}
G(1)=\frac{\gamma+1}{\gamma-1+2 b^{\prime}}, \quad Z(1)=\frac{2 \gamma(\gamma-1+25)^{2}}{(\gamma-1)(\gamma+1)^{2}}, \quad V(1)=\frac{2(1-\bar{b})}{\gamma+1} . \tag{3.6}
\end{equation*}
$$

These imply that behind the shock front

$$
\begin{equation*}
L=\frac{2 \gamma}{\gamma-1}\left[\frac{\gamma-1}{\gamma+1}(1-\bar{\sigma})\right]^{\gamma+1} . \tag{3.7}
\end{equation*}
$$

We adopted a method of solution similar to that expounded in $(17,8107)$, but one that is, for $b \neq 0$, necessarily more intricate. It is therefore prudent to examine first some possibly simpler alternatives, such as the CCW approximation, which has been shown to work well in the case of an ideal gas (15), and which can be modified to give exact results (18). The idea behind the CCW approximation is that the characteristics behind the shock that overtake the front do not influence the structure of the solution. This leads to a simple relationship between changes $d p$ in pressure, $d v$ in fluid velocity normal to the front (relative to the rest frame) and area $d A$ occupied by an infinitesimal area of the front, namely

$$
\begin{equation*}
d p_{2}+\rho_{2} a_{2} d v_{2}+\frac{\rho_{2} a_{2}^{2} v_{2}}{v_{2}+a_{2}} \frac{d A}{A}=0 \tag{3.8}
\end{equation*}
$$

In the present application, $v_{2}=v_{2 r}-v_{1 r}$ and $A=4 \pi r^{2}$ so that $d A / A=2 / r$. From (2.6) and (2.13) we have

$$
\begin{gather*}
d p_{2}=\frac{4 \rho_{1} a_{1}^{2}}{\gamma+1} M d M  \tag{3.9}\\
d v_{2}=\frac{2 a_{1}}{\gamma+1} \cdot \frac{1+M^{2}}{M^{2}}\left(1-b \rho_{1}\right) d M \tag{3.10}
\end{gather*}
$$

so that (3.8) becomes

$$
\begin{equation*}
\frac{\lambda M d M}{M^{2}-1}+\frac{d A}{A}=0 \tag{3.11}
\end{equation*}
$$

[^0]where, after some reductions,
\[

$$
\begin{equation*}
\lambda(M)=\left\{1+\frac{2 \mu\left(M^{2}-1\right)(1-\bar{b})}{(\gamma+1) M^{2}-2\left(M^{2}-1\right)(1-\bar{b})}\right\}\left\{1+\frac{1}{M^{2}}+2 \mu\right\}, \tag{3.12}
\end{equation*}
$$

\]

and

$$
\begin{equation*}
\mu^{2}=\frac{(\gamma-1) M^{2}+2}{2 \gamma M^{2}-(\gamma-1)} \tag{3.13}
\end{equation*}
$$

In the case of strong shocks, (3.12) reduces to

$$
\begin{equation*}
\lambda(M) \sim \lambda_{\infty}=\frac{\left[(\gamma-1)(\gamma+2)+2 \bar{b}+(\gamma+\bar{b})(2 \gamma(\gamma-1))^{\frac{1}{2}}\right]}{\gamma(\gamma-1+2 \overline{5})}, \quad M \rightarrow \infty . \tag{3.14}
\end{equation*}
$$

It follows from (3.11) and (3.14) that for the similarity solutions

$$
\begin{equation*}
\alpha=\frac{\lambda_{\infty}}{2+\lambda_{\infty}} . \tag{3.15}
\end{equation*}
$$

It was pointed out by Chisnell (15) that the approximation (3.13) is a very good one in the case of an ideal gas $(b=0)$. This will be confirmed below. It is less obvious, and it is one of the objects of this paper to discover, if it is an equally good approximation for a van der Waals gas ( $b \neq 0$ ). We note here that in the dense-gas limit

$$
\begin{equation*}
\lambda(M)=1+\frac{1}{M^{2}}+2 \mu \quad \text { if } \quad \bar{b}=1 \tag{3.16}
\end{equation*}
$$

A non-uniformity arises in the double limit $\gamma \rightarrow 1, \bar{b} \rightarrow 0$. It is easily seen from (3.14) and (3.15) that

$$
\left.\begin{array}{lll}
\lambda_{\infty} \rightarrow 1, & \alpha \rightarrow \frac{1}{3}, & \text { for } \gamma \rightarrow 1  \tag{3.17}\\
\lambda_{\infty} \rightarrow \infty, & \alpha \rightarrow 1, & \text { for } \gamma \rightarrow 1
\end{array} \text { when } 0<b \leqslant 1,\right\}
$$

This non-uniformity is also present in the unapproximated theory to which we now return.

It follows from (3.2) to (3.4) that

$$
\begin{align*}
\frac{d Z}{d V} & =\frac{Z}{(1-V)(1-\bar{b})} \\
& \times\left\{\frac{\left[Z-(1-V)^{2}\right][2(1-5 G) / \alpha-(3 \gamma-1+45 G) V]}{[3 V-\kappa(1-\bar{E} G)] Z-V(1-V)(1 / \alpha-V)}+\gamma-1+25 G\right\} \tag{3.18}
\end{align*}
$$

cf. (17, (107.8)), to which (3.18) reduces in the case when $b=0$. There are two critical curves in the ( $V, Z$ )-plane:
curve 1

$$
\begin{equation*}
Z=(1-V)^{2} \tag{3.19}
\end{equation*}
$$

curve 2

$$
\begin{equation*}
[3 V-\kappa(1-5 G)] Z=V(1-V)(1 / \alpha-V) . \tag{3.20}
\end{equation*}
$$

In order that the solution corresponds to a shock that is reffected at O and eventually reaches $r=\infty$, where

$$
\begin{equation*}
V(\infty)=Z(\infty)=0, \tag{3.21}
\end{equation*}
$$

the curve representing the solution of (3.2) to (3.6) must cross both curves 1 and 2 , and in order that the solution itself be physically acceptable (for example, single-valued in $\xi$ ) it must cross curves 1 and 2 at a point where those curves intersect; the tangent to the curve must be continuous everywhere, including at that point of intersection, $\xi=\xi_{c}$ (say). This determines $\alpha . \dagger$ In the Guderley case of an ideal gas, the $\bar{E} G$ terms are absent from (3.18) to (3.20) but, in the van der Waals case, the search for $\alpha$ is a three-dimensional one, in ( $V, Z, G$ )-space, and the curves 1 and 2 are really surfaces which intersect on curves; we shall, however, continue to refer to surfaces 1 and 2 as 'curves' intersecting in 'points'.

We integrated the governing equations numerically, using a fourth-order Runge-Kutta scheme. In most of the cases we examined, the curves 1 and 2 intersect twice. We shall call the intersection of smaller $V$ and larger $Z$ 'the upper intersection point', or UIP for short; the one for larger $V$ and smaller $Z$ will be called 'the lower intersection point' or LIP. For an ideal gas the solutions can only cross at the LIP but for a van der Waals gas either point (or both!) may, depending on $\gamma$ and $b$, be a possible crossing point. If a value of $\alpha$ is found such that the solution for the imploding shock reaches the LIP there is no difficulty in continuing that solution for the reflected shock, the integral curve continuing smoothly until (3.21) is satisfied. A solution successfully crosses at the UIP only if the derivatives of $V, Z$ and $G$ are continuous there, and this consideration (if it can be met) determines $\alpha$. We should add that there appears to be no inherent reason why curves 1 and 2 should intersect at all! We were, for instance, unable with our technique to obtain any similarity solution for $0.0001<b<0.05$ when $\gamma=1 \cdot 2$; perhaps curves 1 and 2 do not intersect in these cases.
Next to $\alpha$, the most significant quantity that must be determined is $A_{0} / A_{i}$, the ratio of scales for the outgoing and ingoing shocks; see (2.15). This ratio decides their relative speeds. The equations governing the ingoing and outgoing shocks are identical but the sign change implied by the reversal of the sign of $t$ in $(2.17)_{3}$ is significant. For the ingoing shock, $t$ and $v$ are

[^1]negative, so that $V$ is positive, corresponding to implosion everywhere. After focusing, $t$ is positive as are $v$ and $V$ behind the shock, but $v$ and $V$ ahead of the shock are negative, since they refer to ingoing material that has not yet met the outgoing shock and had its motion reversed by that encounter. Thus, the state at $t=0+$ is obtained from the state at $t=0-$ by reversing the sign of $V$ but leaving $Z$ and $G$ with their asymptotic forms:
\[

$$
\begin{equation*}
V=O\left(\xi^{-1 / \alpha}\right), \quad Z=O\left(\xi^{-2 / \alpha}\right), \quad G=O(1), \quad \xi \rightarrow \infty \tag{3.22}
\end{equation*}
$$

\]

$L$ is unchanged. The method of determining the reflected shock may be clarified by reference to Fig. $1(\mathrm{~g})$. The curve through $(V, Z)=(0,0)$ corresponds to the upstream state. It is obtained by integration of the basic state, assuming in the first instance that $A_{o}=A_{i}$. The rescaling subsequently necessary to correct this untenable assumption does not change this curve, but it does alter the correspondence between $\xi$ and points on the curve. The curve itself determines the dashed curve on the right-hand side of the figure. This curve delineates possible states immediately behind the shock, as given by the jump conditions (2.24) to (2.27). This dashed curve meets the solution curve (the full line at the top of the figure) obtained by integration starting from the initial conditions

$$
\begin{array}{r}
V(0) \rightarrow \frac{1}{3} \kappa, \quad G(0)=O\left(\xi^{3 \kappa /(3-\kappa)}\right) \rightarrow 0, \quad Z(0)=O\left(\xi^{-2-3 \kappa /(3-\kappa)}\right) \rightarrow \infty \\
\text { as } \xi \rightarrow 0 \tag{3.23}
\end{array}
$$

corresponding (for $\kappa<3$ ) to the reflected shock as it leaves $O$. The intersection of dashed and undashed curves defines the state immediately behind the shock. The second dashed curve on the left connects this point to the corresponding preshocked state on the other full curve, as given by (2.24) to (2.27). It is finally necessary to adjust $A_{o}$ on this curve so that the point of intersection of dashed and undashed curves becomes $\xi=1$, the normalized position of the shock front; see (3.1). This completes the determination of the reflected shock and of the constant $A_{o} / A_{i}$.

Figures 1 to 4 display solutions for $\gamma=\frac{\xi}{3}$ and for various values of $\bar{b}$. Figure 1 shows the Guderley solution ( $b=0$ ). The full curve diagonally across the ( $V, Z$ )-plot of Fig. 1(a) is curve 1 ; see (3.19). The dashed line shows curve 2, defined by (3.20). The other full curve is the solution curve; it clearly passes through the 'lower' point of intersection of curves 1 and 2 ,

Fig. 1. The Guderley (5) solution in the case when $\gamma=\frac{7}{9}$. Panel (a) illustrates the way in which $\alpha$ is determined (see text); panels (b) to (d) show respectively $G, V$ and $Z$. Here $\xi=r / A_{i}(-t)^{\alpha}$ is the similarity variable. The resulting ingoing velocity is shown in panel (e), where $U$ is the velocity of the shock front. The pressure disturbance is graphed in panel (f). Panels (g) to (I) refer to the shock that emerges from the origin after
(1) Imploding shock


Fso. 1. Continued-the implosion. Panel (g) illustrates how that reflected shock is married up to conditions ahead of the shock, where the fluid is still moving inwards; see text. Panels (h) to (l) show for the reflected shock the same quantities as (b) to (f) for the implosion
corresponding to the LIP. (This panel is essentially the same as (17, Fig. 95).) Figures $1(\mathrm{~b})$ to (d) show $G, V$ and $Z$ for the imploding shock, the dashed line being the shock discontinuity. Figure 1(e), a restatement of Fig. 1(c), gives the fluid velocity in terms of the shock speed. The pressure variation is plotted in Fig. $1(\mathrm{f})$. Figures $1(\mathrm{~g})$ to (l) refer to the reflected shock, the significance of Fig. $1(\mathrm{~g})$ having already been described above. Figures 1(h) to (l) are the counterparts of Figs 1(b) to (f).

Similarly organized data are presented in Figs 2 to 4 for $b=0.05,0.05$ and 0.8 . There are two sets of data for $\bar{B}=0.05$ because, for this and neighbouring values of $\bar{b}$, two completely different solutions are possible, one of which (Fig. 2) crosses the LIP in the ( $V, Z$ )-plane, and the other (Fig. 3) crosses the UIP. The corresponding ( $V, Z$ )-plots are shown in Figs 2(a) and 3(a); similar details are shown in Figs 2(g) and 3(g). The UIP solution shows greater compression of the gas than the LIP solution; compare Figs 2(b) and 2(h) with Figs 3(b) and 3(h). In fact, the densities in the former approach closely the limit $\rho_{\infty}=1 / \bar{m}=20$, set by the van der Waals excluded volume. The greater densities in the UIP solution are paralleled by greater pressure jumps across the imploding shock; compare Figs 2(f) and 3(f). Properties of the $\gamma=\{$ solutions are also summarized in Table 1. The third column of this table gives the central compression, $\rho_{c} / \rho_{0}=G$, at the moment of focusing, that is, the $O(1)$ constant of (3.22). The last three columns refer to the reflected shock, and give the upstream ( $\rho_{o u}$ ) and downstream ( $\rho_{\text {od }}$ ) densities at the shock front; $M_{o}$ is the Mach number based on the upstream fluid velocity and sound speed in the shock frame.

