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STRUCTURE FOR NONNEGATIVE SQUARE ROOTS OF UNBOUNDED NONNEGATIVE SELFADJOINT OPERATORS

By

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Abstract. It is well known that, for an unbounded nonnegative selfadjoint operator A on a Hilbert space, there is a unique nonnegative square root $A^{1/2}$, which is frequently associated with the structural damping in many practical vibration systems. In this paper we develop a general theory for the structure of $A^{1/2}$, which includes the expression of $A^{1/2}$ and a program to find the domain of $A^{1/2}$ explicitly from the domain of A. The relationship between $A^{1/2}$ and related differential operators is determined for the selfadjoint differential operator A. Finally, the theoretical results given in this paper are applied to fourth-order "beam" operators and n-dimensional "wave" operators with sufficient complexity for applications to elastic vibration systems.

1. Introduction. Let H be a Hilbert space with the inner product $(\cdot,\cdot)$ and the induced norm $\|\cdot\|$. Let A be an unbounded nonnegative selfadjoint operator on H and $\mathcal{L}$ a closed linear operator on H. In the last two decades, great attention has been focused on the following elastic system:

$$\begin{cases} \ddot{y}(t) + \mathcal{L}\dot{y}(t) + Ay(t) = 0, \\ y(0) = y_0, \ \dot{y}(0) = y_1, \end{cases} \tag{1.1}$$

where a dot denotes $\frac{d}{dt}$, and $y, y_0, y_1 \in H$. The usual procedure for dealing with the system (1.1) is as follows.

Letting $x_1 = A^{1/2}y, x_2 = y$, the system (1.1) can be transformed into an equivalent first-order evolution system

$$\begin{cases} \frac{d}{dt}\tilde{x}(t) = \mathcal{A}\tilde{x}(t), \\ \tilde{x}(0) = Y_0, \end{cases} \tag{1.2}$$

where

$$\tilde{x}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \mathcal{A} = \begin{pmatrix} 0 & A^{1/2} \\ -A^{1/2} & -\mathcal{L} \end{pmatrix}, \qquad Y_0 = \begin{pmatrix} A^{1/2}y_0 \\ y_1 \end{pmatrix},$$

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and $A^{1/2}$ is the nonnegative square root of A. The domain of $\mathcal{A}$ is

$$D(A^{1/2}) \times D(A^{1/2})$$

if $D(\mathcal{L}) \subset D(A^{1/2})$, where $D(\mathcal{L})$ is the domain of $\mathcal{L}$. For many real systems, however, the structure of $D(A^{1/2})$ is not clear except that $D(A)$ has special boundary conditions.

On the other hand, $A^{1/2}\dot{y}$ corresponds to structural damping for elastic vibration systems in [1] and is related to frequency proportional damping in [4]. Thus, it is necessary to understand the relationship between $A^{1/2}$ and differential operators in order that $A^{1/2}\dot{y}$ admits the proper physical interpretation if A is a differential operator. D. L. Russell in [2] gave the relationship between $A^{1/2}$ and differential operators for fourth-order "beam" operators with the "symmetric" boundary conditions but the structure of $D(A^{1/2})$ is not dealt with. A. V. Balakrishnan in [3], [4] obtained $D(A^{1/2})$ and the relationship between $A^{1/2}$ and differential operators for two models of bending of uniform Bernoulli beams by the formula introduced in [5].

In this paper, we develop a general theory for the structure of the nonnegative square root of any unbounded nonnegative selfadjoint operator A. In Sec. 2, a program is given to find the domain of $A^{1/2}$ and the expression of $A^{1/2}$ is studied. Finally, in Secs. 3 and 4, we apply the theoretical results developed in the paper to fourth-order elastic "beam" operators and n-dimensional "wave" operators, respectively.

2. The main results. Throughout this paper we make the following assumptions.

Let H be a complex Hilbert space with the inner product $(\cdot,\cdot)$ and the induced norm $\|\cdot\|$, and let A be an unbounded nonnegative selfadjoint operator on H with compact resolvent and domain $D(A)$; let $A^{1/2}$ be the nonnegative square root of A. Note that A is nonnegative if and only if

$$(Ax, x) \geq 0, \quad \forall x \in D(A).$$

The operator A is said to be positive if

$$(Ax, x) > 0, \quad \forall x \in D(A),\ x \neq 0.$$

Let H_1 be another complex Hilbert space with the inner product $(\cdot,\cdot)_1$ and the induced norm $\|\cdot\|_1$ and $B : D(B) \subset H \to H_1$ be a closed linear operator such that

$$D(B) \supset D(A); \tag{2.1}$$

$$(Ax, x) \geq \|Bx\|_1^2, \quad \forall x \in D(A). \tag{2.2}$$

Set

$$[x, y] = (x, Ay) - (Bx, By)_1, \quad \forall x \in D(B),\ \forall y \in D(A),$$

$$M_0 = \{x \mid x \in D(A), (x, Ax) = \|Bx\|_1^2\}.$$

We introduce several definitions in preparation for development of the structure of $A^{1/2}$. It should be noted that, throughout this paper, the definitions, given specially, all belong to the authors.

DEFINITION 2.1. (B, H_1) is said to be a pseudo-square-root of A if $\overline{M}_0 = H$, where $\overline{M}_0$ is the closure of M_0 in H.

REMARK 2.1. Any nonnegative selfadjoint operator A has at least one pseudo-square-root. For example, $(A^{1/2}, H)$ is a pseudo-square-root of A, where $B = A^{1/2}$ and $H_1 = H$.

EXAMPLE 2.1. Let Ω be a bounded domain of R^n with smooth boundary $\partial\Omega$ of C^2 and $H = L^2(\Omega)$. Consider the following Laplace operator:

$$D(A) = \left\{ u \;\middle|\; u \in H^2(\Omega), \left(\frac{\partial u}{\partial n} + \alpha u \right) \bigg|_{\partial\Omega} = 0 \right\},$$

$$Au = -\Delta u,$$

where $\alpha \geq 0$, $\frac{\partial}{\partial n}$ is the normal derivative, and $\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}$. It is well known that A is an unbounded nonnegative selfadjoint operator on $L^2(\Omega)$.

Set $H_1 = (L^2(\Omega))^n$; H_1 is a product Hilbert space with the inner product

$$\langle \tilde{u}, \tilde{v} \rangle = \sum_{j=1}^{n} (u_j, v_j)_{L^2(\Omega)}, \quad \forall \tilde{u} = (u_1, u_2, \dots, u_n)^\tau,\ \tilde{v} = (v_1, v_2, \dots, v_n)^\tau \in (L^2(\Omega))^n.$$

Define $B : L^2(\Omega) \to (L^2(\Omega))^n$ by

$$D(B) = H^1(\Omega), \qquad Bu = \nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n} \right)^\tau.$$

Conditions (2.1) and (2.2) can easily be verified by using Green's formula, and

$$H_0(\Omega) = \left\{ u \;\middle|\; u \in H^2(\Omega), u|_{\partial\Omega} = \frac{\partial u}{\partial n} \bigg|_{\partial\Omega} = 0 \right\} \subset M_0.$$

Hence $(\nabla, (L^2(\Omega))^n)$ is a pseudo-square-root of A. □

DEFINITION 2.2. Let (B, H_1) be a pseudo-square-root of A. Let H_∂^+ be a Hilbert space with the inner product $(\cdot, \cdot)_\partial^+$ and the induced norm $\|\cdot\|_\partial^+$. If there exist a bounded positive selfadjoint operator $\Upsilon : H_\partial^+ \to H_\partial^+$ and a mapping $\Gamma : D(B) \to H_\partial^+$ such that

$$[x, y] = (\Upsilon\Gamma x, \Gamma y)_\partial^+, \quad \forall x, y \in D(A), \tag{2.3}$$

then we say that $(H_\partial^+, \Upsilon, \Gamma)$ is a positive boundary space of A corresponding to B.

