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Structure Connectivity and Substructure Connectivity of k-Ary n-Cube Networks

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ABSTRACT We present new results on the fault tolerability of k-ary n-cube (denoted Q_n^k) networks. Q_n^k is a topological model for interconnection networks that has been extensively studied since proposed, and this paper is concerned with the *structure/substructure connectivity* of Q_n^k networks, for *paths* and *cycles*, two basic yet important network structures. Let G be a connected graph and T a connected subgraph of G. The T-*structure connectivity* $\kappa(G; T)$ of G is the cardinality of a minimum set of subgraphs in G, such that each subgraph is isomorphic to T, and the set's removal disconnects G. The T-*substructure connectivity* $\kappa^s(G; T)$ of G is the cardinality of a minimum set of subgraphs in G, such that each subgraph is isomorphic to a connected subgraph of T, and the set's removal disconnects G. In this paper, we study $\kappa(Q_n^k; T)$ and $\kappa^s(Q_n^k; T)$ for $T = P_i$, a path on i nodes (resp. $T = C_i$, a cycle on i nodes). Lv *et al.* determined $\kappa(Q_n^k; T)$ and $\kappa^s(Q_n^k; T)$ for $T \in \{P_1, P_2, P_3\}$. Our results generalize the preceding results by determining $\kappa(Q_n^k; P_i)$ and $\kappa^s(Q_n^k; P_i)$. In addition, we have also established $\kappa(Q_n^k; C_i)$ and $\kappa^s(Q_n^k; C_i)$.

INDEX TERMS Interconnection networks, structure connectivity, substructure connectivity, k-ary n-cubes, paths, cycles.

I. INTRODUCTION

Interconnection networks play an important role in large-scale multiprocessor systems. Like most networks, an interconnection network can be represented by a graph $G = (V(G), E(G))$, where nodes in $V(G)$ correspond to processors, and edges in $E(G)$ correspond to communication links.

A. CONNECTIVITY OF INTERCONNECTION NETWORKS

The fault tolerance of interconnection networks has always been an important issue. One crucial parameter to evaluate the fault tolerability of a network is its *connectivity*. The connectivity of a graph G, denoted by $\kappa(G)$, is the minimum cardinality of a node set $F \subseteq V(G)$, such that F's deletion disconnects G. As variants of the classic node-connectivity, several kinds of conditional connectivity were proposed and studied [2], [3], [5], [6], [8], [9], [12]–[16], [18], [23], [25], [26]. Notably among them, Fàbrega and Fiol [4] introduced the g-extra connectivity. The g-extra connectivity $\kappa_g(G)$ of a connected graph G is the minimum cardinality of a set of nodes in G, if such a set exists, whose deletion disconnects G and leaves each remaining component with at least $g + 1$ nodes. Obviously, $\kappa_0(G) = \kappa(G)$, making $\kappa_g(G)$ a generalization of $\kappa(G)$.

Lin *et al.* [17] considered the fault status of a certain *structure*, rather than individual nodes, and proposed *structure connectivity* and *substructure connectivity*. Let G be a connected graph, and T a connected subgraph of G. The T-structure connectivity $\kappa(G; T)$ of G is the cardinality of a minimum set of subgraphs $F = \{T_1, T_2, \ldots, T_m\}$ in G, such that every $T_i \in F$ is isomorphic to T, and F's deletion disconnects G. The T-substructure connectivity $\kappa^s(G; T)$ of G is the cardinality of a minimum set of subgraphs $F = \{H_1, H_2, \ldots, H_m\}$, such that every $H_i \in F$ is isomorphic to a connected subgraph of T, and F's deletion disconnects G. By definition, $\kappa(G; T) \geq \kappa^s(G; T)$. The structure connectivity and substructure connectivity have been studied for some well-known networks [11], [17], [21], [22], [27].

B. APPLICATIONS OF STRUCTURE/SUBSTRUCTURE CONNECTIVITY AND OUR CONTRIBUTIONS

The traditional connectivity assumes that the status of a node is an event independent of the status of nodes around it. However in reality, nodes that are linked could affect each

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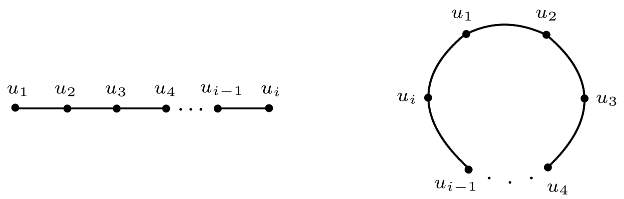

FIGURE 1. A path P_i and a cycle C_i.

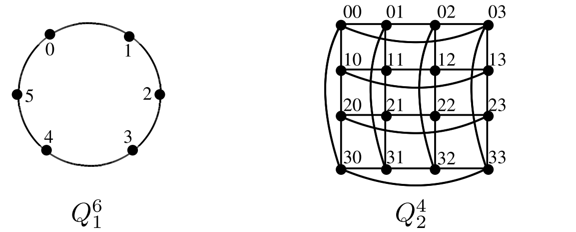

FIGURE 2. Q_1^6 and Q_2^4.

other, and the neighbours of a faulty node are more likely to fail than other nodes. Moreover, in the *NoC technology* (*Network-on-Chip*), part or whole of a network of interest are made on a chip, which means that the failure of any node on the chip can be considered the failure of the whole chip. All these motivated the research on fault tolerance of networks based on some certain structures rather than individual nodes. The study of structure fault tolerance is therefore of both scientific value and practical significance.

In this paper, we focus on two basic structures of all networks: paths and cycles. Let P_i be a path on i nodes, and C_i a cycle on i nodes, respectively (see FIGURE 1). Paths and cycles in a network are very important structures, both for basic network functionality and for implementing many algorithms executing on networks. When nodes in a path or cycle become faulty, the impacted path/cycle cannot function as a whole. So the whole path/cycle can be viewed as faulty. In many cases, it is easier to identify and locate a faulty structure than individual nodes in the structure. There are already many works in the literature studying path/cycle-structure fault tolerance for some well-known networks. For example, Lin *et al.* [17] investigated $\{P_2, P_3, C_4\}$-structure/substructure connectivity for hypercubes. Wang *et al.* [22] established $\{C_3, C_4\}$-structure/substructure connectivity for generalized hypercubes. The general $\{P_i, C_i\}$-structure/substructure connectivity have been studied for hypercubes, folded hypercubes and bubble-sort graphs [21], [27]. In this paper, we determine the path- and cycle-structure/substructure connectivity for k-ary n-cubes. The newfound results further our understanding of k-ary n-cubes, and furnish more parameters to consider when evaluating and selecting an interconnection network.

Lv *et al.* [11] studied $\kappa(Q_n^k; T)$ and $\kappa^s(Q_n^k; T)$ of the k-ary n-cube Q_n^k for $T \in \{P_1, P_2, P_3\}$. In this paper, we generalize the results by establishing $\kappa(Q_n^k; P_i)$ and $\kappa^s(Q_n^k; P_i)$. Also, we establish $\kappa(Q_n^k; C_i)$ and $\kappa^s(Q_n^k; C_i)$. The results in this paper are summarized as follows.

For Q_n^3, we have:

- $\kappa(Q_n^3; P_{3l+s}) = \kappa^s(Q_n^3; P_{3l+s}) = \lceil \frac{2n}{2l+s} \rceil$ for $2l + s \leq 2n$ and $s = 0, 1, 2$;
- $\kappa(Q_n^3; C_{3l}) = \lceil \frac{2n}{2l} \rceil$ for $4 \leq 2l \leq 2n$;
- $\kappa(Q_n^3; C_{3l+2}) = \lceil \frac{2n}{2l+1} \rceil$ for $2l + 1 < 2n$;
- $\kappa^s(Q_n^3; C_{3l+s}) = \lceil \frac{2n}{2l+s} \rceil$ for $2l + s \leq 2n$ and $s = 0, 1, 2$.

For Q_n^k with $k \geq 4$, We have:

- $\kappa(Q_n^k; P_{2l+1}) = \kappa^s(Q_n^k; P_{2l+1}) = \lceil \frac{2n}{l+1} \rceil$ for $1 \leq l + 1 \leq 2n$;
- $\kappa(Q_n^k; P_{2l}) = \kappa^s(Q_n^k; P_{2l}) = \lceil \frac{2n}{l} \rceil$ for $2 \leq l \leq 2n$;
- $\kappa(Q_n^k; C_{2l}) = \kappa^s(Q_n^k; C_{2l}) = \lceil \frac{2n}{l} \rceil$ for $4 \leq l \leq 2n$;
- $\kappa(Q_n^k; C_{2l+1}) \leq 2n - 2$ for $\frac{k-1}{2} \leq l \leq k - 2$; and $\kappa^s(Q_n^k; C_{2l+1}) = \lceil \frac{2n}{l+1} \rceil$ for $\frac{k+1}{2} \leq l + 1 \leq 2n$.

Of particular note is that a definitive structure connectivity for odd-cycles in Q_n^k still remains elusive. Our result of $\kappa(Q_n^k; C_{2l+1}) \leq 2n - 2$ provides an *upper-bound* on the structure connectivity for odd-cycles. This "half-solved" $\kappa(Q_n^k; C_{2l+1})$ and the unknown $\kappa(Q_n^3; C_{3l+1})$ are the two missing pieces for a complete solution to Q_n^k's structure/substructure connectivity for paths and cycles.

The rest of this paper is organized as follows. In Section 2, we introduce definitions and notations used throughout the paper. Section 3 establishes $\kappa(Q_n^3; T)$ and $\kappa^s(Q_n^3; T)$ for $T \in \{P_i, C_i\}$. In Section 4, we determine $\kappa(Q_n^k; T)$ and $\kappa^s(Q_n^k; T)$ for $k \geq 4$, $T \in \{P_i, C_i\}$. Section 5 concludes the paper.

II. PRELIMINARIES

The k-ary n-cube Q_n^k is a popular interconnection network for parallel systems which has been proved to possess many attractive properties such as regularity, node transitivity and link transitivity. A number of parallel systems have been built with a k-ary n-cube forming the underlying topology, such as the J-machine [19], the iWarp [20] and the Cray T3D [10]. In particular, the 3-ary n-cube Q_n^3 has been widely deployed in interconnections of parallel systems like the IBM Blue Gene/Q [1].

The k-ary n-cube Q_n^k ($k \geq 2$ and $n \geq 1$) is a graph consisting of k^n nodes, each of which has the form $u = a_1a_2 \ldots a_n$, where $0 \leq a_i \leq k - 1$ for $1 \leq i \leq n$. Two nodes $u = a_1a_2 \ldots a_n$ and $v = b_1b_2 \ldots b_n$ are adjacent if and only if there exists an integer j, $1 \leq j \leq n$, such that $a_j = b_j \pm 1 \pmod{k}$ and $a_i = b_i$, for every $i \in \{1, 2, \ldots, n\} \setminus \{j\}$. Such a link uv is called a j-dimensional link. For clarity of presentation, we omit writing "(mod k)" in similar expressions for the remainder of the paper. Note that each node has degree $2n$ when $k \geq 3$, and n when $k = 2$. Obviously, Q_1^k is a cycle of length k, Q_n^2 is an n-dimensional hypercube. Q_1^6 and Q_2^4 are depicted in FIGURE 2.