It may be recalled that, according to (2.22), the shock speed is $\alpha R_{s} / t$. The larger $\alpha$ is, the faster (for the same $R$ and $t$ ) the shock moves. The constant $A_{o} / A_{i}$ determines whether the outgoing shock moves slower $\left(A_{o} / A_{i}<1\right)$ or faster $\left(A_{o} / A_{i}>1\right)$ than the ingoing shock at the same radius. In the case of the ideal gas, it moves more slowly; in the other cases shown, it moves more rapidly.

Figure 5 shows the critical exponent as a function of $\bar{b}$ for three values of $\boldsymbol{\gamma}$. The full curves correspond to LIP and UIP solutions, the dashed curves to results from the CCW approximation. In the case when $\gamma=\frac{5}{3}$, the LIP solution arises in the range $0 \leqslant \sigma \leqslant 0.05$; the UIP solution exists in $0.005 \leqslant 5 \leqslant 0.035$ and again in $0.05 \leqslant b \leqslant 1$, as shown in Fig. 5(c). When $\gamma=\frac{7}{3}$ a UIP solution exists for $0.05 \leqslant \bar{b} \leqslant 1$ and a LIP in $0 \leqslant \bar{b}<0.081$. For $\gamma=\frac{8}{3}$, the LIP solution only exists over a tiny range, $0 \leqslant b \leqslant 0 \cdot 0001$; a UIP solution arises in $0.05 \leqslant b \leqslant 1$. We did not obtain a similarity solution in the range $0.0001 \leqslant \bar{b}<0.05$. Computational difficulties arise when $\gamma$ and $\bar{b}$ are small. For example, a satisfactory solution for $\gamma=1.2$ and $\bar{b}=0.0001$ requires $\alpha$ to be determined to seven decimal places.

As was noted in $(6,8)$, an imploding shock, almost independently of how it is excited, approaches similarity form immediately prior to the moment ( $t=0$ ) of collapse. Greenspan and Nadim (9) based their discussion of the
(1) Imploding shock


Fig. 2. The similarity solution of Guderley type for an implosion in which imperfections of the gas are slight ( $\bar{b}=0.05, \gamma=\frac{1}{3}$ ). For an explanation of the panels see the legend to Fig. 1
(1) Imploding shock

(2) Reflected shock






Fig. 3. The similarity solution on the new branch for an implosion in which imperfections of the gas are slight ( $\tilde{b}=0.05, \gamma=\frac{7}{3}$ ). For an explanation of the panels see the legend to Fig. 1
(1) Imploding shock

(2) Reflected shock


Fig. 4. The similarity solution on the new branch for an implosion in which imperfections of the gas are great ( $\tilde{b}=0 \cdot 8, \gamma=\frac{3}{3}$ ). For an explanation of the panels see the legend to Fig. 1

Table 1. Summary of integrations for the case when $\gamma=\boldsymbol{\}}$

| $\sigma$ | $\alpha$ | $\rho_{c} / \rho_{0}$ | $A_{o} / A_{i}$ | $\rho_{o u} / \rho_{0}$ | $\rho_{o d} / \rho_{0}$ | $M_{o}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.0 | 0.717 | 20.1 | 0.492 | 64.2 | 145 | 1.73 |
| 0.05 (LIP) | 0.661 | 12.6 | 2.05 | 15.1 | 17.9 | 2.01 |
| 0.05 (UIP) | 0.494 | 19.5 | 6.44 | 19.6 | 19.9 | 2.88 |
| 0.25 | 0.534 | 3.70 | 3.66 | 3.77 | 3.91 | 1.94 |
| 0.8 | 0.554 | 1.22 | 2.99 | 1.22 | 1.24 | 1.49 |

sonoluminescing bubble on the same idea. Similarity solutions are useful approximations in quite general circumstances.

The approach of solutions to similarity form is illustrated in Figs 6 to 11, which display solutions for $\gamma=\frac{7}{3}$ for spherical implosions driven by a 'piston'. One supposes that a spherical diaphragm separates the gas at $r=\boldsymbol{j}_{3}$ into two domains, the pressure in the outer domain being $10^{4}$ times the pressure in the inner domain. At $t=0$, this diaphragm is removed. Similarly-generated shocks were considered in (6,8). (In the computations, which were performed using the Lax-Friedrich method on a non-uniform grid, the initial discontinuity in $p$ at $r=\frac{2}{3}$ is smoothed out slightly.)

Figures 6 to 8 were derived from the solution of this 'piston problem' for the ideal gas and Figs 9 to 11 are from the solution for the van der Waals gas with $b=0.25$; the moment of collapse is now $t=t_{0}$. The first columns of Figs 6 and 9 show the initial density, velocity and pressure distributions. The second, third and fourth columns show these quantities at subsequent times for the imploding shock; the final columns show them for the reflected shock. In Figs 7 to 9 they are normalized to simplify comparison with the corresponding solutions given in Figs $1(b, e, f, h, k, l)$. The dashed curves in Figs 8 and 11 are similarity solutions.


Fig. 5. The variation of $\alpha$ with $\bar{b}$ for $\gamma=\frac{9}{3}, \frac{3}{3}$ and $\frac{5}{3}$. The longer solid curve in each panel refers to the new 'upper-branch' solution; the shorter solid curve corresponds to 'lower-branch' solutions of Guderley type. The dashed curves give the predictions of CCW theory


Fig. 6. The structure of the solution for both ingoing and outgoing shocks generated at $t=0$ by a 'piston' situated at $r=3$. The gas is ideal and $\gamma=\}$. The rows show density, velocity and pressure. The first column shows these quantities initially, and the next three columns show the subsequent evolution of the solution as the shock implodes. The final column shows the solution when the reflected shock has started to move outwards

Figures $8(\mathrm{a}, \mathrm{b})$ display, for $b=0$ and $\gamma=\xi$, the shock radius as a function of position for the piston problem (full curves) and the results of fitting similarity solutions (dashed curves) to them. Figure 8(b) shows the fit on a magnified scale. It may be seen that, on the extended scale of Fig. 8(a), the similarity solution agrees with the reflected shock only over a small fraction of the time interval shown. The reason is apparent from an inspection of Fig. 6. The compression of the incoming shock covers only a small $r$-range, so that the refiected shock soon encounters the much reduced $\rho$ behind the piston. The corresponding results for $\bar{b}=0.25$ are shown in Figs 11. These exhibit some interesting features. The time dependence of $R_{S}$ cannot initially be well-fitted to a power law corresponding to the $\alpha$ determined from integration of the similarity equation, which is $\alpha \approx 0.534$; see Fig. 11(a). The best fit ( $\alpha \approx 0.59$ ) corresponds to a smaller $\bar{b}$ but is close to the value, namely $\alpha=0.5886$, predicted by the CCW approximation for $\bar{b}=0 \cdot 25 . \dagger$ This suggests that, initially while the strength of the shock is comparatively small, the CCW approximation gives a good account of the evolution of the shock by the piston; it is only as the Mach number approaches $\infty$ that the approximation fails. Close to the moment of collapse, the solution makes a surprisingly abrupt change to the correct exponent; see Figs 11 (b) and (c). The reflected shock also takes that exponent, at least for small $t-t_{0}$. Qualitatively similar results hold for other $\bar{b}$. If $\bar{b}$ is small ( $\bar{b} \in 0 \cdot 1$ ), no abrupt change, of the type seen in Fig. 11(c), can be discerned and, if $\bar{b}$ is close to 1 , the correct exponent (that is, the one derived from the full theory) is achieved early in the collapse. There is a continuous transition between these two extremes; if, for example $0 \cdot 1<5<0 \cdot 25$, the change in exponent occurs closer to the moment of collapse than is seen in Fig. 11(c); if $\overline{5}>0.25$ it occurs nearer to $t=0$.

Some other differences between the $\bar{b}=0$ and $\bar{b}=0.25$ solutions may be noted. The shocks are initiated in an identical fashion, but the one in the non-ideal gas focuses first, at $t=t_{0}=0.004752$, as compared with $t=t_{0}=$ 0.006454 for the case of the ideal gas. This is due to the greater shock speeds in the van der Waals gas. The speed of the ingoing shock when $R_{S}=0.005$ is 510 for $\bar{b}=0$ but 3680 for $\bar{b}=0.25$. This difference may be mostly attributed to the different values of $\alpha$. The speed of the outgoing shock, also at $R_{S}=0.005$, is 190 for $b=0$ but 42,000 for $\bar{b}=0.25$. This is a consequence of the very different values of $A_{o} / A_{i}$, which are 0.492 for $\bar{B}=0$ but 3.66 for $\bar{b}=0.25$.

## 4. Shock stability

We now investigate the linear stability of the spherical shock studied in section 3. We now add a suffix 0 to variables in that solution; a suffix 1 will

[^2]distinguish perturbed quantities. We substitute
\[

$$
\begin{equation*}
G=G_{0}\left(1+G_{1}\right), \quad \mathbf{V}=\mathbf{V}_{0}+\mathbf{V}_{1}, \quad Z=Z_{0}+Z_{1} \tag{4.1}
\end{equation*}
$$

\]

into (2.18) to (2.20). After linearization, we obtain

$$
\begin{align*}
& \frac{t}{\alpha} \frac{\partial G_{1}}{\partial t}=\left(1-V_{0}\right)(\xi \cdot \nabla) G_{1}-\nabla \cdot\left(\xi \mathbf{V}_{1}\right)-\frac{G_{0}^{\prime}}{G_{0}} \xi \cdot \mathbf{V}_{1},  \tag{4.2}\\
& \frac{t}{\alpha} \frac{\partial Z_{1}}{\partial t}=\left(1-V_{0}\right)(\xi \cdot \nabla) Z_{1}+2\left(\frac{1}{\alpha}-V_{0}\right) Z_{1} \\
& -\frac{\left(\xi^{2} Z_{0}\right)^{\prime}}{\xi^{2}} \boldsymbol{\xi} \cdot \mathbf{V}_{1}-\frac{(\gamma+1) \hbar G_{0}}{\left(1-\bar{b} G_{0}\right)^{2}} Z_{0} \nabla \cdot\left(\xi \mathbf{V}_{0}\right) G_{1} \\
& -\frac{\left(\gamma-1+2 \bar{b} G_{0}\right)}{1-\tilde{b} G_{0}}\left[Z_{0} \nabla \cdot\left(\xi \mathbf{V}_{1}\right)+\nabla \cdot\left(\xi \mathbf{V}_{0}\right) Z_{1}\right] \text {, }  \tag{4.3}\\
& \frac{t}{\alpha} \frac{\partial \mathbf{V}_{1}}{\partial t}=\left(1-V_{0}\right)(\xi \cdot \nabla) \mathbf{V}_{1}+\left(\frac{1}{\alpha}-2 V_{0}\right) \mathbf{V}_{1} \\
& -\frac{1}{\gamma \xi}\left[\left(1-5 G_{0}\right) \nabla\left(\xi^{2} Z_{1}\right)+\left(1-25 G_{0}\right) \xi^{2} Z_{0} \nabla G_{1}\right] \\
& -\frac{1}{\gamma \xi}\left[\gamma V_{0}^{\prime}\left(\boldsymbol{\xi} \cdot V_{1}\right)-\frac{\left(\xi^{2} G_{0}^{2} Z_{0}\right)^{\prime}}{\xi G_{0}} \sigma G_{1}+\left(1-25 G_{0}\right) \frac{\xi G_{0}^{\prime}}{G_{0}} Z_{1}\right] \xi . \tag{4.4}
\end{align*}
$$

We suppose that the equation of the shock front after perturbation is given by

$$
\begin{equation*}
\xi=1+f_{1}(\theta, \phi, t) \tag{4.5}
\end{equation*}
$$

where $(r, \theta, \phi)$ are spherical coordinates; the deviation of $\mathbf{n}$ from the unit radial vector $\mathbf{1}_{r}$ is negligible. The velocity of the front along its normal is, by (2.23),

$$
\begin{equation*}
U=U_{0}+U_{1}=1+\frac{t}{\alpha} \frac{\partial f_{1}}{\partial t} . \tag{4.6}
\end{equation*}
$$

Equations (4.2) to (4.6) admit solutions that are the real parts of

$$
\begin{gather*}
G_{1}=G_{1}(\xi)(-t)^{\alpha \beta} P_{n}^{m}(\theta) e^{i m \phi},  \tag{4.7}\\
Z_{1}=Z_{1}(\xi)(-t)^{\alpha \beta} P_{n}^{m}(\theta) e^{i m \phi},  \tag{4.8}\\
\mathbf{V}_{1}=\left[V_{1}(\xi) P_{n}^{m}(\theta), W_{1}(\xi) \frac{d P_{n}^{m}(\theta)}{d \theta}, W_{1}(\xi) \frac{i m P_{n}^{m}(\theta)}{\sin \theta}\right](-t)^{\alpha \beta} e^{i m \phi},  \tag{4.9}\\
f_{1}=a(-t)^{\alpha \beta} P_{n}^{m}(\theta) e^{\iota m \phi}, \tag{4.10}
\end{gather*}
$$

where $P_{n}^{m}$ is the Legendre function; without loss of generality the functions of $\xi$ and the constant $a$ are real. Corresponding to (4.5) and (4.10), the equation of the shock front in physical space is

$$
\begin{equation*}
r=A_{t}(-t)^{\alpha}+A_{t} f_{1}(-t)^{\alpha(\beta+1)} \tag{4.11}
\end{equation*}
$$



Fig. 7. Comparisons between density, velocity and pressure in the piston solution (full curves) and the Guderley solution (dashed lines) for the same cases as shown in Fig. 6

If $\operatorname{Re}(\beta)<-1$, the shock is unstable by any criterion. If $-1<\operatorname{Re}(\beta)<0$, the surface distortion decreases in time but, because the surface is contracting more rapidly, the relative surface distortion increases. We shall therefore regard as unstable any mode for which $\operatorname{Re}(\beta)<0$; if $\operatorname{Re}(\beta)>0$, the mode is linearly stable.