EXAMPLE 2.2. Let H, A, H_1, and B be given by Example 2.1. By Green's formula, we have

$$\begin{aligned} [u, v] &= -\int_{\partial\Omega} u \frac{\partial \overline{v}}{\partial n} d\sigma \\ &= \alpha \int_{\partial\Omega} u\overline{v}\, d\sigma, \quad \forall u \in H^1(\Omega),\ \forall v \in D(A). \end{aligned} \tag{2.4}$$

Set $H_\partial^+ = L^2(\partial\Omega)$ and $\Upsilon = \alpha I$, where I is the identity mapping on $L^2(\partial\Omega)$. Define $\Gamma : H^1(\Omega) \to L^2(\partial\Omega)$ by

$$\Gamma u = u|_{\partial\Omega}, \quad \forall u \in H^1(\Omega).$$

It is easy to verify from (2.4) that $(L^2(\partial\Omega), \alpha I, \Gamma)$ is a positive boundary space of A corresponding to B. □

The following proposition follows from a direct check.

PROPOSITION 2.1. Let (B, H_1) be a pseudo-square-root of A and $(H_\partial^+, \Upsilon, \Gamma)$ a positive boundary space of A corresponding to B. Then

$$|[x,y]| \leq 2\|A^{1/2}x\| \, \|A^{1/2}x\| \, \|A^{1/2}y\|, \quad \forall x, y \in D(A^{1/2}); \tag{2.5}$$

$$|(\Upsilon\Gamma x, \Gamma y)_\partial^+| \leq 2\|A^{1/2}x\| \, \|A^{1/2}y\|, \quad \forall x, y \in D(A). \tag{2.6}$$

It is well known that the spectrum of A consists of eigenvalues:

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \lambda_{n+1} \leq \cdots$$

with $\lim_{n\to\infty} \lambda_n = +\infty$, where the multiple eigenvalues are listed according to their algebraic multiplicities. There exists an orthonormal basis $\{x_n\}$ of H such that

$$Ax_n = \lambda_n x_n, \quad n = 1, 2, \ldots .$$

Let k be the nonnegative integer such that

$$\lambda_j = 0 \quad \text{if } j \leq k; \qquad \lambda_j > 0 \quad \text{if } j \geq k+1.$$

If A is positive, then $k = 0$.

PROPOSITION 2.2. Let (B, H_1) be a pseudo-square-root of A and $(H_\partial^+, \Upsilon, \Gamma)$ a positive boundary space of A corresponding to B. Set

$$g_n = \frac{1}{\sqrt{\lambda_n}} \Gamma x_n, \qquad \psi_n = \frac{1}{\sqrt{\lambda_n}} B x_n, \quad n = k+1, k+2, \ldots . \tag{2.7}$$

Then

$$\left\{ \begin{pmatrix} \Upsilon^{1/2} g_n \\ \psi_n \end{pmatrix} \,\middle|\, n \geq k+1 \right\}$$

is an orthonormal set in the product Hilbert space $H_\partial^+ \times H_1$, where the inner product of $H_\partial^+ \times H_1$ is defined by

$$\left\langle \begin{pmatrix} f \\ \psi \end{pmatrix}, \begin{pmatrix} g \\ \phi \end{pmatrix} \right\rangle = (f, g) + (\psi, \phi)_1, \quad \forall f, g \in H_\partial^+, \ \forall \psi, \phi \in H_1.$$

Proof. For any $n, m \geq k+1$, we have

$$\begin{aligned}
(x_n, x_m) &= \frac{1}{\lambda_m}(x_n, Ax_m) = \frac{1}{\lambda_m}[x_n, x_m] + \frac{1}{\lambda_m}(Bx_n, Bx_m)_1 \\
&= \sqrt{\frac{\lambda_n}{\lambda_m}} [(\Upsilon g_n, g_m)_\partial^+ + (\psi_n, \psi_m)_1] \\
&= \sqrt{\frac{\lambda_n}{\lambda_m}} \left\langle \begin{pmatrix} \Upsilon^{1/2} g_n \\ \psi_n \end{pmatrix}, \begin{pmatrix} \Upsilon^{1/2} g_m \\ \psi_m \end{pmatrix} \right\rangle .
\end{aligned}$$

The orthonormality of $\{x_n\}$ in H yields the orthonormality of

$$\left\{ \begin{pmatrix} \Upsilon^{1/2} g_n \\ \psi_n \end{pmatrix} \,\middle|\, n \geq k+1 \right\}$$

in $H_\partial^+ \times H_1$. $\square$

First, we consider the structure of $D(A^{1/2})$.

THEOREM 2.1. Let (B, H_1) be a pseudo-square-root of A and $(H_\partial^+, \Upsilon, \Gamma)$ a positive boundary space of A corresponding to B. Then

$$D(A^{1/2}) = \{x \mid x \in D(B), [x, y] = (\Upsilon\Gamma x, \Gamma y)_\partial^+,\ \forall y \in D(A)\}.$$

Proof. Let $x \in D(A^{1/2})$. Then there are $\{z_n\}$ in $D(A)$ such that in H

$$z_n \to x \quad \text{and} \quad A^{1/2} z_n \to A^{1/2} x, \quad \text{as } n \to \infty.$$

By Proposition 2.1 and (2.2), we have that $x \in D(B)$ and that

$$\begin{aligned}[x, y] &= \lim_{n\to\infty} [z_n, y] = \lim_{n\to\infty} (\Upsilon\Gamma z_n, \Gamma y)_\partial^+ \\ &= (\Upsilon\Gamma x, y)_\partial^+, \quad \forall y \in D(A).\end{aligned}$$

Conversely, suppose $x \in D(B)$ such that

$$[x, y] = (\Upsilon\Gamma x, y)_\partial^+, \quad \forall y \in D(A). \tag{2.8}$$

Set

$$x = \sum_{j=1}^{\infty} \alpha_n x_n.$$

Then

$$\begin{aligned}\alpha_n &= (x, x_n) = \frac{1}{\lambda_n}(x, Ax_n) \\ &= \frac{1}{\lambda_n}[x, x_n] + \frac{1}{\lambda_n}(Bx, Bx_n)_1 \\ &= \frac{1}{\sqrt{\lambda_n}}\left[x, \frac{1}{\sqrt{\lambda_n}} x_n\right] + \frac{1}{\sqrt{\lambda_n}}(Bx, \psi_n)_1, \quad \forall n \geq k+1,\end{aligned}$$

or equivalently

$$\begin{aligned}\sqrt{\lambda_n}\alpha_n &= \left[x, \frac{1}{\sqrt{\lambda_n}} x_n\right] + (Bx, \psi_n)_1 \\ &= (\Upsilon^{1/2}\Gamma x, \Upsilon^{1/2} g_n)_\partial^+ + (Bx, \psi_n)_1 \\ &= \left\langle \begin{pmatrix} \Upsilon^{1/2}\Gamma x \\ Bx \end{pmatrix}, \begin{pmatrix} \Upsilon^{1/2} g_n \\ \psi_n \end{pmatrix} \right\rangle, \quad \forall n \geq k+1.\end{aligned} \tag{2.9}$$