Two distinct adjacent nodes are neighbours. The set of neighbours of a node v in a graph G is denoted by $N(v)$, that is, $N(v) = \{u \in V(G) : uv \in E(G)\}$. For $W \subseteq V(G)$, denote $N(W) = (\bigcup_{v \in W} N(v)) \setminus W$. Let G_1 and G_2 be two graphs. Denote $G_1 \cong G_2$ when G_1 and G_2 are isomorphic. G_1 and G_2 are disjoint if they have no common node. Let $F_i = \{T_1, T_2, \ldots, T_m : T_j \cong P_i, 1 \leq j \leq m\}$ and $|F_i| = m$, let

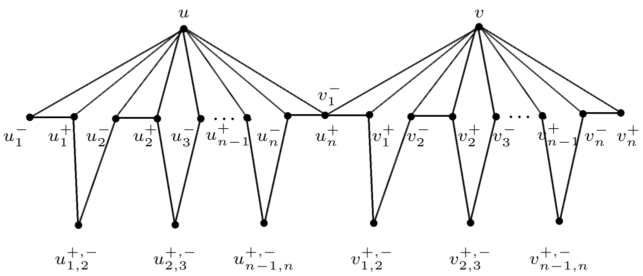

FIGURE 3. The *neighbour structure* of *u* and *v*.

$F_i' = \{T_1, T_2, \ldots, T_m : T_j \cong C_i, 1 \leq j \leq m\}$ and $|F_i'| = m$, and let $Q_n^k - F_i$ (resp. $Q_n^k - F_i'$) be the graph obtained from Q_n^k by deleting the nodes of F_i (resp. F_i') together with their incident links.

The following lemmas are useful in Sections 3 and 4.

Lemma 1: Let P_i, C_i be subgraphs of Q_n^k. Then $\kappa^s(Q_n^k; P_i) \geq \kappa^s(Q_n^k; C_i)$.

Proof: Let $F = \{T_1, T_2, \ldots, T_m\}$ be a set of subgraphs in Q_n^k with $m = \kappa^s(Q_n^k; P_i)$ such that every $T_j \in F$ is isomorphic to a connected subgraph of P_i, and F's deletion disconnects Q_n^k. Then every $T_j \in F$ is isomorphic to a connected subgraph of C_i. By definition of $\kappa^s(Q_n^k; C_i)$, $\kappa^s(Q_n^k; P_i) \geq \kappa^s(Q_n^k; C_i)$. □

Lemma 2 ([3], [7]): $\kappa_1(Q_n^3) = 4n - 2$ for $n \geq 2$, and $\kappa_2(Q_n^k) = 6n - 5$ for $n \geq 5$ and $k \geq 4$.

III. THE STRUCTURE CONNECTIVITY AND SUBSTRUCTURE CONNECTIVITY OF Q_n^3

In this section, we determine $\kappa(Q_n^3; T)$ and $\kappa^s(Q_n^3; T)$ of Q_n^3 for $T \in \{P_i, C_i\}$.

Let $u = a_1a_2 \ldots a_n$ be a node of Q_n^3. For $1 \leq i, j \leq n$, let $u_i^- = a_1 \ldots (a_i - 1) \ldots a_n$, let $u_i^+ = a_1 \ldots (a_i + 1) \ldots a_n$, and let $u_{i,j}^{+,-} = a_1 \ldots (a_i + 1) \ldots (a_j - 1) \ldots a_n$. Similarly, $u_{i,j}^{+,+}, u_{i,j}^{-,-}, u_{i,j}^{-,+}$ can be defined. Let $P(u) = u_1^- u_1^+ u_{1,2}^{+,-} u_2^- u_2^+ u_{2,3}^{+,-} \ldots u_{n-1}^- u_{n-1}^+ u_{n-1,n}^{+,-} u_n^- u_n^+$. Then $P(u)$ is a path and is called the *neighbour structure* of u (see FIGURE 3). Let $P = u_i^- u_i^+ u_{i,i+1}^{+,-} u_{i+1}^- u_{i+1}^+ u_{i+1,i+2}^{+,-} \cdots u_j^- u_j^+$ is a path lying on $P(u)$ for $1 \leq i, j \leq n$. For convenience, we denote such a path P by $[u_i^-, u_j^+]$. Similarly, $[u_i^-, u_j^-]$, $[u_i^+, u_j^+]$ for $1 \leq i, j \leq n$ and $[u_i^-, u_{j,j+1}^{+,-}]$ for $1 \leq i, j \leq n-1$ can be defined. Let $v \in V(Q_n^3)$ with $v_1^- = u_n^+$. Similarly, consider the *neighbour structure* of v. It is easy to see that the *neighbour structure* of u and v has exactly two common nodes $u_n^+ = v_1^-$ and $u_1^+ = v_n^-$(see FIGURE 3).

A. $\kappa(Q_n^3; P_i)$ AND $\kappa^s(Q_n^3; P_i)$

Lv *et al.* [11] proved the following theorem about $\kappa(Q_n^3; P_i)$ and $\kappa^s(Q_n^3; P_i)$ for $i = 1, 2, 3$. In this subsection, we generalize the theorem by establishing $\kappa(Q_n^3; P_i)$ and $\kappa^s(Q_n^3; P_i)$ for $i \geq 1$.

Theorem 1 ([11]): For $n \geq 2$, $\kappa(Q_n^3; P_1) = \kappa^s(Q_n^3; P_1) = 2n$ and $\kappa(Q_n^3; P_2) = \kappa^s(Q_n^3; P_2) = \kappa(Q_n^3; P_3) = \kappa^s(Q_n^3; P_3) = n$.

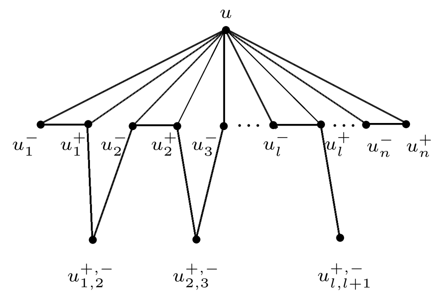

FIGURE 4. A path $P_{3l}^1 = [u_1^-, u_{l,l+1}^{+,-}]$ using $P(u)$.

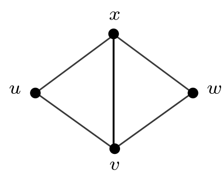

FIGURE 5. The structure *A*.

Lemma 3: Let $n \geq 2$ and $l \geq 0$. Then $\kappa(Q_n^3; P_{3l+s}) \leq \lceil \frac{2n}{2l+s} \rceil$ for $2l + s \leq 2n$ and $s = 0, 1, 2$.

Proof: Let $u = 111 \ldots 11$ and $v = 211 \ldots 12$. Then $u_n^+ = v_1^-$. We will successively find $\lceil \frac{2n}{2l+s} \rceil$ pairwise disjoint P_{3l+s}'s denoted by $P_{3l+s}^1, P_{3l+s}^2, \ldots, P_{3l+s}^{\lceil \frac{2n}{2l+s} \rceil}$ by using $P(u)$ and $P(v)$. We consider the following three cases.

Case 1: $s = 0$.

If $2l = 2n$, then $\lceil \frac{2n}{2l} \rceil = 1$, and let $P_{3l}^1 = [u_1^-, u_n^+]v_1^+$. If $2l < 2n$, then let $2n = p2l + 2q$, and so $2q \geq 2$. Let $P_{3l}^1 = [u_1^-, u_{l,l+1}^{+,-}]$ (see FIGURE 4), $P_{3l}^2 = [u_{l+1}^-, u_{2l,2l+1}^{+,-}]$, $P_{3l}^3 = [u_{2l+1}^-, u_{3l,3l+1}^{+,-}]$, $\ldots$, $P_{3l}^p = [u_{(p-1)l+1}^-, u_{pl,pl+1}^{+,-}]$. By $2l < 2n$ and $u_n^+ = v_1^-$, we can find $P_{3l}^{\lceil \frac{2n}{2l} \rceil}$ lying on $P(u)$ and $P(v)$. By definition of $P(v)$ and $2q \geq 2$, $v_n^- \notin V(P_{3l}^{\lceil \frac{2n}{2l} \rceil})$.

Case 2: $s = 1$.

Let $2n = p(2l+1)+q$. By $2l+1 < 2n$, $q \geq 1$. Let $P_{3l+1}^1 = [u_1^-, u_{l+1}^-]$, $P_{3l+1}^2 = [u_{l+1}^+, u_{2l+1}^+]$, $P_{3l+1}^3 = [u_{2l+2}^-, u_{3l+2}^-]$, $\ldots$, $P_{3l+1}^p = [u_{(p-1)l+\frac{p+1}{2}}^-, u_{pl+\frac{p+1}{2}}^-]$ with p odd, and $P_{3l+1}^p = [u_{(p-1)l+\frac{p}{2}}^+, u_{pl+\frac{p}{2}}^+]$ with p even. If $q = 1$ and $l + 1 = n$, then let $P_{3l+1}^{\lceil \frac{2n}{2l+1} \rceil} = [v_1^-, v_{n-1}^+]v_{n-1,n}^{+,+}v_n^+$. Otherwise, by $2l + 1 < 2n$ and $u_n^+ = v_1^-$, we can find $P_{3l+1}^{\lceil \frac{2n}{2l+1} \rceil}$ lying on $P(u)$ and $P(v)$ with $v_n^- \notin V(P_{3l+1}^{\lceil \frac{2n}{2l+1} \rceil})$.

Case 3: $s = 2$.

If $2l+2 = 2n$, then $\lceil \frac{2n}{2l+2} \rceil = 1$, and let $P_{3l+2}^1 = [u_1^-, u_n^+]$. If $2l + 2 < 2n$, then let $2n = p(2l + 2) + 2q$, and so $2q \geq 2$. Let $P_{3l+2}^1 = [u_1^-, u_{l+1}^+]$, $P_{3l+2}^2 = [u_{l+2}^-, u_{2l+2}^+]$, $P_{3l+2}^3 = [u_{2l+3}^-, u_{3l+3}^+]$, $\ldots$, $P_{3l+2}^p = [u_{(p-1)l+p}^-, u_{pl+p}^+]$. By $2l + 2 < 2n$ and $u_n^+ = v_1^-$, we can find $P_{3l+2}^{\lceil \frac{2n}{2l+2} \rceil}$ lying on $P(u)$ and $P(v)$. By definition of $P(v)$ and $2q \geq 2$, $v_n^- \notin V(P_{3l+2}^{\lceil \frac{2n}{2l+2} \rceil})$.

Let $F = \{P_{3l+s}^1, P_{3l+s}^2, \ldots, P_{3l+s}^{\lceil \frac{2n}{2l+s} \rceil}\}$. Then $Q_n^3 - F$ is disconnected because $\{u\}$ is a component of $Q_n^3 - F$. By definition of $\kappa(Q_n^3; P_{3l+s})$, $\kappa(Q_n^3; P_{3l+s}) \leq \lceil \frac{2n}{2l+s} \rceil$. □

Lemma 4 ([24]): Let C_3 be a cycle in Q_n^3. Then there exists $j \in \{1, 2, \ldots, n\}$ such that C_3 contains only j-dimensional links.