Substituting (4.7) to (4.9) into (4.2) to (4.4), we obtain

$$
\begin{align*}
& \quad\left(1-V_{0}\right) \xi \frac{d G_{1}}{d \xi}-\beta G_{1}=\xi \frac{d V_{1}}{d \xi}+\left(\frac{\xi G_{0}^{\prime}}{G_{0}}+3\right) V_{1}-n(n+1) W_{1}  \tag{4.12}\\
& \left(1-V_{0}\right) \xi \frac{d Z_{1}}{d \xi}-\left(2 V_{0}-\frac{2}{\alpha}+\beta\right) Z_{1}
\end{align*}
$$



Fig. 8. Detailed comparison of the shock position as a function of time for the same case as Fig. 6; the piston solution is the full curve and the Guderley solution is the dashed curve

$$
\begin{aligned}
= & \left(\xi Z_{0}^{\prime}+2 Z_{0}\right) V_{1}+\frac{(\gamma+1) 5 G_{0}}{\left(1-5 G_{0}\right)^{2}}\left(\xi V_{0}^{\prime}+3 V_{0}\right) Z_{0} G_{1} \\
& +\frac{\left(\gamma-1+25 G_{0}\right)}{1-5 G_{0}}\left[Z_{0}\left(\xi \frac{d V_{1}}{d \xi}+3 V_{1}\right)+\left(\xi V_{0}^{\prime}+3 V_{0}\right) Z_{1}-n(n+1) Z_{0} W_{1}\right]
\end{aligned}
$$

$$
\begin{equation*}
\left(1-V_{0}\right) \xi \frac{d V_{1}}{d \xi}-\left(\xi V_{0}^{\prime}+2 V_{0}-\frac{1}{\alpha}+\beta\right) V_{1} \tag{4.13}
\end{equation*}
$$

$$
=\frac{1}{\gamma}\left[\left(1-\tilde{b} G_{0}\right)\left(\xi \frac{d Z_{1}}{d \xi}+2 Z_{1}\right)-\tilde{b} G_{0}\left(\xi Z_{0}^{\prime}+2 Z_{0}\right) G_{1}\right.
$$

$$
\begin{equation*}
\left.+\left(1-25 G_{0}\right)\left(\frac{\xi G_{0}^{\prime}}{G_{0}} Z_{1}+Z_{0} \xi \frac{d G_{1}}{d \xi}\right)-25 G_{0} Z_{0} \frac{\xi G_{0}^{\prime}}{G_{0}} G_{1}\right] \tag{4.14}
\end{equation*}
$$

$$
\begin{equation*}
\left(1-V_{0}\right) \xi \frac{d W_{1}}{d \xi}-\left(2 V_{0}-\frac{1}{\alpha}+\beta\right) W_{1}=\frac{1}{\gamma}\left[\left(1-\delta G_{0}\right) Z_{1}+\left(1-25 G_{0}\right) Z_{0} G_{1}\right] \tag{4.15}
\end{equation*}
$$

When we linearize the boundary conditions, we obtain $a=-G_{1}(1) / G_{0}^{\prime}(1)$ and

$$
\begin{gather*}
V_{1}(1)=K_{1}\left[V_{0}^{\prime}(1)-\beta V_{0}(1)\right]  \tag{4.16}\\
Z_{1}(1)=K_{1}\left[Z_{0}^{\prime}(1)-2 \beta Z_{0}(1)\right]  \tag{4.17}\\
W_{1}(1)=K_{1} V_{0}(1) \tag{4.18}
\end{gather*}
$$

where for brevity we have written $K_{1}=G_{1}(1) G_{0}(1) / G_{0}^{\prime}(1)$. Equation (4.14)


$$
t=4.13 \times 10^{-3}
$$

$$
R_{s}=0.2
$$




Fig. 9. The structure of the solution for both ingoing and outgoing shocks generated at $t=0$ by a piston situated at $r=\frac{\text {. The }}{3}$. Thedium is a van der Waals gas with $b=0.25$ and $\gamma=\frac{7}{3}$. See caption to Fig. 6


Fig. 10. Comparisons between density, velocity and pressure in the piston solution (full curves) and the similarity solution (dashed lines) for the same cases as shown in Fig. 9
and condition (4.18) do not apply in the case of $n=0$, for which $W_{1}=0$. Like (3.2) to (3.4), equations (4.12) to (4.15) possess a singular point at $\xi=\xi_{c}$, where (3.19) and (3.20) hold. The condition that the perturbation is bounded there is

$$
\begin{align*}
(1 & \left.-V_{0}\right)^{2}\left[\alpha \beta\left(1-5 G_{0}\right)+(2-2 \alpha-\alpha \beta) 5 G_{0}\right] G_{1} \\
& +\left[\alpha \gamma\left(1-V_{0}\right) \frac{\xi G_{0}^{\prime}}{G_{0}}-(2-2 \alpha-\alpha \beta)\left(1-5 G_{0}\right)\right] Z_{1} \\
& +\left(1-V_{0}\right)\left[2 \alpha \gamma\left(1-V_{0}\right) \frac{\xi G_{0}^{\prime}}{G_{0}}-2(1-\alpha)\left(1-5 G_{0}\right)-\gamma(1-\alpha \beta)\right. \\
& \left.+\alpha \gamma\left(3-4 V_{0}\right)\right] V_{1}-n(n+1) \alpha \gamma\left(1-V_{0}\right)^{2} W_{1}=0 \quad \text { at } \xi=\xi_{c} \tag{4.19}
\end{align*}
$$



Fio. 11. Detailed comparison of the shock position as a function of time for the same case as Fig. 9; the piston solution is the full curve and the similarity solution is the dashed curve. Panels (a) to (c) show the path of the incoming shock; it is noteworthy how the solution (full curves) switches quite suddenly from one similarity structure to the correct exponent (dashed curves) as the shock approaches the origin. Panel (d) shows that the reflected shock path (full curve) follows the correct similarity path (dashed curve) as it leaves the origin

This condition, together with (4.12) to (4.18), defines an eigenvalue problem for $\beta$.

There are some obvious tests to which one can subject (4.12) to (4.18):
A. Consider the difference between two of the solutions obtained in section 3 corresponding to infinitesimally different $\boldsymbol{A}_{i}$. That difference, which is proportional to

$$
\begin{equation*}
V_{1}=\xi V_{0,}^{\prime} \quad G_{1}=\frac{\xi G_{0}^{\prime}}{G_{0}}, \quad Z_{1}=\xi Z_{0}^{\prime} \tag{4.20A}
\end{equation*}
$$

must satisfy (4.12) to (4.14), (4.16), (4.17) and (4.19), for $n=0$ and $\beta=0$, which it does;
B. Consider the difference between two of the solutions obtained in section 3 corresponding to infinitesimally different origins of $t$. That difference, which is proportional to

$$
\begin{equation*}
V_{1}=\xi V_{0}^{\prime}+\frac{V_{0}}{\alpha}, \quad G_{1}=\frac{\xi G_{0}^{\prime}}{G_{0}}, \quad Z_{1}=\xi Z_{0}^{\prime}+\frac{2 Z_{0}}{\alpha}, \tag{4.20B}
\end{equation*}
$$

must satisfy (4.12) to (4.14), (4.16), (4.17) and (4.19) for $n=0$ and $\beta=-1 / \alpha$, which it does;
C. Consider the difference between two of the solutions obtained in section 3 corresponding to infinitesimally different centres of implosion. That difference, which is proportional to

$$
\begin{equation*}
V_{1}=V_{0}^{\prime}+\frac{V_{0}}{\xi}, \quad G_{1}=\frac{G_{0}^{\prime}}{G_{0}}, \quad Z_{1}=Z_{0}^{\prime}+\frac{2 Z_{0}}{\xi}, \quad W_{1}=\frac{V_{0}}{\xi}, \tag{4.20C}
\end{equation*}
$$

must satisfy (4.12) to (4.19) for $n=1$ and $\beta=-1$, which it does.
Although $\beta<0$ for two of these solutions, they cannot be classed as instabilities. Solution $A$ applies when the initial perturbation slightly changes the energy of the configuration, solution $B$ when the initial perturbation causes the shock to focus at a time slightly different from $t=0$, and solution C when the initial perturbation causes the shock to focus at a point slightly displaced from the origin. We do not consider that these three solutions are of any physical interest. We found them useful, however, as checks on the analytic and numerical accuracy of our work.

The eigenvalue problem just defined was investigated numerically, again by applying a fourth order Runge-Kutta method. In the case of $n=0$, we recovered the two solutions (4.20A) and (4.20B). Based on the analysis of section 5 (which admittedly only applies in the limit $\gamma \rightarrow 1$ ), we conjecture that these are the only discrete spherically symmetric modes, and that the remaining modes belong to a continuum. If the analysis of section 5 is a reliable guide to the case of general $\gamma$, it appears that, for any $n>0$ and for every admissible choice of $\gamma$ and $\bar{b}$, the spectrum of eigenvalues is discrete, with limit point at $\beta=\infty$. In Table 2, we show the most unstable modes, that is, the ones for which $\operatorname{Re}(\beta)$ is smallest, but excluding the mode (4.20C). We also located some higher modes; these are not listed in the table, but are reported in section 5. The two columns labelled $\gamma=0.05$ refer to the two possible types of solution located in section 3. The results in the third column refer to the lower intersection point; those in the fourth and

$$
\begin{aligned}
& 10-1.00 \pm 4.01 i-0.91 \pm 4.61 i \\
& 14-0.99 \pm 5.47 i-0.83 \pm 6.36 i \\
& 18-0.95 \pm 6.96 i-0.72 \pm 8.16 i \\
& 22-0.89 \pm 8.46 i-0.57 \pm 9.97 i \\
& 26-0.82 \pm 9.98 i-0.45 \pm 11.82 i \\
& 30-0.75 \pm 11 \cdot 51 i-0.34 \pm 13 \cdot 68 i \\
& 34-0.68 \pm 13.05 i-0.19 \pm 15.55 i \\
& 38-0.61 \pm 14.61 i-0 \cdot 10 \pm 17.47 i \\
& 42-0.54 \pm 16 \cdot 17 i \quad 0.03 \pm 19.29 i
\end{aligned}
$$

Table 2. Values of $\beta$ for $\gamma=\frac{7}{3}$ for various $\bar{b}$ and $n$
subsequent columns to the upper intersection point. It is evident that, as $n$ is increased for fixed $\overline{\bar{b}}$, the shock becomes increasingly stable. It appears, however, that the Guderley solution ( $b=0$ ) is unstable for every $n$, although $\operatorname{Re}(\beta)$ tends to zero as $n \rightarrow \infty$; see Appendix $B$ and (4.23) below. The Guderley-like LIP solution for $\bar{b}=0.05$ (third column), though very unstable, ultimately becomes stable near $n=41$. In contrast, the instability of the corresponding UIP solution (fourth column) is quenched at $n=6$. In the sense that the transitional value of $n$ at first increases with $\bar{b}$ for the UIP solutions (compare columns 4 and 5), they are at first increasingly unstable with increasing $b$, but this situation is later reversed and all states are linearly stable for $b>0.8$ (approximately).
The linear-stability problem for the ideal gas has also been attacked by Gardner et al. (12) who used the CCW approximation. For $n=1$ they obtained $\beta=1-\alpha^{-1}$, which for $\left.\gamma=\right\}$ gives $\beta=-0.397$ (taking the CCW value of $\alpha$, namely 0.7173 ), which may be compared with the value shown in the second column of Table 2. For $n>1$ they obtained

$$
\begin{equation*}
\beta=\frac{1}{2 \lambda}\left\{-(\lambda+2) \pm \sqrt{ }\left[(\lambda+2)^{2}-4 \lambda n(n+1)\right]\right\} . \tag{4.21}
\end{equation*}
$$

according to which $\operatorname{Re}(\beta)$ is independent of $n$ and negative (instability), and also, using (3.15),

$$
\begin{equation*}
\operatorname{Im}(\beta) \sim \pm n \sqrt{\left[\frac{1-\alpha}{2 \alpha}\right]} \quad n \rightarrow \infty . \tag{4.22}
\end{equation*}
$$