Thus Proposition 2.2 yields

$$\sum_{n=k+1}^{\infty} |\alpha_n|^2 \lambda_n \leq \left\| \begin{pmatrix} \Upsilon^{1/2}\Gamma x \\ Bx \end{pmatrix} \right\|_{H_\partial^+ \times H_1}^2 < +\infty,$$

so that $x \in D(A^{1/2})$. $\square$

When $M_0 = D(A)$, we can choose $H_\partial^+ = \{0\}$. Thus we have

COROLLARY 2.1. Let (B, H_1) be a pseudo-square-root of A. If $M_0 = D(A)$, then

$$D(A^{1/2}) = \{x \mid x \in D(B), [x, y] = 0, \ \forall y \in M_0\}.$$

EXAMPLE 2.3. Let A be a "string vibration" operator on $L^2(0,1)$, i.e., $Au = -u''(\cdot)$, $u \in D(A)$, such that

$$H_0^2(0,1) \subset D(A). \tag{2.10}$$

Set $H_1 = L^2(0,1)$. The closed linear operator $B : L^2(0,1) \to L^2(0,1)$ is defined by

$$D(B) = H^1(0,1), \qquad Bu = iDu = iu'(\cdot).$$

By (2.10) and integration by parts it is easily checked that $(iD, L^2(0,1))$ is a pseudo-square-root of A.

(i) If A has the following boundary conditions,

$$D(A) = \{u \mid u \in H^2(0,1), u(0) = u(1) = 0\}, \tag{2.11}$$

it is easily verified that $M_0 = D(A)$. For any $u \in H^1(0,1)$, it is clear that u satisfies

$$[u, v] = u(0)\overline{v'(0)} - u(1)\overline{v'(1)} = 0, \quad \forall v \in D(A)$$

if and only if $u(0) = u(1) = 0$. By Corollary 2.1, we have

$$D(A^{1/2}) = \{u \mid u \in H^1(0,1), u(0) = u(1) = 0\} = H_0^1(0,1).$$

(ii) If A has the boundary conditions

$$D(A) = \{u \mid u \in H^2(0,1), u'(0) = u'(1) = 0\}, \tag{2.12}$$

then $M_0 = D(A)$. For any $u \in H^1(0,1)$, it is clear that

$$[u, v] = u(0)\overline{v'(0)} - u(1)\overline{v'(1)} = 0, \quad \forall v \in D(A).$$

Therefore, Corollary 2.1 yields

$$D(A^{1/2}) = H^1(0,1).$$

(iii) If A has the boundary conditions

$$D(A) = \{u \mid u \in H^2(0,1), u(0) = u'(1) = 0\}, \tag{2.13}$$

by a similar process, we can obtain

$$D(A^{1/2}) = \{u \mid u \in H^1(0,1), u(0) = 0\}.$$

(iv) For the boundary conditions

$$D(A) = \{u \mid u \in H^2(0,1), u(0) = u'(0), u(1) = 0\}, \tag{2.14}$$

set $H_\partial^+ = \mathbb{C}$ and $\Upsilon = 1$. Define $\Gamma : H^1(0,1) \to C$ by $\Gamma u = u(0)$. Then $(C, 1, \Gamma)$ is a positive boundary space of A corresponding to iD. Since

$$\begin{aligned} [u, v] &= -u(x)\overline{v'(x)}|_0^1 = u(0)\overline{v'(0)} - u(1)\overline{v'(1)} \\ &= (\Upsilon\Gamma u, \Gamma v)_\partial^+ - u(1)\overline{v'(1)}, \quad \forall u \in H^1(0,1), \ \forall v \in D(A), \end{aligned}$$

we have from Theorem 1.1 that

$$D(A^{1/2}) = \{u \mid u \in H^1(0,1), \ u(1) = 0\}. \qquad \square$$

Now we consider explicit representations of $A^{1/2}$.

PROPOSITION 2.3. Let (B, H_1) be a pseudo-square-root of A and $(H_\partial^+, \Upsilon, \Gamma)$ a positive boundary space of A corresponding to B. Let

$$\mathcal{M}_0 = \overline{\operatorname{span}} \left\{ \begin{pmatrix} \Upsilon^{1/2} g_n \\ \psi_n \end{pmatrix} \,\middle|\, n \geq k+1 \right\}$$

be a closed linear subspace of $H_\partial^+ \times H_1$. Then

$$\mathcal{M}_0 = \left\{ \begin{pmatrix} \Upsilon^{1/2} \Gamma x \\ Bx \end{pmatrix} \,\middle|\, x \in D(A^{1/2}) \right\}.$$

Proof. By Theorem 2.1 and (2.2), we have

$$\begin{aligned}(\|\Upsilon^{1/2}\Gamma x\|_\partial^+)^2 &= [x, x] = (x, Ax) - (Bx, Bx)_1 \\ &\leq 2\|A^{1/2}x\|^2, \quad \forall x \in D(A^{1/2}).\end{aligned} \tag{2.15}$$

We have that $A^{1/2}x_n = 0$, $\forall n$, $1 \leq n \leq k$, since $Ax_n = 0$, for any $1 \leq n \leq k$. Therefore, (2.2) yields

$$Bx_n = 0, \quad \forall n, \ 1 \leq n \leq k. \tag{2.16}$$

By (2.6), it follows that

$$\Gamma x_n = 0, \quad \forall n, \ 1 \leq n \leq k. \tag{2.17}$$

Suppose $x \in D(A^{1/2})$. Set $x = \sum_{n=1}^\infty \alpha_n x_n$. From (2.6) together with (2.7), we have

$$\Upsilon^{1/2}\Gamma x = \sum_{n=1}^\infty \alpha_n \Upsilon^{1/2}\Gamma x_n = \sum_{n=k+1}^\infty \alpha_n \Upsilon^{1/2}\Gamma x_n = \sum_{n=k+1}^\infty \sqrt{\lambda_n}\alpha_n \Upsilon^{1/2} g_n. \tag{2.18}$$

By (2.2) and (2.5), it follows that

$$Bx = \sum_{n=1}^\infty \alpha_n Bx_n = \sum_{n=k+1}^\infty \alpha_n Bx_n = \sum_{n=k+1}^\infty \sqrt{\lambda_n}\alpha_n \psi_n. \tag{2.19}$$

Therefore, (2.18) and (2.19) mean that

$$\begin{pmatrix} \Upsilon^{1/2}\Gamma x \\ Bx \end{pmatrix} = \sum_{n=k+1}^\infty \sqrt{\lambda_n}\alpha_n \begin{pmatrix} \Upsilon^{1/2} g_n \\ \psi_n \end{pmatrix} \in \mathcal{M}_0, \quad \forall x = \sum_{n=1}^\infty \alpha_n x_n \in D(A^{1/2}). \tag{2.20}$$