Lemma 5: Q_n^3 contains no structure A.

Proof: By contradiction. Suppose that Q_n^3 contains structure A (see FIGURE 5). Then xuv and xwv are both cycles of length 3 in Q_n^3. By Lemma 4, xuv contains only i-dimensional links and xwv contains only j-dimensional links for $i, j \in \{1, 2, \ldots, n\}$. Thus $i = j$ and so $u = w$, a contradiction. □

Lemma 6: Let $n \geq 2$ and $l \geq 1$. Then $\kappa^s(Q_n^3; P_{3l+s}) \geq \kappa^s(Q_n^3; C_{3l+s}) \geq \lceil \frac{2n}{2l+s} \rceil$ for $2l + s \leq 2n$ and $s = 0, 1, 2$.

Proof: By Lemma 1, $\kappa^s(Q_n^3; P_{3l+s}) \geq \kappa^s(Q_n^3; C_{3l+s})$. Let $F = \cup_{i=1}^{3l+s} F_i \cup F'_{3l+s}$ with $|F| = \sum_{i=1}^{3l+s} |F_i| + |F'_{3l+s}| \leq \lceil \frac{2n}{2l+s} \rceil - 1$. In order to prove that $\kappa^s(Q_n^3; C_{3l+s}) \geq \lceil \frac{2n}{2l+s} \rceil$, it is enough to show that $Q_n^3 - F$ is connected. Suppose, to the contrary, that $Q_n^3 - F$ is disconnected. Let T_0 be a smallest component of $Q_n^3 - F$.

Case 1: $|V(T_0)| = 1$.

Set $V(T_0) = \{x\}$. Thus $N(x) \subseteq V(F)$. To make the number of faulty subgraphs of C_{3l+s} minimum which contain the nodes in $N(x)$, we should construct as many P_{3l+s}'s/C_{3l+s}'s as possible and each P_{3l+s}/C_{3l+s} need to contain as many nodes in $N(x)$ as possible. By Lemma 5, Q_n^3 contains no structure A, and so any three nodes in $N(x)$ are not three consecutive nodes on a path/cycle. Combining this with the definition of the *neighbour structure* of x, each P_{3l+s}/C_{3l+s} contain at most $2l + s$ nodes in $N(x)$. Note that $|N(x)| = 2n$. Then $|F| \geq \lceil \frac{2n}{2l+s} \rceil > \lceil \frac{2n}{2l+s} \rceil - 1 \geq |F|$, a contradiction.

Case 2: $|V(T_0)| \geq 2$.

By $n \geq 2$, $|V(T_0)| \geq 2$ and Lemma 2, $|V(F)| \geq 4n - 2$. Note that $|F| \leq \lceil \frac{2n}{2l+s} \rceil - 1$. Thus $|V(F)| \leq (3l+s)(\lceil \frac{2n}{2l+s} \rceil - 1) \leq (3l + s)(\frac{2n+2l+s-1}{2l+s} - 1) = \frac{(3l+s)}{(2l+s)}(2n - 1) < 4n - 2 \leq |V(F)|$, a contradiction. □

Note that $\kappa(Q_n^3) = 2n$ for $n \geq 2$. Then for any F_1 with $|F_1| \leq 2n - 1$, $Q_n^3 - F_1$ is still connected, and for any $F_1 \cup F_2$ with $|F_1| + |F_2| \leq n - 1$, $Q_n^3 - (F_1 \cup F_2)$ is still connected. Thus $\kappa^s(Q_n^3; P_1) \geq 2n$ and $\kappa^s(Q_n^3; P_2) \geq n$. Combining this with Lemma 6, we have $\kappa^s(Q_n^3; P_{3l+s}) \geq \lceil \frac{2n}{2l+s} \rceil$ for $n \geq 2$ and $l \geq 0$. Recall that $\kappa(Q_n^3; P_{3l+s}) \geq \kappa^s(Q_n^3; P_{3l+s})$. Lemma 3 yields the following result.

Theorem 2: Let $n \geq 2$ and $l \geq 0$. Then $\kappa(Q_n^3; P_{3l+s}) = \kappa^s(Q_n^3; P_{3l+s}) = \lceil \frac{2n}{2l+s} \rceil$ for $2l + s \leq 2n$ and $s = 0, 1, 2$.

Set $3l + s = 1, 2, 3$ in the Theorem 2. Then Theorem 1 given by Lv *et al.* [11] is an immediate corollary of Theorem 2.

B. $\kappa(Q_n^3; C_l)$ AND $\kappa^s(Q_n^3; C_l)$

In this subsection, we investigate the cycle-structure/substructure connectivity for Q_n^3. Let $u = a_1 a_2 \ldots a_n$ be a node of Q_n^k. For $1 \leq i, j, t \leq n$, let $u_{i,j,t}^{+,+,-} = a_1 \ldots (a_i + 1) \ldots (a_j + 1) \ldots (a_t - 1) \ldots a_n$. Similarly, $u_{i,j,t}^{+,+,+}$ and $u_{i,j,t}^{-,-,-}$ can be defined.

Lemma 7: $\kappa(Q_n^3; C_{3l}) \leq \lceil \frac{2n}{2l} \rceil$ for $n \geq 2$ and $4 \leq 2l \leq 2n$, and $\kappa(Q_n^3; C_{3l+2}) \leq \lceil \frac{2n}{2l+1} \rceil$ for $n \geq 3$ and $2l + 1 < 2n$.

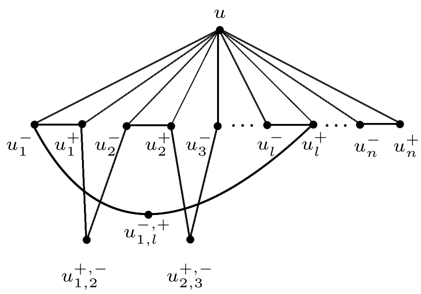

FIGURE 6. A cycle $C_{3l}^1 = [u_1^-, u_l^+]u_{1,l}^{-,+}u_1^-$ using $P(u)$.

Proof: Let $u = 111\ldots11$ and $v = 211\ldots12$. Then $u_n^+ = v_1^-$. Consider the *neighbour structure* of u and v. Let $u_i^-, u_i^+, u_j^-, u_j^+ \in P(u)$ with $1 \leq i, j \leq n$. Then $u_i^- u_{i,j}^{-,-} u_j^-$, $u_i^- u_{i,j}^{-,+} u_j^+$, $u_i^+ u_{i,j}^{+,-} u_j^-$, $u_i^+ u_{i,j}^{+,+} u_j^+$ are P_3's. Let $u_j^- \in P(u)$ and $v_i^+ \in P(v)$ with $2 \leq i < j \leq n - 1$. Then $v_i^+ = u_{1,i,n}^{+,+,+}$, and so $v_i^+ u_{1,i}^{+,+} u_{1,i,j}^{+,+,-} u_{1,j}^{+,-} u_j^-$ is a P_5. Similarly, $v_i^- u_{i,n}^{-,+} u_{i,j,n}^{-,-,+} u_{j,n}^{-,+} u_j^-$ and $v_i^+ u_{1,i}^{+,+} u_{1,i,j}^{+,+,+} u_{1,j}^{+,+} u_j^+$ are P_5's. In the following, such P_3's and P_5's can be used to construct the desired cycles.

For C_{3l}, we will successively find $\lceil \frac{2n}{2l} \rceil$ pairwise disjoint C_{3l}'s denoted by $C_{3l}^1, C_{3l}^2, \ldots, C_{3l}^{\lceil \frac{2n}{2l} \rceil}$ by using $P(u)$ and $P(v)$ with $v_n^- \notin V(C_{3l}^{\lceil \frac{2n}{2l} \rceil})$. If $2l = 2n$, then $\lceil \frac{2n}{2l} \rceil = 1$, and let $C_{3l}^1 = [u_1^-, u_n^+]u_{1,n}^{-,+}u_1^-$. Next consider $2l < 2n$ and assume that $2n = p2l + 2q$. Let $C_{3l}^1 = [u_1^-, u_l^+]u_{1,l}^{-,+}u_1^-$ (see FIGURE 6), $C_{3l}^2 = [u_{l+1}^-, u_{2l}^+]u_{l+1,2l}^{-,+}u_{l+1}^-$, $C_{3l}^3 = [u_{2l+1}^-, u_{3l}^+]u_{2l+1,3l}^{-,+}u_{2l+1}^-$, $\ldots$, $C_{3l}^p = [u_{(p-1)l+1}^-, u_{pl}^+]u_{(p-1)l+1,pl}^{-,+}u_{(p-1)l+1}^-$. If $q = 1$, then let $C_{3l}^{\lceil \frac{2n}{2l} \rceil} = u_n^- u_n^+ v_1^+ v_{1,2}^{+,-} u_{1,2,n}^{-,-,-} u_{1,n}^{-,-} u_n^-$ with $l = 2$, and let $C_{3l}^{\lceil \frac{2n}{2l} \rceil} = u_n^- u_n^+ v[v_1^+, v_{l-1}^+]u_{1,l-1,n}^{+,+,-} u_{1,n}^{+,-} u_n^-$ with $l \geq 3$. Note that $|V(C_{3l}^{\lceil \frac{2n}{2l} \rceil})| = 3 + 3(l - 2) + 3 = 3l$ and $v_{l-1}^+ = u_{1,l-1,n}^{+,+,+}$ by $3 \leq l \leq n$. Then $C_{3l}^{\lceil \frac{2n}{2l} \rceil}$ is indeed a cycle on $3l$ nodes. Next assume that $q \geq 2$. Then $n - q + 1 \leq n - 1$. If $l - q = 1$, then let $C_{3l}^{\lceil \frac{2n}{2l} \rceil} = [u_{n-q+1}^-, u_n^+]v_1^+ v v_{n-q+1}^- u_{1,n-q+1}^{+,-} u_{n-q+1}^-$. Note that $|V(C_{3l}^{\lceil \frac{2n}{2l} \rceil})| = 3q + 3 = 3l$ and $v_{n-q+1}^- = u_{1,n-q+1,n}^{+,-,+}$ by $2 \leq pl + 1 = n - q + 1 \leq n - 1$. Then $C_{3l}^{\lceil \frac{2n}{2l} \rceil}$ is indeed a cycle on $3l$ nodes. Now consider $l - q \geq 2$. By $2l \leq 2n$, $l - q < n - q + 1$. Thus $2 \leq l - q < n - q + 1 \leq n - 1$, and so let $C_{3l}^{\lceil \frac{2n}{2l} \rceil} = [u_{n-q+1}^-, u_n^+][v_1^+, v_{l-q}^+]u_{1,l-q}^{+,+} u_{1,l-q,n-q+1}^{+,+,-} u_{1,n-q+1}^{+,-} u_{n-q+1}^-$. Note that $|V(C_{3l}^{\lceil \frac{2n}{2l} \rceil})| = 3q + 3(l - 1 - q) + 3 = 3l$ and $v_{l-q}^+ = u_{1,l-q,n}^{+,+,+}$. Then $C_{3l}^{\lceil \frac{2n}{2l} \rceil}$ is indeed a cycle on $3l$ nodes.