Our results confirm that all $n>0$ modes are unstable but, in contrast to their result, we find that, for the most unstable mode, $\operatorname{Re}(\beta)$ increases with $n$. An asymptotic analysis (Appendix B) indicates that $\beta$ is purely imaginary as $n \rightarrow \infty$; in fact, instead of (4.22), we find that

$$
\begin{equation*}
\beta \sim \pm i n \sqrt{\left[\frac{\gamma-1}{\gamma+1}\right]}, \quad n \rightarrow \infty . \tag{4.23}
\end{equation*}
$$

According to CCW theory there are at most two discrete modes of instability for any $n$. Our work strongly suggests, however, that there is an infinite spectrum of such modes for all $n>0$.
The analysis of (12) can be applied with almost no change to the case of the van der Waals gas. It is necessary only to use the general expression (3.14) for $\lambda$ rather than the particular case of $\lambda$ for $\bar{b}=0$ used to obtain (4.21). The consequences seem however to conflict badly with the numerical results obtained above. For example, it predicts that $\beta=-2 / \lambda<0$ for $n=1$ and all $b$. We see from Table 2 however that the $n=1$ modes are stable for all sufficiently large $\delta$. Perhaps this disappointing lack of agreement is not surprising in view of the fact (section 3) that the spherical shock itself is poorly represented by CCW theory, for all but small $b$.

One disappointing feature of the linear-stability analysis is that it gives no information about $n=0$ modes. In particular, in parameter domains where both UIP and LIP exist, it does not determine which, if either, is unstable. It was partly for this reason that we examined the piston problem in section 3 . For $\gamma=\xi$, we found that, if $\bar{\delta} \in 0 \cdot 1$, the shock created by the piston approaches an LIP solution at the moment of collapse; for $\overline{b \geqslant 0 \cdot 1 \text {, it }}$ develops into a UIP solution instead. When $\bar{\sigma}=0 \cdot 1$, the central density, $\rho_{c}$, is approximately the same in both solutions, and it is tempting to conjecture that, whenever two solutions are possible, the disturbance will evolve to the one in which the central density is the smaller. We performed an experiment in which, instead of a piston, we set up an initial state where the UIP solution exists everywhere. We followed the evolution of that state during the time that $R_{S}$ decreased from 1 to $10^{-4}$, at which point numerical resolution was lost. We found that during that time the UIP solution remained a UIP solution. When we repeated the experiment at the same value 0.07 of $\bar{b}$ but with an initial LIP state, the solution again did not make a transition to the other solution. We speculate that, when more than one solution exists, each is relatively stable with respect to the other, and that, in other circumstances (as when the shock is generated by a piston), either solution may be the relevant similarity solution at the moment of collapse, depending on the condition of excitation.

## 5. The limit $\gamma \rightarrow 1$

As has been mentioned in section 1 , sonoluminescence has recently been observed from a bubble of ethane, a gas in which the ratio of specific heats, $\gamma$, is close to unity. If ethane were an ideal gas, the shock heating would vanish for $\gamma \rightarrow 1$, for then $p / \rho \propto T=$ constant. But for a van der Waals gas, $\rho_{2}$ is limited by $1 / b$. By equations (2.6) and (2.7) it follows that $T_{2} \propto p_{2} / \rho_{2} \sim M^{2} b \rho_{1} \propto M^{2}$ for $M \rightarrow \infty$. It is therefore reasonable that shock heating, due to the imperfections of ethane, can explain its ionization and the emission of light. This provides a strong motive for studying the similarity shock and its stability in the limit $\gamma \rightarrow 1$ with $\bar{b}$ fixed $(0<b<1)$. The problem is solved by matched asymptotic expansions.

Consider first the unperturbed shock. In the outer, large $\xi$, region, the variables are rescaled in the following way:

$$
\begin{align*}
\xi & =\lambda(\gamma-1)^{-1 /(4-\kappa)} \bar{\xi},  \tag{5.1}\\
V & =(\gamma-1)^{3 /(4-\kappa)} \bar{V}(\bar{\xi}),  \tag{5.2}\\
G & =\frac{1}{b}\left[1-(\gamma-1)^{3 /(4-\kappa)} \bar{G}(\bar{\xi})\right],  \tag{5.3}\\
Z & =\bar{Z}(\bar{\xi}), \tag{5.4}
\end{align*}
$$

where $\lambda$ is a scaling constant that is determined below. We assume that $\bar{G}$,
$\bar{Z}$ and $\bar{V}$ are $O(1)$ in the limit $\gamma \rightarrow 1$, as are all derivatives with respect to $\bar{\xi}$. To leading order, (3.2) and (3.4) become

$$
\begin{gather*}
\bar{\xi} \frac{d \bar{V}}{d \bar{\xi}}+3 \bar{V}=-\bar{\xi} \frac{d \bar{G}}{d \bar{\xi}},  \tag{5.5}\\
\bar{\xi} \frac{d \bar{V}}{d \bar{\xi}}+\frac{1}{2}(\kappa+2) \bar{V}=\frac{\bar{G}}{2}\left[\bar{\xi} \frac{d \bar{Z}}{d \bar{\xi}}-(\kappa-2) \bar{Z}\right] . \tag{5.6}
\end{gather*}
$$

The Larraza invariant (3.5) reduces to

$$
\begin{equation*}
\bar{G}=\bar{C} \bar{\xi}^{-(\kappa+2) / 2} \bar{Z}^{-\frac{1}{2}} \tag{5.7}
\end{equation*}
$$

where $\bar{C}$ is a constant. After some algebraic reductions, it is found that

$$
\begin{align*}
&(1-\bar{Z})\left(\bar{\xi} \frac{d}{d \bar{\xi}}\right)^{2} \bar{Z}-\frac{1}{2 \bar{Z}}(3-\bar{Z})\left(\bar{\xi} \frac{d \bar{Z}}{d \bar{\xi}}\right)^{2} \\
&-\left[\frac{1}{2}(\kappa+2)-(\kappa-3) \bar{Z}\right] \bar{\xi} \frac{d \bar{Z}}{d \bar{\xi}}+\frac{1}{2}(\kappa-2)(4-\kappa) \bar{Z}^{2}=0 \tag{5.8}
\end{align*}
$$

The requirement that $d \bar{Z} / d \bar{\xi}$ be bounded at the critical point, $\bar{\xi}=\bar{\xi}_{c}(\bar{b})$, where $\vec{Z}=1$, determines a unique solution to (5.8). Although (5.8) can, like (3.18), be reduced to the problem of solving an Abel equation of the second kind, it does not appear that that solution can be written in a closed analytic form. It is, however, readily seen that the general solution of the limiting form of (5.8) for small $\bar{\xi}$ is $\bar{Z} \sim A_{0} \bar{\xi}^{\kappa-4}\left(1+\bar{\xi} / \bar{\xi}_{0}\right)$, where $A_{0}$ and $\bar{\xi}_{0}$ are arbitrary constants. It follows that

$$
\begin{equation*}
\bar{Z} \sim A_{0} \bar{\xi}^{\kappa-4}, \quad \bar{\xi} \rightarrow 0 . \tag{5.9}
\end{equation*}
$$

(It also follows from (5.8) that $\bar{Z} \propto \bar{\xi}^{-\kappa-2}$ for $\bar{\xi} \rightarrow \infty$, which agrees with (3.22).)

In the inner solution $\xi$ and $V$ are unscaled, but $G$ and $Z$ are scaled as

$$
\begin{align*}
& G=\frac{1}{6}[1-(\gamma-1) \mathcal{G}],  \tag{5.10}\\
& Z=\frac{Z}{\gamma-1}, \tag{5.11}
\end{align*}
$$

where, according to (3.6),

$$
\begin{equation*}
G(1)=\frac{1-b}{2 b}, \quad \mathcal{Z}(1)=25^{2}, \quad V(1)=1-b \tag{5.12}
\end{equation*}
$$

We assume that $\hat{G}, \hat{Z}$ and $V$ are $O(1)$ in the limit $\gamma \rightarrow 1$, as are all derivatives with respect to $\xi$. To leading order, (3.2) and (3.4) become

$$
\begin{gather*}
\xi \frac{d V}{d \xi}+3 V=0  \tag{5.13}\\
(1-V) \xi \frac{d V}{d \xi}+\left(\frac{1}{\alpha}-V\right) V=\frac{\hat{G}}{2}\left[\xi \frac{d \hat{Z}}{d \xi}+2 \mathcal{Z}-\frac{\kappa \mathcal{Z}}{1-V}\right] . \tag{5.14}
\end{gather*}
$$

The Larraza invariant (3.5) reduces to

$$
\begin{equation*}
\sigma=C \xi^{-1 / \alpha} \mathcal{Z}^{-\frac{1}{2}}(1-V)^{-\kappa / 6}, \tag{5.15}
\end{equation*}
$$

where $\hat{C}$ is a constant that, according to (5.12), is given by

$$
\begin{equation*}
C^{2}=\frac{1}{2} b^{\kappa / 3}(1-b)^{2} . \tag{5.16}
\end{equation*}
$$

It follows from (5.12) ${ }_{3}$ and (5.13) that

$$
\begin{equation*}
V=(1-\zeta) \xi^{-3} . \tag{5.17}
\end{equation*}
$$

Now use (5.15) and (5.17) to reduce (5.14) to an equation for $\mathcal{Z}$ alone. This equation may be integrated to give

$$
\begin{equation*}
Z=2 \xi^{\kappa-2}\left(\frac{1-V}{5}\right)^{\kappa / 3}\left[K+\frac{3 \alpha-1}{\alpha \xi}-\frac{V}{2 \xi}\right]^{2}, \tag{5.18}
\end{equation*}
$$

where the constant $K$ is determined from (5.12) $)_{2}$ as

$$
\begin{equation*}
K=\frac{1}{\alpha}-\frac{5-b}{2} . \tag{5.19}
\end{equation*}
$$

It is now necessary to match the inner solution (for $\xi \rightarrow \infty$ ) to the outer solution (for $\bar{\xi} \rightarrow 0$ ). We readily see from (5.9) and (5.18) that $K=0$, that is,

$$
\begin{equation*}
\alpha=\frac{2}{5-\bar{b}} . \tag{5.20}
\end{equation*}
$$

Values of $\alpha$ obtained by numerical integration of the full equations for values of $\gamma$ close to 1 are shown in Table 3. The final line displays values given by (5.20). The agreement seems satisfactory.