Conversely, suppose $(g, \psi)^\tau \in \mathcal{M}_0$, where $g \in H_\partial^+$ and $\psi \in H_1$. By Proposition 2.2, we have

$$\begin{pmatrix} g \\ \psi \end{pmatrix} = \sum_{n=k+1}^\infty \beta_n \begin{pmatrix} \Upsilon^{1/2} g_n \\ \psi_n \end{pmatrix},$$

where $\sum_{n=k+1}^\infty |\beta_n|^2 < +\infty$. Setting $x = \sum_{n=k+1}^\infty \frac{\beta_n}{\sqrt{\lambda_n}} x_n$, it is easily checked that $x \in D(A^{1/2})$ and $(g, \psi)^\tau = (\Upsilon^{1/2}\Gamma x, Bx)^\tau$. $\square$

PROPOSITION 2.4. Let (B, H_1) be a pseudo-square-root of A and $(H_\partial^+, \Upsilon, \Gamma)$ a positive boundary space of A corresponding to B. Define $\mathcal{B} : H \to H_\partial^+ \times H_1$ by

$$D(\mathcal{B}) = D(A^{1/2}), \qquad \mathcal{B}x = \begin{pmatrix} \Upsilon^{1/2}\Gamma x \\ Bx \end{pmatrix}. \tag{2.21}$$

Then

$$\mathcal{B} : H \to H_\partial^+ \times H_1$$

is a closed linear operator.

Proof. Suppose that $\{z_n\} \subset D(A^{1/2})$ such that in $H, z_n \to x$, as $n \to \infty$, such that in $H_\partial^+, \Upsilon^{1/2}\Gamma z_n \to g$, as $n \to \infty$, and such that in $H_1, Bz_n \to \psi$, as $n \to \infty$, where $x \in H$, $g \in H_\partial^+$, and $\psi \in H_1$. From the closedness of B it follows that $x \in D(B)$ and $Bx = \psi$. From (2.6) and (2.2), it follows that $\Upsilon^{1/2}\Gamma x = g$. Therefore $\mathcal{B}$ is closed. □

Now we consider the expression of $A^{1/2}$.

THEOREM 2.2. Let (B, H_1) be a pseudo-square-root of A and $(H_\partial^+, \Upsilon, \Gamma)$ a positive boundary space of A corresponding to B. Then there exists a bounded linear operator $\mathcal{Q} : H_\partial^+ \times H_1 \to H$ such that

(i) $\|\mathcal{Q}\| \le 1$, $\mathcal{Q} : \mathcal{M}_0 \to H$ is isometric;

(ii) $A^{1/2}x = \mathcal{Q}\mathcal{B}x$, for any $x \in D(A^{1/2})$.

Proof. For any $(g, \psi)^\tau \in \mathcal{M}_0$, set

$$\begin{pmatrix} g \\ \psi \end{pmatrix} = \sum_{n=k+1}^{\infty} \alpha_n \begin{pmatrix} \Upsilon^{1/2} g_n \\ \psi_n \end{pmatrix}.$$

Define a linear operator $\mathcal{Q} : H_\partial^+ \times H_1 \to H$ as shown below: for $(g, \psi)^\tau \in \mathcal{M}_0$, set

$$\mathcal{Q}\begin{pmatrix} g \\ \psi \end{pmatrix} = \sum_{n=k+1}^{\infty} \alpha_n x_n;$$

in the orthogonal complement of $\mathcal{M}_0$ in $H_\partial^+ \times H_1$, define $\mathcal{Q}$ by the zero operator.

It is easily checked that (i) holds. For any $x \in D(A^{1/2})$, by (2.20) we have

$$\mathcal{Q}\mathcal{B}x = \sum_{n=k+1}^{\infty} \sqrt{\lambda_n}\alpha_n x_n = A^{1/2}x. \qquad \square$$

From the proof of Theorem 2.2, we know that, if $\Gamma x = 0$, $\forall x \in D(A^{1/2})$, then $\mathcal{Q}$ can be defined from H_1 to H. In fact, we have the following result.

THEOREM 2.3. Let (B, H_1) be a pseudo-square-root of A. Set

$$\mathbb{R}_0 = \overline{\operatorname{span}}\{\psi_n \mid n \ge k+1\}.$$

Then $M_0 = D(A)$ if and only if there exists a bounded linear operator $Q : H_1 \to H$ such that

(i) $\|Q\| \le 1$, $Q : \mathbb{R}_0 \to H$ is isometric;

(ii) $A^{1/2}x = QBx$, $\forall x \in D(A^{1/2})$.

Proof. The "only if" part follows from Theorem 2.2.

The "if" part. By (ii) we have that $\sqrt{\lambda_n}x_n = A^{1/2}x_n = QBx_n$, $\forall n \geq k+1$, i.e.,

$$Q\psi_n = x_n, \quad n = k+1, k+2, \dots . \tag{2.22}$$

The isometry of $Q : \mathbb{R}_0 \to H$ yields

$$(\psi_n, \psi_m)_1 = (x_n, x_m), \quad \forall n, m \geq k+1. \tag{2.23}$$

From the orthogonality of $\{x_n\}$ in H we obtain that

$$[x_n, x_m] = \lambda_m \left[(x_n, x_m) - \sqrt{\frac{\lambda_n}{\lambda_m}} (\psi_n, \psi_m)_1 \right] = 0, \quad \forall n, m \geq k+1. \tag{2.24}$$

In addition, we have by a calculation that

$$[x_n, x] = \overline{[x, x_n]} = 0, \quad \forall x \in D(A),\ 1 \leq n \leq k. \tag{2.25}$$

For any $x, y \in D(A)$, $x = \sum_{n=1}^{\infty} \alpha_n x_n$, $y = \sum_{n=1}^{\infty} \beta_n x_n$, we have from (2.24) and (2.25) that

$$[x, y] = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \alpha_n \overline{\beta_m} [x_n, x_m] = 0.$$

Thus $M_0 = D(A)$. $\square$

EXAMPLE 2.4. Let A and B be given by Example 2.3.

(i) Let A have any one of the boundary conditions of (2.11), (3.12), and (2.13). Then $M_0 = D(A)$. By Theorem 2.3, there is a bounded linear operator $Q : L^2(0,1) \to L^2(0,1)$ such that

$$A^{1/2}u = Q(iu'), \quad \forall u \in D(A^{1/2}).$$

Set

$$F(t, s) = \sum_{n=1}^{\infty} (Qx_n)(t)\overline{x_n(s)}, \quad 0 \leq t,\ s \leq 1.$$

Then

$$A^{1/2}u = i \int_0^1 F(t, \xi) u'(\xi)\, d\xi, \quad \forall u \in D(A^{1/2}).$$

(ii) Let the boundary conditions of A be given as (2.14). Then $(\mathbb{C}, 1, \Gamma)$ is a positive boundary space of A corresponding to B, where $\Gamma u = u(0)$, $\forall u \in H^1(0,1)$. By Theorem 2.2, there is a bounded linear operator $\mathcal{Q} : \mathbb{C} \times L^2(0,1) \to L^2(0,1)$ such that

$$A^{1/2}u = \mathcal{Q} \begin{pmatrix} u(0) \\ iu'(\cdot) \end{pmatrix}, \quad \forall u \in D(A^{1/2}) = \{u \mid u \in H^1(0,1), u(1) = 0\}.$$