For C_{3l+2}, we will successively find $\lceil \frac{2n}{2l+1} \rceil$ pairwise disjoint C_{3l+2}'s denoted by C_{3l+2}^1, C_{3l+2}^2, $\ldots$, $C_{3l+2}^{\lceil \frac{2n}{2l+1} \rceil}$ by using $P(u)$ and $P(v)$ with $v_n^- \notin V(C_{3l+2}^{\lceil \frac{2n}{2l+1} \rceil})$. Let $2n = p(2l + 1) + q$. By $2l + 1 < 2n$, $q \geq 1$. Let $C_{3l+2}^1 = [u_1^-, u_{l+1}^-]u_{1,l+1}^{-,-}u_1^-$, $C_{3l}^2 = [u_{l+1}^+, u_{2l+1}^+] u_{l+1,2l+1}^{+,+} u_{l+1}^+$, $C_{3l}^3 = [u_{2l+2}^-, u_{3l+2}^-]u_{2l+2,3l+2}^{-,-}u_{2l+2}^-$, $\ldots$, $C_{3l}^p = [u_{(p-1)l+\frac{p+1}{2}}^-, u_{pl+\frac{p+1}{2}}^-]u_{(p-1)l+\frac{p+1}{2},pl+\frac{p+1}{2}}^{-,-} u_{(p-1)l+\frac{p+1}{2}}^-$

with p odd, and $C^2_{3l} = [u^+_{(p-1)l+\frac{p}{2}}, u^+_{pl+\frac{p}{2}}]u^{+,+}_{(p-1)l+\frac{p}{2},pl+\frac{p}{2}} u^+_{(p-1)l+\frac{p}{2}}$ with p even.

If q is even, then assume that $q = 2s$. If $s = 1$, then let $C^{\lceil\frac{2n}{2l+1}\rceil}_{3l+2} = u^-_n u^+_n v v^+_1 u^{-,-}_{1,n} u^-_n$ with $l = 1$, and let $C^{\lceil\frac{2n}{2l+1}\rceil}_{3l+2} = u^-_n u^+_n [v^+_1, v^+_l] u^{+,+,-}_{1,l,n} u^{+,-}_{1,n} u^-_n$ with $l \geq 2$. Note that $|V(C^{\lceil\frac{2n}{2l+1}\rceil}_{3l+2})| = 3 + 3(l-1) + 2 = 3l + 2$ and $v^+_l = u^{+,+,+}_{1,l,n}$ by $2 \leq l \leq n-1$. Then $C^{\lceil\frac{2n}{2l+1}\rceil}_{3l+2}$ is indeed a cycle on $3l+2$ nodes. Next assume that $s \geq 2$. Then $n - s + 1 \leq n - 1$. If $s = l$, then, by $2n = p(2s+1) + 2s$, $n - s + 1 \geq 4$. Let $C^{\lceil\frac{2n}{2l+1}\rceil}_{3l+2} = [u^-_{n-s+1}, u^+_n] v^+_1 u^{-,-,+}_{1,n-s+1,n} u^{-,-}_{1,n-s+1} u^-_{n-s+1}$. Note that $|V(C^{\lceil\frac{2n}{2l+1}\rceil}_{3l+2})| = 3s + 2 = 3l + 2$ and $v^+_1 = u^{-,+}_{1,n}$. Then $C^{\lceil\frac{2n}{2l+1}\rceil}_{3l+2}$ is indeed a cycle on $3l+2$ nodes. If $s \leq l - 1$, then $l - s + 1 \geq 2$. By $2l + 1 < 2n$, $l - s + 1 < n - s + 1$. Thus $2 \leq l - s + 1 < n - s + 1 \leq n - 1$, and so let $C^{\lceil\frac{2n}{2l+1}\rceil}_{3l+2} = [u^-_{n-s+1}, u^+_n][v^+_1, v^-_{l-s+1}] u^{-,+}_{l-s+1,n} u^{-,-,+}_{l-s+1,n-s+1,n} u^{-,+}_{n-s+1,n} u^-_{n-s+1}$. Note that $|V(C^{\lceil\frac{2n}{2l+1}\rceil}_{3l+2})| = 3s + 3(l-s-1) + 5 = 3l+2$ and $v^-_{l-s+1} = u^{+,-,+}_{1,l-s+1,n}$. Then $C^{\lceil\frac{2n}{2l+1}\rceil}_{3l+2}$ is indeed a cycle on $3l+2$ nodes.

If q is odd, then assume that $q = 2s + 1$. First suppose that $s = 0$. By $2l + 1 < 2n$, $l \leq n - 1$. Let $C^{\lceil\frac{2n}{2l+1}\rceil}_{3l+2} = [v^-_1, v^+_{n-1}] v^{+,+}_{n-1,n} v^+_n v v^-_1$ with $l + 1 = n$, and let $C^{\lceil\frac{2n}{2l+1}\rceil}_{3l+2} = [v^-_1, v^-_{l+1}] v^{-,-}_{1,l+1} v^-_1$ with $l + 1 \leq n - 1$. Next suppose that $s \geq 1$. Then $n - s \leq n - 1$. If $l - s = 1$, then let $C^{\lceil\frac{2n}{2l+1}\rceil}_{3l+2} = [u^+_{n-s}, u^+_n] v^+_1 v v^+_{n-s} u^{+,+}_{1,n-s} u^+_{n-s}$. Note that $|V(C^{\lceil\frac{2n}{2l+1}\rceil}_{3l+2})| = 3s + 5 = 3l + 2$ and $v^+_{n-s} = u^{+,+,+}_{1,n-s,n}$ by $2 \leq \frac{p(2l+1)+1}{2} = n - s \leq n-1$. Then $C^{\lceil\frac{2n}{2l+1}\rceil}_{3l+2}$ is indeed a cycle on $3l+2$ nodes. Now consider $l - s \geq 2$. By $2l + 1 < 2n$, $l - s < n - s$. Thus $2 \leq l - s < n - s \leq n - 1$, and so let $C^{\lceil\frac{2n}{2l+1}\rceil}_{3l+2} = [u^+_{n-s}, u^+_n][v^+_1, v^+_{l-s}] u^{+,+}_{1,l-s} u^{+,+,+}_{1,l-s,n-s} u^{+,+}_{1,n-s} u^+_{n-s}$. Note that $|V(C^{\lceil\frac{2n}{2l+1}\rceil}_{3l+2})| = 3s + 3(l-s-1) + 5 = 3l + 2$ and $v^+_{l-s} = u^{+,+,+}_{1,l-s,n}$. Then $C^{\lceil\frac{2n}{2l+1}\rceil}_{3l+2}$ is indeed a cycle on $3l+2$ nodes.

Let $F = \{C^1_{3l}, C^2_{3l}, \ldots, C^{\lceil\frac{2n}{2l}\rceil}_{3l}\}$. Then $Q^3_n - F$ is disconnected because $\{u\}$ is a component of $Q^3_n - F$. By definition of $\kappa(Q^3_n; C_{3l})$, $\kappa(Q^3_n; C_{3l}) \leq \lceil\frac{2n}{2l}\rceil$. Similarly, we can show that $\kappa(Q^3_n; C_{3l+2}) \leq \lceil\frac{2n}{2l+1}\rceil$. $\square$

Lemma 8: *$\kappa(Q^3_n; C_{3l}) \geq \lceil\frac{2n}{2l}\rceil$ for $n \geq 2$ and $4 \leq 2l \leq 2n$, and $\kappa(Q^3_n; C_{3l+2}) \geq \lceil\frac{2n}{2l+1}\rceil$ for $n \geq 3$ and $2l + 1 < 2n$.*

Proof: By Lemma 6, $\kappa^s(Q^3_n; C_{3l}) \geq \lceil\frac{2n}{2l}\rceil$. Note that $\kappa(Q^3_n; C_{3l}) \geq \kappa^s(Q^3_n; C_{3l})$. Then $\kappa(Q^3_n; C_{3l}) \geq \lceil\frac{2n}{2l}\rceil$.

In order to prove that $\kappa(Q^3_n; C_{3l+2}) \geq \lceil\frac{2n}{2l+1}\rceil$, it is enough to show that $Q^3_n - F'_{3l+2}$ is connected for any F'_{3l+2} with $|F'_{3l+2}| \leq \lceil\frac{2n}{2l+1}\rceil - 1$. Suppose, to the contrary, that $Q^3_n - F'_{3l+2}$ is disconnected. Let T_0 be a smallest component of $Q^3_n - F'_{3l+2}$.

Case 1: $|V(T_0)| = 1$.

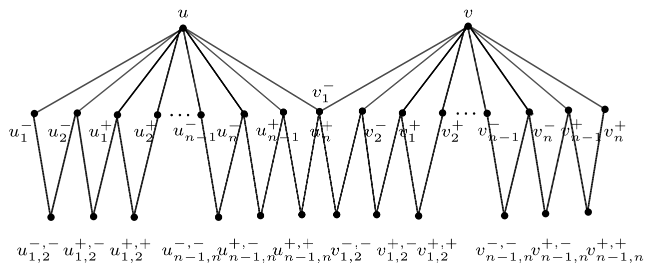

FIGURE 7. The *neighbour structure* of u and v with n even.

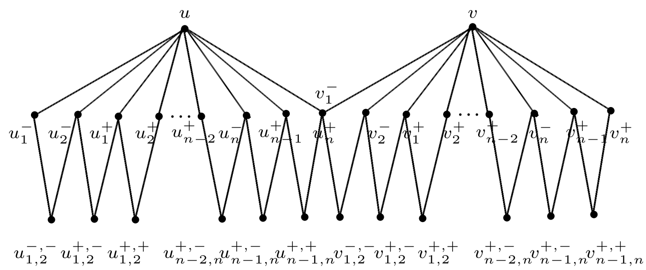

FIGURE 8. The *neighbour structure* of u and v with n odd.

Set $V(T_0) = \{x\}$. Thus $N(x) \subseteq V(F'_{3l+2})$. To make the number of faulty C_{3l+s}'s minimum which contain the nodes in $N(x)$, each C_{3l+2} need to contain as many nodes in $N(x)$ as possible. By Lemma 5, Q^3_n contains no structure A, and so any three nodes in $N(x)$ are not three consecutive nodes on a cycle. Combining this with the definition of the *neighbour structure* of x, a cycle C_{3l+2} contain at most $2l+1$ nodes in $N(x)$. Note that $|N(x)| = 2n$. Then $|F'_{3l+2}| \geq \lceil\frac{2n}{2l+1}\rceil > \lceil\frac{2n}{2l+1}\rceil - 1 \geq |F|$, a contradiction.

Case 2: $|V(T_0)| \geq 2$.