To complete the matching of the solutions we note that, according to (5.18) and (5.20),

$$
\begin{equation*}
z \sim \frac{(1+5)^{2}}{2 b^{\kappa \beta}} \xi^{-1-5}, \quad \bar{G} \sim\left(\frac{1-\bar{b}}{1+\bar{b}}\right) 5^{\kappa / 3} \bar{\xi}^{-2+5}, \quad \xi \rightarrow \infty . \tag{5.21}
\end{equation*}
$$

Comparing (5.21) with (5.9), we see that $\lambda$ and $\bar{C}$ are given by

$$
\begin{gather*}
\lambda=\left[\frac{(1+\bar{b})^{2}}{2 \bar{b}^{\kappa / 3} A_{0}}\right]^{1 /(1+\bar{b})},  \tag{5.22}\\
\bar{C}=C \lambda^{-1 / \alpha} . \tag{5.23}
\end{gather*}
$$

Table 3. Values of $\alpha$ for small $\gamma-1$

| $\gamma$ | $\bar{y}=0.2$ | $\bar{y}=0.6$ | $\boldsymbol{b}=0.999$ |
| :--- | :---: | :---: | :---: |
| 1.667 | 0.595 | 0.563 | 0.571 |
| 1.4 | 0.530 | 0.544 | 0.565 |
| 1.2 | 0.471 | 0.520 | 0.555 |
| 1.1 | 0.447 | 0.501 | 0.545 |
| 1.05 | 0.434 | 0.486 | 0.535 |
| 1.01 | 0.421 | 0.467 | 0.517 |
| 1.005 | 0.419 | 0.463 | 0.512 |
| 1.0001 | 0.418 | 0.456 | 0.502 |
| 1 | 0.417 | 0.455 | 0.500 |

In integrating (5.8) backwards from the critical point $\bar{\xi}_{c}$, we may place that point at any convenient location, but now (5.1) and (5.22) determine $\xi_{c}$ uniquely. We obtained the values shown in Table 4. We also verified, by direct integration of (3.2)-(3.6) that $\xi_{c}$ scales with $\gamma$ as indicated in (5.1). For $\gamma=1.0005$ we obtained the values $\lambda \bar{\xi}_{c}=0.632$ for $\bar{b}=0.2, \lambda \bar{\xi}_{c}=0.551$ for $\bar{B}=0.4, \lambda \bar{\xi}_{c}=0.603$ for $\bar{B}=0.6$, and $\lambda \bar{\xi}_{c}=0.788$ for $\bar{B}=0.999$. The agreement with the values shown in Table 4 seems reasonable when it is recalled (see (5.1)) that $\xi_{c}$ is large for small $\gamma$ and that some precision is necessarily lost when the general equations are integrated to very large $\xi$.
We now turn to the stability problem. We shall henceforth add a suffix 0 to the unperturbed variables, to distinguish them from the corresponding perturbed quantities, which carry the suffix 1 . Consider first the outer region. The scaling is analogous to (5.1):

$$
\begin{gather*}
V_{1}=(\gamma-1)^{3(1+5)} \bar{V}_{1}(\bar{\xi}),  \tag{5.24}\\
W_{1}=(\gamma-1)^{3(1+5)} \bar{W}_{1}(\bar{\xi}),  \tag{5.25}\\
G_{1}=-(\gamma-1)^{3 /(1+5)} \bar{G}_{1}(\bar{\xi}),  \tag{5.26}\\
Z_{1}=\bar{Z}_{1}(\bar{\xi}) . \tag{5.27}
\end{gather*}
$$

Table 4. Values of $\lambda \bar{\xi}_{c}$ for various $\bar{b}$ in the limit $\gamma \rightarrow 1$

| $\sigma$ | $\lambda_{c}$ | $\sigma$ | $\lambda \bar{\xi}_{c}$ | $\sigma$ | $\lambda \bar{\xi}_{c}$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| 0.0001 | 799.83 | 0.2 | 0.64578 | 0.5 | 0.58935 |
| 0.001 | 79.718 | 0.25 | 0.59911 | 0.6 | 0.62202 |
| 0.01 | 7.9279 | 0.3 | 0.57707 | 0.8 | 0.71018 |
| 0.05 | 1.69044 | 0.4 | 0.57036 | 0.9 | 0.76158 |
| 0.1 | 0.96218 | 0.45 | 0.57764 | 0.999 | 0.81593 |

After some reductions, we then find that to leading order

$$
\begin{gather*}
\bar{\xi} \frac{d \bar{V}_{1}}{d \bar{\xi}}+3 \bar{V}_{1}-n(n+1) \bar{W}_{1}=-\bar{\xi} \frac{d \bar{G}_{1}}{d \bar{\xi}_{\xi}}+\beta \bar{G}_{1}  \tag{5.28}\\
{\left[\bar{\xi}\left(\frac{d}{d \bar{\xi}}-\frac{\bar{Z}_{0}^{\prime}}{\bar{Z}_{0}}\right)-\beta\right]\left[\bar{Z}_{1}-\frac{2 \bar{X}_{1}}{\bar{G}_{0}}\right]=0}  \tag{5.29}\\
\bar{\xi} \frac{d \bar{V}_{1}}{d \bar{\xi}}+\left(\frac{1}{\alpha}-\beta\right) \bar{V}_{1}=\bar{\xi} \frac{d \bar{X}_{1}}{d \bar{\xi}}+2 \bar{X}_{1}  \tag{5.30}\\
\bar{\xi} \frac{d \bar{W}_{1}}{d \bar{\xi}}+\left(\frac{1}{\alpha}-\beta\right) \bar{W}_{1}=\bar{X}_{1} \tag{5.31}
\end{gather*}
$$

where

$$
\begin{equation*}
\bar{X}_{1}=\bar{G}_{0} \bar{Z}_{1}+\bar{Z}_{0} \bar{G}_{1} \tag{5.32}
\end{equation*}
$$

It follows from (5.29) and (5.32) that, for some constant $\bar{B}$,

$$
\begin{align*}
& \bar{Z}_{1}=\frac{2 \bar{X}_{1}}{\bar{G}_{0}}+\bar{B} \bar{\xi}^{\beta} \bar{Z}_{0}  \tag{5.33}\\
& \bar{G}_{1}=-\frac{\bar{X}_{1}}{\bar{Z}_{0}}-\bar{B} \bar{\xi}^{\beta} \bar{G}_{0} \tag{5.34}
\end{align*}
$$

Substituting (5.34) into (5.28) we obtain
$\bar{Z}_{0}\left[\bar{\xi} \frac{d \bar{V}_{1}}{d \bar{\xi}}+3 \bar{V}_{1}-n(n+1) \bar{W}_{1}\right]=\left[\bar{\xi}\left(\frac{d}{d \bar{\xi}}-\frac{\bar{Z}_{0}^{\prime}}{\bar{Z}_{0}}\right)-\beta\right] \bar{X}_{1}+\bar{B} \bar{\xi}^{\beta+1} \bar{G}_{0}^{\prime} \bar{Z}_{0}$.
Comparison of (5.30) with (5.35) shows that these equations have a critical point also at $\bar{\xi}=\bar{\xi}_{c}$, where $\bar{Z}_{0}=1$. As for the case of the unperturbed shock, it appears to be impossible to obtain a solution in closed analytic form, but the asymptotic behaviour of the solution for $\bar{\xi} \rightarrow 0$ can be determined.

Consider first the case when $n=0$. It follows from (5.28) and (5.30) that the leading terms in the asymptotic expansion of $\bar{V}_{1}$ and $\bar{X}_{1}$ are

$$
\begin{gather*}
\bar{V}_{1} \sim P \bar{\xi}^{-3}+Q \bar{\xi}^{-2+\bar{b}}, \quad \bar{\xi} \rightarrow 0  \tag{5.36}\\
\bar{X}_{1} \sim \frac{1}{2}(1+5+2 \beta) P \bar{\xi}^{-3}+\frac{1}{2 \bar{b}}(1+\bar{b}-2 \beta) Q \bar{\xi}^{-2+\bar{b}}+\bar{k}_{1} \bar{\xi}^{-2}, \quad \bar{\xi} \rightarrow 0 \tag{5.37}
\end{gather*}
$$

where $P, Q$ and $\bar{k}_{1}$ are constants. The final term in (5.37), which arises from integration of (5.28), is too small to be retained in what follows. By
integrating (5.35) using (5.36), we obtain an alternative expression to (5.37) but one that must be equivalent to it. For some constant $\vec{A}_{1}$, we have

$$
\begin{array}{ll}
\bar{X}_{1} \sim-\bar{B} \bar{\xi}^{\beta} \bar{G}_{0} \bar{Z}_{0}-\frac{(1+\bar{b}) Q}{2-5+\beta} \bar{\xi}^{-2+5} \bar{Z}_{0}+\bar{A}_{1} \bar{\xi}^{\beta} \bar{Z}_{0}, & \bar{\xi} \rightarrow 0 \\
\bar{Z}_{1} \sim-\bar{B} \bar{\xi}^{\beta} \bar{Z}_{0}-\frac{2(1+\bar{b}) Q}{2-\bar{b}+\beta} \bar{\xi}^{-2+5} \frac{\bar{Z}_{0}}{\bar{G}_{0}}+2 \bar{A}_{1} \bar{\xi}^{\beta} \bar{Z}_{0} & \overline{\bar{G}_{0}} \tag{5.39}
\end{array}
$$

The three terms of (5.39) are $O\left(\bar{\xi}^{-1-\bar{b}+\beta}\right), O\left(\bar{\xi}^{-1-\bar{b}}\right)$ and $O\left(\bar{\xi}^{1-2 \bar{b}+\beta}\right)$ respectively. We return to (5.39) after considering the interior solution later.

Consider next the case when $n \neq 0$. It follows from (5.30) and (5.31) that, for some constant $R$,

$$
\begin{equation*}
\bar{V}_{1}=\left(\bar{\xi} \frac{d}{d \bar{\xi}}+2\right) \bar{W}_{1}+R \bar{\xi}^{-\frac{1}{2}(\varsigma-5-2 \beta)} \tag{5.40}
\end{equation*}
$$

We now have, for some constants $Q$ and $S$,

$$
\begin{gather*}
\bar{W}_{1} \sim Q \bar{\xi}^{n-2}+\frac{1}{2}(1+5+2 \beta) S \bar{\xi}^{-k(5-\bar{b}-2 \beta)}, \quad \bar{\xi} \rightarrow 0,  \tag{5.41}\\
\bar{V}_{1} \sim n Q \bar{\xi}^{n-2}+\left[R-\frac{1}{4}(1+5+2 \beta)^{2} S\right] \bar{\xi}^{-\frac{1}{2}(5+\bar{b}-2 \beta)}, \quad \bar{\xi} \rightarrow 0,  \tag{5.42}\\
\bar{X}_{1} \sim\left[n-\frac{1}{2}(1+\bar{b}+2 \beta)\right] Q \bar{\xi}^{n-2}, \quad \bar{\xi} \rightarrow 0, \tag{5.43}
\end{gather*}
$$

where (5.42) and (5.43) follow from (5.41) by (5.31) and (5.40). By (5.41) and (5.42) we have

$$
\begin{align*}
& \bar{\xi} \frac{d \bar{V}_{1}}{d \bar{\xi}}+3 \bar{V}_{1}-n(n+1) \bar{W}_{1} \\
& \quad \sim \frac{1}{2}(1+\bar{b}+2 \beta)\left[R-\frac{1}{4}(1+5+2 \beta)^{2} S-n(n+1) S\right] \bar{\xi}^{-\frac{2}{2(S-6-2 \beta)}}, \quad \bar{\xi} \rightarrow 0 . \tag{5.44}
\end{align*}
$$

When this is substituted into (5.28) it is apparent that it will dominate that equation unless

$$
\begin{equation*}
R=\left[4(1+b+2 \beta)^{2}+n(n+1)\right] S \tag{5.45}
\end{equation*}
$$

a relation that is henceforth subsumed. If $\operatorname{Re}(\beta)>\frac{1}{2}(2 n+1-\bar{b})$, the dominant terms in (5.41) and (5.42) are $O\left(\bar{\xi}^{n-2}\right.$ ) but we shall later find that these cannot be matched to the inner solution in this range of $\beta$, so that in this case $Q=0$ and the dominant terms of $\bar{V}_{1}$ and $\bar{W}_{1}$ are $O\left(\bar{\xi}^{\left.-\frac{f(s-\bar{b}-2 \beta)}{}\right) \text {. If }}\right.$ $\operatorname{Re}(\beta)<\frac{1}{2}(2 n+1-\bar{b})$ the $O\left(\bar{\xi}^{-\frac{1}{2}(5-\bar{\sigma}-2 \beta)}\right)$ terms dominate the $O\left(\bar{\xi}^{n-2}\right)$ terms. Thus, in all circumstances we have

$$
\begin{equation*}
\bar{V} \sim n(n+1) S \bar{\xi}^{-\frac{1}{2}(5-\delta-2 \beta)}, \quad \bar{W} \sim \frac{1}{2}(1+\bar{b}+2 \beta) S \bar{\xi}^{-\frac{1}{2}(5-\delta-2 \beta)}, \quad \bar{\xi} \rightarrow 0 \tag{5.46}
\end{equation*}
$$

Consider now the inner region. The scaling is similar to (5.10) and (5.11),

$$
\begin{align*}
& G_{1}=-(\gamma-1) \hat{G}_{1}  \tag{5.47}\\
& Z_{1}=Z_{1} /(\gamma-1) \tag{5.48}
\end{align*}
$$

After some reductions we find that, to leading order,

$$
\begin{gather*}
\xi \frac{d V_{1}}{d \xi}+3 V_{1}-n(n+1) W_{1}=0  \tag{5.49}\\
{\left[\left(1-V_{0}\right) \xi\left(\frac{d}{d \xi}-\frac{\mathcal{Z}_{0}^{\prime}}{\mathcal{Z}_{0}}\right)-\beta\right]\left[\mathcal{Z}_{1}-\frac{2 \hat{X}_{1}}{\hat{G}_{0}}\right]=\frac{\kappa Z_{0}}{\left(1-V_{0}\right)} V_{1}}  \tag{5.50}\\
\left(1-V_{0}\right) \xi \frac{d V_{1}}{d \xi}+\left(\frac{1}{\alpha}-\beta+V_{0}\right) V_{1}=\xi \frac{d \hat{X}_{1}}{d \xi}+2 \hat{X}_{1}  \tag{5.51}\\
\left(1-V_{0}\right) \xi \frac{d W_{1}}{d \xi}+\left(\frac{1}{\alpha}-\beta-2 V_{0}\right) W_{1}=\hat{X}_{1} \tag{5.52}
\end{gather*}
$$

where

$$
\begin{equation*}
\hat{X}_{1}=\hat{G}_{0} \mathcal{Z}_{1}+\hat{Z}_{0} \hat{G}_{1} \tag{5.53}
\end{equation*}
$$