Let $\{x_n(\cdot)\}$ be the orthonormal basis of $L^2(0,1)$ consisting of the eigenfunctions of A. Set

$$\mathcal{Y}_n(t) = \mathcal{Q} \begin{pmatrix} x_n(0) \\ 0 \end{pmatrix} \in L^2(0,1), \quad n \geq 1,\ 0 \leq t \leq 1,$$

$$\mathcal{Z}_n(t) = \mathcal{Q}\begin{pmatrix} 0 \\ x_n \end{pmatrix} \in L^2(0,1), \quad n \geq 1,\ 0 \leq t \leq 1,$$

$$F_\partial(t,s) = \sum_{n=1}^{\infty} \mathcal{Y}_n(t)\overline{x_n(s)}, \qquad F(t,s) = \sum_{n=1}^{\infty} \mathcal{Z}_n(t)\overline{x_n(s)}, \quad 0 \leq t, s \leq 1.$$

Then we have that

$$\begin{aligned} A^{1/2}u &= \mathcal{Q}\begin{pmatrix} u(0) \\ 0 \end{pmatrix} + \mathcal{Q}\begin{pmatrix} 0 \\ iu'(\cdot) \end{pmatrix} \\ &= \sum_{n=1}^{\infty}(u, x_n)_{L^2(0,1)}\mathcal{Y}_n(t) + \sum_{n=1}^{\infty}(iu', x_n)_{L^2(0,1)}\mathcal{Z}_n(t) \\ &= \int_0^1 F_\partial(t,\xi)u(\xi)d\xi + i\int_0^1 F(t,\xi)u'(\xi)d\xi, \quad \forall u \in D(A^{1/2}). \quad \square \end{aligned} \tag{2.26}$$

Finally, we consider the relationship between $A^{1/2}$ and B from a different point of view. For the elastic "beam" operator A, i.e., $Au = u^{(4)}(\cdot)$, $u \in D(A) \subset H^4(0,1)$, and the assumption that the boundary conditions of A are "symmetric" (see Section 3, (3.3)), D. L. Russell [2] proved that there is a bounded linear operator $P : L^2(0,1) \to L^2(0,1)$ such that

$$PA^{1/2}u = -u''(\cdot), \quad \forall u \in D(A^{1/2}).$$

Here we have the following general result.

THEOREM 2.4. Let (B, H_1) be a pseudo-square-root of A. Then there is a bounded linear operator $P : H \to H_1$ with $\|P\| \leq 1$ such that

$$PA^{1/2}x = Bx, \quad \forall x \in D(A^{1/2}). \tag{2.27}$$

Proof. Define $P : H \to H_1$ by

$$Px = \sum_{n=k+1}^{\infty} \alpha_n\psi_n, \quad \forall x = \sum_{n=1}^{\infty}\alpha_n x_n \in H. \tag{2.28}$$

Let $(H_\partial^+, \Upsilon, \Gamma)$ be a positive boundary space of A corresponding to B. Set

$$\begin{pmatrix} g \\ \psi \end{pmatrix} = \sum_{n=k+1}^{\infty} \alpha_n \begin{pmatrix} \Upsilon^{1/2}g_n \\ \psi_n \end{pmatrix}, \quad \forall x = \sum_{n=1}^{\infty}\alpha_n x_n \in H.$$

Since $\sum_{n=1}^{\infty}|\alpha_n|^2 < +\infty$ we know that $(g, \psi)^\tau \in H_\partial^+ \times H_1$. Hence

$$Px = \psi \in H_1, \quad \forall x \in H,$$

and

$$\|Px\|_1^2 \leq \|g\|_{\partial_+}^2 + \|\psi\|_1^2 = \sum_{n=k+1}^{\infty} |\alpha_n|^2 \leq \|x\|^2, \quad \forall x = \sum_{n=1}^{\infty}\alpha_n x_n \in H,$$

that is, $\|P\| \leq 1$. Furthermore, for any $x = \sum_{n=1}^{\infty} \alpha_n x_n \in D(A^{1/2})$, we have

$$PA^{1/2}x = \sum_{n=k+1}^{\infty} \sqrt{\lambda_n}\alpha_n P x_n = Bx. \quad \square$$

By Theorem 2.4, if $P : H \to \mathbb{R}_0$ has a bounded inverse $P^{-1} : \mathbb{R}_0 \to H$, then $A^{1/2}$ can be written as $A^{1/2} = P^{-1}B$, where $P^{-1} : \mathbb{R}_0 \to H$ is not isometric in general. In particular, if $\{\psi_n | n \geq k+1\}$ is an orthogonal set in H_1 (which is not true in general), then the above assertion holds. In the following we give a necessary condition on the orthogonality of $\{\psi_n | n \geq k+1\}$ if $\dim(H_\partial^+) < +\infty$ by which an error in [2] can be corrected (see Sec. 3).

PROPOSITION 2.5. Let (B, H_1) be a pseudo-square-root of A and $(H_\partial^+, \Upsilon, \Gamma)$ a positive boundary space of A corresponding to B with $\dim(H_\partial^+) < \infty$. If $\{\psi_n | n \geq k+1\}$ is an orthogonal set in H_1, then there is a positive integer n_0 such that

$$[x_n, x_n] = 0, \quad \forall n \geq n_0. \tag{2.29}$$

Proof. Suppose $\dim(H_\partial^+) = m$. By an argument similar to the proof of Proposition 2.2, we have

$$(x_n, x_j) = \sqrt{\frac{\lambda_n}{\lambda_j}} \left[(\Upsilon^{1/2} g_n, \Upsilon^{1/2} g_j)_\partial^+ + (\psi_n, \psi_j)_1 \right], \quad \forall n, j \geq k+1. \tag{2.30}$$

Thus $\{\Upsilon^{1/2} g_n | n \geq k+1\}$ is an orthogonal set in H_∂^+. Since $\dim(H_\partial^+) = m$, there are at most m nonzero elements in $\{\Upsilon^{1/2} g_n | n \geq k+1\}$. Therefore, there exists a positive integer $n_0 \geq k+1$ such that

$$\Upsilon^{1/2} g_n = 0, \quad \forall n \geq n_0.$$

From (2.24), we have

$$[x_n, x_n] = \lambda_n [(x_n, x_n) - (\psi_n, \psi_n)_1] = 0, \quad \forall n \geq n_0. \quad \square$$

3. Applications to elastic "beam" operators. Let $H = L^2(0,1)$. In this section, we shall consider elastic "beam" operators defined by

$$Au = u^{(4)}(\cdot), \quad u \in D(A) \subset H^4(0,1). \tag{3.1}$$

Since

$$(Au, u)_{L^2(0,1)} = u'''\overline{u}|_0^1 - u''\overline{u'}|_0^1 + \int_0^1 |u''|^2 dt, \quad \forall u \in D(A),$$

it is supposed that

$$u'''\overline{u}|_0^1 - u''\overline{u'}|_0^1 \geq 0, \quad \forall u \in D(A). \tag{3.2}$$

Set $H_1 = L^2(0,1)$. Define $B : L^2(0,1) \to L^2(0,1)$ by

$$D(B) = H^2(0,1), \qquad Bu = -D^2 u = -u''(\cdot).$$

The following proposition can be easily verified.

PROPOSITION 3.1. Let A be a nonnegative selfadjoint operator that satisfies (3.1) and (3.2). Then $(-D^2, L^2(0,1))$ is a pseudo-square-root of A.