By $n \geq 3$, $|V(T_0)| \geq 2$ and Lemma 2, $|V(F'_{3l+2})| \geq 4n-2$. Note that $|F'_{3l+2}| \leq \lceil\frac{2n}{2l+1}\rceil - 1$. Thus $|V(F'_{3l+2})| \leq (3l+2)(\lceil\frac{2n}{2l+1}\rceil - 1) \leq (3l+2)(\frac{2n+2l}{2l+1} - 1) = \frac{3l+2}{2l+1}(2n-1) < 4n - 2 \leq |V(F'_{3l+2})|$, a contradiction. $\square$

Lemmas 7 and 8 yield the following result.

Theorem 3: *$\kappa(Q^3_n; C_{3l}) = \lceil\frac{2n}{2l}\rceil$ for $n \geq 2$ and $4 \leq 2l \leq 2n$, and $\kappa(Q^3_n; C_{3l+2}) = \lceil\frac{2n}{2l+1}\rceil$ for $n \geq 3$ and $2l + 1 < 2n$.*

By Lemma 6, $\kappa^s(Q^3_n; P_{3l+s}) \geq \kappa^s(Q^3_n; C_{3l+s}) \geq \lceil\frac{2n}{2l+s}\rceil$. By Theorem 2, $\kappa^s(Q^3_n; P_{3l+s}) = \lceil\frac{2n}{2l+s}\rceil$. Thus we have the following theorem.

Theorem 4: *Let $n \geq 2$. Then $\kappa^s(Q^3_n; C_{3l+s}) = \lceil\frac{2n}{2l+s}\rceil$ for $2l + s \leq 2n$ and $s = 0, 1, 2$.*

IV. THE STRUCTURE CONNECTIVITY AND SUBSTRUCTURE CONNECTIVITY OF Q^k_n

In this section, we determine $\kappa(Q^k_n; T)$ and $\kappa^s(Q^k_n; T)$ of Q^k_n for $k \geq 4$ and $T \in \{P_i, C_i\}$.

Let $u = a_1 a_2 \ldots a_n$ be a node of Q^k_n, let $u^-_j = a_1 \ldots (a_j - 1) \ldots a_n$, $u^+_j = a_1 \ldots (a_j + 1) \ldots a_n$, $u^{-,-}_{j,j+1} = a_1 \ldots (a_j - 1)(a_{j+1} - 1) \ldots a_n$. Similarly, $u^{+,-}_{j,j+1}$ and $u^{+,+}_{j,j+1}$ can be defined. Let $P(u) = u^-_1 u^{-,-}_{1,2} u^-_2 u^{+,-}_{1,2} u^+_1 u^{+,+}_{1,2} u^+_2 u^{+,-}_{2,3} u^-_3 \ldots u^-_{n-1} u^{-,-}_{n-1,n} u^-_n u^{+,-}_{n-1,n} u^+_{n-1} u^{+,+}_{n-1,n} u^+_n$ with n even (see FIGURE 7), and $P(u) = u^-_1 u^{-,-}_{1,2} u^-_2 u^{+,-}_{1,2} u^+_1 u^{+,+}_{1,2} u^+_2 u^{+,-}_{2,3} u^-_3 \ldots u^+_{n-2} u^{+,-}_{n-2,n} u^-_n$

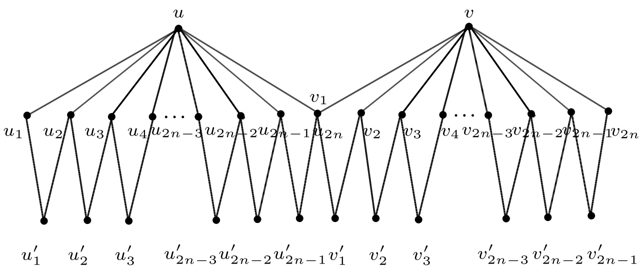

FIGURE 9. The ***neighbour structure*** of ***u*** and ***v***.

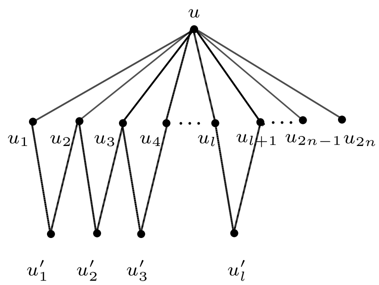

FIGURE 10. A path $P^1_{2l+1} = [u_1, u_{l+1}]$ using $P(u)$.

$u^{+,-}_{n-1,n}u^{+}_{n-1}u^{+,+}_{n-1,n}u^{+}_{n}$ with n odd (see FIGURE 8). Then $P(u)$ is a path and is called the *neighbour structure* of u. Let $v \in V(Q^k_n)$ with $u^+_n = v^-_1$. Similarly, consider the *neighbour structure* of v. It is easy to see that the *neighbour structure* of u and v has exactly two common nodes $u^+_n = v^-_1$ and $u^+_1 = v^-_n$ (see FIGURE 7 and FIGURE 8). For convenience, no matter what the parity of n is, the above *neighbour structure* of u and v is denoted by $P(u) = u_1u_1'u_2u_2'u_3\ldots u_{2n-1}u_{2n-1}'u_{2n}$ and $P(v) = v_1v_1'v_2v_2'v_3\ldots v_{2n-1}v_{2n-1}'v_{2n}$ with $u_{2n} = v_1$ (see FIGURE 9). Let $P = u_iu_i'u_{i+1}u_{i+1}'u_{i+2}\ldots u_{j-1}u_{j-1}'u_j$ be a path lying on $P(u)$ for $1 \le i, j \le 2n$. For convenience, we denote such a path P by $[u_i, u_j]$. Similarly, $[u_i, u_j']$ can be defined for $1 \le i, j \le 2n-1$.

A. $\kappa(Q^k_n; P_i)$ AND $\kappa^s(Q^k_n; P_i)$

Lv *et al.* [11] proved the following theorem about $\kappa(Q^k_n; P_i)$ and $\kappa^s(Q^k_n; P_i)$ for $i = 1, 3$. In this subsection, we generalize the theorem by establishing $\kappa(Q^k_n; P_i)$ and $\kappa^s(Q^k_n; P_i)$.

Theorem 5 ([11]): For $n \ge 2$ and $k \ge 4$, $\kappa(Q^k_n; P_1) = \kappa^s(Q^k_n; P_1) = 2n$ and $\kappa(Q^k_n; P_3) = \kappa^s(Q^k_n; P_3) = n$.

Lemma 9: Let $u = 111\ldots11$ and $v = 211\ldots12$. If $[v_1, w] \subseteq P(v)$ with $w \ne v^+_n$ and $v^-_n \in V([v_1, w])$, then there exist a path P starting at v_1 such that $v^-_n \notin V(P)$, $|V(P)| = |V([v_1, w])|$ and $(V(P) \setminus v_1) \cap V(P(u)) = \emptyset$.

Proof: If $w = v^-_n$, then, by definition of $P(v)$, let $P = [v_1, v^-_{n-1}]v^{-,+}_{n-1,n}v^+_n$ with n even, and $P = [v_1, v^+_{n-2}]v^{+,+}_{n-2,n}v^+_n$ with n odd. Then P satisfies the conditions. If $w \ne v^-_n$, then $w \in \{v^{+,-}_{n-1,n}, v^+_{n-1}, v^{+,+}_{n-1,n}\}$. When n is even, let $P = [v_1, v^-_{n-1}]v^{-,+}_{n-1,n}v^+_nv^{+,+}_{n-1,n}$ with $w = v^{+,-}_{n-1,n}$, let $P = [v_1, v^-_{n-1}]v^{-,+}_{n-1,n}v^+_nv^{+,+}_{n-1,n}v^+_{n-1}$ with $w = v^+_{n-1}$, and let $P = [v_1, v^-_{n-1}]v^{-,+}_{n-1,n}v^+_nv^{+,+}_{n-1,n}v^+_{n-1}v^{+,-}_{n-1,n}$ with $w = v^{+,+}_{n-1,n}$. When n is odd, let $P = [v_1, v^+_{n-2}]v^{+,+}_{n-2,n}v^+_nv^{+,+}_{n-1,n}$ with $w = v^{+,-}_{n-1,n}$, let $P = [v_1, v^+_{n-2}]v^{+,+}_{n-2,n}v^+_nv^{+,+}_{n-1,n}v^+_{n-1}$ with $w = v^+_{n-1}$, and let $P = [v_1, v^+_{n-2}]v^{+,+}_{n-2,n}v^+_nv^{+,+}_{n-1,n}v^+_{n-1}v^{+,-}_{n-1,n}$ with $w = v^{+,+}_{n-1,n}$. Then P satisfies the conditions. □

Lemma 10: Let $n \ge 2$ and $k \ge 4$. Then $\kappa(Q^k_n; P_{2l+1}) \le \lceil \frac{2n}{l+1} \rceil$ for $1 \le l+1 \le 2n$, and $\kappa(Q^k_n; P_{2l}) \le \lceil \frac{2n}{l} \rceil$ for $l \le 2n$.

Proof: Let $u = 111\ldots11$ and $v = 211\ldots12$. Then $u^+_n = v^-_1$, that is, $u_{2n} = v_1$. Consider the *neighbour structure* $P(u)$ and $P(v)$ of u and v.

For P_{2l+1}, We will successively find $\lceil \frac{2n}{l+1} \rceil$ pairwise disjoint P_{2l+1}'s denoted by $P^1_{2l+1}, P^2_{2l+1}, \ldots, P^{\lceil \frac{2n}{l+1} \rceil}_{2l+1}$ using $P(u)$ and $P(v)$. If $l+1 = 2n$, then $\lceil \frac{2n}{l+1} \rceil = 1$, and let $P^1_{2l+1} = [u_1, u_{2n}]$. If $l+1 < 2n$, then let $2n = p(l+1)+q$, and let $P^1_{2l+1} = [u_1, u_{l+1}]$ (see FIGURE 10), $P^2_{2l+1} = [u_{l+2}, u_{2l+2}]$, $P^3_{2l+1} = [u_{2l+3}, u_{3l+3}]$, $\ldots$, $P^p_{2l+1} = [u_{(p-1)l+p}, u_{pl+p}]$. By $l+1 < 2n$ and Lemma 9, we can find $P^{\lceil \frac{2n}{l+1} \rceil}_{2l+1}$ with $v^-_n \notin V(P^{\lceil \frac{2n}{l+1} \rceil}_{2l+1})$.

For P_{2l}, We will successively find $\lceil \frac{2n}{l} \rceil$ pairwise disjoint P_{2l}'s denoted by $P^1_{2l}, P^2_{2l}, \ldots, P^{\lceil \frac{2n}{l} \rceil}_{2l}$ using $P(u)$ and $P(v)$. If $l = 2n$, then $\lceil \frac{2n}{l} \rceil = 1$, and let $P^1_{2l} = [u_1, u_{2n}]v_1'$. If $l < 2n$, then let $2n = pl+q$, and let $P^1_{2l} = [u_1, u_l']$, $P^2_{2l} = [u_{l+1}, u_{2l}']$, $P^3_{2l} = [u_{2l+1}, u_{3l}']$, $\ldots$, $P^p_{2l} = [u_{(p-1)l+1}, u_{pl}']$. By $l < 2n$ and Lemma 9, we can find $P^{\lceil \frac{2n}{l} \rceil}_{2l}$ with $v^-_n \notin V(P^{\lceil \frac{2n}{l} \rceil}_{2l})$.