Initial conditions equivalent to (4.16) to (4.18) are

$$
\begin{equation*}
V_{1}(1)=-3-\beta, \quad \mathcal{Z}_{1}(1)=\frac{9-7 \tilde{b}}{4 \tilde{b}^{2}}, \quad \hat{X}_{1}(1)=\frac{1}{2}(3-9 \tilde{b}-4 \tilde{b} \beta), \quad W_{1}(1)=1 \tag{5.54}
\end{equation*}
$$

Consider first the case when $n=0 . \mathrm{By}(5.49)$ and (5.54) ${ }_{1}$ we have

$$
\begin{equation*}
V_{1}=-(3+\beta) \xi^{-3} \tag{5.55}
\end{equation*}
$$

Substituting this into (5.50) and (5.51) and integrating, taking note of (5.54), we obtain

$$
\begin{gather*}
\widehat{X}_{1}=\left[\frac{1}{2}(1+5+2 \beta)-V_{0}\right] V_{1}+\frac{1}{2} \beta(5-\bar{b}+2 \beta) \xi^{-2}  \tag{5.56}\\
\mathcal{Z}_{1}=\frac{2 \hat{X}_{1}}{G_{0}}+\left(\frac{3-\bar{b}}{1-\bar{b}}\right) \frac{V_{0} \mathcal{Z}_{0}}{1-V_{0}}+\frac{5-5+2 \beta}{b^{\beta / 3}(1-\overline{5})^{1-\beta / 3}}\left(\frac{1-V_{0}}{V_{0}}\right)^{\beta / 3} Z_{0} \tag{5.57}
\end{gather*}
$$

The final term in (5.56) dominates as $\xi \rightarrow \infty$ unless its coefficient is zero, that is, unless $\beta=0$ or $\beta=-1 / \alpha$. These correspond, in fact, to the two special solutions (4.20A) and (4.20B). For these $\bar{B}=(5-\bar{b}) /(1-\bar{b})$ and 0 , respectively, as may be verified by matching (5.57) to (5.38). Consider now other values of $\beta$. Then (5.57) shows that $\mathcal{Z} \propto \xi^{-\bar{b}}$ if $\operatorname{Re}(\beta)<1$ and $\mathcal{Z} \propto \xi^{-1-\bar{b}+\beta}$ if $\operatorname{Re}(\beta)>1$, for $\xi \rightarrow \infty$. These cannot be matched to (5.39). We conclude that the two special solutions (4.20A) and (4.20B) are the only solutions
belonging to the discrete spectrum of the linear stability problem when $n=0$.

Consider next the case when $n \neq 0$. Eliminating $W_{1}$ and $X_{1}$ between (5.49), (5.51) and (5.52), we obtain

$$
\begin{align*}
& \left(1-V_{0}\right)\left(\xi \frac{d}{d \xi}\right)^{3} V_{1}+\frac{1}{2}\left(15-5-2 \beta-8 V_{0}\right)\left(\xi \frac{d}{d \xi}\right)^{2} V_{1} \\
& \quad+\frac{1}{2}\left(37-5 \bar{b}-10 \beta-2 V_{0}\right) \xi \frac{d V_{1}}{d \xi}+3\left(5-\bar{b}-2 \beta+2 V_{0}\right) V_{1} \\
& \quad=n(n+1)\left[\left(1-V_{0}\right) \xi \frac{d V_{1}}{d \xi}+\frac{1}{2}\left(5-\bar{b}-2 \beta+2 V_{0}\right) V_{1}\right] \tag{5.58}
\end{align*}
$$

On solving this subject to the initial conditions implied by (5.54), we obtain

$$
\begin{align*}
V_{1}= & \frac{n(n+1)}{2 n+1}\left\{\left[1-\left(\frac{3+\beta}{n+1}\right)\right] \xi^{n-2}-\left[1+\left(\frac{3+\beta}{n}\right)\right] \xi^{-n-3}\right. \\
& \left.+\frac{(1+\beta)(1-5)}{\bar{b}} \int_{1}^{\xi}\left[\frac{1}{\bar{b}}-\frac{(1-5)}{5 u^{3}}\right]^{\mu-1}\left[\xi^{n-2} u^{-n-3}-\xi^{-n-3} u^{n-2}\right] u^{3 \mu+2} d u\right\}, \tag{5.59}
\end{align*}
$$

where $\mu=\frac{1}{8}(2 \beta+5-1)$.
The case of greatest interest is

$$
\begin{equation*}
-\frac{1}{2}(2 n+1-\bar{b})<\operatorname{Re}(\beta)<\frac{1}{2}(2 n+1-\overline{5}), \tag{5.60}
\end{equation*}
$$

when it follows from (5.49) and (5.59) that

$$
\begin{align*}
V_{1} & \sim \frac{n \mathscr{C}}{2 n+1} \xi^{n-2}-\frac{4 n(n+1)(1+\beta)(1-\bar{b})}{(2 n+1)^{2}-(\overline{5}+2 \beta)^{2}}\left(\frac{\xi}{\bar{b}}\right)^{-\frac{1}{2}(5-\bar{b}-2 \beta)}, \quad \xi \rightarrow \infty, \quad(5.61  \tag{5.61}\\
W_{1} & \sim \frac{\mathscr{E}}{2 n+1} \xi^{n-2}-\frac{2(1+\beta)(1-\bar{b})(1+\bar{b}+2 \beta)}{(2 n+1)^{2}-(5+2 \beta)^{2}}\left(\frac{\xi}{\bar{b}}\right)^{-\frac{\xi(5-\bar{L}-2 \beta)}{},} \quad \xi \rightarrow \infty, \tag{5.62}
\end{align*}
$$

where

$$
\begin{equation*}
\mathscr{C}=n-2-\beta+\frac{1}{3}(n+1)(1+\beta) 5^{-\mu}(1-5)^{1+\mu-\frac{1}{2} n} B_{1-\delta}\left(\frac{1}{3} n-\mu, \mu\right), \tag{5.63}
\end{equation*}
$$

and $B_{x}(a, b)$ is the incomplete beta function, which is defined for $\operatorname{Re}(a)>0$ by

$$
\begin{equation*}
B_{x}(a, b)=\int_{0}^{x} t^{a-1}(1-t)^{b-1} d t, \quad 0 \leqslant x<1 \tag{5.64}
\end{equation*}
$$

and which can be analytically continued to all complex $a$ and $b$, apart from non-positive integral $a$. To match (5.61) and (5.62) to (5.46), we require that

$$
\begin{equation*}
\mathscr{C}=0 \tag{5.65}
\end{equation*}
$$

We note that this is satisfied by the special solution (4.20C) for which $n=1$ and $\beta=-1$.

The result of solving (5.65) for $n=1$ is shown in Table 5. Entries marked with an asterisk were confirmed, to three-decimal-point accuracy, by direct integration of (5.49) to (5.54), together with the condition (see (5.61)) that $V_{1}=O\left(\xi^{-1(s-\bar{b}-2 \beta)}\right)$ for $\xi \rightarrow \infty$. A few values of $\beta$, obtained by direct integration of (4.12) to (4.18) for $\gamma=1.005$, agreed with those shown, to an accuracy of two decimal places. These cases are marked with a dagger. Higher modes (that is, modes with smaller growth rates) were located, and are shown in the lower portion of Table 5. As $\bar{b}$ increases between 0.35 and 0.355 , the complex roots coalesce and become real. It is easily demonstrated from (5.65) that, for the most unstable mode, $\beta \rightarrow-1.5$ as $\bar{b} \rightarrow 0$ and that $\beta \rightarrow 1$ as $b \rightarrow 1$. The latter limit is a uniform one, and the results derived for $\bar{b} \rightarrow 1$ are in fact valid for $\bar{B}=1$; see Appendix A. The marginal case, separating unstable and stable solutions, occurs at $\bar{b}=0.46537$ (approximately).
Values of $\beta$ for $n>1$ obtained by solving (5.65) are shown in Table 6. The first column for $\bar{b}=0.001$ shows the most unstable mode; the second shows the next most unstable mode. The values displayed for other $\bar{b}$ belong to the most unstable mode. It is evident that, as $\bar{\zeta}$ increases from zero for any given $n, \operatorname{Re}(\beta)$ starts negative (instability) and finishes positive (linear stability). The marginal $\bar{b}$, separating stable and unstable solutions, decreases with increasing $n$, that is, the $n=2$ mode is the last to remain unstable as $\bar{b}$ is increased, but that too becomes stable once $\bar{b}$ exceeds 0.446652 (approximately). As 5 increases for any fixed $n$, the pair of

Table 5. Values of $\beta$ for $\gamma \rightarrow 1$ for various $\bar{b}$ and $n=1$

| b | $\beta$ | b | $\beta$ | 5 | $\beta$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0.005 | $-1.4398$ | 0.105* | -0.9879 | 0.6* $\dagger$ | 0.2913 |
| 0.01 | -1.4030 | 0.11* | -0.9703 | $0 \cdot 8^{*}$ | 0.6732 |
| 0.03 | -1.2909 | 0.12* | -0.9359 | 0.9* | 0.8433 |
| 0.05* | -1.1994 | 0.15* | -0.8370 | 0.95 | 0.9233 |
| 0.07* | -1.1176 | 0.2* $\dagger$ | -0.6831 | 0.99* | 0.9849 |
| 0.08* | -1.0790 | 0.25 | -0.5397 | 0.995 | 0.9925 |
| 0.1* | -1.0056 | 0.4 ${ }^{\text {h }} \dagger$ | -0.1530 | 0.999 | 0.9985 |
| 5 | $\beta$ |  | b | $\beta$ | $\beta$ |
| 0.001 | $1 \cdot 4168 \pm 2 \cdot 2508 i$ |  | 0.355 | 5.5697 | 5.8888 |
| 0.01 | $2 \cdot 3231 \pm 2.7531 i$ |  | 0.4 | 4.9816 | $6 \cdot 5652$ |
| 0.05 | $3 \cdot 5999 \pm 2 \cdot 8599 i$ |  | 0.5 | 4.5731 | 6.9988 |
| $0 \cdot 1$ | $4 \cdot 3825 \pm 2 \cdot 6030 i$ |  | 0.6 | $4 \cdot 3656$ | $7 \cdot 1149$ |
| 0.25 | $5.4652 \pm 1.4496 i$ |  | 0.7 | 4.2297 | $7 \cdot 1245$ |
| $0 \cdot 3$ | $5.6245 \pm 0.9766 i$ |  | 0.8 | $4 \cdot 1315$ | 7.0944 |

Table 6. Values of $\beta$ for $\gamma \rightarrow 1$ for various $\bar{b}$ and $n(>1)$