First, we introduce a main result given in [2]. Consider the real Hilbert space $L^2(0,1)$. Set

$$B(u,v) = u'''(\cdot)v(\cdot) - u''(\cdot)v' + u'(\cdot)v''(\cdot) - u(\cdot)v'''(\cdot), \quad \forall u, v \in D(A);$$
$$\psi_n(\cdot) = -(\sqrt{\lambda_n})^{-1} D^2 x_n(\cdot), \quad n \geq k+1,$$

where λ_n, x_n, and k are the same as those in Sec. 2. D. L. Russell in [2] has shown the following result.

Let A be a nonnegative selfadjoint operator on $L^2(0,1)$ with the compact resolvent such that (3.1) and (3.2) hold. Suppose that

$$B(u,v) = 0, \quad \forall u, v \in D(A), \tag{3.3}$$

at $x = 0$ and at $x = 1$. Then

(i) there is a bounded linear operator $P : L^2(0,1) \to L^2(0,1)$ such that

$$PA^{1/2}u = -D^2 u, \quad \forall u \in D(A^{1/2}). \tag{3.4}$$

(ii) Set $\mathbb{R}_0 = \overline{\operatorname{span}}\{\psi_n | n \geq k+1\}$. Then P has a bounded inverse on $\mathbb{R}_0$, which extends to a bounded linear operator Q on $L^2(0,1)$, so that likewise

$$A^{1/2}u = Q(-D^2)u, \quad \forall u \in D(A^{1/2}). \quad \square$$

From the proof of the above assertion (ii) in [2, pp. 761–765] we know that the assertion (ii) is based on the assertion that $\{\psi_n | n \geq k+1\}$ is an orthogonal set in $L^2(0,1)$. Unfortunately, Proposition 3.2 below shows that in general $\{\psi_n | n \geq k+1\}$ is not an orthogonal set in $L^2(0,1)$ under the assumption (3.3) only. Therefore, the proof of the assertion (ii) given in [2] failed. By Theorem 2.2 and Theorem 2.3, the assertion (ii) can be revised (see Example 3.1 below).

EXAMPLE 3.1. Let us consider a cantilever with elastic forces applying at the free end, $x = 1$. The boundary conditions become

$$u(0) = u'(0) = 0, \quad u'''(1) - \alpha u(1) = 0, \quad u''(1) + \beta u'(1) = 0, \tag{3.5}$$

where $\alpha > 0$, $\beta > 0$.

Set $H_\partial^+ = \mathbb{C}^2$. Define $\Upsilon : \mathbb{C}^2 \to \mathbb{C}^2$ by

$$\Upsilon = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

Define $\Gamma : H^2(0,1) \to \mathbb{C}^2$ by

$$\Gamma u = \begin{pmatrix} u(1) \\ u'(1) \end{pmatrix}, \quad \forall u \in H^2(0,1).$$

It can easily be verified that

$$[u,v] = \alpha u(1)\overline{v(1)} + \beta u'(1)\overline{v'(1)}, \quad \forall u,v \in D(A). \tag{3.6}$$

Then $(\mathbb{C}^2, \Upsilon, \Gamma)$ is a positive boundary space of A corresponding to $-D^2$. Since

$$\begin{aligned}[u,v] &= \alpha u(1)\overline{v(1)} + \beta u'(1)\overline{v'(1)} - u(0)\overline{v'''(0)} + u'(0)\overline{v''(0)} \\ &= (\Upsilon\Gamma u, \Gamma v)_{\mathbb{C}^2} - u(0)\overline{v'''(0)} + u'(0)\overline{v''(0)}, \quad \forall u \in H^2(0,1),\ v \in D(A),\end{aligned}$$

from Theorem 2.1, we have

$$\begin{aligned} D(A^{1/2}) &= \{u \mid u \in H^2(0,1), [u,v] = (\Upsilon\Gamma u, \Gamma v)_{\mathbb{C}^2},\ \forall v \in D(A)\} \\ &= \{u \mid u \in H^2(0,1), u(0) = u'(0) = 0\}.\end{aligned}$$

By Theorem 2.2, there is a bounded linear operator $\mathcal{Q} : \mathbb{C}^2 \times L^2(0,1) \to L^2(0,1)$ such that

$$A^{1/2}u = \mathcal{Q}\begin{pmatrix} \sqrt{\alpha}u(1) \\ \sqrt{\beta}u'(1) \\ -u''(\cdot) \end{pmatrix}, \quad \forall u \in D(A^{1/2}).$$

By an argument similar to Example 2.4, we can find $F_{\partial_1}(t,\xi)$, $F_{\partial_2}(t,\xi)$, and $F(t,\xi)$, for $0 \le t, \xi \le 1$, such that

$$A^{1/2}u = \sqrt{\alpha}\int_0^1 F_{\partial_1}(t,\xi)u(\xi)d\xi + \sqrt{\beta}\int_0^1 F_{\partial_2}(t,\xi)u'(\xi)d\xi - \int_0^t F(t,\xi)u''(\xi)d\xi,$$
$$\forall u \in D(A^{1/2}). \quad \square$$

PROPOSITION 3.2. Let A be the "beam" operator on $L^2(0,1)$ with the boundary conditions given by (3.5). Then (3.3) holds but $\{\psi_n | n \ge k+1\}$ is not an orthogonal set in $L^2(0,1)$, where $\psi_n = -(\sqrt{\lambda_n})^{-1}x_n''(\cdot)$, λ_n is the eigenvalue of A, and $x_n(\cdot)$ is the eigenfunction of A corresponding to λ_n, for all $n \ge 1$.

Proof. It is easily checked that (3.3) holds.

Suppose that $\{\psi_n | n \ge k+1\}$ is an orthogonal set in $L^2(0,1)$. By Proposition 2.5, there is a positive integer n_0 such that

$$[x_n, x_n] = \alpha|x_n(1)|^2 + \beta|x_n'(1)|^2 = 0, \quad \forall n \ge n_0.$$

Therefore, x_n is a solution of the following boundary value problem:

$$\begin{cases} x_n^{(4)}(t) = \lambda_n x_n(t), \quad 0 < t < 1, \\ x_n(0) = x_n'(0) = x_n(1) = x_n'(1) = x_n''(1) = x_n'''(1) = 0, \end{cases} \tag{3.7}$$

for all $n \ge n_0$. It is obvious that the problem (3.7) has the unique zero solution, for all $n \ge n_0$. Thus

$$x_n = 0, \quad \forall n \ge n_0.$$

This is a contradiction. $\square$

REMARK 3.1. It should be noted that there is a modified inner product, as shown in Proposition 2.2, relative to which ψ_n may be considered orthogonal. Thus, the statement made in [2] is correct if restricted to "SDE" boundary conditions (strictly distributed energy [2]). Without the SDE assumption, the potential energy form associated with the operator A includes some boundary terms.

Finally, we conclude this section by giving an example derived from the pointwise control of a flexible manipulator arm [6].