Let $F = \{P^1_{2l+1}, P^2_{2l+1}, \ldots, P^{\lceil \frac{2n}{l+1} \rceil}_{2l+1}\}$. Then $Q^k_n - F$ is disconnected because $\{u\}$ is a component of $Q^k_n - F$. By definition of $\kappa(Q^k_n; P_{2l+1})$, $\kappa(Q^k_n; P_{2l+1}) \le \lceil \frac{2n}{l+1} \rceil$. Similarly, we can show that $\kappa(Q^k_n; P_{2l}) \le \lceil \frac{2n}{l} \rceil$. □

Lemma 11: Let $n \ge 5$ and $k \ge 4$. Then $\kappa^s(Q^k_n; P_{2l+1}) \ge \lceil \frac{2n}{l+1} \rceil$ for $1 \le l+1 \le 2n$ and $\kappa^s(Q^k_n; C_{2l+1}) \ge \lceil \frac{2n}{l+1} \rceil$ for $3 \le l+1 \le 2n$.

Proof: We only show that $\kappa^s(Q^k_n; C_{2l+1}) \ge \lceil \frac{2n}{l+1} \rceil$. The proof of $\kappa^s(Q^k_n; P_{2l+1}) \ge \lceil \frac{2n}{l+1} \rceil$ is similar. Let $F = \cup_{i=1}^{2l+1} F_i \cup F'_{2l+1}$ with $|F| = \sum_{i=1}^{2l+1} |F_i| + |F'_{2l+1}| \le \lceil \frac{2n}{l+1} \rceil - 1$. In order to prove that $\kappa^s(Q^k_n; C_{2l+1}) \ge \lceil \frac{2n}{l+1} \rceil$, it is enough to show that $Q^k_n - F$ is connected. Suppose, to the contrary, that $Q^k_n - F$ is disconnected. Let T_0 be a smallest component of $Q^k_n - F$.

Case 1: $|V(T_0)| = 1$.

Set $V(T_0) = \{x\}$. Thus $N(x) \subseteq V(F)$. To make the number of faulty subgraphs of C_{2l+1} minimum which contain the nodes in $N(x)$, we should construct as many P_{2l+1}'s/C_{2l+1}'s as possible and each P_{2l+1}/C_{2l+1} need to contain as many nodes in $N(x)$ as possible. Since Q^k_n contains no triangles for $k \ge 4$, any two nodes in $N(x)$ are not two consecutive nodes on a path/cycle. Combining this with the definition of the *neighbour structure* of x, each P_{2l+1}/C_{2l+1} contain at most $l+1$ nodes in $N(x)$. Note that $|N(x)| = 2n$. Then $|F| \ge \lceil \frac{2n}{l+1} \rceil > \lceil \frac{2n}{l+1} \rceil - 1 \ge |F|$, a contradiction.

Case 2: $|V(T_0)| = 2$.

Set $V(T_0) = \{\{x, y\} | xy \in E(Q^k_n)\}$. Thus $N(\{x, y\}) \subseteq V(F)$, and so $|V(F)| \ge |N(\{x, y\})| = 4n - 2$. Note that $|F| \le \lceil \frac{2n}{l+1} \rceil - 1$. Then $|V(F)| \le (2l+1)(\lceil \frac{2n}{l+1} \rceil - 1) \le (2l+1)(\frac{2n+l}{l+1} - 1) = \frac{(2l+1)}{(l+1)}(2n-1) < 4n-2 \le |V(F)|$, a contradiction.

Case 3: $|V(T_0)| \geq 3$.

By $n \geq 5$, $|V(T_0)| \geq 3$ and Lemma 2, $|V(F)| \geq 6n-5$. Recall that $|V(F)| < 4n-2 < 6n-5 \leq |V(F)|$, a contradiction. □

Lemma 12: Let $n \geq 5$ and $k \geq 4$. Then $\kappa^s(Q_n^k; P_{2l}) \geq \kappa^s(Q_n^k; C_{2l}) \geq \lceil \frac{2n}{l} \rceil$ for $2 \leq l \leq 2n$.

Proof: By Lemma 1, $\kappa^s(Q_n^k; P_{2l}) \geq \kappa^s(Q_n^k; C_{2l})$. Let $F = \cup_{i=1}^{2l} F_i \cup F'_{2l}$ with $|F| = \sum_{i=1}^{2l} |F_i| + |F'_{2l}| \leq \lceil \frac{2n}{l} \rceil - 1$. In order to prove that $\kappa^s(Q_n^k; C_{2l+1}) \geq \lceil \frac{2n}{l+1} \rceil$, it is enough to show that $Q_n^k - F$ is connected. Suppose, to the contrary, that $Q_n^k - F$ is disconnected. Let T_0 be a smallest component of $Q_n^k - F$.

Case 1: $|V(T_0)| = 1$.

Set $V(T_0) = \{x\}$. Thus $N(x) \subseteq V(F)$. To make the number of faulty subgraphs of C_{2l} minimum which contain the nodes in $N(x)$, we should construct as many P_{2l}'s/C_{2l}'s as possible and each P_{2l}/C_{2l} need to contain as many nodes in $N(x)$ as possible. Since Q_n^k contains no triangles for $k \geq 4$, any two nodes in $N(x)$ are not two consecutive nodes on a path/cycle. Combining this with the definition of the *neighbour structure* of x, each P_{2l}/C_{2l} contain at most l nodes in $N(x)$. Note that $|N(x)| = 2n$. Then $|F| \geq \lceil \frac{2n}{l} \rceil > \lceil \frac{2n}{l} \rceil - 1 \geq |F|$, a contradiction.

Case 2: $|V(T_0)| = 2$.

Set $V(T_0) = \{\{x, y\} | xy \in E(Q_n^k)\}$. Thus $N(\{x, y\}) \subseteq V(F)$, and so $|V(F)| \geq |N(\{x, y\})| = 4n-2$. Note that $|F| \leq \lceil \frac{2n}{l} \rceil - 1$. Thus $|V(F)| \leq (2l)(\lceil \frac{2n}{l} \rceil - 1) \leq (2l)(\frac{2n+l-1}{l} - 1) = 4n-2$. We have $|V(F)| = 4n-2$, and so $V(F) = N(\{x, y\})$. Note that Q_n^k contains no C_3 for $k \geq 4$. Thus any two nodes in $N(x)$ or $N(y)$ are not adjacent. Without loss of generality, assume that $x = 11\ldots1$ and $y = 01\ldots1$. For the two nodes $u = 21\ldots1$ and $v = (k-1)1\ldots1$ in $N(\{x, y\})$, we see that u, v are not adjacent to the nodes in $N(\{x, y\}) \setminus \{u, v\}$. Thus u, v are not on a path P_k with $k \geq 4$ and $V(P_k) \subseteq N(\{x, y\})$. It follows that $|F| \geq \lceil \frac{4n-4}{2l} \rceil + 1$. Recall that $|F| \leq \lceil \frac{2n}{l} \rceil - 1$. Then $|F| \leq \frac{2n+l-1}{l} - 1 < \frac{4n-4}{2l} + 1 \leq \lceil \frac{4n-4}{2l} \rceil + 1 \leq |F|$ by $l \geq 2$, a contradiction.

Case 3: $|V(T_0)| \geq 3$.

By $n \geq 5$, $|V(T_0)| \geq 3$ and Lemma 2, $|V(F)| \geq 6n-5$. Recall that $|V(F)| \leq 4n-2 < 6n-5 \leq |V(F)|$, a contradiction. □

Note that $\kappa(Q_n^k; P_i) \geq \kappa^s(Q_n^k; P_i)$. Lemmas 10, 11 and 12 yield following result.

Theorem 6: Let $n \geq 5$ and $k \geq 4$. Then $\kappa(Q_n^k; P_{2l+1}) = \kappa^s(Q_n^k; P_{2l+1}) = \lceil \frac{2n}{l+1} \rceil$ for $1 \leq l+1 \leq 2n$ and $\kappa(Q_n^k; P_{2l}) = \kappa^s(Q_n^k; P_{2l}) = \lceil \frac{2n}{l} \rceil$ for $2 \leq l \leq 2n$.

Set $2l+1 = 1, 3$ in the Theorem 6. Then Theorem 5 given by Lv *et al.* [11] is an immediate corollary of Theorem 6.

B. $\kappa(Q_n^k; C_l)$ AND $\kappa^s(Q_n^k; C_l)$

In this subsection, we investigate the cycle-structure/substructure connectivity for Q_n^k.

Lemma 13: Let $n \geq 5$ and $k \geq 4$. Then $\kappa(Q_n^k; C_{2l}) \leq \lceil \frac{2n}{l} \rceil$ for $4 \leq l \leq 2n$.

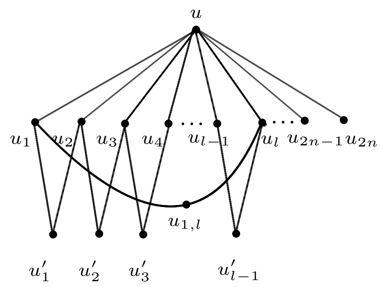

FIGURE 11. A cycle $C_{2l}^1 = [u_1, u_l]u_{1,l}u_1$ using $P(u)$.

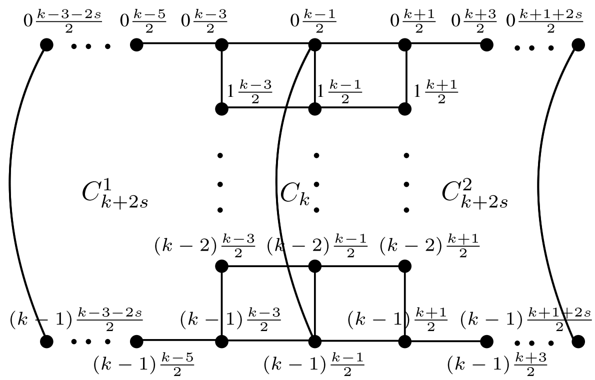

FIGURE 12. An example of C_k, C_{k+2s}^1 and C_{k+2s}^2 in Q_2^k.

Proof: Let $u = 111\ldots11$ and $v = 211\ldots12$. Then $u_n^+ = v_1^-$, that is, $u_{2n} = v_1$. Consider the *neighbour structure* $P(u)$ and $P(v)$ of u and v. Note that $P(u) = u_1u'_1u_2u'_2u_3\ldots u_{2n-1}u'_{2n-1}u_{2n}$. In the following, we first give a claim which can be used to construct the desired cycles.