| $n$ | $5=0.001$ | $\overline{5}=0.001$ | $\overline{5}=0.01$ | $\tilde{b}=0.05$ | $\overline{5}=0 \cdot 1$ | $5=0.25$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 2 | $-1.27 \pm 1 \cdot 22 i$ | $1 \cdot 14 \pm 2 \cdot 62 i$ | $-1.24 \pm 1.30 i$ | $-1.09 \pm 1.44 i$ | $-0.93 \pm 1.51 i$ | -0.49 ${ }^{\text {a }} 1.55 i$ |
| 3 | $-1.28 \pm 1.72 i$ | $0.98 \pm 2.86 i$ | $-1.28 \pm 1.84 i$ | $-1.11 \pm 2.07 i$ | $-0.89 \pm 2.22 i$ | $-0.26 \pm 2.36 i$ |
| 4 | $-1.28 \pm 2.09 i$ | $0.86 \pm 3.02 i$ | $-1.32 \pm 2 \cdot 24 i$ | $-1.13 \pm 2.57 i$ | $-0.86 \pm 2.79 i$ | $-0.06 \pm 3.05 i$ |
| 5 | $-1.27 \pm 2.41 i$ | $0 \cdot 76 \pm 3 \cdot 14 i$ | $-1.35 \pm 2.57 i$ | $-1.15 \pm 2.99 i$ | $-0.83 \pm 3.28 i$ | $0 \cdot 13 \pm 3.67 i$ |
| 6 | $-1 \cdot 26 \pm 2 \cdot 69 i$ | $0 \cdot 67 \pm 3.23 i$ | $-1.37 \pm 2 \cdot 86 i$ | $-1.16 \pm 3 \cdot 37 i$ | $-0.80 \pm 3.74 i$ | $0 \cdot 32 \pm 4 \cdot 25 i$ |
| 8 | $-1 \cdot 22 \pm 3 \cdot 19 i$ | $0 \cdot 52 \pm 3 \cdot 34 i$ | $-1.40 \pm 3 \cdot 37 i$ | $-1 \cdot 18 \pm 4.05 i$ | $-0.73 \pm 4.57 i$ | $0.69 \pm 5.33 i$ |
| 10 | $-1 \cdot 19 \pm 3 \cdot 66 i$ | $0.41 \pm 3.40 i$ | $-1.42 \pm 3.80 i$ | $-1.18 \pm 4 \cdot 66 i$ | $-0.65 \pm 5.32 i$ | $1.08 \pm 6.34 i$ |
| 15 | $-1 \cdot 21 \pm 4 \cdot 65 i$ | $0 \cdot 31 \pm 3.44 i$ | $-1.41 \pm 4.72 i$ | $-1.14 \pm 6.02 i$ | $-0.40 \pm 7.04 i$ | $2.07 \pm 8.69 i$ |
| 20 | $-1.27 \pm 5.43 i$ | $0 \cdot 31 \pm 3.52 i$ | $-1.36 \pm 5 \cdot 53 i$ | $-1.04 \pm 7 \cdot 24 i$ | $-0.08 \pm 8.63 i$ | $3 \cdot 13 \pm 10 \cdot 92 i$ |
| 30 | $-1 \cdot 30 \pm 6 \cdot 65 i$ | $0 \cdot 34 \pm 3 \cdot 70 i$ | $-1 \cdot 16 \pm 7.03 i$ | $-0.70 \pm 9.53 i$ | $0 \cdot 70 \pm 11 \cdot 65 i$ | $5 \cdot 39 \pm 15 \cdot 24 i$ |
| 40 | $-1 \cdot 26 \pm 7 \cdot 70 i$ | $0.37 \pm 3.87 i$ | $-1.02 \pm 8 \cdot 56 i$ | $-0.25 \pm 11.76 i$ | $1 \cdot 62 \pm 14.62 i$ | $7.79 \pm 19.50 i$ |
| 50 | $-1.21 \pm 8.68 i$ | $0 \cdot 40 \pm 4 \cdot 03 i$ | $-1.02 \pm 9.96 i$ | $0 \cdot 25 \pm 14 \cdot 00 i$ | $2 \cdot 61 \pm 17 \cdot 60 i$ | $10 \cdot 26 \pm 23.78 i$ |
| $n$ | $5=0.4$ | $5=0.6$ | $\overline{5}=0 \cdot 8$ | $5=0.9$ | $5=0.99$ | $5=0.999$ |
| 2 | $-0.11 \pm 1.44 i$ | $0.33 \pm 1.11 i$ | $0.71 \pm 0.38 i$ | 0.20 (1.52) | 0.02 (1.96) | 0.002 (1.996) |
| 3 | $0 \cdot 30 \pm 2.28 i$ | $0.97 \pm 1.92 i$ | $1.56 \pm 1.20 i$ | $1.80 \pm 0.45 i$ | 1.04 (2.92) | 1.004 (2.992) |
| 4 | $0.68 \pm 3.01 i$ | $1 \cdot 59 \pm 2 \cdot 62 i$ | $2 \cdot 39 \pm 1 \cdot 78 i$ | $2.73 \pm 1.03 i$ | 2.08 (3.87) | 2.008 (3.988) |
| 5 | $1.04 \pm 3.67 i$ | $2 \cdot 18 \pm 3 \cdot 26 i$ | $3 \cdot 21 \pm 2 \cdot 31 i$ | $3.65 \pm 1.47 i$ | $3 \cdot 13$ (4.81) | 3.012 (4.982) |
| 6 | $1 \cdot 40 \pm 4 \cdot 30 i$ | $2.77 \pm 3.88 i$ | $4.02 \pm 2.82 i$ | $4 \cdot 57 \pm 1.86 i$ | $4 \cdot 20$ (5.73) | 4.018 (5.976) |
| 8 | $2 \cdot 11 \pm 5 \cdot 49 i$ | $3.92 \pm 5.05 i$ | $5 \cdot 62 \pm 3.77 i$ | $6.39 \pm 2.60 i$ | 6.43 (7.48) | 6.032 (7.959) |
| 10 | $2.81 \pm 6.61 i$ | $5.06 \pm 6.16 i$ | $7 \cdot 21 \pm 4 \cdot 69 i$ | $8 \cdot 20 \pm 3 \cdot 30 i$ | $8.94 \pm 0.35 i$ | 8.051 (9.938) |
| 15 | $4 \cdot 59 \pm 9.26 i$ | $7.91 \pm 8.81 i$ | $11 \cdot 15 \pm 6 \cdot 88 i$ | $12 \cdot 69 \pm 4.97 i$ | $13.91 \pm 1.21 i$ | $13 \cdot 121$ (14.864) |
| 20 | $6 \cdot 41 \pm 11 \cdot 78 i$ | $10 \cdot 78 \pm 11 \cdot 36 i$ | $15.08 \pm 8.99 i$ | $17 \cdot 16 \pm 6 \cdot 57 i$ | $18 \cdot 87 \pm 1 \cdot 83 i$ | 18.229 (19.780) |
| 30 | $10 \cdot 19 \pm 16 \cdot 69 i$ | $16 \cdot 59 \pm 16.31 i$ | $22.95 \pm 13.09 i$ | $26 \cdot 10 \pm 9 \cdot 69 i$ | $28.78 \pm 2.95 i$ | 28.731 (29.236) |
| 40 | $14 \cdot 08 \pm 21 \cdot 54 i$ | $22 \cdot 48 \pm 21 \cdot 20 i$ | $30 \cdot 87 \pm 17 \cdot 12 i$ | $35 \cdot 04 \pm 12 \cdot 74 i$ | $38.69 \pm 4.01 i$ | $38.978 \pm 0.809 i$ |
| 50 | $18.03 \pm 26.39 i$ | $28.43 \pm 26.08 i$ | $38 \cdot 82 \pm 21 \cdot 13 i$ | $44 \cdot 00 \pm 15 \cdot 76 i$ | $48 \cdot 58 \pm 5 \cdot 05 i$ | $48.971 \pm 1.256 i$ |

complex-conjugate $\beta$ ultimately become real, and take the values $\beta=n-2$ and $\beta=n$ as $\bar{b} \rightarrow 1$, a fact that is easily verified from (5.65). In agreement with the CCW approximation, $\operatorname{Re}(\beta)$ is almost constant for small $\bar{b}$; in fact, the real part of the displayed $\beta$ are, provided $n$ is not too large, in approximate agreement with the CCW values; the imaginary parts of $\beta$ are, however, quite different, which may not be surprising since our solution is for the UIP branch while the CCW approximation is an LIP solution.

We used the full equations, (4.12) to (4.18), to check answers displayed in Table 6. We obtained the following results: $\beta=-1 \cdot 13 \pm 2 \cdot 56 i$ for $\bar{b}=0.05$ and $n=4 ; \beta=-1.01 \pm 7.26 i$ for $b=0.05$ and $n=20 ; \beta=0.35 \pm 14.12 i$ for $\bar{b}=0.05$ and $n=50 ; \beta=0.33 \pm 1 \cdot 115 i$ for $b=0.6$ and $n=2$; and $\beta=$ $3.38 \pm 5 \cdot 16 i$ for $\bar{b}=0.05$ and $n=4$. In all these cases, we took $\gamma=1.005$, and the final result (for which (5.65) gave $\beta=3 \cdot 4031 \pm 5 \cdot 1348 i$ ) confirms the existence of higher-order modes; this solution appears to correspond to the modes with the second most negative growth rates for that value of $\bar{B}$ and $n$. Higher-order modes, obtained from (5.65) for this case, were $\beta=6.6096 \pm 8.3818 i, \beta=9.7785 \pm 11.4969 i, \quad \beta=12.9455 \pm 14.5675 i$ and $\beta=16 \cdot 1149 \pm 17.6167$ i. It seems probable that there is an infinity of possible $\beta$ for each $n$ and $b$. It is easily shown from (5.63) that $\beta \sim[-5 \pm i \vee(24 n-25)] / 4$ for $b \rightarrow 0$.

It may be worth reiterating that in this section we have examined the limit $\gamma \rightarrow 1$ for any fixed non-zero $\bar{b}$, no matter how small, and have found only the UIP solution exists. This agrees with the numerical work of section 3, which indicated that, as $\gamma$ decreases, the LIP, Guderley-type branch exists in an ever smaller range of $\bar{b}$ contiguous to $\bar{\sigma}=0$. The LIP branch appears to exist for the ideal case ( $\bar{b}=0$ ) for all $\gamma$ including $\gamma \rightarrow 1$ (for which $\alpha \rightarrow 1$ and $\xi_{c}=1$ ). This also is consistent with the trends revealed by the numerical work of section 3 . The limiting $\bar{b} \rightarrow 0$ form of the $\gamma \rightarrow 1$ solution is a UIP solution and does not coincide with the limiting $\gamma \rightarrow 1$ form of the $\bar{b} \rightarrow 0$ solution, which is an LIP solution; that is, the order in which the limits are taken is significant; see also (3.17). Concerning the limit $\gamma \rightarrow 1$ for $\bar{b}=0$, it may be seen, by substituting

$$
\begin{equation*}
\alpha \sim 1-\varepsilon^{\frac{1}{\alpha}} \dot{\alpha}, \quad V \sim 1-\varepsilon^{\frac{1}{V}} \check{V}, \quad Z \sim \frac{1}{2} \varepsilon-\varepsilon^{\frac{1}{2} \check{Z}}, \quad \text { for } \varepsilon=\gamma-1 \rightarrow 0, \tag{5.66}
\end{equation*}
$$

into (3.18) that, to leading order, $\breve{\alpha}=\sqrt{ } 2, \breve{V}_{c}=1 / \sqrt{ } 2$ and

$$
\begin{equation*}
\frac{d \check{Z}}{d \check{V}} \sim \frac{(1+\sqrt{ } 2 \breve{V})(\sqrt{ } 2+\breve{V})}{3 \breve{V}(1+\sqrt{ } 2 \breve{V} / 3)} \tag{5.67}
\end{equation*}
$$

This value of $\breve{\alpha}$ agrees with the prediction of CCW theory: see (3.14) and (3.15). Integrating (5.67) and applying the boundary conditions (3.6), we find that

$$
\check{Z}-\check{V}+\frac{\sqrt{ } 2}{3} \ln \left(\frac{2 \breve{V} / \varepsilon^{\frac{1}{2}}}{1+\sqrt{ } 2 \check{V} / 3}\right)
$$

Table 7. The limit $\gamma \rightarrow 1$ for the Guderley solution ( $\bar{\sigma}=0$ )

| $\gamma$ | $\alpha$ | $\xi_{c}$ | $V_{c}$ | $G_{c}$ |  |
| :--- | :---: | :---: | :---: | ---: | :---: |
| 1.01 | $0.9019(0.8586)$ | $1.0264(1.0300)$ | $0.9385(0.9293)$ | $382 \cdot 35(355.56)$ |  |
| 1.001 | $0.9614(0.9553)$ | $1.0089(1.0095)$ | $0.9791(0.9776)$ | 3758.8 | $(3555.6)$ |
| 1.0001 | $0.9866(0.9859)$ | $1.0029(1.0030)$ | $0.9931(0.9929)$ | 36533 | $(35556)$ |

Returning to (3.2) and (3.3) or (3.4), we deduce, again using (3.6), that

$$
\begin{gather*}
\xi \sim 1+\varepsilon^{1}[\check{V}-\sqrt{ } 2 \ln (1+\sqrt{ } 2 \check{V} / 3)]  \tag{5.68}\\
G \sim 2 \varepsilon^{-1}(1+\sqrt{ } 2 \check{V} / 3)^{2} \tag{5.69}
\end{gather*}
$$

The predictions concerning the critical point,

$$
\begin{equation*}
\xi_{c} \sim 1+0.3003(\gamma-1)^{\frac{1}{2}}, \quad V_{c} \sim 1-0.7071(\gamma-1)^{\frac{1}{2}}, \quad G_{c} \sim 3.5556 /(\gamma-1) \tag{5.70}
\end{equation*}
$$

that follow from (5.68), (5.69) and from $\breve{V}_{c}=1 / \sqrt{ } 2$ are in good agreement with results obtained from our numerical integrations which are shown in Table 7; the results given by the present theory are shown in parentheses.

## 6. Conclusions

We have demonstrated in this paper that similarity solutions exist for spherically converging and diverging shocks in a van der Waals gas. These represent a generalization of the corresponding and well-known Guderley solutions for an ideal gas. For small van der Waals excluded volume, $b$, there is only one branch of solutions, and these differ little from the Guderley solution ( $b=0$ ). For larger $b$, a second branch of similarity solutions arises that is distinct from the Guderley-type branch but may coexist with it. It is characterized by large compressions, the density approaching the van der Waals limit, $1 / b$. These solutions cannot be obtained by the CCW approximation. For larger $b$, they are the only ones available; the Guderley-type solutions do not then exist.