EXAMPLE 3.2. Let $H = \mathbb{C}^3 \times L^2(0,1)$ with the inner product

$$(\Phi_1, \Phi_2) = \alpha_1\overline{\alpha_2} + \mu_1\overline{\mu_2} + \xi_1\overline{\xi_2} + \int_0^1 \varphi_1\overline{\varphi_2}dt$$

where $\Phi_j = (\alpha_j, \mu_j, \xi_j, \varphi_j)^\tau \in \mathbb{C}^3 \times L^2(0,1)$. Define the linear operator A by

$$A\tilde{\varphi} = \begin{pmatrix} -\varphi''(0) \\ -\varphi'''(1) \\ \varphi''(1) \\ \varphi^{(4)}(\cdot) \end{pmatrix}, \qquad \tilde{\varphi} = \begin{pmatrix} \varphi'(0) \\ \varphi(1) \\ \varphi'(1) \\ \varphi(\cdot) \end{pmatrix},$$

$$D(A) = \{\tilde{\varphi} = (\varphi'(0), \varphi(1), \varphi'(1), \varphi(\cdot))^\tau \mid \varphi \in H^4(0,1), \varphi(0) = 0\}.$$

It is easily checked that A is a nonnegative selfadjoint operator on $\mathbb{C}^3 \times L^2(0,1)$.

Set $H_1 = L^2(0,1)$. Define $B : \mathbb{C}^3 \times L^2(0,1) \to L^2(0,1)$ by

$$B\tilde{\varphi} = -D^2\varphi = -\varphi''(\cdot), \qquad \tilde{\varphi} = \begin{pmatrix} \varphi'(0) \\ \varphi(1) \\ \varphi'(1) \\ \varphi(\cdot) \end{pmatrix},$$

$$D(B) = \{\tilde{\varphi} = (\varphi'(0), \varphi(1), \varphi'(1), \varphi(\cdot))^\tau \mid \varphi \in H^2(0,1)\}.$$

It is easily checked that $(B, L^2(0,1))$ is a pseudo-square-root of A. Since

$$[\tilde{\psi}, \tilde{\varphi}] = 0, \quad \forall \tilde{\psi}, \tilde{\varphi} \in D(A),$$

then $M_0 = D(A)$.Since

$$[\tilde{\psi}, \tilde{\varphi}] = -\psi(0)\overline{\varphi'''(0)}, \quad \forall \tilde{\psi} \in D(B), \tilde{\varphi} \in D(A),$$

by Corollary 2.1, we have

$$\begin{aligned} D(A^{1/2}) &= \{\tilde{\psi} \mid \tilde{\psi} \in D(B), [\tilde{\psi}, \tilde{\varphi}] = 0, \forall \tilde{\varphi} \in D(A)\} \\ &= \{\tilde{\psi} = (\psi'(0), \psi(1), \psi'(1), \psi(\cdot))^\tau \mid \psi \in H^2(0,1), \psi(0) = 0\}. \end{aligned}$$

Furthermore, by Theorem 2.3, there is a bounded linear operator $Q : L^2(0,1) \to \mathbb{C}^3 \times L^2(0,1)$ such that

$$A^{1/2}\tilde{\psi} = -Q(\psi''), \quad \forall \tilde{\psi} = (\psi'(0), \psi(1), \psi'(1), \psi(\cdot))^\tau \in D(A^{1/2}).$$

Since $Q : L^2(0,1) \to \mathbb{C}^3 \times L^2(0,1)$ is bounded, Q has the form

$$Qu = \begin{pmatrix} g_1(u) \\ g_2(u) \\ g_3(u) \\ Q_0 u \end{pmatrix}, \quad \forall u \in L^2(0,1),$$

where $g_j(\cdot)$ are bounded linear functionals, for $j = 1, 2, 3$, and $Q_0 : L^2(0,1) \to L^2(0,1)$ is a bounded linear operator. By the Riesz theorem and an argument similar to Example 2.4, we obtain $f_j \in L^2(0,1)$ and $F(t,s)$, $j = 1, 2, 3$, $0 \le t, s \le 1$, such that

$$A^{1/2}\tilde{\varphi} = \begin{pmatrix} -\int_0^1 f_1(\xi)\varphi''(\xi)d\xi \\ -\int_0^1 f_2(\xi)\varphi''(\xi)d\xi \\ -\int_0^1 f_3(\xi)\varphi''(\xi)d\xi \\ -\int_0^1 F(\cdot,\xi)\varphi''(\xi)d\xi \end{pmatrix}, \quad \forall \tilde{\varphi} = (\varphi'(0), \varphi(1), \varphi'(1), \varphi(\cdot))^\tau \in D(A^{1/2}). \quad \square$$

4. Applications to n-dimensional "wave" operators. In this section we shall apply the results given in previous sections to a variety of boundary value problems of n-dimensional "wave" operators. To our knowledge, there presently is no available method for readily calculating the square root of high-dimensional Laplace operators. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ of class C^2 and $\frac{\partial}{\partial n}$ the normal derivative.

EXAMPLE 4.1. Let $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. Consider the Laplace operator on $L^2(\Omega)$

$$D(A) = \left\{ u \;\middle|\; u \in H^2(\Omega), u|_{\Gamma_1} = 0, \left.\frac{\partial u}{\partial n}\right|_{\Gamma_2} = 0, \left.\left(\frac{\partial u}{\partial n} + \alpha u\right)\right|_{\Gamma_3} = 0 \right\},$$

$$Au = -\Delta u, \tag{4.1}$$

where $\alpha > 0$.

Set $H_1 = (L^2(\Omega))^n$. Define the closed linear operator $B : L^2(\Omega) \to (L^2(\Omega))^n$ by

$$D(B) = H^1(\Omega), \qquad Bu = \nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n} \right)^\tau .$$

It is easily checked that $(\nabla, (L^2(\Omega))^n)$ is a pseudo-square-root of A. By Green's formula, we have

$$\begin{aligned} [u, v] &= -\int_{\partial\Omega} u \frac{\partial \overline{v}}{\partial n}\, d\sigma \\ &= -\int_{\Gamma_1} u \frac{\partial \overline{v}}{\partial n}\, d\sigma + \alpha \int_{\Gamma_3} u\overline{v}\, d\sigma, \quad \forall u \in H^1(\Omega),\ \forall v \in D(A). \end{aligned} \tag{4.2}$$

Set $H_\partial^+ = L^2(\Gamma_3)$ and $\Upsilon = \alpha I$, where I is the identity mapping on $L^2(\Gamma_3)$. Define $\Gamma : H^1(\Omega) \to L^2(\Gamma_3)$ by

$$\Gamma u = u|_{\Gamma_3}, \quad \forall u \in H^1(\Omega).$$

By (4.2), we have

$$[u, v] = \alpha \int_{\Gamma_3} u\overline{v}\, d\sigma = (\Upsilon \Gamma u, \Gamma v)_{L^2(\Gamma_3)}, \quad \forall u, v \in D(A).$$

Thus $(L^2(\Gamma_3), \alpha I, \Gamma)$ is a positive boundary space of A corresponding to ∇. From Theorem 2.1 and (4.2), we have

$$D(A^{1/2}) = \{u \mid u \in H^1(\Omega), u|_{\Gamma_1} = 0\}. \tag{4.3}$$

By Theorem 2.2, there is a bounded linear operator $\mathcal{Q} : L^2(\Gamma_3) \times (L^2(\Omega))^n \to L^2(\Omega)$ such that

$$A^{1/2} u = \mathcal{Q} \begin{pmatrix} \sqrt{\alpha} u|_{\Gamma_3} \\ \nabla u \end{pmatrix}, \quad \forall u \in D(A^{1/2}). \tag{4.4}$$