Claim 1. *For any $u_i, u_j \in V(P(u))$ with $1 \leq i < j \leq 2n$ and $j - i \geq 3$, there exists $u_{i,j} \in V(Q_n^k)$ with $u_{i,j} \notin V(P(u))$ such that $u_iu_{i,j}u_j$ is a P_3.*

By the definition of u_i, we have $u_i, u_j \in \{u_1^-, u_2^-, u_1^+, u_2^+, u_3^-, \ldots, u_n^-, u_{n-1}^+, u_n^+\}$. Without loss of generality, assume that $u_i = u_s^-$ and $u_j = u_t^+$. By the definition of $P(u)$ and $j-i \geq 3$, we have $s < t$. Let $u_{i,j} = u_{s,t}^{-,+}$. Then $u_{i,j} \notin V(P(u))$ and $u_iu_{i,j}u_j$ is a P_3. The claim holds.

We will successively find $\lceil \frac{2n}{l} \rceil$ pairwise disjoint C_{2l}'s denoted by $C_{2l}^1, C_{2l}^2, \ldots, C_{2l}^{\lceil \frac{2n}{l} \rceil}$ by using $P(u)$ and $P(v)$ with $v_n^- \notin V(C_{2l}^{\lceil \frac{2n}{l} \rceil})$. If $l = 2n$, then $\lceil \frac{2n}{l} \rceil = 1$. By Claim 1, let $C_{2l}^1 = [u_1, u_{2n}]u_{1,2n}u_1$. Next consider $l < 2n$ and assume that $2n = pl + q$. Then $p + 1 = \lceil \frac{2n}{l} \rceil$. By Claim 1, let $C_{2l}^1 = [u_1, u_l]u_{1,l}u_1$ (see FIGURE 11), $C_{2l}^2 = [u_{l+1}, u_{2l}]u_{l+1,2l}u_{l+1}$, $C_{2l}^3 = [u_{2l+1}, u_{3l}]u_{2l+1,3l}u_{2l+1}$, $\ldots$, $C_{2l}^p = [u_{(p-1)l+1}, u_{pl}]u_{(p-1)l+1,pl}u_{(p-1)l+1}$. If $q = 1$, then, by Lemma 9 and Claim 1, let $C_{2l}^{\lceil \frac{2n}{l} \rceil} = [v_1, v_l]v_{1,l}v_1$ with $v_n^- \notin V(C_{2l}^{\lceil \frac{2n}{l} \rceil})$. Next assume that $q \geq 2$. Then $2n - q + 1 \leq 2n - 1$. If $l - q = 1$, then $u_{2n-q+1} \neq u_n^-$. If not, then $2n - q + 1 = 2n - 2$, and so $q = 3$ and $l = 4$. Note that $2n = pl + q$, that is, $2n = 4p + 3$, a contradiction. Let $C_{2l}^{\lceil \frac{2n}{l} \rceil} = [u_{2n-q+1}, u_{2n}]vv_{2n-q+1}(v_{2n-q+1})_n^-u_{2n-q+1}$.

Note that $|V(C_{2l}^{\lceil \frac{2n}{l} \rceil})| = 2q-1+3 = 2q+2 = 2l$ and $v_{2n-q+1} \neq v_n^-$. Then $C_{2l}^{\lceil \frac{2n}{l} \rceil}$ is indeed a cycle on $2l$ nodes. Now consider $l-q \geq 2$. By $l < 2n$, $2n-q+1-(l-q) = 2n-l+1 \geq 2$. If $u_{2n-q+1} = u_n^-$ and $v_{l-q} \in \{v_2^-, v_2^+\}$, then $q = 3$ and $l-q \in \{2, 4\}$. Thus $l = 5$ or $l = 7$. For $l = 5$, let $x = 31\ldots 11$, $y = 31\ldots 10$, $z = 21\ldots 10$ and $C_{2l}^{\lceil \frac{2n}{l} \rceil} = [u_n^-, u_n^+]vv_1^+xyzu_n^-$. Then $|V(C_{2l}^{\lceil \frac{2n}{l} \rceil})| = 10$. For $l = 7$, let $C_{2l}^{\lceil \frac{2n}{l} \rceil} = [u_n^-, u_n^+]v_{1,2}^{-,-}v_2^-vv_n^+v_{n-1,n}^{+,+}v_{n-1}^+u_{1,n-1}^{+,+}u_{1,n-1,n}^{+,+,-}u_{1,n}^{+,-}u_n^-$ by $v_{n-1}^+ = u_{1,n-1,n}^{+,+,+}$. Then $|V(C_{2l}^{\lceil \frac{2n}{l} \rceil})| = 14$.

Claim 2. *There exist $x, y, z \in V(Q_n^k)$ with $x, y, z \notin V(\cup_{i=1}^{\lceil \frac{2n}{l} \rceil - 1} C_{2l}^i)$, such that $v_{l-q}xyzu_{2n-q+1}$ is a P_5, as long as $u_{2n-q+1} \neq u_n^-$ or $v_{l-q} \notin \{v_2^-, v_2^+\}$.*

Assume that uu_{2n-q+1} and vv_{l-q} are j-dimensional and i-dimensional links, respectively. By $2n-q+1-(l-q) \geq 2$, $i \leq j$. If $i = j$, then $2n-q+1-(l-q) = 2$, $u_{2n-q+1} = u_j^+$ and $v_{l-q} = v_j^-$. By $l \geq 4$, $2n-q+1 = pl+1 \geq 5$, and so $j \geq 3$. Recall that $q \geq 2$. Then $j \leq n-1$. Let $x = v$, $y = v_j^+$ and $z = u_{1,j}^{+,+}$. Then $x, y, z \notin V(\cup_{i=1}^{\lceil \frac{2n}{l} \rceil - 1} C_{2l}^i)$ and $v_{l-q}xyzu_{2n-q+1}$ is a P_5. Next assume that $i < j$. Then $2n-q+1-(l-q) \geq 3$. If $1 < i < j < n$, then, without loss of generality, assume that $u_{2n-q+1} = u_j^-$ and $v_{l-q} = v_i^+$. Let $x = u_{i,n}^{+,+}$, $y = u_{i,j,n}^{+,-,+}$ and $z = u_{j,n}^{-,+}$. By definition of $P(u)$ and $P(v)$, $x, y, z \notin V(\cup_{i=1}^{\lceil \frac{2n}{l} \rceil - 1} C_{2l}^i)$. Note that $v_i^+ = u_{1,i,n}^{+,+,+}$. Then $v_{l-q}xyzu_{2n-q+1}$ is a P_5. Now consider $i = 1$ or $j = n$, which is equivalent to $v_{l-q} = v_1^+$ or $u_{2n-q+1} = u_n^-$ by $l-q \geq 2$ and $q \geq 2$. If $v_{l-q} = v_1^+$ and $u_{2n-q+1} = u_n^-$, then let $x = 31\ldots 11$, $y = 31\ldots 10$ and $z = 21\ldots 10$. By definition of $P(u)$ and $P(v)$, $x, y, z \notin V(\cup_{i=1}^{\lceil \frac{2n}{l} \rceil - 1} C_{2l}^i)$ and $v_{l-q}xyzu_{2n-q+1}$ is a P_5. If $v_{l-q} \neq v_1^+$ and $u_{2n-q+1} = u_n^-$, then, without loss of generality, assume that $v_{l-q} = v_i^-$. Note that the hypothesis that $u_{2n-q+1} \neq u_n^-$ or $v_{l-q} \notin \{v_2^-, v_2^+\}$. Thus $i \geq 3$ by $l-q \geq 2$. Let $x = u_{1,i}^{+,-}$, $y = u_{1,i,n}^{+,-,-}$ and $z = u_{1,n}^{+,-}$. By definition of $P(u)$ and $P(v)$, $x, y, z \notin V(\cup_{i=1}^{\lceil \frac{2n}{l} \rceil - 1} C_{2l}^i)$. Note that $v_i^- = u_{1,i,n}^{+,-,+}$. Then $v_{l-q}xyzu_{2n-q+1}$ is a P_5. If $v_{l-q} = v_1^+$ and $u_{2n-q+1} \neq u_n^-$, then, without loss of generality, assume that $u_{2n-q+1} = u_j^-$ with $j \leq n-1$. Then $u_j^- = 11\ldots 101\ldots 11$ and $v_1^+ = 31\ldots 111\ldots 12$. Let $x = 31\ldots 111\ldots 11$, $y = 31\ldots 101\ldots 11$ and $z = 21\ldots 101\ldots 11$. By definition of $P(u)$ and $P(v)$, $x, y, z \notin V(\cup_{i=1}^{\lceil \frac{2n}{l} \rceil - 1} C_{2l}^i)$. Then $v_{l-q}xyzu_{2n-q+1}$ is a P_5. The claim holds.

By Claim 2, let $C_{2l}^{\lceil \frac{2n}{l} \rceil} = [u_{2n-q+1}, u_{2n}][v_2, v_{l-q}]xyz\, u_{2n-q+1}$. Note that $|V(C_{2l}^{\lceil \frac{2n}{l} \rceil})| = 2q+2(l-1-q)+2 = 2l$. Then $C_{2l}^{\lceil \frac{2n}{l} \rceil}$ is indeed a cycle on $2l$ nodes.

Let $F = \{C_{2l}^1, C_{2l}^2, \ldots, C_{2l}^{\lceil \frac{2n}{l} \rceil}\}$. Then $Q_n^k - F$ is disconnected because $\{u\}$ is a component of $Q_n^k - F$. By definition of $\kappa(Q_n^k; C_{2l})$, $\kappa(Q_n^k; C_{2l}) \leq \lceil \frac{2n}{l} \rceil$. $\square$

Note that $\kappa(Q_n^k; C_{2l}) \geq \kappa^s(Q_n^k; C_{2l})$. Lemmas 12 and 13 yield the following result.

Theorem 7: Let $n \geq 5$ and $k \geq 4$. Then $\kappa(Q_n^k; C_{2l}) = \kappa^s(Q_n^k; C_{2l}) = \lceil \frac{2n}{l} \rceil$ for $4 \leq l \leq 2n$.

Next, we consider the case that Q_n^k contains odd cycles. Note that Q_n^k is bipartite if and only if k is even. Thus Q_n^k contains odd cycles only if k is odd. The minimum odd cycle in Q_n^k is C_k, which implies that the general odd cycle in Q_n^k can be denoted by C_{k+2s} for $s \geq 0$.

Lemma 14: Let $n \geq 2$ and odd $k \geq 5$. Then $\kappa(Q_n^k; C_{k+2s}) \leq 2n-2$ for $0 \leq s \leq \frac{k-3}{2}$.