We have investigated the linear stability of the similarity shocks with respect to non-spherically-symmetric perturbations. We have confirmed the conclusions of Gardner et al. (12) that the shock is unstable in an ideal gas for perturbations of all harmonic number $n$. We find, however, that the rate of growth of the instabilities diminishes with increasing $n$ and approaches zero for $n \rightarrow \infty$. We found that, as a general rule, the shocks are more stable in a van der Waals gas than in an ideal gas. In the limit $\gamma \rightarrow 1$ we found that
 initial uncompressed density. For $\gamma=\xi$, we found that all modes are stable
for $\bar{\sigma}>0.8$ (approximately). It seems plausible that this transitional value of $\bar{b}$ increases with $\gamma$, indicating that, if other factors are the same, shocks are more stable in a gas of small $\gamma$ than in a gas of large $\gamma$.
Our linear-stability analysis fails for spherically-symmetric perturbations ( $n=0$ ). We could show, in the case when $\gamma \rightarrow 1$, that there exist only two discrete modes, neither of which is physically significant. We also failed to find any other discrete modes when we investigated the case where $\gamma=\frac{7}{\xi}$ numerically. It seems probable that, for all $\gamma$ and $b$, the physically meaningful, spherically-symmetric modes belong to a continuum, which is inaccessible to the methods described here. The $n=0$ modes are, however, of special interest for those ranges of $b$ in which both the Guderley and the new branch coexist, and in which their relative stability comes into question. We therefore undertook a separate investigation in which spherical shocks were generated by a piston, and in which we sought evidence that one similarity solution was preferred to the other as the resulting shock imploded at the origin. We found no such evidence.
As noted in the introduction, a desire to understand better one aspect of the physics of sonoluminescence provided the motivation for our study. And we may now reflect on how far our findings can help in the search for configurations that provide greater luminosities or, in cases with deuterium and/or tritium present, can result in greater fusion rates (19). It seems possible (20) that a natural limit to these processes will be set by the stability of the imploding shock. And it is natural to seek ways of stabilizing the shocks. As pointed out by Wu and Roberts (2), the replacement of the air in the bubble by gaseous $\mathrm{D}_{2}$ or DT is potentially less promising than the use of compounds, such as ethane, but in which D and/or T atoms replace H atoms. The greater atomic weight of such compounds lowers the sound speed and leads to greater compressions during the implosion. This idea gains additional support from the present study, which shows that shocks in a material, such as ethane, are more stable than they would be in gaseous $\mathrm{D}_{2}$ or DT, which have a larger $\gamma$. The following point is also of interest. During the acoustic compression of the bubble, the trapped gas is first compressed adiabatically. The shock that forms later cuts off the central part of the bubble from its outside, that is, the central compression no longer increases. It now appears, as a result of the present study, that it is desirable to prolong the initial adiabatic compression as much as possible, in order that the density $\rho_{0}$ ahead of the shock, and hence $\bar{b}$, is as large as possible, so promoting the stability of the shock. Such a prolongation might be achieved by shaping the wave form of the acoustic driver. It may also be recalled that, in many situations of practical interest, the compression of the bubble is accompanied by two shocks in succession (1,2). The second of these, which travels across the gas compressed by the first shock, is usually responsible for the greater light emission and, conceivably in the thermonuclear application, by the larger fusion rate. It is interesting that the
greater value of $\bar{b}$ for the second shock implies, according to the present investigation, that it will also be the more stable.

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We owe to Dr Seth Putterman the suggestion that similarity solutions of Guderley type would exist for the van der Waals gas. The first author gratefully acknowledges support from NSF grant ATM-92-01662 and NASA grant NAGW-2848. Computing facilities were provided by the San Diego Supercomputer Center, which is sponsored by NSF.

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## APPENDIX A

The dense gas limit, $\bar{b} \rightarrow 1$
We consider the solutions to the spherical shock equations (3.2), (3.3), (3.5) and (3.6) in the limit $\tilde{b} \rightarrow 1$. We substitute

$$
\begin{equation*}
\bar{b}=1-\varepsilon, \quad V=\varepsilon \tilde{V}, \quad G=1+\varepsilon-\varepsilon \tilde{Z}, \tag{A1}
\end{equation*}
$$

into those equations and retain only the $O(1)$ parts of $\tilde{V}, \tilde{G}$ and $Z$, so obtaining

$$
\begin{gather*}
\xi \frac{d \bar{V}}{d \xi}+3 \bar{V}=-\xi \frac{d \bar{G}}{d \xi}  \tag{A2}\\
\xi \frac{d \bar{V}}{d \xi}+\frac{\bar{V}}{\alpha}=\frac{\tilde{\zeta}}{\gamma+1}\left[\xi \frac{d Z}{d \xi}+\frac{2}{\gamma}\left(\gamma+1-\frac{1}{\alpha}\right) Z\right],  \tag{A3}\\
\tilde{V}(1)=\frac{2}{\gamma+1}, \quad Z(1)=\frac{2 \gamma}{\gamma-1}, \quad \tilde{G}(1)=\frac{\gamma-1}{\gamma+1} . \tag{A4}
\end{gather*}
$$

The Larraza invariant (3.5) is now

$$
\begin{equation*}
\tilde{L}=\xi^{2 / a} Z \tilde{G}^{\gamma+1} . \tag{A5}
\end{equation*}
$$

We integrated this reduced set of equations and have obtained results indistinguishable from those displayed in the last column of Table 3 for $\bar{b}=0.999$.

## APPENDIX B

The limit $n \rightarrow \infty$ for the ideal gas
In this Appendix we derive the WKBJ solution for the limit $n \rightarrow \infty$ in the case ( $\bar{\sigma}=0$ ) of an ideal gas. We use $\boldsymbol{x}=\ln \xi$ in place of $\xi$ as independent variable, and write

$$
\begin{equation*}
V_{0}(\xi)=v_{0}(x) e^{S(x)}, \quad G_{0}(\xi)=g_{0}(x) e^{S(x)}, \quad Z_{0}(\xi)=z_{0}(x) e^{S(x)}, \tag{B1}
\end{equation*}
$$

with similar notation for the perturbation variables. On substituting into (4.12) to (4.15) and anticipating that $S^{\prime}$ and $\beta$ are $O(n)$, we obtain in leading order

$$
\begin{gather*}
{\left[\left(1-v_{0}\right) S^{\prime}-\beta\right] g_{1}-\left[S^{\prime} v_{1}-n(n+1) w_{1}\right]=0,}  \tag{B2}\\
{\left[\left(1-v_{0}\right) S^{\prime}-\beta\right] z_{1}-\left(\frac{\gamma-1-2 \tilde{b} g_{0}}{1-\tilde{b} g_{0}}\right) z_{0}\left[S^{\prime} v_{1}-n(n+1) w_{1}\right]=0,}  \tag{B3}\\
{\left[\left(1-v_{0}\right) S^{\prime}-\beta\right] v_{1}-\frac{1}{\gamma} S^{\prime}\left[\left(1-\tilde{b} g_{0}\right) z_{1}+\left(1-2 \tilde{b} g_{0}\right) z_{0} g_{1}\right]=0,}  \tag{B4}\\
{\left[\left(1-v_{0}\right) S^{\prime}-\beta\right] w_{1}-\frac{1}{\gamma}\left[\left(1-\tilde{b} g_{0}\right) z_{1}+\left(1-2 \tilde{b} g_{0}\right) z_{0} g_{1}\right]=0,} \tag{B5}
\end{gather*}
$$

where

$$
\begin{equation*}
v_{1}(0)=-\beta w_{1}(0), \quad z_{1}(0)=-2 \beta z_{0}(0) w_{1}(0) / v_{0}(0), \quad g_{1}(0)=0 . \tag{B6}
\end{equation*}
$$

If we introduce

$$
\begin{align*}
& u_{1}=\frac{1}{\gamma}\left[\left(1-\tilde{b} g_{0}\right) z_{1}+\left(1-2 \tilde{b} g_{0}\right) z_{0} g_{1}\right],  \tag{B7}\\
& p_{1}=v_{1}-S^{\prime} w_{1},  \tag{B8}\\
& y_{1}=\frac{1}{\gamma}\left[z_{1}-\left(\frac{\gamma-1-2 \tilde{b} g_{0}}{1-\tilde{b} g_{0}}\right) z_{0} g_{1}\right], \tag{B9}
\end{align*}
$$

we reduce these to

$$
\begin{align*}
& {\left[\left(1-v_{0}\right) S^{\prime}-\beta\right] u_{1}=z_{0}\left\{S^{\prime} p_{1}+\left[S^{\prime 2}-n(n+1)\right] w_{1}\right\}}  \tag{B10}\\
& {\left[\left(1-v_{0}\right) S^{\prime}-\beta\right] w_{1}=u_{1},}  \tag{B11}\\
& {\left[\left(1-v_{0}\right) S^{\prime}-\beta\right] p_{1}=0}  \tag{B12}\\
& {\left[\left(1-v_{0}\right) S^{\prime}-\beta\right] y_{1}=0 .} \tag{B13}
\end{align*}
$$

Inspecting (B10) to (B14), we see that the eigenvalue problem factorizes; we need only solve (B10) to (B12) to determine $\beta$ and find $y_{1}$ subsequently, if desired.

Eliminating between (B10) and (B11), we obtain two functions $S^{\prime}$, the roots of

$$
\begin{equation*}
\left[z_{0}-\left(1-v_{0}\right)^{2}\right] S^{\prime 2}+2 \beta\left(1-v_{0}\right) S^{\prime}-\left[\beta^{2}+n(n+1) z_{0}\right]=0 \tag{B14}
\end{equation*}
$$

but only one of these is acceptable, the other leading to unbounded solutions at the critical point $x=x_{c}$. The only permissible choice is

$$
\begin{equation*}
S_{1}^{\prime}=-\beta \frac{1-v_{0}-R}{z_{0}-\left(1-v_{0}\right)^{2}}, \tag{B15}
\end{equation*}
$$

where the sign of the square root,

$$
\begin{equation*}
R=\sqrt{ }\left\{\left(1-v_{0}\right)^{2}+\left[z_{0}-\left(1-v_{0}\right)^{2}\right]\left[\beta^{2}+n(n+1) z_{0}\right] / \beta^{2}\right\}, \tag{B16}
\end{equation*}
$$

has to be positive at $x=x_{c}$, where $z_{0}=\left(1-v_{0}\right)^{2}$. For this choice of $S^{\prime}, p_{1}=0$. For the other possible $S^{\prime}$, namely

$$
\begin{equation*}
S_{2}^{\prime}=\frac{\beta}{1-v_{0}} \tag{B17}
\end{equation*}
$$

$u_{1}=0$ and $S^{\prime} p_{1}+\left[S^{\prime 2}-n(n+1)\right] \omega_{1}=0$.
We may now write the leading-order solution of (4.12) to (4.15) in an obvious notation as

$$
\begin{gather*}
W_{1}=A_{1} e^{S_{1}}+A_{2} e^{S_{2}},  \tag{B18}\\
U_{1}=\left[\left(1-v_{0}\right) S_{1}^{\prime}-\beta\right] A_{1} e^{s_{1}},  \tag{B19}\\
P_{1}=-\left[S_{2}^{\prime 2}-n(n+1)\right] A_{2} e^{s_{2}} / S_{21}^{\prime} \tag{B20}
\end{gather*}
$$

where $A_{1}$ and $A_{2}$ are slowly-varying functions of $x$. The boundary conditions require that

$$
\begin{gather*}
P_{1}+\left(\frac{d}{d x}+\beta\right) W_{1}=0,  \tag{B21}\\
U_{1}+\left[\frac{2 \beta z_{0}}{\gamma v_{0}}\left(1-\beta g_{0}\right)\right] W_{1}=0 \tag{B22}
\end{gather*}
$$

at $x=0$. If $\bar{b}=0$, these conditions are satisfied by

$$
\begin{equation*}
\beta^{2}=-n(n+1)\left(\frac{\gamma-1}{\gamma+1}\right), \tag{B23}
\end{equation*}
$$

for then $S_{1}^{\prime}(0)=-\beta$ and (B21) is automatically satisfied; see also (3.6). This is not true when $\bar{b} \neq 0$ and it appears that $\beta^{2}$ is then complex, and that the turning point defined by the zero of $R$ (which falls at $x=0$ when $\bar{b}=0$ ) no longer lies on the real $x$-axis. The function $z_{0}$ is not currently available for complex $x$.


[^0]:    $\dagger$ We are grateful to Andres Larraza for informing us of the existence of this invariant for the case of an ideal gas.

[^1]:    $\dagger$ Equation (3.18) apparently presents a second opportunity, namely crossing at the intersection of curve 2 with the curve $2(1-\overline{-G}) / \alpha=(3 \gamma-1+4 \bar{b} G) V$, but this turns out to be illusory.

[^2]:    $\dagger$ It is only for constant $\lambda(M)$ that (3.11) will lead to a similarity solution. The value of $\alpha=0.5886 \approx 0.59$ cited here is derived from $\lambda(\infty)$, but $\lambda(M)$ does not depend sensitively on $M$.