Suppose that $\{x_j(\zeta)\}$ is the orthonormal basis of $L^2(\Omega)$ consisting of the eigenfunctions of A and $e_j = (0, \ldots, 1, \ldots, 0)^\tau \in \mathbb{R}^{n+1}$, $j = 0, 1, \ldots, n$. Setting

$$y_j(\zeta) = \mathcal{Q}(x_j|_{\Gamma_3} e_0) \in L^2(\Omega), \quad Z_{j,m}(\zeta) = \mathcal{Q}(x_j e_m) \in L^2(\Omega), \quad j \geq 1,\ 1 \leq m \leq n,\ \zeta \in \Omega,$$

$$F_\partial(\zeta,\xi)=\sum_{j=1}^{\infty}y_j(\zeta)\overline{x_j(\xi)},\qquad F_m(\zeta,\xi)=\sum_{j=1}^{\infty}Z_{j,m}(\zeta)x_j(\xi),\quad \forall\zeta,\ \xi\in\Omega,\ 1\le m\le n.$$

We finally obtain

$$A^{1/2}u=\sqrt{\alpha}\int_\Omega F_\partial(\zeta,\xi)u(\xi)\,d\xi+\sum_{m=1}^{n}\int_\Omega F_m(\zeta,\xi)\frac{\partial}{\partial x_m}u(\xi)\,d\xi,\quad \forall u\in D(A^{1/2}).\quad\square$$

EXAMPLE 4.2. Let A be the Laplace operator with the domain

$$D(A)=\left\{u\ \middle|\ u\in H^2(\Omega),\int_{\partial\Omega}u\,d\sigma=0,\frac{\partial u}{\partial n}=\frac{1}{|\partial\Omega|}\int_{\partial\Omega}\Delta u\,d\sigma\right\},$$

where $|\partial\Omega|$ is the measure of $\partial\Omega$ in $\mathbb{R}^{n-1}$.

Let $(\nabla,(L^2(\Omega))^n)$ be given by Example 4.1. Then $(\nabla,(L^2(\Omega))^n)$ is a pseudo-square-root of A. Since

$$[u,v]=-\int_\partial u\frac{\partial\overline{v}}{\partial n}d\sigma=\frac{-1}{|\partial\Omega|}\int_{\partial\Omega}\Delta\overline{v}\,d\sigma\int_\partial u\,d\sigma=0,\quad \forall u,v\in D(A),$$

then $M_0=D(A)$. By Corollary 2.1, we have that

$$\begin{aligned}D(A^{1/2})&=\{u\mid u\in H^1(\Omega),[u,v]=0,\ \forall v\in D(A)\}\\&=\{u\mid u\in H^1(\Omega),\int_{\partial\Omega}u\,d\sigma=0\}.\end{aligned}\tag{4.5}$$

By Theorem 2.3, there exists a bounded linear operator $Q:(L^2(\Omega))^n\to L^2(\Omega)$ such that

$$A^{1/2}u=Q(\nabla u),\quad u\in D(A^{1/2}).$$

By an argument similar to Example 4.1, there exist $F_m(\zeta,\xi)$, for $m=1,2,\dots,n$ such that

$$A^{1/2}u=\sum_{m=1}^{n}\int_\Omega F_m(\zeta,\xi)\frac{\partial u(\xi)}{\partial x_m}d\xi,\quad \forall u\in D(A^{1/2}).\quad\square\tag{4.6}$$

EXAMPLE 4.3. Let A be the Laplace operator with the domain

$$D(A)=\{u\mid u\in H^2(\Omega),\int_{\partial\Omega}\frac{\partial u}{\partial n}d\sigma=0,u|_{\partial\Omega}=\text{constant}\}.$$

Then $(\nabla,(L^2(\Omega))^n)$ is a pseudo-square-root of A.

Since

$$[u,v]=-\int_{\partial\Omega}u\frac{\partial\overline{v}}{\partial n}d\sigma=-u|_{\partial\Omega}\int_{\partial\Omega}\frac{\partial\overline{v}}{\partial n}d\sigma=0,\quad \forall u,v\in D(A),$$

we know that $M_0=D(A)$. Suppose $u\in H^1(\Omega)$ such that

$$[u,v]=-\int_{\partial\Omega}u\frac{\partial\overline{v}}{\partial n}d\sigma=0,\quad \forall v\in D(A).$$

It is easily checked by the trace operator theorem that

$$\int_{\partial\Omega}ug\,d\sigma=0,\quad \forall g\in L^2(\partial\Omega),\quad \int_{\partial\Omega}g\,d\sigma=0.$$

Thus $u|_{\partial\Omega}=$ constant. Therefore, by Corollary 2.1, we have

$$\begin{aligned}D(A^{1/2})&=\{u\mid u\in H^1(\Omega),[u,v]=0,\ \forall v\in D(A)\}\\&=\{u\mid u\in H^1(\Omega),u|_{\partial\Omega}=\text{constant}\}.\end{aligned}$$

In addition, by Theorem 2.3, $A^{1/2}$ has the form of (4.6). $\square$

EXAMPLE 4.4. Let

$$a(\zeta) = (a_{ij}(\zeta))_{n\times n}, \quad a_{ij} \in C^1(\overline{\Omega}),\ 1 \le i,j \le n,$$

be a symmetric matrix on $\mathbb{C}^n$ and suppose that there is $\delta > 0$ such that

$$\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}(\zeta)\mu_i\overline{\mu_j} \ge \delta|\mu|^2, \quad \forall \mu = (\mu_1, \mu_2, \dots, \mu_n)^{\tau} \in \mathbb{C}^n,\ \zeta \in \overline{\Omega}.$$

Define A by

$$D(A) = \{u \mid u \in H^2(\Omega), u|_{\partial\Omega} = 0\},$$
$$Au = -\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(\sum_{i=1}^{n} a_{ij}(\zeta)\frac{\partial u}{\partial x_i}\right).$$

It is easily checked that A is a nonnegative selfadjoint operator on $L^2(\Omega)$.

Set $H_1 = (L^2(\Omega))^n$. Define $B : L^2(\Omega) \to (L^2(\Omega))^n$ by

$$D(B) = H^1(\Omega), \qquad Bu = (a(\zeta))^{1/2}\nabla u.$$

It is easily checked that $((a(\zeta))^{1/2}\nabla, (L^2(\Omega))^n)$ is a pseudo-square-root of A. By Green's formula, we have

$$[u,v] = -\int_{\partial\Omega} u \sum_{j=1}^{n}\left(\sum_{i=1}^{n}\overline{a_{ij}}\frac{\partial\overline{v}}{\partial x_i}\right) n_j\, d\sigma = 0, \quad \forall u, v \in D(A),$$

where $n = (n_1, n_2, \dots, n_n)$ is the unit normal direction of $\partial\Omega$. Thus $M_0 = D(A)$. Hence

$$D(A^{1/2}) = \{u \mid u \in H^1(\Omega), [u,v] = 0,\ \forall v \in D(A)\}$$
$$= \{u \mid u \in H^1(\Omega), u|_{\partial\Omega} = 0\}.$$

By Theorem 2.3, there is a bounded linear operator $Q : (L^2(\Omega))^n \to L^2(\Omega)$ such that

$$A^{1/2}u = Q((a(\zeta))^{1/2}\nabla u), \quad \forall u \in D(A^{1/2}).$$

By an argument similar to Example 4.2, $A^{1/2}$ also has a form of (4.6).

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