Proof: Let $C_k = (0\tfrac{k-1}{2}\tfrac{k-1}{2}\ldots\tfrac{k-1}{2})(1\tfrac{k-1}{2}\tfrac{k-1}{2}\ldots\tfrac{k-1}{2})\ldots((k-1)\tfrac{k-1}{2}\tfrac{k-1}{2}\ldots\tfrac{k-1}{2})(0\tfrac{k-1}{2}\tfrac{k-1}{2}\ldots\tfrac{k-1}{2})$ be a cycle of Q_n^k. We will find $2n-2$ pairwise disjoint C_{k+2s}'s denoted by $C_{k+2s}^1, C_{k+2s}^2, \ldots, C_{k+2s}^{2n-2}$ by using $N(V(C_k))$ (see FIGURE 12 for an example of C_k, C_{k+2s}^1 and C_{k+2s}^2 in Q_2^k). Let

$$C_{k+2s}^1 = (0\tfrac{k-3}{2}\tfrac{k-1}{2}\ldots\tfrac{k-1}{2})(0\tfrac{k-5}{2}\tfrac{k-1}{2}\ldots\tfrac{k-1}{2})\ldots(0\tfrac{k-1-2s}{2}\tfrac{k-1}{2}\ldots\tfrac{k-1}{2})(0\tfrac{k-3-2s}{2}\tfrac{k-1}{2}\ldots\tfrac{k-1}{2})((k-1)\tfrac{k-3-2s}{2}\tfrac{k-1}{2}\ldots\tfrac{k-1}{2})((k-1)\tfrac{k-1-2s}{2}\tfrac{k-1}{2}\ldots\tfrac{k-1}{2})\ldots((k-1)\tfrac{k-5}{2}\tfrac{k-1}{2}\ldots\tfrac{k-1}{2})((k-1)\tfrac{k-3}{2}\tfrac{k-1}{2}\ldots\tfrac{k-1}{2})((k-2)\tfrac{k-3}{2}\tfrac{k-1}{2}\ldots\tfrac{k-1}{2})\ldots(1\tfrac{k-3}{2}\tfrac{k-1}{2}\ldots\tfrac{k-1}{2})(0\tfrac{k-3}{2}\tfrac{k-1}{2}\ldots\tfrac{k-1}{2}),$$

$$C_{k+2s}^2 = (0\tfrac{k+1}{2}\tfrac{k-1}{2}\ldots\tfrac{k-1}{2})(0\tfrac{k+3}{2}\tfrac{k-1}{2}\ldots\tfrac{k-1}{2})\ldots(0\tfrac{k-1+2s}{2}\tfrac{k-1}{2}\ldots\tfrac{k-1}{2})(0\tfrac{k+1+2s}{2}\tfrac{k-1}{2}\ldots\tfrac{k-1}{2})((k-1)\tfrac{k+1+2s}{2}\tfrac{k-1}{2}\ldots\tfrac{k-1}{2})((k-1)\tfrac{k-1+2s}{2}\tfrac{k-1}{2}\ldots\tfrac{k-1}{2})\ldots((k-1)\tfrac{k+3}{2}\tfrac{k-1}{2}\ldots\tfrac{k-1}{2})((k-1)\tfrac{k+1}{2}\tfrac{k-1}{2}\ldots\tfrac{k-1}{2})((k-2)\tfrac{k+1}{2}\tfrac{k-1}{2}\ldots\tfrac{k-1}{2})\ldots(1\tfrac{k+1}{2}\tfrac{k-1}{2}\ldots\tfrac{k-1}{2})(0\tfrac{k+1}{2}\tfrac{k-1}{2}\ldots\tfrac{k-1}{2}),$$

$$C_{k+2s}^3 = (0\tfrac{k-1}{2}\tfrac{k-3}{2}\ldots\tfrac{k-1}{2})(0\tfrac{k-1}{2}\tfrac{k-5}{2}\ldots\tfrac{k-1}{2})\ldots(0\tfrac{k-1}{2}\tfrac{k-1-2s}{2}\ldots\tfrac{k-1}{2})(0\tfrac{k-1}{2}\tfrac{k-3-2s}{2}\ldots\tfrac{k-1}{2})((k-1)\tfrac{k-1}{2}\tfrac{k-3-2s}{2}\ldots\tfrac{k-1}{2})((k-1)\tfrac{k-1}{2}\tfrac{k-1-2s}{2}\ldots\tfrac{k-1}{2})\ldots((k-1)\tfrac{k-1}{2}\tfrac{k-5}{2}\ldots\tfrac{k-1}{2})((k-1)\tfrac{k-1}{2}\tfrac{k-3}{2}\ldots\tfrac{k-1}{2})((k-2)\tfrac{k-1}{2}\tfrac{k-3}{2}\ldots\tfrac{k-1}{2})\ldots(1\tfrac{k-1}{2}\tfrac{k-3}{2}\ldots\tfrac{k-1}{2})(0\tfrac{k-1}{2}\tfrac{k-3}{2}\ldots\tfrac{k-1}{2}),$$

$$C_{k+2s}^4 = (0\tfrac{k-1}{2}\tfrac{k+1}{2}\ldots\tfrac{k-1}{2})(0\tfrac{k-1}{2}\tfrac{k+3}{2}\ldots\tfrac{k-1}{2})\ldots(0\tfrac{k-1}{2}\tfrac{k-1+2s}{2}\ldots\tfrac{k-1}{2})(0\tfrac{k-1}{2}\tfrac{k+1+2s}{2}\ldots\tfrac{k-1}{2})((k-1)\tfrac{k-1}{2}\tfrac{k+1+2s}{2}\ldots\tfrac{k-1}{2})((k-1)\tfrac{k-1}{2}\tfrac{k-1+2s}{2}\ldots\tfrac{k-1}{2})\ldots((k-1)\tfrac{k-1}{2}\tfrac{k+3}{2}\ldots\tfrac{k-1}{2})((k-1)\tfrac{k-1}{2}\tfrac{k+1}{2}\ldots\tfrac{k-1}{2})((k-2)\tfrac{k-1}{2}\tfrac{k+1}{2}\ldots\tfrac{k-1}{2})\ldots(1\tfrac{k-1}{2}\tfrac{k+1}{2}\ldots\tfrac{k-1}{2})(0\tfrac{k-1}{2}\tfrac{k+1}{2}\ldots\tfrac{k-1}{2}),$$

$\ldots$

$$C_{k+2s}^{2n-3} = (0\tfrac{k-1}{2}\tfrac{k-1}{2}\ldots\tfrac{k-3}{2})(0\tfrac{k-1}{2}\ldots\tfrac{k-1}{2}\tfrac{k-5}{2})\ldots(0\tfrac{k-1}{2}\tfrac{k-1}{2}\ldots\tfrac{k-1-2s}{2})(0\tfrac{k-1}{2}\tfrac{k-1}{2}\ldots\tfrac{k-3-2s}{2})((k-1)\tfrac{k-1}{2}\tfrac{k-1}{2}\ldots\tfrac{k-3-2s}{2})((k-1)\tfrac{k-1}{2}\tfrac{k-1}{2}\ldots\tfrac{k-1-2s}{2})\ldots((k-1)\tfrac{k-1}{2}\tfrac{k-1}{2}\ldots\tfrac{k-5}{2})((k-1)\tfrac{k-1}{2}\tfrac{k-1}{2}\ldots\tfrac{k-3}{2})((k-2)\tfrac{k-1}{2}\tfrac{k-1}{2}\ldots\tfrac{k-3}{2})\ldots(1\tfrac{k-1}{2}\tfrac{k-1}{2}\ldots\tfrac{k-3}{2})(0\tfrac{k-1}{2}\tfrac{k-1}{2}\ldots\tfrac{k-3}{2}),$$

$$C_{k+2s}^{2n-2} = (0\tfrac{k-1}{2}\tfrac{k-1}{2}\ldots\tfrac{k+1}{2})(0\tfrac{k-1}{2}\tfrac{k-1}{2}\ldots\tfrac{k+3}{2})\ldots(0\tfrac{k-1}{2}\tfrac{k-1}{2}\ldots\tfrac{k-1}{2}\tfrac{k-1+2s}{2})(0\tfrac{k-1}{2}\tfrac{k-1}{2}\ldots\tfrac{k+1+2s}{2})((k-1)\tfrac{k-1}{2}\tfrac{k-1}{2}\ldots\tfrac{k+1+2s}{2})((k-1)\tfrac{k-1}{2}\tfrac{k-1}{2}\ldots\tfrac{k-1+2s}{2})\ldots((k-1)\tfrac{k-1}{2}\tfrac{k-1}{2}\ldots\tfrac{k+3}{2})((k-1)\tfrac{k-1}{2}\tfrac{k-1}{2}\ldots\tfrac{k+1}{2})((k-2)\tfrac{k-1}{2}\tfrac{k-1}{2}\ldots\tfrac{k+1}{2})\ldots(1\tfrac{k-1}{2}\tfrac{k-1}{2}\ldots\tfrac{k+1}{2})(0\tfrac{k-1}{2}\tfrac{k-1}{2}\ldots\tfrac{k+1}{2}).$$

Let $F = \{C_{k+2s}^1, C_{k+2s}^2, \ldots, C_{k+2s}^{2n-2}\}$. Then $Q_n^k - F$ is disconnected because C_k is a component of $Q_n^k - F$. By definition of $\kappa(Q_n^k; C_{k+2s})$, $\kappa(Q_n^k; C_{k+2s}) \leq 2n-2$. $\square$

Set $2l+1 = k+2s$ in the Lemma 14. Then $s = \frac{2l+1-k}{2}$, and so $0 \leq s \leq \frac{k-3}{2}$ is equivalent $\frac{k-1}{2} \leq l \leq k-2$. We have:

Theorem 8: Let $n \geq 2$ and odd $k \geq 5$. Then $\kappa(Q_n^k; C_{2l+1}) \leq 2n-2$ for $\frac{k-1}{2} \leq l \leq k-2$.

Let $n \geq 5$ and $k \geq 4$ with $l+1 \leq 2n$. By Lemmas 1 and 11, $\kappa^s(Q_n^k; P_{2l+1}) \geq \kappa^s(Q_n^k; C_{2l+1}) \geq \lceil \frac{2n}{l+1} \rceil$. By Theorem 6, $\kappa^s(Q_n^k; P_{2l+1}) = \lceil \frac{2n}{l+1} \rceil$. Thus we obtain the following result.

Theorem 9: Let $n \geq 5$ and odd $k \geq 5$. Then $\kappa^s(Q_n^k; C_{2l+1}) = \lceil \frac{2n}{l+1} \rceil$ for $\frac{k+1}{2} \leq l+1 \leq 2n$.

V. CONCLUSION

In a given network, how many of a particular *structure* can go faulty, and the network still remains connected? That is the question this paper tried to address. It established structure connectivity $\kappa(Q_n^k; T)$ and substructure connectivity $\kappa^s(Q_n^k; T)$, where $k \geq 3$, and T is a path or cycle, both being basic yet important structures in all computer networks. Our work not only generalized the known result on path structures [11], but also extended it to cycle structures. These results reveal new characteristics of Q_n^k, affording more insights into this important network.

The paper leaves a few unresolved open questions. (1) For Q_n^3 and C_{3l+1}, cycles on $3l+1$ nodes, $\kappa(Q_n^3; C_{3l+1})$ is yet to be determined; and (2) The paper's result on structure connectivity for odd-node cycles, $\kappa(Q_n^k; C_{2l+1})$ with odd $k \geq 5$, is an *upper-bound*, instead of a definitive connectivity. These two sub-problems proved to be challenging, and solving them will completely solve the Q_n^k's structure/substructure connectivity for paths and cycles. New and more innovative approaches, different than ours used in this paper, might be in order.

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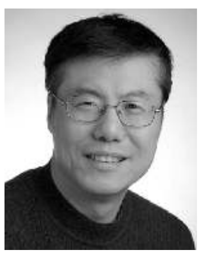

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